ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ
ПО МАТЕМАТИКА И ИНФОРМАТИКА

КНИГА 2 — МЕХАНИКА

Том 83
1989

ANNUAIRE

DE

L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

LIVRE 2 — MECANIQUE

Tome 83
1989

СОФИЯ. 1992

SOFIA. 1992

ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ
"СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ
ПО МАТЕМАТИКА И ИНФОРМАТИКА

КНИГА 2 — МЕХАНИКА

Том 83
1989

ANNUAIRE

DE

L'UNIVERSITE DE SOFIA
"ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

LIVRE 2 — MECANIQUE

Tome 83
1989

СОФИЯ, 1992, SOFIA
УНИВЕРСИТЕТСКО ИЗДАТЕЛСТВО "СВ. КЛИМЕНТ ОХРИДСКИ"
PRESSES UNIVERSITAIRES "ST. KLIMENT OHRIDSKI"

51+531/534(05)

СОДЕРЖАНИЕ

CONTENTS

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 83, 1989

APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. I. THERMAL LOADING

KLAUS HERRMANN, IVAN MIHOVSKY

Клаус Херман, Иван Миховски. ПРИБЛИЖЕННОЕ АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ УПРУГОПЛАСТИЧЕСКОГО ПОВЕДЕНИЯ ВОЛОКНИСТЫХ КОМПОЗИТОВ. I. ТЕРМИЧЕСКОЕ НАГРУЖЕНИЕ.

Предложена математико-механическая модель упругопластического поведения класса волокнистых композитов с пластической матрицей и параллельными упругими волокнами с низким объемным содержанием последних. Наряду с качественными заключениями относительно механизмов пластифицирования матрицы получен ряд количественных оценок поведения композитов в условиях термического и механического нагружения (части I и II соответственно).

Klaus Herrmann, Ivan Mihovsky. APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. PART I. THERMAL LOADING.

A mechano-mathematical model of the elastic-plastic response of a class of fibrous composites is proposed. It concerns low fibre volume fraction composites with a ductile matrix and parallel elastic fibres. Along with the qualitative-conclusions about the mechanisms of matrix plastification a series of quantitative results is derived as well, concerning the composites response under thermal and mechanical loading conditions (Parts I and II, respectively).

INTRODUCTION

Reinforcement of complaint materials by parallelly aligned continuous strong fibres provides an essential increase in their strength and stiffness and makes the fibrous composites thus obtained attractive for various load-bearing applications. On the other hand such applications involve, as a rule, high fracture resistance requirements. Fibrous composites with ductile matrices prove to satisfy these requirements sufficiently well.

Thus, matrix plasticity appears to be a desired property of the composites. It reduces their sensitivity to a variety of typical structural defects which are either introduced by the fabrication processes or created artificially. The plasticity of the matrix material improves the resistance of the composites to initiation of modes of local fracture, associated with the stress concentration effects due to such structural defects. At the same time, matrix plasticity is known to change essentially the overall thermomechanical response of the composites and, in particular, to reduce considerably their overall strength. In other words, matrix plasticity leads to an overall behaviour of the composite material and to the development of modes of failure, which are much less sensitive to the local structural defects. Therefore, this phenomenon should be considered to be due to the very nature of the plastic deformation process developing within the matrix phase. The mechanisms, involved in this process, change the entire pattern of fibre-matrix interactions and, correspondingly, the basic features of the phenomena of load transfer and distribution, respectively, developing within the composite structures. Thus, it is of definite interest to clear up the nature of these mechanisms and in addition the trends in their development, their dependence on the structural parameters and the loading status of the composites, and accordingly their influence on the overall thermomechanical response of the latter. An attempt in this regard is made in the present study which concerns also the associated questions of how these mechanisms affect the failure phenomena in the composites, and how and to what extent they reduce their sensitivity to the typical structural defects.

A general approach to the problem is developed and an approximate analytical version of this approach is realized. The approach concerns the class of unidirectionally fibre reinforced composites of relatively low fibre volume fraction and with continuous strong elastic fibres perfectly bonded to a matrix of a weaker ductile material. Furthermore, the class of thermal and mechanical loading conditions is considered under which axisymmetric stress-strain states develop within a composite unit cell consisting of a circular cylindrical fibre with a coaxial cylindrical matrix coating. Numerous aspects of the basic problem considered in the following have been already successfully studied, for example, in the works of Hill [1], Spencer [2], Mulhern et al. [3], Ebert et al. [4], Thomason [5], Dvorak & Rao [6], Strife & Prevo [7], Min [8], Morley [9]. It should be immediately underlined that these references exhaust by no means the large list of publications on the problem but, at the same time, the present study aims neither at describing the state of the arts nor at reviewing the existing literature. Reference is made to these articles since they, even in such a restricted amount, clearly indicate how different the approaches to the problem may be and, in addition, how this variety of approaches is derivable from practically the same adoptions about the composite structures as well as by means of the same basic concepts of the plasticity theory. The distinguishing features of these approaches concern, in fact, the ways in which they account for (or neglect) the specific effects of the continuous fibre reinforcement, namely the strengthening (including the stiffening), the stress concentration, and the shrinkage effect. From the point of view of this distinguishing criterion one may specify the approach below as an attempt for a more rigorous account for each of these effects as well as for the simultaneous account of all of them. The remaining adoptions and concepts involved in the analysis do not differ in their nature from these of the works just cited.

In essence, the approach itself is a direct further development of the matrix plastification model previously proposed by the authors in [10, 11]. This basically qualitative model has proved to imply a series of useful conclusions concerning, for example, the development of the matrix plastification process (existence of a maximum plastic zone size), the mechanisms and the modes of failure of the composites (plastic instability of the matrix), and, in addition, the fibre-matrix cracks interactions phenomena (applicability of the Dugdale crack model, cf. [11, 12]). The development of the model in the present study leads to further conclusions concerning both the qualitative and the quantitative aspects of the considered problem. When coupled with appropriate numerical methods the general approach allows to achieve an improved accuracy of the results as well as an enlargement of the classes of the considered composite structures and loading conditions without principal changes in the structure of the governing equations. At the same time the object of the present investigation is not to deliver quantitative estimations of high accuracy but rather to bring a sufficient understanding of the very nature of the processes of matrix plastification and of their influence on the overall response of the composites. To clear up these questions is the principal aim and to this respect the general approach proves to be an effective tool even in its simplified approximate analytical version. The latter simulates adequately enough the specific features and trends of development of the matrix plastification process. The analysis predicts an overall response which is consistent with the commonly adopted understanding of the composites behaviour in the "rule of mixtures" sense.

Two model problems are considered in detail. These are the problems of matrix cooling (a simplified version of the cooling of the entire composite structure) and longitudinal extension. They simulate loading conditions which are typically involved in the processes of fabrication of the composites (thermal treatment) and in their load-bearing applications, respectively. The study is divided into two parts. This is due to the fact that the general approach reveals quite different specific patterns of the elastic-plastic response of the composites when applied to each of the two model problems considered. Each of these patterns proves to deserve due attention from the point of view of the corresponding analysis, its predictions, and the practical applications of the latter. The first part of the study deals with the thermally induced elastic-plastic behaviour of the considered class of fibrous composites.

STATEMENT OF THE PROBLEM

The class of composites and the composite unit cell, considered in the following, are as specified in the introduction. When referred to a cylindrical coordinate system $\{r, \theta, z\}$, where the z-axis coincides with the axis of the fibre, the cross-sections of the fibre and the matrix occupy the regions $\{0 \leqq r \leqq r_f, 0 \leqq \theta \leqq 2\pi\}$ and $\{r_f < r \leqq r_m, 0 \leqq \theta \leqq 2\pi\}$, respectively.

The fibre material is linearly elastic with Young's modulus E_f, Poisson's ratio ν_f, and linear thermal expansion coefficient α_f. The material of the matrix is elastic (E_m, ν_m, α_m) — perfectly plastic and obeys the von Mises yield condition. The thermoelastic properties of the fibre and the matrix as well as the tensile yield stress σ_y of the latter are considered as temperature independent.

The thermal loading is specified as matrix cooling, that means as a process of monotonous quasi- static decrease of the itself negative matrix temperature T_m, which is measured from the temperature of the initially unstressed state of the composite. The same scheme of loading has been considered in [11]. The generelization of the analysis of this model scheme with respect to the process of cooling of the entire cell, which is practically always involved in the fabrication of the composites, as well as to other more realistic modes of thermal loading is almost straightforward. No external loads are applied to the cell. Thus, the corresponding thermally induced stress-strain state of the cell is axisymmetric and, due to the assumed perfect fibre-matrix bond, allows to be treated by applying the plane cross-sections hypothesis. Correspondingly, the normal stresses in both the fibre and matrix phases are principal ones and depend upon the radial coordinate only.

THE MATRIX PLASTIFICATION MODEL

It was already mentioned that the analysis in the present investigation is based upon the matrix plastification model, developed in previous works of the authors [10, 11]. Thus, a brief general description of the model and of the associated basic concepts would be usefull both for the better understanding of the analysis and for its concise presentation. As it should be expected, the basic concepts of the model concern the principal features of the considered composites and, firstly, the main effect of the fibre reinforcement, namely the strengthening one. In fact, due to the associated decrease of the compliance of the composites connected with this effect, the longitudinal strains ε_z in the latters remain relatively small, i.e. comparable with the themselves small purely elastic strains in the stiff fibres. Then the elastic ε_z^e and the plastic ε_z^p-components of the itself small total ε_z-strain in the plastified matrix region are also small enough for a comparison, using relations like "much larger" or "negligibly small". Accordingly, the model states first of all that by considering the matrix plastification process one should permanently account for the current ε_z^e-strain instead of neglecting it with respect to the ε_z^p-strain, as it is the usual case in the common plasticity approaches. The way, in which the latter account is carried out, is associated with another principal feature of the considered composites, namely the limited elastic response of the matrix material. The natural development of a given process of progressive plastification in a point, i.e. in an elementary volume of such a material, involves, most generally speaking, trends of progressive decrease and increase in the elastic and the plastic strain increments, respectively. One may thus generally relate such a process with a certain specific instant of its development upon which the elastic strains may be viewed as keeping approximately constant values, since their further increments become small enough to prevent (upon superposing) further substantial changes in the values which they have achieved at this instant. In accordance with these mostly qualitative but realistic considerations the model assumes the following. For a given composite structure, given loading status, for a given elementary volume of the matrix phase, a specific value $\overset{*}{\varepsilon}{}_z^e$ of the ε_z^e-strain exists such that upon a certain transitional regime of plastification, at the end of which the ε_z^e-strain in this volume achieves the value $\overset{*}{\varepsilon}{}_z^e$, a second regime starts developing for which the relation $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ holds true. A further simplifying assumption of the model concerns the dependence of the $\overset{*}{\varepsilon}{}_z^e$-value on the location of the elementary volume, i.e. on the spesific and actually unknown pattern of the transitional plastic stress redistribution which depends itself on this location. The model actually deals with the same $\overset{*}{\varepsilon}{}_z^e$-value in the entire matrix region, where the second regime has started developing. The quantity $\overset{*}{\varepsilon}{}_z^e$ may be thus considered as an average overall measure of the limited elastic response of a given composite under a given loading status. The determination of this quantity is, of course, a part of the analysis of the elastic-plastic response of the composites.

When specified with respect to the considered composite unit cell these basic concepts of the model imply the following qualitative description of the development of the matrix plastification process for both the model problems mentioned. Due to the stress concentration effect of the fibre, plastic deformations appear in the matrix at first at the fibre-matrix interface and a transitional regime of matrix plastification starts developing. The plastic zone associated with this regime has, due to the symmetry, the form of an annulus $r_f \leqq r \leqq r_c$ and spreads into the matrix phase. At the instant when $\varepsilon_z^e|_{r=r_f} = \overset{*}{\varepsilon}{}_z^e$, i.e. when the ε_z^e-strain achieves its limiting value (and this instant is first achieved at the fibre-matrix interface), the second regime starts developing with a plastic zone $r_f \leqq r \leqq R_c$, $R_c \leqq r_c$, within which the relation $\varepsilon_z^e = \overset{*}{\varepsilon}{}_z^e$ holds true. The second plastic zone spreads into the matrix phase as well having the first one, which occupies now the annulus $R_c \leqq r \leqq r_c$ at its front $r = R_c$. Thereby the transitional plastic zone $R_c \leqq r \leqq r_c$ is further considered as a thin layer, i.e. $R_c \approx r_c$. The latter plays the role of an elastic-plastic boundary, to which a softened version of fulfillment of the standard elastic-plastic transitional conditions is applicable (cf. [10, 11]).

Finally, the following remark is due with respect to the thermal problem considered below. The elastic part ε_z^e of the total axial strain ε_z in this case involves itself a part $\varepsilon_z^{e,sts}$, due to the thermal stresses, and a part $\varepsilon_z^{e,temp}$, due to the thermal contraction or expansion, respectively. When referred to the thermal problem the considerations, made above with respect to the ε_z^e-strain, should be now viewed as concerning not the entire ε_z^e-strain but its $\varepsilon_z^{e,sts}$-part only. Moreover, the strain

$\varepsilon_z^{e,temp}$ is stress independent.

ELASTIC BEHAVIOUR AND ELASTIC-PLASTIC TRANSITION

The assumptions, specifying the class of fibrous composites under consideration, allow to treat the products and the powers of the ratios E_m/E_f and r_f/r_m as small quantities. Appropriate simplifications are carried out accordingly in the following sections and the results derived are presented in forms, containing the principal terms only.

The linear-elastic solution of the considered problem is obtainable as a simple generalization of the plane strain ($\varepsilon_z \equiv 0$) solution of Herrmann [13]. The process of matrix cooling implies the following elastic distribution of the stresses σ_i^{me} and σ_i^{fe}, $i = r, \theta, z$, in the matrix and in the fibre respectively:

$$
\begin{aligned}
\left.\begin{matrix}\sigma_r^{me} \\ \sigma_\theta^{me}\end{matrix}\right\} &= \frac{E_m}{1+\nu_m}\frac{C}{r_m^2}\left(1 \mp \frac{r_m^2}{r^2}\right), \\
\sigma_z^{me} &= E_m(\varepsilon_z - \alpha_m T_m) + \nu_m(\sigma_r^{me} + \sigma_\theta^{me}), \\
\sigma_r^{fe} &= \sigma_\theta^{fe} = \sigma_r^{me}|_{r=r_f}, \\
\sigma_z^{fe} &= E_f \varepsilon_z + 2\nu_f \sigma_r^{me}|_{r=r_f},
\end{aligned}
\tag{1}
$$

where

$$C = -r_f^2 \alpha_m T_m (1 + \nu_m). \tag{2}$$

In fact, eqn (2) represents the exact value of the principal term of C for a composite with $\nu_m = \nu_f$. Generally, this term involves the multiplying factor $[1 - (\nu_m - \nu_f)/(1+\nu_m)(1+E_c)]$ as well, where

$$E_c = E_f r_f^2 / E_m r_m^2. \tag{3}$$

The latter factor is neglected in the following analysis, since, as one may actually prove, it does not affect substantially the basic features of composite's behaviour. Along with the self-equilibrium condition of the axial stresses σ_z^{ie}, $i = f$, m, the stress distribution from eqns (1) implies the relations

$$\varepsilon_z = \alpha_m T_m/(1 + E_c), \tag{4}$$

$$\varepsilon_z^{sts} = -\alpha_m T_m E_c/(1 + E_c), \tag{5}$$

where $\varepsilon_z^{sts} = \varepsilon_z - \varepsilon_z^{temp}$ is the part of the ε_z-strain, due to the stresses, and $\varepsilon_z^{temp} = \alpha_m T_m$. Eqns (4) and (5) are obviously approximations of the thermoelastic response of the composite unit cell in the common "rule of mixtures" sense. Furthermore, in accordance with the von Mises' yield condition, the foregoing relations define the *temperature of initial matrix plastification* T_m^{pl} at the fibre-matrix interface as

$$T_m^{pl} = -\sigma_y/\sqrt{3}\alpha_m E_m. \tag{6}$$

The corresponding ε_z^{pl}- and $\varepsilon_z^{sts,pl}$-values are

$$\varepsilon_z^{pl} = -\sigma_y/\sqrt{3}E_m(1+E_c), \tag{7}$$

$$\varepsilon_z^{sts,pl} = \sigma_y E_c/\sqrt{3}E_m(1+E_c). \tag{8}$$

ANALYSIS OF THE ELASTIC-PLASTIC BEHAVIOUR

According to the matrix plastification model described above the $\varepsilon_z^{e,sts}$-strain at the fibre-matrix interface achieves upon a certain transitional regime the value $\overset{*e}{\varepsilon}_z$. Its initial value is the value $\varepsilon_z^{sts,pl}$ defined by eqn (8). At this instant the second plastic zone $r_f \leqq r \leqq R_c$ starts spreading into the matrix phase. The relation $\varepsilon_z^{e,sts} = \overset{*e}{\varepsilon}_z$ holds true within this zone. In accordance with the generalized thermoelastic Hooke's law the stresses σ_i^{mp}, $i = r, \theta, z$, in the plastic zone satisfy the relation

$$\sigma_z^{mp} = E_m \overset{*e}{\varepsilon}_z + \nu_m(\sigma_r^{mp} + \sigma_\theta^{mp}). \tag{9}$$

Eqn (9) allows a reduction of the von Mises' yield criterion to the form

$$\left(\frac{\sigma_\theta^{mp} - \sigma_r^{mp}}{2}\right)^2 + \left(\frac{\sigma_\theta^{mp} + \sigma_r^{mp}}{2} - \frac{E_m \overset{*e}{\varepsilon}_z}{1-2\nu_m}\right)^2 \frac{(1-2\nu_m)^2}{3} - \frac{\sigma_y^2}{3} = 0. \tag{10}$$

The latter equation is identically satisfied by stresses of the form

$$\left.\begin{matrix}\sigma_r^{mp}\\ \sigma_\theta^{mp}\end{matrix}\right\} = \frac{E_m \overset{*e}{\varepsilon}_z}{1-2\nu_m} + \frac{\sigma_y}{\sqrt{3}\sin\Phi}\cos(\omega \pm \Phi) \tag{11}$$

where

$$\sin\omega = \frac{\sigma_\theta - \sigma_r}{2} \Big/ \frac{\sigma_y}{\sqrt{3}}, \tag{12}$$

(13) $$\tan\Phi = (1-2\nu_m)/\sqrt{3}.$$

Due to the elastic restriction specified by eqn (9) the yield condition defines an ellipse in the $(\sigma_\theta, \sigma_r)$-plane, eqn (10) or eqns (11), respectively. The points of the yield ellipse have coordinates $(\sigma_\theta^{mp}, \sigma_r^{mp})$ and are representative points in the stress-space for the stress-states in the points of the plastified matrix phase. Thus, a specific process of plastic stress redistribution in a point of the matrix phase defines via the angle ω, eqn (12), a specific law of motion along the yield ellipse of a corresponding representative point. Thereby the angle ω is easily seen to be a function of the loading parameter, i.e. T_m, as well as of the radial coordinate r and is further depending on both the geometrical and the mechanical characteristics of the composite constituents. The r-dependence of the angle ω is obtainable upon integrating the equilibrium equation

(14) $$\frac{d\sigma_r^{mp}}{dr} + \frac{\sigma_r^{mp} - \sigma_\theta^{mp}}{r} = 0$$

in the interval $r_f \leqq r \leqq R_c$ with the boundary condition

(15) $$\omega|_{r=R_c} \equiv \omega_{R_c} = \arccos\left[-E_m \overset{*}{\varepsilon}{}_z^{e}/\sigma_y(1+\nu_m)\right].$$

The latter condition reflects the assumption (cf. Herrmann & Mihovsky [10, 11]) that $\varepsilon_z^{e,sts}$ is the only non-negligible elastic strain in the second plastic zone (where, as adopted, $\varepsilon_z^{e,sts} = \overset{*}{\varepsilon}{}_z^{e}$) and that the matrix material is plastically incompressible.

The result of the integration reads

(16) $$\frac{R_c^2}{r^2} = \frac{\sin\omega}{\sin\omega_{R_c}} \exp[(\omega - \omega_{R_c})\text{cotan }\Phi],$$

where the plastic zone radius R_c is to be further determined as a function of the loading parameter T_m.

With respect to the values of ω at the fibre-matrix interface eqn (16) implies

(17) $$\frac{R_c^2}{r_f^2} = \frac{\sin\omega_{r_f}}{\sin\omega_{R_c}} \exp[(\omega_{r_f} - \omega_{R_c})\text{cotan }\Phi],$$

where the notation is introduced

(18) $$\omega_{r_f} = \omega(r)|_{r=r_f}.$$

It is clear from the very nature of the considered thermal loading process that progressive matrix cooling should result in progressive shrinkage, i.e. in progressive decrease of the itself negative radial stress acting over the fibre-matrix interface. At the same time the shrinkage effect is limited itself in the sense that, as eqns (11) prove, a maximum shrinkage, i.e. a minimum value of the latter stress is achievable at the instant when $\omega_{r_f} = \pi - \Phi$. This specific instant for the composite unit cell is shown in [10, 11] to correspond to a critical state of the cell when failure modes start developing in the latter due to the plastic instability of the matrix at the fibre-matrix interface. Further, these considerations imply the conclusion that with progressive thermal loading the angle ω_{r_f} increases (cf. the structure of the $\sigma_r^{mp}|_{r=r_f}$-stress, eqns (11)), running actually within the interval

$$\omega_{R_c} \leqq \omega_{r_f} \leqq \pi - \Phi. \tag{19}$$

The latter conclusion is meaningful if, of course, the angles ω_{R_c} and Φ satisfy the relation $\omega_{R_c} < \pi - \Phi$. Since the quantity $\overset{*e}{\varepsilon}_z$ should be expected to belong actually to the interval $[\varepsilon_z^{sts,pl}, \sigma_y/E_m]$, then eqns (13) and (15) prove immediately that the latter relation is valid if $\nu_m > 0.1$, which is the practical case for the commonly used matrix materials. Moreover, in accordance with this conclusion eqn (17) proves the existence of a maximum plastic zone size R_c^* and defines the latter as

$$R_c^{*2} = r_f^2 \frac{\sin\Phi}{\sin\omega_{R_c}} \exp[(\pi - \Phi - \omega_{R_c})\text{cotan}\,\Phi]. \tag{20}$$

For reasons of simplicity the analysis below is restricted to cases for which $R_c^* < r_m$. From its quantitative side this analysis aims at the prediction of the thermally induced elastic-plastic response of the unit cell, i.e. the $\varepsilon_z(T_m)$-dependence. This aim is achieved in the following in a step-wise way, which involves at first the determination of the $\omega_{r_f}(\varepsilon_z)$- and the $R_c(\varepsilon_z)$-dependences.

The procedure of obtaining the $\omega_{r_f}(\varepsilon_z)$-dependence involves the following basic steps. First, the condition of continuity of the radial displacements u_r^i, $i = f, m$ at the fibre-matrix interface is constructed by the aid of the known axisymmetric relations $u_r^i|_{r=r_f} = r_f\varepsilon_\theta^i|_{r=r_f}$, where ε_θ^i, $i = f, m$, are the circumferential strains in the fibre and the matrix, respectively. Further, the strain rates ξ_θ^i, $i = f, m$, are obtained as derivatives of the strains ε_θ^i with respect to the loading parameter T_m. Thereby the elastic part ξ_θ^{me} of the ξ_θ^m-strain rate at the interface $r = r_f$ is neglected (cf. the text following eqn (15)). The strain rate ξ_θ^f is defined via the generalized Hooke's law and eqns (1), now with $\sigma_r^{mp}|_{r=r_f}$ instead of $\sigma_r^{me}|_{r=r_f}$ for the stresses at the fibre-matrix interface. The plastic part ξ_θ^{mp} of the ξ_θ^m-strain rate is defined in accordance with the associated flow rule concept along with the yield function, used as a plastic potential (cf. [11]). Moreover, the thermal part of the ξ_θ^m-strain rate can be neglected without affecting the basic trends of the ω_{r_f}-behaviour. The u_r-continuity condition is thus reduced to the form

$$\Lambda d\varepsilon_z = f(\omega_{r_f})d\omega_{r_f}, \tag{21}$$

where the notations are introduced

$$\Lambda = \frac{E_f\sqrt{3}}{2\sigma_y(1+\nu_f)(1-2\nu_f)}, \tag{22}$$

$$f(\omega_{r_f}) = \frac{\sin(\omega_{r_f}+\Phi)\cos\omega_{r_f}}{\sin(\omega_{r_f}+\Phi) - 2\nu_f \sin\Phi\cos\omega_{r_f}}. \tag{23}$$

Eqn (21) has to be solved in the interval $[\omega_{R_c}, \pi - \Phi]$ with the approximate boundary condition

$$\varepsilon_z|_{\omega_{r_f}=\omega_{R_c}} = \tilde{\varepsilon}_z^{pl} = -\overset{*}{\varepsilon}{}_z^{e}/E_c. \tag{24}$$

This boundary condition results from the assumption that the behaviour of the unit cell in the interval between the initial matrix plastification and the occurrence of the second plastic zone, i.e. in the transitional régime, is not substantially affected by the only presence of the corresponding transitional plastic zone and thus may be considered as following the linear-elastic dependence, given by eqn (4) or eqn (5) respectively. Such an assumption practically identifies the $\overset{*}{\varepsilon}{}_z^{e}$ and $\varepsilon_z^{sts,pl}$ strains and further defines by means of eqns (6) and (8) (the latter with $\varepsilon_z^{sts,pl} = \overset{*}{\varepsilon}{}_z^{e}$ now) the instant of occurrence of the second plastic zone (when $\omega_{r_f} = \omega_{R_c}$, cf. eqn (24)), as corresponding to the values $\tilde{T}_m^{pl}$ and $\tilde{\varepsilon}_z^{pl}$ of T_m and ε_z^{pl} respectively, which are

$$\tilde{T}_m^{pl} = -\overset{*}{\varepsilon}{}_z^{e}(1+E_c)/\alpha_m E_c, \tag{25}$$

$$\tilde{\varepsilon}_z^{pl} = -\overset{*}{\varepsilon}{}_z^{e}/E_c. \tag{26}$$

An approximate series expansion procedure for solving the boundary value problem, specified by eqns (21) and (24), is applied. It consists of the following steps. Eqn (21) is first solved for values of ω_{r_f}, close to $\pi - \Phi$, upon an expansion of the function $f(\omega_{r_f})$ into the powers of the small differences $(\pi - \Phi - \omega_{r_f})$. The solution thus obtained is then extrapolated over the entire interval $[\omega_{R_c}, \pi - \Phi]$ in order to fit the boundary condition, eqn (24). Accordingly, the following form of the desired approximate dependence is obtained

$$\omega_{r_f}(\Delta\varepsilon_z) = \pi - \Phi - \left[(\pi - \Phi - \omega_{R_c})^2 + \frac{2b\Lambda}{\cos\Phi}\Delta\varepsilon_z\right]^{1/2}, \tag{27}$$

where

(28) $$b = 2\sqrt{3}\nu_f(1-2\nu_m)/[3+(1-2\nu_m)^2],$$

(29) $$\Delta\varepsilon_z = \varepsilon_z - \tilde{\varepsilon}_z^{pl}.$$

The quantity $\Delta\varepsilon_z$ is thus the part of the total axial strain ε_z which develops upon the occurrence of the second plastic zone. The critical value $\Delta\overset{*}{\varepsilon}_z$ of $\Delta\varepsilon_z$ at which the unit cell undergoes a transition to failure, follows from eqn (27) with $\omega_{r_f} = \pi - \Phi$ to be

(30) $$\Delta\overset{*}{\varepsilon}_z \doteq -(\pi - \Phi - \omega_{R_c})^2 \frac{\cos\Phi}{2b\Lambda}.$$

With the aid of a similar expansion technique one obtains upon introducing ω_{r_f} from eqn (27) into eqn (17) the $R_c(\varepsilon_z)$-dependence in the form

(31) $$R_c^2(\Delta\varepsilon_z) = R_c^{*2}\left[1 - \left(1 - \frac{r_f^2}{R_c^{*2}}\right)\left(1 - \frac{\Delta\varepsilon_z}{\Delta\varepsilon_z^*}\right)\right].$$

It should be pointed out that eqns (27) and (31) approximate the actual $\omega_{r_f}(\varepsilon_z)$- and $R_c(\varepsilon_z)$-dependences rather roughly but, at the same time, they keep and clearly indicate the basic features of the latter, due to their simple analytical forms.

Further, the determination of the $\varepsilon_z(T_m)$-dependence is a matter of simple computations, based upon the condition of self-equilibrium of the axial stresses

(32) $$r_f^2\sigma_z^f + (r_m^2 - R_c^2)\sigma_z^{me} + 2\int_{r_f}^{R_c} \sigma_z^{mp} r\,dr = 0.$$

Thereby the stress σ_z^f is to be defined from eqns (1) with $\sigma_r^{mp}|_{r=r_f}$ instead of $\sigma_r^{me}|_{r=r_f}$ and with $\sigma_r^{mp}|_{r=r_f}$ given by eqns (11) with $\omega = \omega_{r_f}$ along with eqn (27) for $\omega_{r_f}(\Delta\varepsilon_z)$. The axial stress σ_z^{me} in the elastically deformed matrix region $R_c \leqq r \leqq r_m$ is obtainable from eqns (1) upon definition of a new C-value from the σ_r-continuity condition at the elastic-plastic boundary $r = R_c$. The latter condition reflects the softened version of the fulfillment of the elastic-plastic transitional conditions mentioned above (cf. [10, 11]).

The axial stress σ_z^{mp} in the plastic zone and the radius R_c of the latter are defined by eqns (9) and (31) respectively.

Upon corresponding computations and appropriate simplifications eqn (32) implies the relation

(33) $$\Delta\varepsilon_z = \alpha_m \Delta T_m \frac{1 - R_c^2/r_m^2}{1 + E_c - R_c^2/r_m^2},$$

and by introducing for R_c from eqn (31) it is obtained

(34) $$\Delta\varepsilon_z = \alpha_m \Delta T_m \left[1 + E_c + \alpha_m \Delta T_m \frac{E_c}{1+E_c}\frac{R_c^{*2}}{r_m^2}\left(1 - \frac{r_f^2}{R_c^{*2}}\right)\frac{1}{\Delta\varepsilon_z^*}\right]^{-1},$$

where the notation is used

(35) $$\Delta T_m = T_m - \widetilde{T}_m^{pl}.$$

The explicite form of the $\varepsilon_z^{sts}(T_m)$-dependence is obtainable straightforwardly from eqn (35) and the relation $\Delta\varepsilon_z^{sts} = \Delta\varepsilon_z - \alpha_m \Delta T_m$.

The critical temperature of failure of the unit cell $T_m^* = \widetilde{T}_m^{pl} + \Delta T_m^*$ follows from eqns (25) and (34) (with $\Delta\varepsilon_z = \Delta\varepsilon_z^*$ for ΔT_m^*) respectively. Both quantities $\Delta\varepsilon_z^*$ and ΔT_m^* and therefore T_m^* are dependent on the specific value of $\overset{*e}{\varepsilon}_z$ for the unit cell and thus for the composite structure also. Consequently, eqn (34) represents the desired approximate analytical form of the thermally induced elastic-plastic response of the composite unit cell in the considered model problem of matrix cooling.

BASIC FEATURES OF THE COMPOSITE BEHAVIOUR

The basic features of the elastic-plastic response of the composite predicted by the foregoing analysis will be briefly considered in this section. It should be mentioned, first of all, that with the aid of the obvious relation $\Delta\varepsilon_z^{sts} = \Delta\varepsilon_z - \alpha_m \Delta T_m$ one may immediately transform eqn (33) into the relation $E_m(r_m^2 - R_c^2)\Delta\varepsilon_z^{sts} + E_f r_f^2 \Delta\varepsilon_z = 0$. Thereby the latter relation is nothing else but an explicite representation of the predicted composite response in the "rule of mixtures" sense. In accordance with this representation the plastified matrix region influences the redistribution of the axial forces via its radius R_c but does not explicity contribute to this redistribution. Its own contribution appears to be just negligible within the frame of the present approximate analysis. Furthermore, the following statement should be made with respect to the structure of the $\Delta\varepsilon_z(\Delta T_m)$-dependence obtained above. The strain $\Delta\varepsilon_z$ defined by eqn (34) is easily seen to decrease monotonically as a concave negative function when the itself negative temperature difference ΔT_m decreases. The curve $\Delta\varepsilon_z(\Delta T_m)$ proves to deviate smoothly from the linear elastic $\varepsilon_z(T_m)$-dependence defined by eqn (4). With the formal limit transition $\Delta T_m \longrightarrow -\infty$ the strain $\Delta\varepsilon_z$ approaches asymptotically a limit value $\Delta\hat{\varepsilon}_z$ which may be easily shown to satisfy the relation $\Delta\hat{\varepsilon}_z < \Delta\varepsilon_z^*$ (with $\Delta\varepsilon_z^* < 0$, cf. eqn (30)). The latter means that the composite cell achieves its critical state of failure at finite values of ΔT_m^* and T_m^* respectively.

A purely qualitative schematic illustration of the total elastic-plastic response, derived above, is presented in Fig.1 where the straight line I describes the behaviour of a homogeneous cylinder of the matrix material under the considered cooling process. No thermal stresses develop in such a cylinder and its axial strain is due to the thermal contraction only. The line II corresponds to purely elastic fibre and matrix materials, eqn (4). Each of the series of the concave curves III,$_i$ corresponds to eqn (34) with an initially specified $\overset{*}{\varepsilon}{}^{e}_{z,i}$-value. Each of these lines coincides with the line II over the corresponding interval $[0, \tilde{T}^{pl}_{m,i}]$ or $[0, \tilde{\varepsilon}^{pl}_{z,i}]$, respectively (cf. eqns

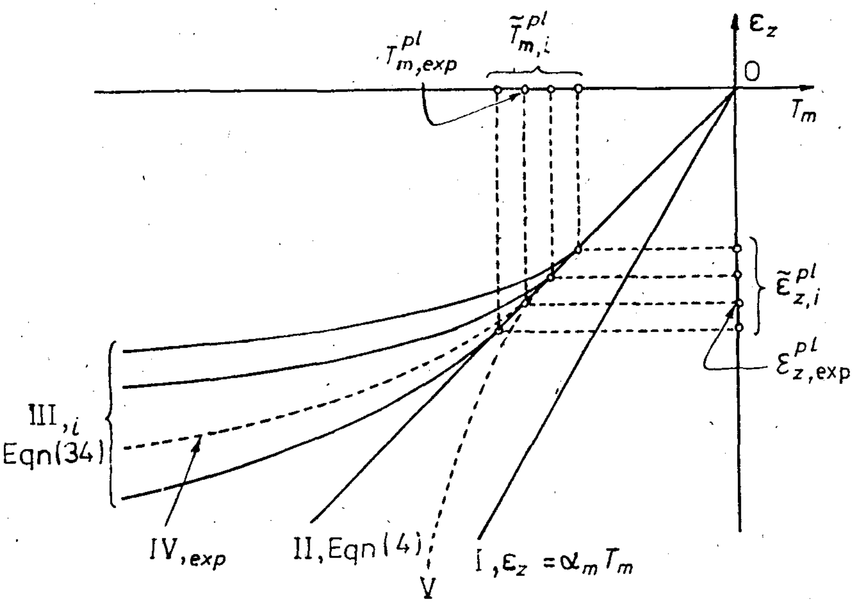


Fig. 1. Schematic qualitative illustration of the elastic-plastic response of fibrous composites due to matrix cooling

(25) and (26)) and smoothly deviates from this line in the way, shown in the graph, at the corresponding points $(\tilde{T}^{pl}_{m,i}, \tilde{\varepsilon}^{pl}_{z,i})$. The line IV is the assumed experimentally obtained $\varepsilon_z(T_m)$-curve for the considered composite. As it is usually accepted in the engineering practice, the linear part of this curve is constructed in accordance with the linear-elastic "rule of mixture" approach. Thus it coincides over this part with the straight line II. Let the line IV deviate from the line II at the point $(T^{pl}_{m,exp}, \varepsilon^{pl}_{z,exp})$, where "*exp*" stays for the experimentally measured values of T^{pl}_m and ε^{pl}_z. Then, upon identifying these values with the $\tilde{T}^{pl}_m$ and $\tilde{\varepsilon}^{pl}_z$ values in eqns (25) and (26), respectively, one defines a corresponding, say $\overset{*}{\varepsilon}{}^{e}_{z,exp}$-value of the quantity $\overset{*}{\varepsilon}{}^{e}_{z}$. It

is this latter value of $\overset{*e}{\varepsilon}_z$ to deal with when applying the foregoing general approach to a given composite structure.

A more sophisticated approach to the identification of the actual value of the $\overset{*e}{\varepsilon}_z$-strain involves a comparison between the actual $\Delta\varepsilon_z(\Delta T_m)$-curve and the series of theoretical curves $III,_i$. Upon introducing an appropriate best fitting criterion and by means of a corresponding processing of these curves one may define the theoretical curve which fits the experimental one in the best way with respect to the chosen criterion. The value of $\overset{*e}{\varepsilon}_z$, to which this theoretical curve corresponds, will then be the actual one for the considered composite. It should be mentioned that the strain $\Delta\varepsilon_z^{sts}$ increases as a concave positive function when the negative temperature difference ΔT_m decreases. One may easily derive the basic features of the $\Delta\varepsilon_z^{sts}(\Delta T_m)$-dependence by the aid of the foregoing equations. Furthermore, in some cases the linearization of the composite response in the elastic-plastic range may be of interest. A simple linearized version of eqn (34) is presented, for example, by the relation $\Delta\varepsilon_z/\Delta\varepsilon_z^* = \Delta T_m/\Delta T_m^*$. Such a linearization replaces the family of concave curves $III,_i$ in Fig.1 by a corresponding family of straight lines with the same points of deviation from the line II. The approaches to the identification of the actual $\overset{*e}{\varepsilon}_z$-value, described above, apply to the linearized case as well.

CONCLUDING REMARKS

The results, obtained in the previous sections, represent in the whole an approximate analytical solution of the considered problem of thermal loading of a composite structure. The general approach, developed in the study, involves a specific parameter $\overset{*e}{\varepsilon}_z$ for the composite structure as well as the loading status and reveals the ways to its identification under the implicit assuption that the real thermomechanical response of the composite corresponds to a concave strain-temperature curve (cf. curve IV in Fig. 1). Whether this is the actual case or, in other words, whether the predictions of the approach (the curve $III,_i$ in Fig. 1) are at least in qualitative agreement with the real composite response is a principal question. A positive answer to this question would not only support the validity of the approach in the whole but would obviously reveal further possibilities for achieving a better quantitative fitting between the predicted and the actual response. Thereby the following statement could be made with regard to this problem. To the authors' knowledge there exist at present no experimental data which could be used in a reliable way for a comparison with the prediction for the model problem considered. At the same time the behaviour of the composite under thermal loading with a concave $\varepsilon_z(T_m)$-curve is explainable in quite a natural way. The progressive matrix plastification results in a softening of the matrix in the sense that the stresses in the itself expanding plastic zone remain limited. This implies a corresponding relative increase in the strengthening effect of the fibre and thus of the overall stiffness of the composite structure. The concave curves $III,_i$ and IV in Fig. 1 reflect, in fact, exactly the latter effect. It is difficult to explain in a similar way an imaginary behaviour of the composite to which a convex curve, such as curve V in Fig. 1, would correspond. It should be mentioned in addition that the theoretically predicted response of the composite allows for a direct realistic interpretation in the "rule of mixture" sense. This fact may be considered to confirm to a further extent the potential of the developed approach for reliable predictions of the elastic-plastic response of the composites. Finally it should be noted that, as Part II of the present study proves, when applied to the problem of longitudinal extension of a fibrous composite the same approach predicts a stress-strain curve which is in entire qualitative agreement with the typical experimental observations. Certain additional aspects of the thermally induced response of the composites will be considered and simultaneously compared with the corresponding aspects of the behaviour of such composites under longitudinal extension in the closing section of Part II of the present study. These aspects concern basically the general features of the matrix plastification processes and their influence on the fracture phenomena in fibrous composites.

ACKNOWLEDGEMENT

The authors thank the Deutsche Forschungsgemeinschaft for its financial support.

REFERENCES

1. Hill, R. Theory of Mechanical Properties of Fibre-Strengthened Materials: II. Inelastic Behaviour. — J. Mech. Phys. Solids, **12**, 1964, 213 - 218.
2. Spencer, A. J. M. A Theory of the Failure of Ductile Materials Reinforced by Elastic Fibres. — Int. J. Mech. Sci., **7**, 1965, 197 - 209.
3. Mulhern, J. F., T. G. Rogers, A. J. M. Spencer. Cyclic Extension of an Elastic Fibre with an Elastic- Plastic Coating. — J. Inst. Maths. Applics., **3**, 1967, 21 - 40.
4. Ebert, L. J., S. S. Hecker, C. H. Hamilton. The Stress-Strain Behaviour of Concentric Composite Cylinders. — J. Comp. Materials, **2**, 1968, 458 - 476.
5. Thomason, P. F. Plastic Flow and Fracture in Fibre Composites. — J. Mech. Phys. Solids, **20**, 1972, 19 - 32.
6. Dvorak, G. J., M. S. M. Rao. Thermal Stresses in Heat Treated Fibrous Composites. — Trans. ASME, Ser. E, JAM, **43**, 1976, 619 - 624.
7. Strife, J. R., K. M. Prevo. The Thermal Expansion Behaviour of Unidirectional and Bidirectional Kevlar/Epoxy Composites. — J. Comp. Materials, **13**, 1979, 264 - 277.
8. Min, B. K. A Plane Stress Formulation for Elastic-Plastic Deformation of Unidirectional Composites. — J. Mech. Phys. Solids, **29**, 1981, 327 - 352.
9. Morley, J. G. On the Development of a General Theory of the Mechanics of Tensile Fracture of Fibre-Reinforced Materials. — J. Mater. Sci., **20**, 1985, 1794 - 1806.
10. Herrmann, K., I. M. Mihovsky. Plastic Behaviour of Fibre-Reinforced Composites and Fracture Effects. — Rozprawy Inzynierskie-Engn. Trans., **31** (No 2), 1983, 165 - 177.

11. H e r r m a n n, K., I. M. M i h o v s k y. Plastic Zone Size of Dugdale Type Cracks in a Self-Stressed Two-Phase Medium with Partially Plastified Matrix Material. — Mechanika Teoretyczna i Stosowana, **22** (No 1/2), 1984, 25 - 39.
12. H e r r m a n n, K., I. M. M i h o v s k y. On the Applicability of the Dugdale Crack Model to Fibre-Reinforced Composites. — ZAMM, **67** (No 4), 1987, T196 - T197.
13. H e r r m a n n, K. Über Eigenspannungen im diskontinuierlich inhomogenen Festkörper. (S c h r ö d e r, K. — ed.) — Beiträge zur Spannungs- und Dehnungsanalyse, Akademie Verlag, Berlin, Bd. 6, 1970, 21 - 52.

Received 2. VII. 1990

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 83, 1989

APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. II. EXTERNAL LOADING

KLAUS HERRMANN, IVAN MIHOVSKY

Клаус Херман, Иван Миховски. ПРИБЛИЖЕННОЕ АНАЛИТИЧЕСКОЕ ИССЛЕДОВАНИЕ УПРУГОПЛАСТИЧЕСКОГО ПОВЕДЕНИЯ ВОЛОКНИСТЫХ КОМПОЗИТОВ. II. ВНЕШНЕЕ НАГРУЖЕНИЕ.

Работа продолжает исследование композитов, рассмотренных в части I, посвященной их поведению в условиях термического нагружения. Здесь исследован случай чисто механического нагружения и точнее — продольного растяжения. Показано, что предложенный в части I подход ведет к надежным качественным и количественным заключениям и оценкам относительно поведения рассматриваемых композитов.

Klaus Herrmann, Ivan Mihovsky. APPROXIMATE ANALYTICAL INVESTIGATION OF THE ELASTIC-PLASTIC BEHAVIOUR OF FIBROUS COMPOSITES. II. EXTERNAL LOADING.

The paper continues the investigation of the composites specified in Part I. While the latter part is devoted to the thermally induced response the present one deals with the purely mechanical problem of longitudinal extension. The approach developed in Part I is shown to lead to realistic (both qualitative and quantitative) predictions of the overall response of the composites considered.

INTRODUCTION

The basic aspects of the influence of the matrix plasticity on the overall thermomechanical response of the fibrous composites are considered in sufficient detail in the introductory section of Part I of the present study along with the specific features of the general approach developed in the latter.

In the present part the same class of fibrous composites is considered by the aid of the same models of the composite unit cell and the process of matrix plastification (Herrmann & Mihovsky [1, 2], cf. p. I). The loading is specified as longitudinal extension, which is a typical operational loading for fibrous composites. Therefore it is quite natural that their response in such load-bearing applications has been intensively studied in the past and that a good understanding of the overall characteristics of this response already exists nowadays.

Following Kelly [3] one may summarize that there are two stages in the behaviour of the considered composites. They reflect the initial purely elastic elongation of the fibres and the matrix, respectively, as well as the following plastic flow in the matrix. The transition to the second stage occurs when the matrix material starts yielding. This process begins at a value of the axial strain which is "a little less" than the yield strain of the matrix, and is complete at "a slightly larger" strain. The contribution of the matrix to the stress-strain curve in the second stage is negligible. The lower bound to this slope, as derived by Hill [4], is (in the notation introduced in p. I) $E_f r_f^2 / r_m^2$.

This brief general description of the elastic-plastic response of the fibrous composites clearly indicates that the entire matrix plastification is a sudden phenomenon. Such an effect of a sudden entire matrix plastification is not involved in the thermally induced composite response, considered in p. I. The plastic zone size in the latter case has been shown to increase monotonically with progressive thermal loading. To clear up the response of the composites in the interval between the initial and the complete matrix plastification, respectively, as well as the very mechanism of the latter appears to be an interesting problem. In fact, this interval corresponds to a very small change of the axial strain from "a little less" to "a slightly larger" value, as stated by Kelly [3]. Thus, at fisrt sight, the details of the composite behaviour in this short interval do not seem to be of essential significance. But, from the view point of the influence of the matrix plasticity on the response (including the failure) of the composites, it is important to get a better understanding of this initial stage of matrix plastification. The real nature of this stage indicates that it is governed by specific mechanisms. Accordingly, one should expect that the latter may further contribute to the occurrence of specific trends of the development of the plastic deformation process in the completely plastified matrix. These trends are of interest with respect to the determination of the overall response of the composites, especially for the occurrence and the development of failure modes.

In this paper, the specific aspects of the matrix plastification process developing in a longitudinally extended fibrous composite are considered. Corresponding conclusions of both qualitative and quantitative character are derived by means of an approximate analytical version of the general approach already used in Part I. This version predicts an elastic-plastic response of the composite material which is consistent with the lower bound estimation obtained by Hill [4]. Certain general conclusions are derived in the closing section of the article with respect to the distinguishing features of the composite response under thermal and mechanical loading conditions. The specific influence of the matrix plasticity on the fracture resistance of the composites for both loading schemes as well as the significance of special structural defects are considered.

STATEMENT OF THE PROBLEM

The class of composites, the composite unit cell, and the mechanical properties of the fibre and the matrix materials, respectively, are the same as already specified in p. I. The loading is specified as longitudinal extension of the composite and therefore as axial extension of the unit cell. The lateral surface of the latter is traction free. Accordingly, the plane cross-sections hypothesis applies, the stress-strain field in the cell is axisymmetric, and the normal stresses in both the fibre and the matrix are principal ones and depend upon the radial coordinate r only. Further, the powers and the products of the ratios E_m/E_f and r_f/r_m are considered again as small quantities and, like in p. I, the final results are presented in forms, containing the principal terms only.

ELASTIC BEHAVIOUR AND ELASTIC-PLASTIC TRANSITION

In accordance with the known elastic solution of the problem (cf., for example, Ebert et al. [5]) the stresses in the matrix and the fibre are of the same form as in eqns (1), p. I, but with a new value of the constant C. This value reads

(1) $$C = \varepsilon_z \Delta\nu r_f^2,$$

where the notation $\Delta\nu$ means

(2) $$\Delta\nu = \nu_m - \nu_f.$$

Moreover, the relation

(3) $$\Delta\nu > 0$$

applies for the commonly used fibrous composites and is assumed to be valid in the following considerations. It provides, in fact, the occurrence of compressive radial stresses at the fibre-matrix interface, i.e. the well-known shrinkage effect. Further, by considering the equilibrium condition for the axial forces, the elastic stress distribution from eqns (1), p. I, together with the new C-value from eqn (1) now implies the relation

(4) $$\sigma_z^c = \varepsilon_z E_m(1 + E_c),$$

where E_c is the same as in eqn (3), p. I, while

(5) $$\sigma_z^c = P/\pi r_m^2$$

is the axial composite stress, induced in the unit cell by the applied axial tensile force P. Eqn (4) represents the well-known "rule of mixtures" approximation of the linear elastic response of a composite.

Now, in accordance with the classical von Mises' yield condition, the matrix starts yielding at the fibre-matrix interface when the stress σ_z^c and the axial strain ε_z, respectively, achieve the values

(6) $$\sigma_z^{c,pl} = \sigma_y(1 + E_c)\left[1 - \frac{3}{2}\frac{(\Delta\nu)^2}{(1+\nu_m)^2}\right],$$

(7) $$\varepsilon_z^{pl} = \frac{\sigma_y}{E_m}\left[1 - \frac{3}{2}\frac{(\Delta\nu)^2}{(1+\nu_m)^2}\right].$$

Due to the smallness of $(\Delta\nu)^2$ one may really view the strain ε_z^{pl} of the initial matrix plastification, given in eqn (7), as "a little less" than the matrix yield strain $\varepsilon_y = \sigma_y/E_m$, as stated by Kelly [3]. In addition, a simple comparison with the thermal problem, considered in p. I, shows that in the present case the initial matrix plastification takes place at a much larger value of the ε_z-strain than it was stated for the corresponding ε_z^{sts}-value (cf. p. I, eqn (8)). On the contrary, the value of the radial stress at the fibre-matrix interface at the instant of initial plastification is much smaller than the corresponding stress value in the thermal problem.

It should be mentioned that in accordance with the sense of the quantity $\overset{*}{\varepsilon}{}_z^e$, involved in the matrix plastification model, its actual value should be expected to be close to the ε_z^{pl}-value or, respectively, to the value $\varepsilon_z^{sts,pl}$ in the thermal problem (cf. p. I). Therefore, the plastic behaviour of the matrix in the present case will be associated with a yield ellipse, which is of the same geometry as that in the thermal case (p. I, eqns (10), (11)) but with its center removed from the origin of the (σ_θ, σ_r)-plane along the line $\sigma_r = \sigma_\theta$ over a distance which is "a little less" than the length $\sigma_y/(1-2\nu_m)$ of its larger principal half-axis. In the thermal case the center of the yield ellipse almost coincides with the origin of the (σ_θ, σ_r)-plane. In the present problem it is its vertex $\omega = \pi$ (p. I, eqn (12)) that almost coincides with the same origin. These general observations will prove to be useful for the following analysis.

ANALYSIS OF THE ELASTIC-PLASTIC BEHAVIOUR

In accordance with the basic adoptions of the matrix plastification model (cf. p. I) one may immediately conclude that the series of equations given in p. I, namely eqns (9) — (18), should hold true in the present problem as well. Further, when keeping the structure of the thermal problem analysis one should consider as a next step the way, in which the shrinkage influences the behaviour of the angle ω_{r_f}, introduced in eqn (18), p. I. Generally speaking, the shrinkage is a desired effect in the load-bearing applications of the fibrous composites, since it prevents the occurrence of delamination phenomena in the latter. Practically, the shrinkage in the present problem results from the larger cross-sectional contraction of the matrix with respect to the fibre. Relation (3) simply proves the validity of this conclusion in the elastic range. Furthermore, the usual assumption of plastic incompressibility implies the natural conclusion that the progressive plastification of the matrix will effectively result in a further increase of Poisson's ratio in the matrix phase. The latter increase will then contribute to the further increase of the shrinkage as it is adopted in the matrix plastification model of Herrmann & Mihovsky [1]. In addition, one should accordingly accept that as in the thermal problem the progressive loading will cause monotonous increase of the angle ω_{r_f} within the interval defined by eqn (19), p. I. But by considering the u_r-continuity condition at the fibre-matrix interface (p. I, eqn (21)), it can be easily seen that this foregoing adoption is not realistic. Because in the present case the function $f(\omega_{r_f})$ has again negative values, whereas, in contrast to the thermal one, $d\varepsilon_z$ is positive due to the elongation of the cell. Therefore, the equation cited predicts negative $d\omega_{r_f}$-values, that means a decrease of ω_{r_f} and thus of the shrinkage with progressive loading, i.e. with increasing ε_z-strain. To clear up the reason for this inconsistency between the adoption mentioned above and the prediction of the u_r-continuity condition appears to be the first necessary step that distinguishes the present analysis from that of the thermal problem.

It should be recalled in this regard that the u_r-continuity condition, eqn (21), p. I, as well as the boundary condition, eqn (15), p. I, are derived under the assumption that the $\varepsilon_i^{e,sts}$- strain ($i = r, \theta$) are neligible (cf. the text following eqn (15), p. I). One may simply prove that, in fact, it is this assumption that leads to the inconsistency mentioned just above. Actually, in the present case, the cross-sectional elastic strain components in the plastic zone should not be neglected since, due to the relatively small stress concentration effect of the fibre (cf. the remarks following eqn (7)), the corresponding plastic strain components will be also small enough. Finally, by means of the procedure, described in sec. 5, p. I, and with the ε_θ^e-strain introduced into the u_r-continuity condition, the following relation holds true

(8) $$\Lambda_1 d\varepsilon_z = f_1(\omega_{r_f})d\omega_{r_f},$$

where

(9) $$\Lambda_1 = \frac{\sqrt{3}E_m}{4\sigma_y(1+\nu_m)(1+\alpha)\sin\Phi},$$

$$f_1(\omega_{r_f}) = \frac{\sin\left(\frac{\pi}{6} - \omega_{r_f}\right)\cos\omega_{r_f}}{\sin(\omega_{r_f} + \Phi) - 2\nu_f \sin\Phi\cos\omega_{r_f}}, \tag{10}$$

$$\alpha = \frac{(1+\nu_f)(1-2\nu_f)E_m}{(1+\nu_m)E_f}. \tag{11}$$

Thereby eqn (8) is to be further coupled with a corresponding interval within which the angle ω_{r_f} changes, as well as with an appropriate boundary condition. It can be proved that the angle ω_{R_c}, as defined by eqn (15), p. I (with the elastic strain neglected), will not apply to the present case. Thus, the account for the elastic strain, involved in eqn (8), requires a more accurate determination of the initial value of ω_{r_f}, i.e. of ω_{R_c}. The latter represents itself the value of ω_{r_f} at the instant of the occurence of the second plastic zone. As in the thermal case it can be assumed that up to this instant the transitional matrix plastification process (cf. p. I) does not affect substantially the linear elastic behaviour of the composite. In the present case this behaviour allows to be considered as satisfying the condition

$$(\sigma_r^{me} + \sigma_\theta^{me})|_{r=r_f} = 0, \tag{12}$$

since the left-hand side of eqn (12) is proportional to r_f^2/r_m^2 and the terms of this order of magnitude are, as adopted, neglected in the present analysis.

Moreover, since the value $\overset{*}{\varepsilon}{}_z^e$ of the ε_z-strain, at which the second plastic zone occurs, should not differ substantially from the value of the ε_z^{pl}-strain (cf. eqn (7)), one may further accept that

$$\overset{*}{\varepsilon}{}_z^e = \varepsilon_z^{pl}. \tag{13}$$

Regarding the yield ellipse (p. I, eqns (10), (11) now with $\overset{*}{\varepsilon}{}_z^e$ from eqn (13)) eqns (12) and (13) simply prove that the stress state at the fibre-matrix interface, at which the second plastic zone occurs, corresponds to the intersection point of this ellipse and of the straight line defined by eqn (12). Accordingly, the value ω_{R_c}, which defines the position of this intersection point over the yield ellipse, can be determined by using eqns (10), p. I, as well as eqns (7) and (13), respectively

$$\omega_{R_c} = \arccos\left[-1 + \frac{3}{2}\frac{(\Delta\nu)^2}{(1+\nu_m)^2}\right]. \tag{14}$$

Due to the smallness of $(\Delta\nu)^2$ the value of ω_{R_c} is approximately π. The same statement is valid for the angle $\pi - \Phi$ (cf. p. I, eqn (13)). Now, in the framework of the model of the matrix plastification process (cf. p. I) the angle $\pi - \Phi$ represents the critical value of ω_{r_f}, at which failure of the composite takes place, whereas ω_{R_c} is the initial value of ω_{r_f}. Therefore, the establishment of an accurate relationship between the angles $\pi - \Phi$ and ω_{R_c} is absolutely necessary. In fact, such a relation follows immediately from eqns (13), p. I, and eqn (14), respectively, and reads

$$(15) \qquad \omega_{R_c} < \pi - \Phi, \qquad \text{if} \qquad \Delta\nu > (1+\nu_m)(1-2\nu_m)/3,$$

$$(16) \qquad \omega_{R_c} > \pi - \Phi, \qquad \text{if} \qquad \Delta\nu < (1+\nu_m)(1-2\nu_m)/3.$$

At this place some additional remarks should be made before proceeding with the further analysis. First of all, it can be stated that the general approach developed in the present study, reduces the entire problem of the elastic-plastic response of the considered fibrous composites to a plane plasticity problem, which has a close analogy to the well-known classical plane stress perfect plasticity problem (cf. for example Kachanov [6]). The latter problem also involves a yield ellipse, for which formally $\overset{*}{\varepsilon}{}_z^e \equiv 0$ and $\Phi = \pi/6$ hold true. By this analogy, the present problem can be approached in the following way. It is in fact a matter of routine procedures to prove that the yield ellipse from eqn (10), p. I, involves arcs of hyperbolicity and ellipticity as well as points of parabolicity. In particular, the arcs $\Phi < \omega < \pi - \Phi$ and $\pi - \Phi < \omega \leqq \pi$ of the latter ellipse are arcs of hyperbolicity and ellipticity, respectively, and the point $\omega = \pi - \Phi$ is a point of parabolicity. As the mathematical plasticity theory shows (Kachanov [6]), the regime of plastic redistribution of stresses, associated with these arcs and points, possess both overall and local specific features. Thus, one should necessarily distinguish between the latter regimes. This general conclusion reveals itself the importance of the above derived relations (14) — (16). They clearly indicate that different regimes of plastic deformation may develop in the matrix phase depending upon the value of the difference $\Delta\nu$ of Poisson's ratio. Thereby for large values of $\Delta\nu$ the relations (14) and (15) predict that a hyperbolic stress state will initially develop in the second plastic zone. In the case of small $\Delta\nu$-values (cf. relation (16)) there exists an elliptic state of stress in the plastic zone. Furthermore, the analysis from p. I proves that the process of matrix cooling induces a hyperbolic regime of plastic deformation in the matrix.

Thus in accordance with the foregoing considerations it would be reasonable to separate the analysis of the cases, corresponding to the relations (15) and (16) respectively. Thereby the terms "hyperbolic" and "elliptic" will be used in the following just in order to distinguish between these cases. The analysis itself will keep the structure of p. I. Certain general considerations, concerning the analogy with the plane stress perfect plasticity problem, are to be found in the authors' article Herrmann & Mihovsky [7].

HYPERBOLIC CASE

It is clear from the conclusions derived above that in this case eqn (8) is to be solved within the interval

$$(17) \qquad \omega_{R_c} \leqq \omega_{r_f} \leqq \pi - \Phi$$

along with the boundary condition

(18) $$\varepsilon_z|_{\omega_{r_f}=\omega_{R_c}} = \overset{*e}{\varepsilon_z},$$

where ω_{R_c} is defined by eqn (14) and satisfies relation (15), while the value $\overset{*e}{\varepsilon_z}$ follows from eqns (7), (13). In this case eqn (8) implies positive $d\omega_{r_f}$-values, i.e. the increasing loading, and thus the increasing axial strain ε_z leads to an increase in the angle ω_{r_f}. By applying the procedure from p. I the $\omega_{r_f}(\varepsilon_z)$-dependence can be obtained as an approximate solution of the problem, specified by eqns (8), (17), (18), respectively. This solution reads (when the principal terms are considered only)

(19) $$\omega_{r_f} = \omega_{R_c} + b_1 \Lambda_1 \Delta\varepsilon_z,$$

where

(20) $$b_1 = 2\nu_f \frac{\tan\Phi}{1-\nu_f},$$

(21) $$\Delta\varepsilon_z = \varepsilon_z - \overset{*e}{\varepsilon_z}.$$

The axial strain difference $\Delta\varepsilon_z^*$, at which failure of the unit cell takes place, follows formally from eqn (19) with $\omega_{r_f} = \pi - \Phi$ to be

(22) $$\Delta\varepsilon_z^* = \frac{\pi - \Phi - \omega_{R_c}}{b_1\Lambda_1}.$$

As it will be explained below, eqn (22) is a formal one. It assumes implicitly that the failure takes place before the entire plastification of the matrix, which is not the actual case in the considered problem (in contrast to the thermal one).

Equations (16), (17), (20) from p. I for the plastic zone radius R_c apply in the present case as well with the new ω_{R_c}-value from eqn (14). Therefore, by analogy with the thermal case and with eqn (19) the $R_c(\varepsilon_z)$-dependence reads

(23) $$R_c^2(\Delta\varepsilon_z) = R_c^{*2}\left[1 - \left(1 - \frac{r_f^2}{R_c^{*2}}\right)\left(1 - \frac{\Delta\varepsilon_z}{\Delta\varepsilon_z^*}\right)^2\right].$$

Finally, the condition of equilibrium of the axial forces

(24) $$r_f^2\sigma_z^f + (r_m^2 - R_c^2)\sigma_z^{me} + 2\int_{r_f}^{R_c} \sigma_z^{mp} r\,dr = r_m^2\sigma_z^c$$

leads to the following forms of the $\sigma_z^c(\varepsilon_z)$-dependence again by the only consideration of principal terms

(25) $$\Delta\sigma_z^c = \Delta\varepsilon_z E_m \left(1 + E_c - \frac{R_c^2}{r_m^2}\right),$$

or

(26) $$\Delta\sigma_z^c = \Delta\varepsilon_z \frac{E_m(1+E_c)}{1 + \frac{2}{1+E_c}\frac{R_c^{*2} - r_f^2}{r_m^2}\frac{\Delta\varepsilon_z}{\Delta\varepsilon_z^*}}.$$

The function $\Delta\sigma_z^c(\Delta\varepsilon_z)$ is easily seen to be convex and to deviate smoothly from the straight $\sigma_z(\varepsilon_z)$-line defined by eqn (4). Further, the analysis of the composite response, eqn (26), in the sense of that from p. I, is now performable straightforwardly. But in the present case such a detailed analysis is actually not necessary for the following reason. Eqn (26) reflects the response of the composite within the short interval $[\overset{*e}{\varepsilon}_z, \varepsilon_z^{t.pl.}]$, where $\varepsilon_z^{t.pl.}$ stays for the value of the ε_z-strain at which total (complete) matrix plastification takes place.

Upon introducing

(27) $$\Delta\varepsilon_z^{t.pl.} = \varepsilon_z^{t.pl.} - \overset{*e}{\varepsilon}_z$$

it is clear that eqns (22), (26) would be of actual importance if the failure of the composite takes place before the total plastification of the matrix, i.e. if $\Delta\varepsilon_z^* < \Delta\varepsilon_z^{t.pl.}$. Thus, the next specific question that needs to be cleared up is which of the two latter phenomena takes place at first. This question concerns, first of all, the determination of the $\varepsilon_z^{t.pl.}$-value.

In solving this question it should be firstly mentioned that in the present case the plastification of the matrix will lead to a reduction of the cross-sectional stresses in the remaining elastically deforming region. The plastic zone radius remains (in contrast to the thermal case) always much smaller than r_m, i.e. $R_c^* \ll r_m$, which is simply due to the smallness of the coefficient $(\pi - \Phi - \omega_{R_c})$ in the exponent in eqn (20), p. I. Accordingly, the themselves small σ_i^{me}-stresses ($i = r, \theta$), acting in the elastic matrix region $R_c \leqq r \leqq r_m$, decrease further. This result allows a consideration of the stress state in this region as approaching a state of pure axial tension with $\sigma_z^{me} = E_m\varepsilon_z$. Thus the plastification of this entire region takes suddenly place when $\sigma_z^{me} = \sigma_y$. This result defines the value $\varepsilon_z^{t.pl.}$ as

(28) $$\varepsilon_z^{t.pl.} = \varepsilon_y = \frac{\sigma_y}{E_m}.$$

The foregoing equations allow to prove that the relation $\Delta\varepsilon_z^* > \Delta\varepsilon_z^{t.pl.}$ holds practically always true. This is consistent with the typical observations of the behaviour of the fibrous composites. Thereby the occurrence of the stage of elastic-plastic behaviour with a completely plastified matrix precedes the failure of the composite. Thus eqn (19) (respectively eqn (26)) is valid only in the interval [$\overset{*e}{\varepsilon}_z$, $\varepsilon_z^{t.pl.}$]. The quantity $\Delta\varepsilon_z^*$, defined by eqn (22), is itself not a real characteristic of the composite. Furthermore, the values of $\Delta\varepsilon_z^{t.pl.}$ and of the corresponding stress of total plastification $\sigma_z^{c,t.pl.}$ (respectively $\Delta\sigma_z^{c,t.pl.}$) follow from eqn (7), (13), (25), (27), (28) by considering principal terms only

(29) $$\Delta\varepsilon_z^{t.pl.} = \frac{\sigma_y}{E_m}\frac{3(\Delta\nu)^2}{2(1+\nu_m)^2},$$

(30) $$\Delta\sigma_z^{c,t.pl.} = \sigma_z^{c,t.pl.} - \sigma_z^{c,pl.} = \sigma_y(1+E_c)\frac{3(\Delta\nu)^2}{2(1+\nu_m)^2}.$$

Upon introducing the notations

(31) $$\Delta\varepsilon_{z,2} = \varepsilon_z - \varepsilon_y,$$

(32) $$\Delta\sigma_{z,2}^c = \sigma_z^c - \sigma_z^{c,t.pl.},$$

and by applying the equilibrium condition of the axial forces the $\sigma_z^c(\varepsilon_z)$-dependence for the considered stage can be given in the form

(33) $$\Delta\sigma_{z,2}^c = E_m E_c \Delta\varepsilon_{z,2}.$$

Thereby eqn (33) is nothing else but the known lower bound estimation of the elastic-plastic response of the considered class of fibrous composites, derived by Hill [4]. The interpretation of eqn (33) in the "rule of mixture" sense with a negligible contribution of the plastified matrix phase (cf. Kelly [3] and the introduction to p. II) is straightforward.

A qualitative purely schematic illustration of the overall response of the considered composites, as predicted by the present analysis, is given in Fig. 1. Thereby the straight lines I and III correspond to eqns (4) and (33), respectively, while the straight line II is the linear approximation of the dotted one, to which eqn (26) corresponds. The strain $\varepsilon_z^* = \varepsilon_z^{t.pl.} + \Delta\varepsilon_{z,2}^*$ with $\varepsilon_z^{t.pl.}$ defined by eqn (28), is the strain at which the failure modes, predicted by the model of Herrmann & Mihovsky [1], start actually developing in the composite cell. How these failure modes occur upon the complete matrix plastification is a problem with the solution of which the analysis of the hyperbolic case will be entirely closed. To this regard eqn (8) proves that with further loading of the composite, i.e. upon $\sigma_z^{c,t.pl.}$, the angle ω_{r_f} further increases and finally approaches the angle $(\pi - \Phi)$. In addition, it can be shown that in accordance with the u_r-continuity condition at $r = R_c$, i.e. at the boundary between the two plastic zones, the angle ω_{R_c} increases as well but it remains, at

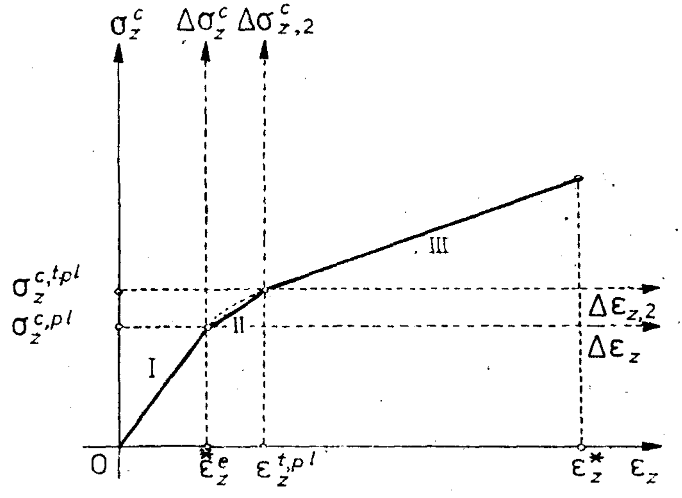


Fig. 1. Qualitative schematic illustration of the response of fibrous composites under longitudinal extension

the same time, smaller than ω_{r_f}. Thus, it is the latter angle that first achieves the critical value $(\pi - \Phi)$ to which the occurrence of the failure modes of the composites at the fibre-matrix interface corresponds.

It should be pointed out that these conclusions result, in fact, from a relatively complicated analysis. The latter involves a second yield ellipse (with $\overset{*}{\varepsilon}{}_z^e = \varepsilon_y$), the σ_r-continuity condition at $r = R_c$, as well as a jump in the σ_θ-stress at $r = R_c$. The occurrence of this jump results from the sudden change in the process of plastic stress redistribution, caused by the sudden entire matrix plastification. The stress state in the suddenly plastified matrix annulus $R_c \leqq r \leqq r_m$ with $\sigma_r \approx \sigma_\theta \approx 0$ corresponds to the vertex $\omega = \pi$ of the second yield ellipse. This vertex belongs to an arc of ellipticity of the latter. The necessity of introducing this ellipse reflects the fact that a sudden plastification of the elastically deformed matrix region corresponds to a value $\overset{*}{\varepsilon}{}_z^e$ of the axial strain in this region, which is equal to ε_y.

Further, eqn (17), p. I, with ω_{R_c} from eqn (14) now proves that with the behaviour of ω_{r_f} and ω_{R_c} described above the plastic zone radius R_c decreases with progressive loading. This is a natural result since the large plastic zone $R_c \leqq r \leqq r_m$ should be really expected to reduce the stress concentration within the thin plastic layer $r_f \leqq r \leqq R_c$, surrounding the fibre, and to reduce in this way its size R_c as well.

When solved in the terms of $\Delta\varepsilon_{z,2}$ and $\Delta\omega_{r_{f,2}} = \omega_{r_f} - \omega_{r_f}^{t.pl.}$, (cf. eqns (19), (29)) the u_r-continuity condition, eqn (8), will imply with $\omega_{r_f} = \pi - \Phi$ the actual critical $\Delta\varepsilon^*_{z,2}$-strain difference, respectively, the critical ε^*_z-strain (cf. Fig. 1) at which modes of failure of the composite will start developing. The determination of this critical strain difference $\Delta\varepsilon^*_{z,2}$ is a matter of simple computations, and the strain ε^*_z is equal to $\varepsilon^{t.pl.}_z + \Delta\varepsilon^*_{z,2}$ (cf. Fig. 1).

ELLIPTIC CASE

This case corresponds to relatively small values of $\Delta\nu$, for which the inequality (16) holds true. The stress concentration effect is smaller than in the foregoing case. The initial value of ω_{r_f}, i.e. ω_{R_c}, belongs to the interval $[\pi - \Phi, \pi]$. The speciality of this case is associated with the behaviour of the function $f_1(\omega_{r_f})$ from eqn (8). This function changes its sign when ω_{r_f} runs through the value

$$(34) \qquad \overline{\omega}_{r_f} = \arctan[-(1 - 2\nu_f)\tan\Phi].$$

The angle $\overline{\omega}_{r_f}$ obviously belongs to the interval $[\pi - \Phi, \pi]$. It is easy to prove at the same time that irrespectively of whether the ω_{R_c}-value is larger or smaller than $\overline{\omega}_{r_f}$, the change in the sign of the function $f_1(\omega_{r_f})$ at $\omega_{r_f} = \overline{\omega}_{r_f}$ guarantees that upon the occurence of the second plastic zone the angle ω_{r_f} in any case will achieve the value $\overline{\omega}_{r_f}$, and that further development of the matrix plastification process will be possible with this constant value $\overline{\omega}_{r_f}$ of ω_{r_f}. It is reasonable to accept that the progressive loading causes again an increase of the plastic zone radius R_c. Then, in accordance with eqn (17), p. I (with $\omega_{r_f} = \overline{\omega}_{r_f}$ now), this increase should be due to the increase of the angle ω_{R_c} in the interval $[\pi - \Phi, \pi]$. At the instant when $\omega_{R_c} = \pi$ the plastic zone radius R_c becomes infinitely large, i.e. sudden total matrix plastification takes place. Note, that the point $\omega = \pi$ belongs to an arc of ellipticity of the yield ellipse (p. I, eqn (11)).

The effect of constancy of the angle ω_{r_f} and of the sudden entire matrix plastification can be explained in the following way. As eqn (8) shows, in this case the angle ω_{r_f} changes upon the occurrence of the second plastic zone $r_f \leqq r \leqq R_c$ between the themselves close values ω_{R_c} and $\overline{\omega}_{r_f}$. The plastic zone radius R_c remains small again, i.e. comparable with the fibre radius r_f (p. I, eqn (17)). At the same time the thinness of the plastic zone reflects the very low stress concentration effect of the fibre. Since the plastification itself further reduces the latter effect, it can be assumed that at a certain instant of the plastification process the radial dependence of the stresses in the thin plastic zone becomes negligible. Then, due to the σ_r-continuity condition at $r = r_f$, the fibre and the thin plastic coating around it could be considered at this instant just as forming an "elastic" core $r_f \leqq r \leqq R_c$ of the composite cell, which expands with an "increasing" Poisson's ratio ν_c. The core is "elastic" in the sense that the existing radial stress in it satisfies, as in the homogeneous linearly elastic fibre material, the relation $\sigma_r = \sigma_r|_{r=R_c}$ (p. I, eqns (1)). The "increase" of the ν_c-ratio is due to the plastic incompressibility of the strain in the thin plastic coating as well as due to the expansion of the latter, i.e. due to the increase of its volume fraction. With the concept of the core formation the u_r-continuity condition at $r = r_f$, i.e. within the core now, may be considered as identically satisfied irrespectively of the values of the angle ω_{r_f}. The latter keeps actually the value $\overline{\omega}_{r_f}$. The core spreads into the matrix phase with the increasing ν_c-value and thus reduces the stresses σ_i^{me}, $i = r, \theta$, in the remaining elastic region, since the latter are proportional to the itself decreasing difference $\nu_m - \nu_c$, cf. eqns (1), p. I and eqn (1), respectively. Obviously, this is the above considered stage of deformation with increasing ω_{R_c}-values ($\omega_{R_c} \longrightarrow \pi$). Thus the instant of complete matrix plastification $\omega_{R_c} = \pi$ corresponds to that one at which ν_c becomes equal to ν_m (cf. eqn (14) with $\Delta\nu = \nu_m - \nu_c$).

It should be mentioned without discussing the details that the analysis of the unit cell behaviour upon the instant of complete matrix plastification follows the same basic lines as in the foregoing case. It predicts, as it should be expected, the same response (cf. eqn (33)). The distinguishing feature between the two cases concerns in fact the length of the transitional interval $[\overset{*}{\varepsilon}{}_z^{e}, \varepsilon_z^{t\cdot pl\cdot}]$ (cf. Fig. 1). This interval proves to be even shorter in the present case than the itself short interval from the hyperbolic case.

The basic features of the development of the plastic deformation process, as well as of its influence on the overall response of a fibrous composite, have been considered above in sufficient detail and do not need to be additionally analyzed in a separate section as in p. I. Nevertheless, it would be of interest to summarize both the common and the specific features of the thermal and the mechanical response of the considered class of composites from the view-point of the general approach developed in this study. Such a summary is presented in the next section along with a brief consideraton of these features, concerning the possible applications of the general approach to some problems of the practice of the fibrous composites.

CONCLUDING REMARKS

The general approach, developed in the present study, predicts a realistic elastic-plastic response of the considered class of fibrous composites under both thermal and mechanical loading condition. Quantitatively, the predicted response reflects both the geometrical and the mechanical properties of the composite structure. It is consistent with the "rule of mixtures" description of the composites behaviour commonly adopted in the engineering practice. The response itself is derived as an overall quantitative estimation of the characteristics of the processes of elastic and plastic deformation, respectively, developing simultaneously within the composite structure.

In accordance with the analysis of both the overall and these specific features of these processes different regimes in the development of the matrix plastification process may occur, depending upon the loading status or/and the properties of the constituents. The response of a fibrous composite under the condition of matrix cooling corresponds to a regime of monotonous increase of the plastic zone size. Similar regimes develop initially under longitudinal extension as well. The latter cover, as a rule, a short interval of axial strain changes and are followed by the phenomenon of the sudden entire matrix plastification. The general approach allows to draw a clear analogy between these regimes and the regimes developing in the classical plane stress perfect plasticity problem. From the point of view of this analogy the phenomena of progressive increase of the plastic zone size and entire sudden matrix plastification just reflect the essential properties of the corresponding set of governing equations, if the latter are of the hyperbolic or elliptic type, respectively. Further, the approach relates the change of the type of this set of equations to a parabolic one with the occurrence of specific failure phenomena, connected with the considered class fibrous composites.

The approach described above accounts in a special way for the mutually conquering effects of the matrix ductility and the fibre stiffness. The quantity $\overset{*}{\varepsilon}{}_z^e$ involved in this approach proves to be a reliable average measure of the interactions between these effects. Its identification in the thermal problem is of importance. To this regard corresponding practical procedures are proposed.

As it was mentioned in the introduction to p. I, the basic effects of the matrix plasticity concern its influence on the overall composite response and the improvement of the fracture resistance of the composites to existing structural defects. Along with the clarification of the first of these effects the present approach allows to derive definite conclusions with respect to the second one as well. Thereby it is clear to this regard that in the case of longitudinal extension the plasticity of the matrix reduces the cross-sectional stresses. Accordingly, if defects are present in the matrix phase, which are sensitive to these stresses, then one should account for the possible growth of such defects only within the linear elastic stage of the composite behaviour. The plasticity of the matrix really improves the resistance of the composites to such defects. Typical defects of this type are, for example, the relatively short cracks which, when reffered to the unit cell cross-section, may be considered as radial cracks. Such cracks occur very often during the processes of thermal treatment involved in the fabrication of the composites.

The plasticity of the matrix does not reduce the sensitivity of the composites to such cracks under the conditions of matrix cooling. In this case the circumferential stress at the front $r = R_c$ of the plastic zone is relatively large. The enlargement of this zone results in a relative increase of this stress in the points, traversed by the front. Therefore, if a radial crack exists in the elastic matrix region then with progressive matrix cooling the elastic-plastic boundary will approach the crack tip and imply larger stress concentration there. Such a crack, even if it was in equilibrium in the elastic stage of the composite behaviour, may start propagating due to the progressive process of matrix plastification. Thus, in that case the plasticity of the matrix does not improve the fracture resistance of a composite. Moreover, the same relatively large circumferential stress, carried by the propagating plastic zone front, may be considered as the reason for the occurrence of such cracks during the processes of thermal treatment involved in the fabrication of the composites.

Thereby, as it was mentioned, the problem of matrix cooling is considered as modelling the real fabrication problem of cooling the entire composite structure. In fact, the basic lines of the analysis from p. I apply to the latter problem as well if, roughly speaking, the term α_m in the thermal analysis is replaced by $\Delta\alpha = \alpha_m - \alpha_f$.

One may then expect by analogy with the case of longitudinal extension that depending upon the specific value of $\Delta\alpha$ different regimes of plastic deformation may develop in the matrix phase during the fabrication process of cooling of the entire composite structure. Accordingly, by using the present approach a development of fabrication technologies should be possible which would at least reduce, if not entirely prevent, the undesired radial cracking of the composites and therefore also the propagation of existing radial cracks respectively. Similar applications of the approach, based upon the suitable choice of the $(\Delta\alpha, \Delta\nu)$-combinations, may be of importance in problems concerning both the load-bearing capacities and the crack sensitivity of the considered composites at low temperatures.

ACKNOWLEDGEMENT

The authors thank the Deutsche Forschungsgemeinschaft for its financial support.

REFERENCES

1. H e r r m a n n, K., I. M. M i h o v s k y. — Plastic Behaviour of Fibre-Reinforced Composites and Fracture Effects. — Rozprawy Inzinierskie-Engn. Trans., **31** (No 2), 1983, 165 - 177.
2. H e r r m a n n, K., I. M. M i h o v s k y. — Plastic Zone Size of Dugdale Type Cracks in a Self-Stressed Two-Phase Medium with Partially Plastified Matrix Material. — Mechanika Teoretyczna i Stosowana, **22** (No 1/2), 1984, 25 - 39.
3. K e l l y, A. — Strong Solids. Clarendon Press, Oxford, 1973
4. H i l l, R. — Theory of Mechanical Properties of Fibre-Strengthened Materials: II. Inelastic Behaviour. — J. Mech. Phys. Solids, **12**, 1964, 213 - 218.
5. E b e r t, L. J., S. S. H e c k e r, C. H. H a m i l t o n. — The Stress-Strain Behaviour of Concentric Composite Cylinders. — J. Comp. Materials, **2**, 1968, 458 - 476.
6. K a c h a n o v, L. M. — Foundations of the Theory of Plasticity. North-Holland, Amsterdam — London, 1971.
7. H e r r m a n n, K., I. M. M i h o v s k y. — On the Elastic-Plastic Transition in Fibrous Composites Under Mechanical and Thermal Loading Conditions. — ZAMM, **68** (4), 1988, T193 - T194.

Received 2. VII. 1990

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 83, 1989

ON THE OPTIMAL THIRD-ORDER BOUNDS ON THE EFFECTIVE ELASTIC MODULI OF RANDOM DISPERSIONS OF SPHERES

KONSTANTIN MARKOV, KRASSIMIR ZVYATKOV

Константин Марков, Красимир Цвятков. ОБ ОПТИМАЛЬНЫХ ГРАНИЦ ТРЕТЬЕГО ПОРЯДКА ДЛЯ ЭФФЕКТИВНЫХ УПРУГИХ МОДУЛЕЙ СЛУЧАЙНЫХ СУСПЕНЗИЙ СФЕР.

Исследуется вопрос оптимальности вариационных границ Берана-Молине, Маккоя и др. для эффективных модулей упругости двухфазных сред. Оптимальность понимается в смысле получения наиболее узких границ при учете только статистической информации, необходимой для подсчета этих границ, именно двух- и трехточечных корреляционных функций. На примере случайной суспензии сфер показано, что аналогично скалярному случаю, эти границы в общем случае неоптимальны. Оптимальность имеет место лишь до порядка c^2, где c объемная концентрация сфер. Для суспензий границы Берана-Молине и Маккоя подсчитаны явно до порядка c^2 и полученные результаты использованы для исследования применимости некоторых эвристических методов механики композитных материалов.

Konstantin Markov, Krassimir Zvyatkov. ON THE OPTIMAL THIRD-ORDER BOUNDS ON THE EFFECTIVE ELASTIC MODULI OF RANDOM DISPERSIONS OF SPHERES.

The problem of optimality of the variational bounds, due to Beran-Molyneux, McCoy, et al., on the effective elastic moduli of two-phase random media is considered. Optimality

is understood in the sense that bounds should be the tightest ones that use the statistical information needed for their evaluation; for the said bounds these are the two- and three-point correlation functions for the medium. For random dispersion of spheres it is shown that the bounds are optimal to the order c^2 only, where c is the volume fraction of the spheres. The Beran-Molyneux and McCoy bounds are then explicitly calculated to the order c^2 for the dispersions and used for a study of applicability of some known schemes of mechanics of composite media.

INTRODUCTION

The paper is devoted to the problem of variational bounding of the effective elastic moduli of two-phase random media. Generalizing the scalar conductivity arguments of [1] we first rederive the Beran-Molyneux [2] and the McCoy [3] bounds on the effective bulk and shear moduli of the media, respectively, as simple Ritz-type approximation within the frame of the general variational procedure given in [4]. Then we pose the central for the paper problem of the optimality of the said bounds. Optimality is understood here in the sense that they should be the tightest ones that use the statistical information needed for their evaluation. For the said bounds these are the two- and three- point correlation functions for the medium. Similarly to the scalar conductivity case [1], it appears that the Beran-Molyneux and the McCoy bounds are not optimal in general. For random dispersions of spheres, however, they are optimal to the order c^2, where c denotes the volume fraction of the spheres. We next calculate explicitly the said bounds to the order c^2. The so-obtained c^2-bounds represent, in particular, a rigorous basis for a comparison with the predictions of some heuristic models in mechanics of composite materials. In this way certain conclusions (mostly negative), concerning the applicability of some known formulas in elasticity of random dispersions, are finally reached.

THE BOUNDING PROCEDURE IN THE ELASTIC CASE

Consider a two-phase elastic random medium, which is statistically homogeneous and isotropic. For definiteness in this moment only we shall call constituents filler and matrix. We assume the constituents isotropic, so that the fourth-rank tensor of elastic moduli of the medium, $\boldsymbol{L}(\boldsymbol{x})$, is a random field of the form

$$\boldsymbol{L}(\boldsymbol{x}) = 3k(\boldsymbol{x})\boldsymbol{J}' + 2\mu(\boldsymbol{x})\boldsymbol{J}'', \tag{2.1a}$$

where $\boldsymbol{J}'$ and $\boldsymbol{J}''$ are the basic isotropic fourth-rank tensors with the Cartesian components

$$J'_{ijkl} = \frac{1}{3}\delta_{ij}\delta_{kl}, \qquad J''_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right),$$

and

(2.1b)
$$k(\boldsymbol{x}) = \langle k \rangle + k'(\boldsymbol{x}), \qquad k'(\boldsymbol{x}) = [k] I'(\boldsymbol{x}),$$
$$\mu(\boldsymbol{x}) = \langle \mu \rangle + \mu'(\boldsymbol{x}), \qquad \mu'(\boldsymbol{x}) = [\mu] I'(\boldsymbol{x}),$$

$[k] = k_f - k_m$, $[\mu] = \mu_f - \mu_m$, k and μ stand everywhere for the bulk and shear modulus, respectively. Hereafter, all quantities, pertaining to the filler, are supplied with the subscript "f" and those for the matrix — with "m", the volume fraction of the filler and matrix are respectively c and $1 - c$. In (2.1b) $I'(\boldsymbol{x}) \doteq I(\boldsymbol{x}) - c$ is the fluctuating part of the indicator function $I(\boldsymbol{x})$ for the region, occupied by the filler constituent, i.e.

(2.2)
$$I(\boldsymbol{x}) = I_f(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in \text{filler}, \\ 0, & \text{if } \boldsymbol{x} \in \text{matrix}, \end{cases}$$

The Lame equations for the medium, at the absense of body forces, read

(2.3a)
$$\nabla \cdot \sigma(\boldsymbol{x}) = 0, \qquad \sigma(\boldsymbol{x}) = \boldsymbol{L}(\boldsymbol{x}) : \varepsilon(\boldsymbol{x}),$$

where σ denotes the stress tensor, $\varepsilon = \frac{1}{2}(\nabla \boldsymbol{u} + \boldsymbol{u}\nabla)$ is the small strain tensor generated by the displacement field $\boldsymbol{u}(\boldsymbol{x})$, the colon denotes contraction with respect to two pairs of indices. In the isotropic case under consideration we have

(2.4)
$$\sigma(\boldsymbol{x}) = k(\boldsymbol{x})\theta(\boldsymbol{x})\boldsymbol{I} + 2\mu(\boldsymbol{x})\boldsymbol{d}(\boldsymbol{x}),$$

(2.5)
$$\varepsilon(\boldsymbol{x}) = \frac{1}{3}\theta(\boldsymbol{x})\boldsymbol{I} + \boldsymbol{d}(\boldsymbol{x}), \qquad \theta(\boldsymbol{x}) = \operatorname{tr} \varepsilon(\boldsymbol{x}),$$

(cf. (2.1)) so that (2.5) is the decomposition of the strain tensor as a sum of its spherical and deviatoric parts, $\boldsymbol{I}$ stands here for the unit second-rank tensor.

We prescribe also the average strain tensor $\boldsymbol{E}$, imposed on the medium

(2.3b)
$$\langle \varepsilon(\boldsymbol{x}) \rangle = \boldsymbol{E},$$

where $\boldsymbol{E}$ is a given symmetrical second-rank tensor, the brackets $\langle \cdot \rangle$ hereafter denote ensemble averaging. Eqns (2.3) represent the basic random problem (with respect to displacements) in elasticity of composite media. This is the elastic counterpart of the scalar problem, considered in [1].

The random problem (2.3) is equivalent to the variational problem

(2.6)
$$W_A[\boldsymbol{u}(\cdot)] = \langle \varepsilon(\boldsymbol{x}) : \boldsymbol{L}(\boldsymbol{x}) : \varepsilon(\boldsymbol{x}) \rangle \longrightarrow \min.$$

The functional W_A is considered over the class of random fields $\boldsymbol{u}(\boldsymbol{x})$, which generate strain fields $\varepsilon(\boldsymbol{x})$ satisfying (2.3b). Moreover, $\min W_A = \boldsymbol{E} : \boldsymbol{L}^* : \boldsymbol{E}$, where $\boldsymbol{L}^*$ is the tensor of effective elastic moduli for the medium. In the statistically isotropic case under consideration $\boldsymbol{L}^* = 3k^*\boldsymbol{J}' + 2\mu^*\boldsymbol{J}''$, where k^* and μ^* are the effective bulk and shear modulus of the random medium respectively.

In order to obtain bounds on the effective properties of a random medium it was proposed in [4] to employ certain truncated functional series as classes of trial fields for the respective variational principles. For an elastic medium the class of such trial fields, in the simplest nontrivial case of interest, is

$$K^{(1)} = \{\boldsymbol{u}(\boldsymbol{x}) | \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E} \cdot \boldsymbol{x} + \int \boldsymbol{T}(\boldsymbol{x} - \boldsymbol{y}) I'(\boldsymbol{y}) d^3\boldsymbol{y}\}. \tag{2.7}$$

Hereafter the integrals are over the whole R^3, if the integration domain is not explicity indicated.

The energy functional (2.6), when restricted over the class (2.7), becomes an usual functional of the nonrandom kernel $\boldsymbol{T}(\boldsymbol{x})$, namely

$$W_A[\boldsymbol{T}(\cdot)] = \langle\lambda\rangle \mathrm{tr}^2 \boldsymbol{E} + 2\langle\mu\rangle \boldsymbol{E} : \boldsymbol{E}$$

$$+2 \int \{[\lambda] \nabla \cdot \boldsymbol{T}(\boldsymbol{y}) + 2[\mu] \boldsymbol{E} : \mathrm{def}\ \boldsymbol{T}(\boldsymbol{y})\} M_2(\boldsymbol{y}) d^3\boldsymbol{y}$$

$$+2 \int\int \nabla \cdot \boldsymbol{T}(\boldsymbol{y}_1) \nabla \cdot \boldsymbol{T}(\boldsymbol{y}_2) \{\langle\lambda\rangle M_2(\boldsymbol{y}_1 - \boldsymbol{y}_2) + [\lambda] M_3(\boldsymbol{y}_1, \boldsymbol{y}_2)\} d^3\boldsymbol{y}_1 d^3\boldsymbol{y}_2$$

$$+2 \int\int \mathrm{def}\ \boldsymbol{T}(\boldsymbol{y}_1) : \mathrm{def}\ \boldsymbol{T}(\boldsymbol{y}_2) \{\langle\mu\rangle M_2(\boldsymbol{y}_1 - \boldsymbol{y}_2) + [\mu] M_3(\boldsymbol{y}_1, \boldsymbol{y}_2)\} d^3\boldsymbol{y}_1 d^3\boldsymbol{y}_2,$$

where $\mathrm{def}\ \boldsymbol{T}(\boldsymbol{x}) = \frac{1}{2}(\nabla \boldsymbol{T}(\boldsymbol{x}) + \boldsymbol{T}(\boldsymbol{x}) \nabla)$,

$$M_2(\boldsymbol{x}) = \langle I'(0) I'(\boldsymbol{x}) \rangle, \qquad M_3(\boldsymbol{x}, \boldsymbol{y}) = \langle I'(0) I'(\boldsymbol{x}) I'(\boldsymbol{y}) \rangle$$

are the two- and three-point moments of the indicator field $I(\boldsymbol{x})$, defined in (2.2). Hereafter the differentiation is with respect to $\boldsymbol{x}$.

The Euler-Lagrange equation for the functional $W_A[\boldsymbol{T}(\cdot)]$ reads

$$\boldsymbol{E} : [\boldsymbol{L}] \cdot \nabla M_2(\boldsymbol{x}) + \int \nabla M_2(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{L}_m : \mathrm{def}\ \boldsymbol{T}(\boldsymbol{y}) d^3\boldsymbol{y}$$
$$+ \int \nabla M_3(\boldsymbol{x}, \boldsymbol{y}) \cdot [\boldsymbol{L}] : \mathrm{def}\ \boldsymbol{T}(\boldsymbol{y}) d^3\boldsymbol{y} = 0. \tag{2.8}$$

It could be shown, employing simple convexity arguments, that the solution of eqn (2.8) does exist and is unique. The solution, $\boldsymbol{T}(\boldsymbol{y})$, is to be inserted into the second equation of (2.3a) which, upon averaging, will bring forth certain upper bounds $k^{(3)}$ and $\mu^{(3)}$ on the effective bulk and shear moduli of the medium. The superscript "3" indicates that the evaluation of the bounds $k^{(3)}$ and $\mu^{(3)}$ requires knowledge of the r-point moments for the field $I(\boldsymbol{x})$ up to $r = 3$. In this sense these bounds are called third-order, similarly to the scalar conductivity case [1, 4]. More important, it could be shown, extending the scalar conductivity arguments of [4], that $k^{(3)}$ and $\mu^{(3)}$ are the optimal third-order bounds in the sense that they are the best ones which can be obtained, making use of the said statistical information, i.e. M_2 and M_3 only.

The explicit solution of the integro-differential equation (2.8) is very difficult in general. That is why we introduce, after [5], a simpler procedure. Let $\widetilde{\boldsymbol{T}}(\boldsymbol{x})$ be a fixed kernel. Consider the set of trial fields

$$
\widetilde{K}^{(1)} = \{\boldsymbol{u}(\boldsymbol{x}) | \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E} \cdot \boldsymbol{x} + \alpha \int \widetilde{\boldsymbol{T}}(\boldsymbol{x} - \boldsymbol{y}) I'(\boldsymbol{y}) d^3\boldsymbol{y}\} \subset K^{(1)}, \tag{2.9}
$$

where $\alpha \in R^1$ is adjustable parameter. The functional W_A, when restricted on $\widetilde{K}^{(1)}$, becomes a quadratic function of α, whose monimization brings forth certain third-order bounds $\widetilde{k}^{(3)}$ and $\widetilde{\mu}^{(3)}$ on the effective bulk and shear moduli. Such bounds, due to obvious reasons, are called in [5] Ritz-type ones. Though not optimal in general, the bounds $\widetilde{k}^{(3)}$ and $\widetilde{\mu}^{(3)}$ could be explicitly evaluated, if the kernel $\widetilde{\boldsymbol{T}}(\boldsymbol{x})$ is skillfully chosen. As a matter of fact, Beran and Molineux [2] and McCoy [3] have pointed out that such a choice of the kernel is supplied by the first-order terms in the perturbation solution of the basic elastic problem (2.3). (A similar observation in the scalar conductivity case is due again to Beran ([6].) In our terminology the above mentioned authors have calculated the bounds $\widetilde{k}^{(3)}$ and $\widetilde{\mu}^{(3)}$ for the said choice of the kernel $\widetilde{\boldsymbol{T}}(\boldsymbol{x})$. Their derivations will be repeated below in the frame of our scheme and then the problem of optimality of the respective bounds for random dispersions of spheres will be addressed. But before this it is necessary that the perturbation solution of the elastic problem (2.3) should be considered at some length.

PERTURBATION SOLUTION OF THE BASIC ELASTIC PROBLEM (8.3)

Let the medium be weakly inhomogeneous, i.e. the ratios

$$
\delta k = \max_{\boldsymbol{x}} \frac{|k'(\boldsymbol{x})|}{\langle k \rangle}, \qquad \delta\mu = \max_{\boldsymbol{x}} \frac{|\mu'(\boldsymbol{x})|}{\langle \mu \rangle} \tag{3.1}
$$

are small, δk, $\delta\mu \ll 1$, noting, however, that δk, $\delta\mu$ may be small of different orders of magnitude. Consider the perturbation series for the displacement field that solves the problem (2.3)

$$u(x) = u^{(0,0)}(x) + u^{(1,0)}(x) + u^{(0,1)}(x) + \sum_{p,q=1}^{\infty} u^{(p,q)}(x), \tag{3.2}$$

where $u^{(0,0)}(x) = E \cdot x$ and $u^{(p,q)}(x)$ has the order of magnitude $(\delta k)^p(\delta\mu)^q$, besides, $\langle u^{(p,q)}(x)\rangle = 0$, p, $q = 0, 1, \ldots$, $p^2 + q^2 \neq 0$.

On introducing (3.2) into (2.3a), we get straightforwardly

$$u^{(1,0)}(x) = \frac{3\text{tr } E}{3\langle k\rangle + 4\langle\mu\rangle} \int \nabla \frac{1}{4\pi|x-y|} k'(y) d^3y, \tag{3.3}$$

$$u^{(0,1)}(x) = \frac{1}{2\pi\langle\mu\rangle} E_d : \int \left\{ \nabla \frac{1}{|x-y|} \otimes I + \overline{\varkappa} \nabla\nabla\nabla |x-y| \right\} \mu'(y) d^3y, \tag{3.4}$$

where $E_d = E - \frac{1}{3} I \text{tr } E$ is the deviatoric average strain, hereafter all gradients are with respect to x, $\nabla = \nabla_x$, and

$$\overline{\varkappa} = -\frac{1}{4(1-\overline{\nu})}, \qquad \overline{\nu} = \frac{3\langle k\rangle - 2\langle\mu\rangle}{6\langle k\rangle + 2\langle\mu\rangle}, \tag{3.5}$$

so that $\overline{\nu}$ is the Poisson ratio of a medium with elastic moduli $\langle k\rangle$ and $\langle\mu\rangle$. (Note that $\overline{\nu} \neq \langle\nu\rangle$.) The well-known Green tensor for the Lame equation in the isotropic case is used in an obvious manner, when deriving (3.3) and (3.4).

The reformulation of the problem (2.3) for the stress field is well-known (cf. [7]):

$$\sigma(x) = \nabla \times \Phi(x) \times \nabla; \tag{3.6a}$$

$$\nabla \times (M(x) : \sigma(x)) \times \nabla = 0. \tag{3.6b}$$

Here $M(x) = L^{-1}(x)$ is the fourth-rank compliance tensor field for the medium, and $\Phi(x)$ is the symmetrical second-rank "tensor potential" field for the stress — the stress function of Maxwell and Morrera, which assures that the equilibrium equation (2.3a) is identically satisfied. Similarly to (2.3b), we prescribe the mean value, Σ, for the stress tensor

$$\langle\sigma(x)\rangle = \Sigma. \tag{3.6c}$$

Eqns (3.6) represent the basic random elastic problem, with respect to stress, in elasticity of composite media. The variational formulation of this problem is the principle of minimum complementary energy. Consider the functional

$$W_B[\Phi(\cdot)] = \langle \sigma(\boldsymbol{x}) : \boldsymbol{M}(\boldsymbol{x}) : \sigma(\boldsymbol{x}) \rangle \longrightarrow \min, \tag{3.7}$$

where the field $\sigma(\boldsymbol{x})$ is the birotor of $\Phi(\boldsymbol{x})$, cf. (3.6a), such that (3.6c) holds. Then the solution $\Phi^*(\boldsymbol{x})$ of the problem (3.6) minimizes W_B, so that $\sigma(\boldsymbol{x}) = \nabla \times \Phi^*(\boldsymbol{x}) \times \nabla$ is the real stress field in the medium. Moreover,

$$\min\ W_B = \Sigma : \boldsymbol{M}^* : \Sigma, \tag{3.8}$$

where $\boldsymbol{M}^* = \boldsymbol{L}^{*-1}$ is the effective compliance tensor of the medium.

In the isotropic case under study we have

$$W_B[\Phi(\cdot)] = \frac{1}{9}\left\langle \frac{1}{k(\boldsymbol{x})} \mathrm{tr}^2 \sigma(\boldsymbol{x}) \right\rangle + \frac{1}{2}\left\langle \frac{1}{\mu(\boldsymbol{x})} \boldsymbol{s}(\boldsymbol{x}) : \boldsymbol{s}(\boldsymbol{x}) \right\rangle, \tag{3.9}$$

$$\min\ W_B[\Phi(\cdot)] = \frac{1}{9k^*} \mathrm{tr}^2 \Sigma + \frac{1}{2\mu^*} \Sigma_d : \Sigma_d, \tag{3.10}$$

where $\boldsymbol{s}(\boldsymbol{x}) = \sigma(\boldsymbol{x}) - \frac{1}{3} \boldsymbol{I} \mathrm{tr}\ \sigma(\boldsymbol{x})$ is the stress deviator and Σ_d is the deviatoric part of the macrostress tensor Σ.

The construction of the Ritz-type lower bounds, similar to the upper ones of Beran and Molineux, needs the first-order perturbation terms in the solution of the random problem (3.6), i.e. the counterparts of the fields $\boldsymbol{u}^{(1,0)}(\boldsymbol{x})$ and $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$, given in (3.3) and (3.4) respectively . As noted by McCoy [3], the straightforward construction of these terms is however lengthy and tedioús. That is why we shall use another scheme of arguments, suggested and, as a matter of fact employed in the same paper [3]. The scheme consists in the following.

Let us insert the perturbation solution (3.2) into the Hooke law

$$\begin{aligned} \sigma(\boldsymbol{x}) &= \boldsymbol{L}(\boldsymbol{x}) : \varepsilon(\boldsymbol{x}) \\ &= \{\langle \boldsymbol{L} \rangle + \boldsymbol{L}'(\boldsymbol{x})\} : \{\boldsymbol{E} + \nabla \boldsymbol{u}_1(\boldsymbol{x}) + o(\delta \boldsymbol{L})\} \\ &= \langle \boldsymbol{L} \rangle : \boldsymbol{E} + \sigma_1(\boldsymbol{x}) + o(\delta \boldsymbol{L}), \end{aligned} \tag{3.11}$$

where

$$\begin{gathered} \sigma_1(\boldsymbol{x}) = \boldsymbol{L}'(\boldsymbol{x}) : \boldsymbol{E} + \langle \boldsymbol{L} \rangle : \nabla \boldsymbol{u}_1(\boldsymbol{x}), \\ \delta \boldsymbol{L} = \max_{\boldsymbol{x},i,j,k,l} |L'_{ijkl}(\boldsymbol{x})|/L, \quad \boldsymbol{L}^2 = \langle L_{\alpha\beta\gamma\delta} L_{\alpha\beta\gamma\delta} \rangle, \end{gathered} \tag{3.12a}$$

$L'(x) = L(x) - \langle L \rangle$ being the fluctuating part of the field $L(x)$. In the isotropic case under consideration we have $\delta L = \max(\delta k, \ \delta\mu)$ and

$$\text{(3.12b)} \qquad u_1(x) = u_1^{(1,0)}(x) + u_1^{(0,1)}(x).$$

On averaging (3.11) we get

$$\text{(3.13)} \qquad \Sigma = \langle L \rangle : E + o(\delta L),$$

so that the field $\sigma_1(x)$, to the order $o(\delta L)$, has the form

$$\text{(3.14)} \qquad \sigma_1(x) = L'(x) : \langle L \rangle^{-1} : \Sigma + \langle L \rangle : \nabla u_1(x).$$

Since $\langle \sigma_1(x) \rangle = 0$, we have in virtue of (3.12) that

$$\sigma(x) = \Sigma + \sigma_1(x) + o(\delta L)$$

and thus $\sigma_1(x)$ is the needed first-order term in the perturbation expansion of the solution of the problem (3.6).

In the isotropic case

$$\sigma_1(x) = \sigma^{(1,0)}(x) + \sigma^{(0,1)}(x),$$

where $\sigma^{(1,0)}(x)$ and $\sigma^{(0,1)}(x)$ have the orders of magnitude δk and $\delta\mu$ respectively. Moreover $\sigma^{(1,0)}(x) = 0$ if $\mu'(x) = 0$, i.e. if the constituents have the same shear modulus, and $\sigma^{(0,1)}(x) = 0$ if $k'(x) = 0$, i.e. if the bulk modulus is the same. The analytic forms of $\sigma^{(1,0)}(x)$ and $\sigma^{(0,1)}(x)$ easily follow from (3.7) and (3.8):

$$\text{(3.15)} \qquad \sigma^{(1,0)}(x) = \frac{1}{3}\frac{k'(x)}{\langle k \rangle} I \operatorname{tr} \Sigma + \frac{1}{3}(3\langle k \rangle - 2\langle \mu \rangle) I \nabla \cdot u^{(1,0)} + \langle \mu \rangle (\nabla u^{(1,0)} + u^{(1,0)} \nabla),$$

$$\text{(3.16)} \qquad \sigma^{(0,1)}(x) = \frac{\mu'(x)}{\langle \mu \rangle} \Sigma_d + \frac{1}{3}(3\langle k \rangle - 2\langle \mu \rangle) I \nabla \cdot u^{(0,1)} + \langle \mu \rangle (\nabla u^{(0,1)} + u^{(0,1)} \nabla).$$

The eventual form of these fields would be obtained, if the expressions (3.3) and (3.4) for $u^{(1,0)}(x)$ and $u^{(0,1)}(x)$ are inserted into (3.15) and (3.16) respectively, and transition from E to Σ is made according to (3.13). Such explicit formulas are not needed in what follows, however, because we can use the respective expressions from the evaluation of the upper bounds, which involve contractions of tensors like $\nabla u^{(1,0)}$ and $\nabla u^{(0,1)}$. Therefore the evaluation of the lower Ritz-type bounds can be readily performed if the respective upper bounds are already calculated. In this way the difficulties that appear, due to the presence of birotors in (3.6), are avoided. That is why we shall give in the following the formulas for the lower bounds without any comments.

BERAN — MOLINEUX (BM) BOUNDS ON THE EFFECTIVE BULK MODULUS

We start with the construction of certain Ritz-type bounds on the effective elastic moduli of the two-phase material, making use of the above constructed first-order perturbation fields $\boldsymbol{u}^{(1,0)}$, $\boldsymbol{u}^{(0,1)}$, $\sigma^{(1,0)}$ and $\sigma^{(0,1)}$. The discussion of the problem of their optimality, in the above explained sense, will be postponed till sec. 6.

Let us consider, after Beran and Molineux [2], the class of trial displacement fields

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E}\cdot\boldsymbol{x} + \alpha\boldsymbol{u}^{(1,0)}(\boldsymbol{x}), \tag{4.1}$$

where $\boldsymbol{E}$ is spherical, $\boldsymbol{u}^{(1,0)}(\boldsymbol{x})$ is given in (3.3) and α is adjustable scalar parameter, cf. (2.9). On inserting (4.1) into the energy functional (2.6) and minimizing the result with respect to α, one gets the following upper bound on the effective bulk modulus k^*, obtained by the above authors:

$$k^* \leqq k^u_{BM}, \qquad k^u_{BM} = \langle k\rangle\{1 - \langle k'^2\rangle^2/K_u\}, \tag{4.2a}$$

where

$$K_u = \{[\langle\lambda\rangle + 2\langle\mu\rangle]\langle k'^2\rangle + \langle\lambda' k'^2\rangle + 2J\}\langle k\rangle, \tag{4.3}$$

$$J = \int\int \langle\mu'(0),\ k'(\boldsymbol{z})k'(\boldsymbol{w})\rangle\nabla\nabla\frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla\frac{1}{4\pi|\boldsymbol{w}|}d^3\boldsymbol{z}d^3\boldsymbol{w}$$

is a certain statistical parameter and $\lambda = k - \frac{2}{3}\mu$ is the Lame constant. Hereafter the prime denotes the fluctuating part of the respective random fields. Simple analysis, based on the relations (2.2), shows that

$$J = [\mu][k]^2 A, \tag{4.4}$$

where A is the dimensionless statistical parameter, introduced as follows

(4.5)
$$A = \int\int i(\boldsymbol{z},\ \boldsymbol{w})\nabla\nabla\frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla\frac{1}{4\pi|\boldsymbol{w}|}d^3\boldsymbol{z}d^3\boldsymbol{w}, \qquad i(\boldsymbol{z},\ \boldsymbol{w}) = \langle I'(0,\ I'(\boldsymbol{z})I''(\boldsymbol{w})\rangle.$$

The lower BM-bound is obtained when the functional (3.9) is minimized over the class of trial stress fields

$$\sigma(\boldsymbol{x}) = \Sigma + \alpha\sigma^{(1,0)}(\boldsymbol{x}), \qquad \alpha\in R^1,$$

with a spherical Σ and $\sigma^{(1,0)}(\boldsymbol{x})$, defined in (3.15). The final result, in the original Beran — Molyneux form, reads

$$k^l_{BM} \leqq k^*, \; (k^l_{BM})^{-1} = \left\langle \frac{1}{k} \right\rangle - \langle k'^2 \rangle^2 / K_l, \qquad (4.2b)$$

$$K_l = \left\langle \frac{k'^2}{k} \right\rangle - \frac{3}{8} \left\langle \frac{k'^2}{\mu} \right\rangle + \frac{9}{8} J',$$

where J' is the statistical parameter

$$J' = \int\int \left\langle \frac{k'(\boldsymbol{z})k'(\boldsymbol{w})}{\mu(0)} \right\rangle \nabla\nabla \frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla \frac{1}{4\pi|\boldsymbol{w}|} d^3\boldsymbol{z} d^3\boldsymbol{w}. \qquad (4.6)$$

Since

$$\frac{1}{\mu(\boldsymbol{x})} = \left\langle \frac{1}{\mu} \right\rangle + \left[\frac{1}{\mu} \right] I'(\boldsymbol{x}), \qquad (4.7a)$$

we have

$$J' = \left\langle \frac{1}{\mu} \right\rangle \langle k'^2 \rangle + \left[\frac{1}{\mu} \right] [k]^2 A, \qquad (4.7b)$$

and thus the upper and lower BM-bounds (4.2) depend on the same statistical parameter A. This parameter appears also in the Beran bounds [6] on the effective conductivity, as it could be easily shown. In turn, BM-bounds may be expressed in a concise form [8] by means of the Milton parameters ξ_1 and ξ_2, defined as

$$\xi_2 = 1 - \xi_2 \qquad (4.8)$$

$$= \frac{9}{c(1-c)} \int\int \frac{d^3\boldsymbol{z} d^3\boldsymbol{w}}{16\pi^2|\boldsymbol{z}|^3|\boldsymbol{w}|^3} \left\{ S_3(\boldsymbol{z}, \; \boldsymbol{w}) - \frac{S_2(\boldsymbol{z})S_2(\boldsymbol{w})}{c} \right\} P_2(u),$$

where $u = \cos\varphi$, φ being the angle between the vectors $\boldsymbol{z}$ and $\boldsymbol{w}$, $P_2(u) = \frac{1}{2}(3u^2 - 1)$ is the Legendre polynomial of order two, and

$$S_2(\boldsymbol{x}) = \langle I(0)I(\boldsymbol{x}) \rangle, \qquad S_3(\boldsymbol{x}, \; \boldsymbol{y}) = \langle I(0)I(\boldsymbol{x})I(\boldsymbol{y}) \rangle \qquad (4.9)$$

are the so-called [9] two- and three-point functions respectively. Let us recall that the quantities $S_2(\boldsymbol{x})$ and $S_3(\boldsymbol{x}, \; \boldsymbol{y})$ are, respectively, the probabilities of finding in the filler phase (phase "2" in our case) the end points O (the origin, chosen arbitrarily) and O' of the line segment $\overrightarrow{OO'} = \boldsymbol{x}$ and the vertices of the triangle $OO'O''$, where $\overrightarrow{OO'} = \boldsymbol{x}$, $\overrightarrow{OO''} = \boldsymbol{y}$.

Note that the relation between the parameters ξ_2 and A is readily deducible from their definitions, if one takes into account that

$$(4.10) \qquad \nabla\nabla\frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla\frac{1}{4\pi|\boldsymbol{w}|} = \frac{6P_2(u)}{|\boldsymbol{z}|^3|\boldsymbol{w}|^3}, \qquad u = \frac{\boldsymbol{z}\cdot\boldsymbol{w}}{|\boldsymbol{z}|\,|\boldsymbol{w}|},$$

and it reads

$$(4.11) \qquad \xi_2 = 1 - \xi_1 = \frac{1}{2}\left(4c + \frac{3A}{c(1-c)} - 1\right).$$

Let the medium have constant shear modulus, i.e. $\mu_f = \mu_m$ and $\mu'(\boldsymbol{x}) = 0$, so that the bulk modulus only varies in position. In this case the lower and the upper BM-bounds coincide yielding the exact values of the effective bulk modulus, namely

$$(4.12) \qquad k^* = \langle k\rangle - \frac{\langle k'^2\rangle}{\langle k\rangle + \frac{4}{3}\langle\mu\rangle + [k](1-2c)}.$$

The same value of k^* can be obtained from the Hashin-Shtrikman bounds [10] on k^*, which also coincide if $\mu_f = \mu_m$. Note that the exact value (4.12) of k^* in the case under consideration was first pointed out by Hill [11].

McCOY (MC) BOUNDS ON THE EFFECTIVE SHEAR MODULUS

The reasoning of McCoy [3] is fully similar to that in sec. 4, namely we assume that $\operatorname{tr}\boldsymbol{E} = 0$, i.e. the macrostrain tensor is deviatoric, and then take the class of trial displacement fields

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E}\cdot\boldsymbol{x} + \alpha\boldsymbol{u}^{(0,1)}(\boldsymbol{x}), \qquad \alpha \in R^1,$$

for the energy functional (2.3). Minimization of the latter with respect to α yields the upper MC-bound on the effective shear modulus of the medium, which we write in the form

$$(5.1a) \qquad \mu^* \leqq \mu^u_{MC}, \qquad \mu^u_{MC} = \langle\mu\rangle - \frac{4(4-5\overline{\nu})^2\langle\mu'^2\rangle^2}{15M_u},$$

$$M_u = 2(1-\overline{\nu})(4-5\overline{\nu}\langle\mu\rangle\langle\mu'^2\rangle + \frac{1}{2}(1-2\overline{\nu})^2\langle k'\mu'^2\rangle I_{k\mu\mu}$$
$$+\langle\mu'^3\rangle\left\{I_\mu\frac{1}{3}(13\overline{\nu}^2 - 22\overline{\nu} + 10)\right\},$$

with the dimensionless statistical parameters, defined as follows

$$(5.2)\quad I_{k\mu\mu} = \frac{1}{\langle k'\mu'^2\rangle}\int\int \langle k'(0)\mu'(z)\mu'(w)\rangle \nabla\nabla\frac{1}{4\pi|z|} : \nabla\nabla\frac{1}{4\pi|w|} d^3z d^3w,$$

$$I_\mu = \frac{3}{4}I_\mu^{(2)} + (7\overline{\nu}^2 - 10\overline{\nu} + 1)I_\mu^{(1)},$$

$$I_\mu^{(1)} = \frac{1}{\langle \mu'^3\rangle}\int\int M_3^\mu(z,\ w)\nabla\nabla\frac{1}{4\pi|z|} : \nabla\nabla\frac{1}{4\pi|w|} d^3z d^3w,$$

$$I_\mu^{(2)} = \frac{1}{16\pi^2\langle \mu'^3\rangle}\int\int M_3^\mu(z,\ w)\nabla\nabla\nabla\nabla|z| \bullet \nabla\nabla\nabla\nabla|w| d^3z d^3w,$$

Where $M_3^\mu(z,\ w) = \langle \mu'(0)\mu'(z)\mu'(w)\rangle$ is the three-point correlation function for the random field $\mu(x)$, and ν is defined in (3.5). The bold-faced point in (5.2) denotes full contraction, i.e. contraction with respect to all four pairs of indices.

The parameters $I_{k\mu\mu}$ and $I_\mu^{(1)}$ obviously coincide, being proportional to the above introduced parameter A:

$$(5.3a)\qquad I_\mu^{(1)} = I_{k\mu\mu} = \frac{A}{c(1-c)(1-c)}.$$

For the parameter $I_\mu^{(2)}$ we have

$$(5.3b)\qquad I_\mu^{(2)} = \frac{4A_1}{c(1-c)(1-2c)},$$

where

$$(5.4)\qquad A_1 = \frac{1}{64\pi^2}\int\int i(z,\ w)\nabla\nabla\nabla\nabla|z| \bullet \nabla\nabla\nabla\nabla|w| d^3z d^3w$$

is another statistical parameter for the medium, independent of A, $i(z,\ w) = \langle I'(0)I'(z)I'w)\rangle$. The parameter A_1 is introduced by Milton and Phan-Thien [12], eqn (63). Milton [12, 13] has employed the statistical parameters η_1, η_2, similar to the ξ's, defined in (4.8):

$$(5.5)\qquad \eta_2 = 1 - \eta_1 = \frac{5}{21}\xi_2$$

$$+\frac{150}{c(1-c)}\int\int \frac{d^3z d^3w}{16\pi^2|z|^3|w|^3}\left\{\dot{S}_3(z,\ w) - \frac{S_2(z)S_2(w)}{c}\right\} P_4(u),$$

the same notations being used here as those in (4.8), where $P_4(u) = \frac{1}{8}(35u^4 - 20u^2 + 3)$ is the Legendre polynomial of order four.

The relation between the parameters $\eta_{1,2}$ and A_1 is given in [12], eqn (29). It reads

$$\eta_2 = 1 - \eta_1 = \frac{5}{6}\left(\frac{c + 4A_1 - 3A}{c(1-c)} + \frac{1-c}{5}\right) \tag{5.6}$$

and follows from the formula

$$\nabla\nabla\nabla\nabla|\boldsymbol{z}| \bullet \nabla\nabla\nabla\nabla|\boldsymbol{w}| = \frac{72P_4(u)}{|\boldsymbol{z}|^3|\boldsymbol{w}|^3}, \qquad u = \frac{\boldsymbol{z}\cdot\boldsymbol{w}}{|\boldsymbol{z}|\,|\boldsymbol{w}|}, \tag{5.7}$$

similar to (4.10).

It is important to point out that both statistical parameters ξ_1 and η_1 (and thus ξ_2 and η_2 as well) lie in the interval [0, 1]. Moreover, they satisfy the inequality

$$21\eta_2 - 5\xi_2 \geqq 0, \tag{5.8}$$

as it is shown by Milton and Phan-Thien [12], eqn (52). This inequality is a consequence of the fact that the upper MC-bound should be always greater than the lower one.

The lower MC-bound μ^l_{MC} on the shear modulus μ^* of the medium is obtained when minimizing the functional (3.9) over the class of trial stress fields

$$\sigma(\boldsymbol{x}) = \Sigma + \alpha\sigma^{(0,1)}(\boldsymbol{x}), \qquad \alpha \in R^1,$$

with a deviatoric Σ and $\sigma^{(0,1)}(\boldsymbol{x})$, defined in (3.16). The final result reads

$$\begin{aligned} &\mu^l_{MC} \leqq \mu^*, \quad (\mu^l_{MC})^{-1} = \left\langle\frac{1}{\mu}\right\rangle - \frac{(7-5\overline{\nu})^2\langle\mu'/\mu\rangle^2}{5M_l}, \\ &M_l = \frac{2}{3}(1+\overline{\nu})^2\left\{3J_{kk\mu} - \langle\mu'^2/k\rangle\right\} + \frac{9}{4}J^{(2)}_{k\mu\mu} \\ &+3(7\overline{\nu}^2 - 10\overline{\nu} + 1)J^{(1)}_{\mu} - (2\overline{\nu}^2 + 4\overline{\nu} - 7)\langle\mu'/\mu\rangle, \end{aligned} \tag{5.1b}$$

with the following statistical parameters

$$\begin{aligned} J_{k\mu\mu} &= \int\!\!\int\left\langle\frac{\mu'(\boldsymbol{z})\mu'(\boldsymbol{w})}{k(0)}\right\rangle\nabla\nabla\frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla\frac{1}{4\pi|\boldsymbol{w}|}d^3\boldsymbol{z}d^3\boldsymbol{w}, \\ J^{(1)}_{\mu} &= \int\!\!\int\left\langle\frac{\mu'(\boldsymbol{z})\mu'\boldsymbol{w})}{\mu(0)}\right\rangle\nabla\nabla\frac{1}{4\pi|\boldsymbol{z}|} : \nabla\nabla\frac{1}{4\pi|\boldsymbol{w}|}d^3\boldsymbol{z}d^3\boldsymbol{w}, \\ J^{(2)}_{\mu} &= \frac{1}{16\pi^2}\int\!\!\int\left\langle\frac{\mu'(\boldsymbol{z})\mu'(\boldsymbol{w})}{\mu(0)}\right\rangle\nabla\nabla\nabla\nabla|\boldsymbol{z}| \bullet \nabla\nabla\nabla\nabla|\boldsymbol{w}|d^3\boldsymbol{z}d^3\boldsymbol{w}, \end{aligned} \tag{5.9}$$

similar to (5.2).

Simple analysis, based on the relations of the type (4.7), shows that the lower MC-bound (5.1) depends on the same statistical parameters as the upper one. They may be chosen either as A and A_1, defined in (4.5) and (5.4) respectively, or as the Milton parameters ξ_1 and η_1, defined in (4.8) and (5.6) (cf. [8]).

THE CLUSTER BOUNDS FOR DISPERSIONS OF SPHERES

Let the medium be a random dispersion of equisized nonoverlapping spheres of radius a and let $\boldsymbol{x}_j$ be the set of random points that serve as centers of the spheres. The random constitution of the dispersion is exhaustively described by the Stratonovich random density function [1, 4]

$$\psi(\boldsymbol{x}) = \sum \delta(\boldsymbol{x} - \boldsymbol{x}_j). \tag{6.1}$$

Then

$$k'(\boldsymbol{x}) = [k] \int h(\boldsymbol{x} - \boldsymbol{y})\psi'(\boldsymbol{x})d^3\boldsymbol{x}, \tag{6.2}$$

$$\mu'(\boldsymbol{x}) = [\mu] \int h(\boldsymbol{x} - \boldsymbol{y})\psi'(\boldsymbol{x})d^3\boldsymbol{x},$$

where $\psi'(\boldsymbol{x})$ is the fluctuating part of $\psi(\boldsymbol{x})$ and $h(\boldsymbol{x})$ is the characteristic function of a single sphere of radius a, located at the origin. On introducing (6.2) into (3.3) and (3.4), we make, similarly to that in [1, 5], a transition from the basic random field $I(\boldsymbol{x})$ (cf. (2.2)) to the random density field $\psi(\boldsymbol{x})$. The first-order perturbation fields $\boldsymbol{u}^{(1,0)}(\boldsymbol{x})$ and $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$ then become

$$\boldsymbol{u}^{(1,0)}(\boldsymbol{x}) = \frac{3[k]}{3\langle k\rangle + 4\langle\mu\rangle} \operatorname{tr} \boldsymbol{E} \int \Gamma_s(\boldsymbol{x} - \boldsymbol{y})\psi'(\boldsymbol{y})d^3\boldsymbol{y}, \tag{6.3}$$

$$\boldsymbol{u}^{(0,1)}(\boldsymbol{x}) = 2\frac{[\mu]}{\langle\mu\rangle}\boldsymbol{E}_d : \int \Gamma_d(\boldsymbol{x} - \boldsymbol{y})\psi'(\boldsymbol{y})d^3\boldsymbol{y}, \tag{6.4}$$

where

$$\Gamma_s(\boldsymbol{x}) = \nabla\varphi(\boldsymbol{x}), \quad \Gamma_d(\boldsymbol{x}) = \nabla\varphi(\boldsymbol{x}) \otimes \boldsymbol{I} + \overline{\varkappa}\nabla\nabla\nabla\chi(\boldsymbol{x}), \quad \overline{\varkappa} = -\frac{1}{4(1-\overline{\nu})}, \tag{6.5}$$

and $\varphi = h * \frac{1}{4\pi|\boldsymbol{x}|}$, $\chi = h * \frac{1}{4\pi}|\boldsymbol{x}|$ are respectively the harmonic (Newtonian) and the biharmonic potentials for a single sphere of radius a, located at the origin.

Obviously, the kernel $\Gamma_s(\boldsymbol{x})$ in (6.3) is proportional to the disturbance of the displacement field in an unbounded elastic matrix (of moduli k_m and μ_m), introduced by a single spherical inhomogeneity (of moduli k_f and μ_f), when the strain tensor at infinity is spherical (cf. [14]). This means that in the latter case the class of trial fields (4.1) is just the superposition of such disturbances, multiplied by an adjustable scalar parameter, over the set of spheres in the dispersion. Therefore the BM-bound (4.2a) on the bulk modulus k^*, which corresponds to the class (4.1), coincides with the first-order cluster bound in the sense of Torquato [15] — a conclusion fully similar to that, already reached in [5] for the scalar conductivity case.

The situation with the displacement field $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$ is a bit more involved. Recalling again the Eshelby result [14], one can easily notice that $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$ is proportional to the disturbance of the displacement field in an unbounded matrix with shear modulus μ_m and the Poisson ratio $\overline{\nu}$, introduced by a single spherical inhomogeneity with elastic moduli k_m and μ_m, when deviatoric strain is applied at infinity. Thus the MC-bound (5.1a) represents a first-order cluster bound in the sense that the field (3.4) is proportional to the disturbance, generated by a single spherical inhomogeneity. Strictly speaking, however, it is not a cluster bound in the sense of Torquato [15], because the field $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$ is not the single-sphere disturbance, generated in the matrix material with the moduli k_m and μ_m, i.e. with the Poisson ratio ν_m. The reason is that $\overline{\nu} \neq \nu_m$ and thus $\overline{\varkappa} \neq \varkappa_m$ as well. It could be easily seen, however, that $\overline{\nu} - \nu_m = O(c)$ and thus $\overline{\varkappa} - \varkappa_m = O(c)$ as well. That is why the kernel $\boldsymbol{E}_d : \Gamma_d(\boldsymbol{x})$ in (6.4) is proportional, to the order $O(c)$, to the single-sphere disturbance in the matrix material. This fact, as we shall see in the following sec. 8, suffices to claim that the MC-bounds together with the BM-ones are optimal to the order c^2 for the random dispersions under study.

A GENERALIZATION OF THE McCOY BOUNDS

The very form (3.4) of the field $\boldsymbol{u}^{(0,1)}(\boldsymbol{x})$ hints the following idea. Consider the class of trial displacements

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E}_d \cdot \boldsymbol{x} + \boldsymbol{E}_d : \int \left\{ \alpha_1 \nabla \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \otimes \boldsymbol{I} + \alpha_2 \nabla\nabla\nabla \frac{1}{4\pi}|\boldsymbol{x}-\boldsymbol{y}| \right\} I'(\boldsymbol{y}) d^3\boldsymbol{y}, \tag{7.1}$$

where $\boldsymbol{E}_d$ is deviatoric and α_1, α_2 are two adjustable scalar parameters. The minimization of the energy functional (2.6) with respect to α_1 and α_2 brings forth a certain upper bound $\widetilde{\mu}_{MC}$ on the effective shear modulus μ^* of the random medium.

This bound could be called g e n e r a l i z e d M C-b o u n d. Obviously, the latter coincides with the upper MC-bound, $\widetilde{\mu}_{MC} = \mu_{MC}$, if

$$A_{\min} = \frac{\alpha_2^{\min}}{\alpha_1^{\min}} = \overline{\varkappa}, \tag{7.2}$$

where $\alpha_1^{\min}$, $\alpha_2^{\min}$ are respectively the values of the parameters α_1 and α_2 that minimize the functional (2.6) in the class (7.1), $\overline{\varkappa}$ is defined in (3.5).

As a matter of fact, the class of trial fields (7.1) has been introduced by Milton and Phan-Thien [12], sec. 5a, who considered two-phase random media of periodic internal constitution and employed the Fourier transform of the fields from the class (7.1).

Let the medium be a random dispersion of spheres. On making transition to the random density field $\psi(\boldsymbol{x})$, cf. (6.1), we recast the trial fields (7.1) as

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{E}_d \cdot \boldsymbol{x} + \boldsymbol{E}_d : \int \{\alpha_1 \nabla\varphi(\boldsymbol{x}-\boldsymbol{y}) \otimes \boldsymbol{I} \\ + \alpha_2 \nabla\nabla\nabla\chi(\boldsymbol{x}-\boldsymbol{y})\}\psi'(\boldsymbol{y})d^3\boldsymbol{y}.\end{aligned} \tag{7.3}$$

Using once more the arguments from sec. 6, we note that $\widetilde{\mu}_{MC}$ resembles again the cluster bound of Torquato, because the best kernel in the integral of (7.3) is proportional to the field

$$\boldsymbol{E}_d : \{\nabla\varphi(\boldsymbol{x}-\boldsymbol{y}) \otimes \boldsymbol{I} + A_{\min}\nabla\nabla\nabla\chi(\boldsymbol{x}-\boldsymbol{y})\}, \tag{7.4}$$

with $A_{\min}$, defined in (7.2). In turn, the field (7.4) is proportional to the single-sphere disturbance with the deviatoric strain $\boldsymbol{E}_d$, acting at infinity. However, this disturbance could exist in an elastic matrix material only if $-0.5 \leqq A_{\min} \leqq -0.25$, because the Poisson ratio $\nu \in (0, 0.5)$.

A detailed study with many examples and figures, concerning the Beran-Molyneux, McCoy, generalized McCoy and other new and more restrictive bounds (of fourth-order) on the effective moduli of random elastic media is performed in the above mentioned paper [12], to which we refer the reader for further information. We shall turn now to the problem of optimality of the aforementioned bounds for random dispersions of spheres and their explicit evaluation to the order c^2.

THE OPTIMAL THIRD-ORDER BOUNDS ON THE ELASTIC MODULI

At mentioned in sec. 2, the optimal third-order bounds on the effective elastic moduli could be obtained by solving the Euler-Lagrange equation (2.8). The foregoing Ritz-type bounds will be optimal if the respective kernels satisfy eqn (2.8). The scalar conductivity arguments, presented in [1], can be easily extended to the elastic case as well, so that we could claim that the Beran-Molyneux, the McCoy and the generalized McCoy bounds are not the optimal third-order bounds.

To show however that the said bounds are optimal to the order c^2 for a dispersion of spheres, we shall use again the scheme of arguments of [1]. The arguments for the moment hold for anisotropic constituents with tensors of elastic moduli $\boldsymbol{L}_M$ (for the matrix) and $\boldsymbol{L}_f$ (for the filler particles). Let

$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{T}(\boldsymbol{x}; n) = \boldsymbol{T}_0(\boldsymbol{x}) + n\boldsymbol{T}_1(\boldsymbol{x}) + n^2\boldsymbol{T}_2(\boldsymbol{x}) + \cdots \tag{8.1}$$

be the virial expansion of the optimal kernel $\boldsymbol{T}(\boldsymbol{x})$. We have to underline that it depends on the number density of the spheres n.

Let us insert (8.1) into the functional (2.6), restricted over the class of trial fields (2.7), and expand the result in powers of n:

$$W[\boldsymbol{T}(\cdot)] = \boldsymbol{E} : \langle \boldsymbol{L} \rangle : \boldsymbol{E} + nW_1[\boldsymbol{T}_0(\cdot)] + n^2 W_2[\boldsymbol{T}_0(\cdot),\ \boldsymbol{T}_1(\cdot)] + o(n^2). \tag{8.2}$$

The functionals W_1 and W_2 depend on the indicated virial coefficients as follows:

$$W_1(\boldsymbol{T}_0(\cdot)] = \int \varepsilon_0(\boldsymbol{x}) : \boldsymbol{L}_m : \varepsilon_0(\boldsymbol{x}) d^3\boldsymbol{x} + \int h(\boldsymbol{x})\{\varepsilon_0(\boldsymbol{x}) + 2\boldsymbol{E}\} : [\boldsymbol{L}] : \varepsilon_0(\boldsymbol{x}) d^3\boldsymbol{x}; \tag{8.3}$$

$$W_2[\boldsymbol{T}_0(\cdot),\ \boldsymbol{T}_1(\cdot)] = \overline{W}_2[\boldsymbol{T}_0(\cdot)] +2\int \{\varepsilon_0(\boldsymbol{x}) : \boldsymbol{L}_m + h(\boldsymbol{x})[\varepsilon_0(\boldsymbol{x}) + \boldsymbol{E} : [\boldsymbol{L}]\} : \varepsilon_1(\boldsymbol{x}) d^3\boldsymbol{x}; \tag{8.4}$$

$$\begin{aligned} \overline{W}_2[\boldsymbol{T}_0(\cdot)] &= V_a \int \varepsilon_0(\boldsymbol{x}) : [\boldsymbol{L}] : \varepsilon_0(\boldsymbol{x}) d^3\boldsymbol{x} \\ &- \int\int \varepsilon_0(\boldsymbol{x} - \boldsymbol{y}_1) : \boldsymbol{L}_m : \varepsilon_0(\boldsymbol{x} - \boldsymbol{y}_2) R_0(\boldsymbol{y}_1 - \boldsymbol{y}_2) d^3\boldsymbol{y}_1 d^3\boldsymbol{y}_2 \\ &- \int\int h(\boldsymbol{x} - \boldsymbol{y}_1) R_0(\boldsymbol{y}_1 - \boldsymbol{y}_2)[2\boldsymbol{E} + 2\varepsilon_0(\boldsymbol{x} - \boldsymbol{y}_1) + \varepsilon_0(\boldsymbol{x} - \boldsymbol{y}_2)] \\ &: [\boldsymbol{L}] : \varepsilon_0(\boldsymbol{x} - \boldsymbol{y}_2) d^3\boldsymbol{y}_1 d^3\boldsymbol{y}_2. \end{aligned} \tag{8.5}$$

When deriving (8.3) — (8.5), the well-known formulas for the moments of the random density field $\psi(\boldsymbol{x})$, correct to the order n^2, are used, namely

$$\langle\psi(\boldsymbol{y})\rangle = n, \qquad \langle\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\rangle = n\delta(\boldsymbol{y}_2-\boldsymbol{y}_1)+n^2g_0(\boldsymbol{y}_1-\boldsymbol{y}_2)+o(n^2),$$

$$\begin{aligned}\langle\psi(\boldsymbol{y}_1)\psi(\boldsymbol{y}_2)\psi(\boldsymbol{y}_3)\rangle &= n\delta(\boldsymbol{y}_2-\boldsymbol{y}_1)\delta(\boldsymbol{y}_3-\boldsymbol{y}_1)\\ &+3n^2\{\delta(\boldsymbol{y}_1-\boldsymbol{y}_2)g_0(\boldsymbol{y}_2-\boldsymbol{y}_3)\}_s+o(n^2).\end{aligned} \tag{8.6}$$

Here g_0 is the zero-density limit of the two-point probability density function for the random set $\boldsymbol{x}_j$ of sphere centers, $\{\cdot\}_s$ denotes symmetrization with respect to all different combination of the indices in the braces, $R_0(\boldsymbol{y}) = 1-g_0(\boldsymbol{y})$, cf. [4, 16], also $\varepsilon_i(\boldsymbol{x}) = \frac{1}{2}(\nabla\boldsymbol{T}_i+\boldsymbol{T}_i\nabla)$ for $i=0,\ 1$, $[\boldsymbol{L}] = \boldsymbol{L}_f-\boldsymbol{L}_m$.

The minimizing kernel satisfies the equation $\delta W = 0$, so that we have in particular $\delta W_1[\boldsymbol{T}_0(\cdot)] = 0$, $\delta W_2[\boldsymbol{T}_0(\cdot),\ \boldsymbol{T}_1(\cdot)] = 0$. The first of these equations yields straightforwardly

$$\nabla\cdot\{[\boldsymbol{L}_m+[\boldsymbol{L}]h(\boldsymbol{x})][\boldsymbol{E}+\varepsilon_0(\boldsymbol{x})]\} = 0, \tag{8.7}$$

which is just the equation for the disturbance in the displacement field in an unbounded matrix of moduli $\boldsymbol{L}_m$, introduced by a single spherical inhomogeneity of moduli $\boldsymbol{L}_f$, when the strain at infinity is $\boldsymbol{E}$. The functional W_2 is then independent of $\boldsymbol{T}_1(\boldsymbol{x})$ (see (8.5), (8.7)):

$$W_2[\boldsymbol{T}_0(\cdot),\ \boldsymbol{T}_1(\cdot)] = \overline{W}_2[\boldsymbol{T}_0(\cdot)].$$

This means that for the bounds to be optimal to the order c^2 it suffices the zero-order coefficient $\boldsymbol{T}_0(\boldsymbol{x})$ in (8.1) to be proportional to the single-sphere disturbance field — the solution of eqn (8.7). (Obviously, this conclusion holds also for dispersions of identical and identically oriented inclusions, randomly and nonoverlappingly distributed in a matrix.) Since the first-order perturbation kernels in (6.3) and (6.4) in the isotropic case either coincide ($\Gamma_s(\boldsymbol{x})$) or coincide to the order $O(c)$ ($\Gamma_d(\boldsymbol{x})$), with the spherical and deviatoric parts, respectively, of the single-sphere disturbance field in the matrix material, we can claim that the BM- and MC-bounds are c^2-optimal. This fact implies, in turn, that the generalized MC-bound $\widetilde{\mu}^u_{MC}$ (cf. sec. 6) coincides to the order c^2 with the MC-bound μ^u_{MC}, given in (5.1a):

$$\widetilde{\mu}^u_{MC} = \mu^u_{MC}+o(c^2). \tag{8.7}$$

The reason is that when evaluating the bound $\widetilde{\mu}_{MC}$ we employ statistical information, given by the two- and three-point moments and thus it cannot be better than the optimal third-order bound. The latter, however, coincides to the order c^2 with the MC-bound $\widetilde{\mu}^u_{MC}$.

EVALUATION OF THE BERAN-MOLYNEUX AND THE McCOY BOUNDS TO ORDER c^2

In order to obtain the explicit forms of the BM- and the MC-bounds for a dispersion of spheres to the order c^2, it suffices to calculate the statistical parameters A and A_1 to the same order of accuracy.

To the order n^2, i.e. c^2, the three-point correlation function $i(\boldsymbol{z},\ \boldsymbol{w})$ for the field $I(\boldsymbol{x})$, see (2.2), has the form

$$(9.1)\qquad i(\boldsymbol{z},\ \boldsymbol{w}) = n\int h(\boldsymbol{x})h(\boldsymbol{z}-\boldsymbol{x})h(\boldsymbol{w}-\boldsymbol{x})d^3\boldsymbol{x}$$

$$-n^2\int\int h(\boldsymbol{z}-\boldsymbol{x}_1)[h(\boldsymbol{w}-\boldsymbol{x}_1)h(\boldsymbol{x}_2)+h(\boldsymbol{w}-\boldsymbol{x}_2)h(\boldsymbol{x}_2)$$

$$+h(\boldsymbol{w}-\boldsymbol{x}_2)h(\boldsymbol{x}_1)]R_0(\boldsymbol{x}_1-\boldsymbol{x}_2)d^3\boldsymbol{x}_1d^3\boldsymbol{x}_2+o(n^2),$$

because $I'(\boldsymbol{x}) = \int h(\boldsymbol{x}-\boldsymbol{y})\psi'(\boldsymbol{x})d^3\boldsymbol{y}$. On inserting (9.1) into (4.5), we get

$$(9.2a)\qquad A = (a_0 - a_1 c)c + o(o^2),$$

where

$$(9.2b)\qquad a_0 = \frac{1}{V_a}\int h(\boldsymbol{x})\nabla\nabla\varphi(\boldsymbol{x}) : \nabla\nabla\varphi(\boldsymbol{x})d^3\boldsymbol{x} = \frac{1}{3},$$

$$(9.2c)\qquad a_1 = \frac{1}{V_a^2}\int F_0(\boldsymbol{x})\nabla\nabla\varphi(\boldsymbol{x}) : \nabla\nabla\varphi(\boldsymbol{x})d^3\boldsymbol{x}$$

$$+\frac{2}{V_a^2}\int R_0(\boldsymbol{x}_1-\boldsymbol{x}_2)h(\boldsymbol{x}_1)\nabla\nabla\varphi(\boldsymbol{x}_1) : \nabla\nabla\varphi(\boldsymbol{x}_2)d^3\boldsymbol{x}_1d^3\boldsymbol{x}_2 = \frac{5}{3} - m_2,$$

making use of the well-known properties of the Newtonian potential $\varphi(\boldsymbol{x})$ for a sphere. In (9.2c)

$$F_0(\boldsymbol{x}) = \int h(\boldsymbol{x}-\boldsymbol{y})R_0(\boldsymbol{y})d^3\boldsymbol{y},$$

and

$$(9.3)\qquad m_2 = 2\int_2^\infty \frac{\lambda^2}{(\lambda^2-1)^3}g_0(\lambda a)d\lambda, \qquad \lambda = r/a,$$

is the statistical parameter for the dispersion, which appears in the Beran bounds on the effective conductivity of the dispersion.

The relations (9.2) can be summarized as follows

$$A = \frac{1}{3}[1 - (5 - 3m_2)c]c + o(c^2), \tag{9.4}$$

and thus the first two coefficients in the virial expansion of the statistical parameter A, defined in (4.5), are calculated for the dispersion under study.

It is worth mentioning that if we insert (9.4) into (4.11), we shall obtain

$$\xi_2 = \frac{3}{2}m_2 c + o(c), \tag{9.5}$$

so that the statistical parameter (9.3) appears to be proportional to the coefficient of the leading c-term in the virial expansion of the Milton parameter (4.8) for the dispersion.

The c^2-evaluation of the statistical parameter A_1, defined in (5.4), is similar. We have

$$A_1 = (b_0 - b_1 c)c + o(c^2). \tag{9.6a}$$

On introducing (9.1) into (5.4), we get

$$b_0 = \frac{1}{4V_a}\int h(\boldsymbol{x})\nabla\nabla\nabla\nabla\chi(\boldsymbol{x}) \bullet \nabla\nabla\nabla\nabla\chi(\boldsymbol{x})d^3\boldsymbol{x} = \frac{1}{5}, \tag{9.6b}$$

availing of the well-known properties of the biharmonic potential $\lambda(\boldsymbol{x})$ for a sphere. In turn, after simple algebra, we find

$$b_1 = \frac{1}{4V_a^2}\left\{\int F_0(\boldsymbol{x})\nabla\nabla\nabla\nabla\chi(\boldsymbol{x}) \bullet \nabla\nabla\nabla\nabla\chi(\boldsymbol{x})d^3\boldsymbol{x}\right.$$
$$\left. + \frac{1}{2\pi}\int F_0(\boldsymbol{x}_1 - \boldsymbol{x}_2)h(\boldsymbol{x}_1)\nabla\nabla\nabla\nabla|\boldsymbol{x}_2| \bullet \nabla\nabla\nabla\nabla\chi(\boldsymbol{x}_1)d^3\boldsymbol{x}_1 d^3\boldsymbol{x}_2\right\} = \frac{7}{5} - m_2', \tag{9.6c}$$

where

$$m_2' = \frac{9}{2\pi}\int_{2a}^{\infty}\frac{F_0'(r)}{r^3}\left\{\frac{a^4}{5r^4} - \frac{2a^2}{5r^2} + \frac{1}{3}\right\}dr \tag{9.7}$$

is a new statistical parameter for the dispersion, similar to m_2.

Since the function $F_0'(\boldsymbol{x})$ depends linearly on $g_0(r)$, the parameter m_2' will be also a linear functional of $g_0(r)$. Its explicit form could be derived by using the method of [4, sec. 11], by means of which the relation (9.3) was reached. The final result reads

$$(9.8) \qquad m_2' = \frac{6}{5}\int_2^\infty \frac{\lambda^2(5\lambda^8 - 30\lambda^6 + 51\lambda^4 - 4\lambda^2 + 2)}{(\lambda^2-1)^7} g_0(\lambda a)d\lambda, \qquad \lambda = r/a.$$

Note that in the so-called well-stirred case, for which $g_0(r) = g(r) = 1$ if $r \geqq 2a$, and vanishes otherwise, we have

$$(9.9) \qquad m_2 = \frac{5}{18} - \frac{1}{8}\ln 3 \approx 0.14045, \qquad m_2' \approx 0.25016.$$

The relations (9.6) can be now summarized as follows

$$(9.10) \qquad A_1 = \frac{1}{5}[1 - (7 - 5m_2')c]c + o(c^2),$$

which is the counterpart of (5.3). In turn, for the second Milton parameter η_2, defined in (5.5), we obtain

$$(9.11) \qquad \eta_2 = \frac{6}{5}M_2c + o(c), \qquad M_2 = 4m_2' - 3m_2,$$

as a consequence of (5.6) and (9.10).

Note that the inequality (5.8) together with (9.5) and (9.11) yields

$$(9.12) \qquad m_2' \geqq \frac{6}{7}m_2.$$

A simple inspection of the kernels in the integral representations (9.3) and (9.11) for the statistical parameters m_2 and m_2' shows that the stronger inequality

$$(9.13) \qquad m_2' \geqq \frac{6}{5}m_2$$

holds for dispersions of nonoverlapping spheres, since $g_0(r)$ is nonnegative. Moreover, the inequality (9.13) is the best in the sense that the constant 6/5 cannot be made bigger. This fact implies that the equality in (9.12) is never realizable for dispersions, so that the equality in the Phan-Thien-Milton inequality (5.8) is never attainable whatever be the random distribution of the spheres.

Let

$$\frac{k^*}{k_m} = 1 + a_{1k}c + a_{2k}c^2 + \cdots \tag{9.14}$$

be the virial expansion for the effective bulk modulus of the dispersion. Making use of eqns (4.4), (4.5) and (9.3), we get as a consequence of the BM-bounds (4.2)

$$a_{1k} = \frac{[k]}{k_m + \alpha_m[k]}, \qquad \alpha_m = \frac{3k_m}{3k_m + 4\mu_m}, \tag{9.15}$$

so that the upper and lower BM-bounds coincide to the order c and for the c^2-coefficient the following inequalities hold

$$a_{2k}^l \leqq a_{2k} \leqq a_{2k}^u, \tag{9.16a}$$

$$a_{2k}^l = \alpha_m a_{1k}^2 \left\{ 1 + 2\alpha_m \frac{\mu_m[\mu]}{\mu_f k_m} m_2 \right\}, \tag{9.16b}$$

$$a_{2k}^u = \alpha_m a_{1k}^2 \left\{ 1 + 2\alpha_m \frac{[\mu]}{k_m} m_2 \right\},$$

where m_2 is the statistical parameter (9.3).

Let

$$\frac{\mu^*}{\mu_m} = 1 + a_{1\mu}c + a_{2\mu}c^2 + \cdots \tag{9.17}$$

be the virial expansion for the effective shear modulus of the dispersion. Making use of eqns (5.2), (5.3) and (9.10), we get as a consequence of the MC-bounds (5.1)

$$a_{1\mu} = \frac{[\mu]}{\mu_m + \beta_m[\mu]}, \qquad \beta_m = \frac{6(k_m + 2\mu_m)}{5(3k_m + 4\mu_m)}, \tag{9.18}$$

so that the upper and lower MC-bounds, similarly to the BM-ones, coincide to the order c. For the c^2-coefficient $a_{2\mu}$ we get the inequalities

$$a_{2\mu}^l \leqq a_{2\mu} \leqq a_{2\mu}^u, \tag{9.19a}$$

where

$$a_{2\mu}^l = \beta_m a_{1\mu}^2 \left\{ 1 + f(\nu_m) \frac{k_m}{k_f} \frac{[k]}{\mu_m} m_2 + \chi \frac{[\mu]}{\mu_f} \right\}, \tag{9.19b}$$

$$a_{2\mu}^l = \alpha_m a_{1k}^2 \left\{ 1 + f(\nu_m) \frac{[k]}{\mu_m} m_2 + \chi \frac{[\mu]}{\mu_m} \right\}$$

with the notations

$$f(\nu_m) = \frac{3(1-2\nu_m)^2}{4(1-\nu_m)(4-5\nu_m)}, \tag{9.19c}$$

$$\chi = \chi(\nu_m,\ m_2,\ m_2') = \frac{3m_2' + (7\nu^2 - 10\nu_m + 1)m_2}{4(1-\nu_m)(4-5\nu_m)},$$

where ν_m is the Poisson ratio of the matrix.

The c^2-bounds (9.16) and (9.19) on the effective elastic moduli of a random dispersion of spheres have been reported in [17], using slightly different notations.

In the case of an incompressible matrix, $\nu_m = 0.5$, the bounds (9.19) are significantly simplified

$$a_{2\mu}^l = \frac{2}{5}\left(\frac{5[\mu]}{3\mu_m + 2\mu_f}\right)^2 \left(1 + \frac{[\mu]}{2\mu_f}M_2\right), \tag{9.20}$$

$$a_{2\mu}^u = \frac{2}{5}\left(\frac{5[\mu]}{3\mu_m + 2\mu_f}\right)^2 \left(1 + \frac{[\mu]}{2\mu_m}M_2\right).$$

The c^2-bounds on μ^* in this case depend on a single statistical parameter $M_2 = 4m_2' - 3m_2$ — the same that appeared in the c-term of the virial expansion (9.11) of the second Milton parameter η_2. The situation is thus fully similar to that for the c^2-bounds on the effective conductivity, considered in detail in [4], with the only difference that a new statistical parameter appears.

Let us assume that the spheres are rigid, so that $\mu_f = \infty$ as well. The upper bound (9.19) then degenerates since $M_2 > 0$, cf. (9.13), and thus

$$\frac{5}{2}\left(1 + \frac{1}{2}M_2\right) \leqq a_{2\mu} < \infty. \tag{9.21}$$

We can conclude from (9.21) that the value 2.5 for $a_{2\mu}$ is never attainable for dispersions with incompressible constituents. For a well-stirred dispersion we get, moreover, that $3.2241 \leqq a_{2\mu} < \infty$, in virtue of (9.9).

SOME IMPLICATIONS OF THE c^2-BOUNDS

The foregoing c^2-bounds (9.16) and (9.19) are third-order also in the sense that they coincide for a weakly inhomogeneous dispersion to the order $([k]/k_m)^p([\mu]/\mu_m)^q$, $p + q = 3$. For instance, for the bounds (9.16) on the bulk modulus we have

$$a_{2k} = \alpha_m a_{1k}^2 \left\{1 + 2\alpha_m \frac{[\mu]}{k_m} m_2\right\} + o\left(\left(\frac{[k]}{k_m}\right)^2 \frac{[\mu]}{\mu_m}\right). \tag{10.1}$$

In turn, the bounds (9.19) coincide to the order $([\mu]/\mu_m)^3$. These facts allow to check on the applicability of the known theories in mechanics of composite media for the case of random dispersions of spheres, making use of the method, proposed in [4, p. II], when studying effective scalar conductivity. The basic idea of the method is to consider the formulae for the effective properties, predicted by some of these theories, to the order c^2 and in the limiting case of a weakly inhomogeneous medium, and to compare the results with the relations of the type of (10.1). In this way the values of the statistical parameters m_2 and m_2', which correspond to the theory under examination, could be obtained. To illustrate the method we shall consider here only two examples: the well-known self-consistent theory of elastic composites, due to Hill [18] and Budiansky [19], and the approximate c^2-theory of random elastic dispersion, due to Willis and Acton [20].

10.1. The self-consistent theory of random dispersions is based on the assumption that each sphere is embedded in an unbounded matrix material that possesses the unknown effective moduli k^*, μ^*, see for more details [19, 20]. This assumption eventually yields the following system for the moduli k^* and μ^*:

(10.2a)
$$k^* = k_m + \frac{[k]k^*c}{k^* + \alpha^*(k_f - k^*)},$$
$$\mu^* = \mu_m + \frac{[\mu]\mu^*c}{\mu^* + \beta^*(\mu_f - \mu^*)},$$

where

(10.2b)
$$\alpha^* = \frac{3k^*}{3k^* + 4\mu^*}, \qquad \beta^* = \frac{6(k^* + 2\mu^*)}{5(3k^* + 4\mu^*)}.$$

Let

(10.3)
$$\frac{k^*}{k_m} = 1 + a_{1k}c + a_{2k}c^2 + \cdots,$$
$$\frac{\mu^*}{\mu_m} = 1 + a_{1\mu}c + a_{2\mu}c^2 + \cdots$$

be the virial expansions of the solution $k^* = k^*(c)$, $\mu^* = \mu^*(c)$ of the system (10.2) at $c \ll 1$. It is easily seen that the c-coefficients a_{1k} and μ_{1k} in (10.3) coincide with those, given in (9.14) and (9.17) respectively. In the case of incompressible spheres, $k_f = \infty$, we get the following expression for a_{2k}:

(10.4)
$$a_{2k} = \alpha_m a_{1k}^2 \left\{ 1 + \frac{4\mu_m[\mu]}{3\alpha_m k_m(\mu_m + \beta_m[\mu])} \right\}$$

which meets the bounds (9.16) only if

$$m_2 = m_2^{sc} = \frac{2}{3}. \tag{10.5}$$

This value of m_2 may be also obtained by comparing (10.1) and (10.4) in the weakly inhomogeneous case $[\mu]/\mu_m \ll 1$. It is noteworthy that the same value (10.5) for m_2 has been found in [4] when analyzing the applicability of the self-consistent theory of effective scalar conductivity for random dispersions.

Suppose that the matrix is also incompressible, $k_m = \infty$, so that $k^* = \infty$ as well. The second equation (10.2a) then simplifies and one easily obtains

$$a_{2\mu} = \frac{2}{5} a_{1\mu}^2 \left(1 + \frac{3[\mu]}{3\mu_m + 2\mu_f}\right), \tag{10.6}$$

$$= \frac{2}{5} a_{1\mu}^2 \left(1 + \frac{3[\mu]}{5\mu_m}\right) + o(([\mu]/\mu_m)^3).$$

Having compared (10.6) and (9.18), and taking into account (10.5), we get the value of the second statistical parameter m_2', corresponding to the self-consistent theory, to be

$$m_2' = \frac{4}{5}. \tag{10.7}$$

However, the values (10.5) and (10.7) of the parameters m_2 and m_2' respectively do not satisfy the inequality (9.13), which should hold for any dispersion of nonoverlapping spheres. We therefore conclude that the predictions of the self-consistent theory, eqns (10.2), are not applicable in general to such dispersions even to the order c^2, whatever be the random distribution of the spheres.

10.2. The approximate c^2-theory of Willis and Acton. In the theory of Willis and Acton [20] the effective elastic moduli of the dispersion are expressed in terms of the solution of an integral equation for the so-called *polarization field*. It is proposed that the equation be solved by iterations and the first two such iterations are analytically found, yielding approximate formulas for the c^2-coefficients a_{1k} and $a_{1\mu}$ of the virial expansions (9.14) and (9.17) of the effective moduli k^* and μ^* respectively.

The formula for a_{2k} of the said authors, in our notations, reads (cf. [20], eqn (5.20)):

$$a_{2k} = \alpha_m a_{1k}^2 \left\{1 + \frac{4}{5}\Lambda\alpha_m \frac{[\mu]}{k_m} \frac{\mu_m}{\mu_m + \beta_m[\mu]}\right\}, \tag{10.8}$$

where

$$\Lambda = 3 \int_2^\infty \frac{g_0(\lambda a)}{\lambda^4} d\lambda \tag{10.9}$$

is the statistical parameter, introduced in [20, eqn (5.18)] and denoted there by λ. (In the well-stirred case $\Lambda = 1/8$.) The expression (10.8) meets the bounds (9.16) only if

$$m_2 = \frac{2}{3}\Lambda. \tag{10.10}$$

A simple inspection of the kernels in the integral representations (9.3) and (10.9) for the parameters m_2 and Λ, respectively, shows that

$$m_2 > \frac{2}{3}\Lambda, \tag{10.11}$$

since $g_0(r) > 0$. This means that the c^2- approximation (10.8) for a_{2k} violates the bounds (9.16) whatever be the function $g_0(r)$.

For incompressible constituents, $k_m = k_f = \infty$, the approximate formula of Willis and Acton for a_{2k} is (cf. [20], eqn (6.1)):

$$a_{2k} = \frac{2}{5} a_{1\mu}^2 \left\{ 1 + 15\Lambda \frac{[\mu]}{3\mu_m + 2\mu_f} \right\}, \qquad a_{1\mu} = \frac{5[\mu]}{3\mu_m + 2\mu_f}. \tag{10.12}$$

When compared to (9.20), eqn (10.12) yields

$$6\Lambda = 4m_2' - 3m_2, \tag{10.13}$$

which is violated for the well-stirred dispersion, cf. (9.9). Unlike the case of bulk modulus there exist, however, random constitutions, i.e. functions $g_0(r)$, for which (10.13) holds.

CONCLUDING REMARKS

The method of truncated functional series [4] has been systematically applied in this paper, in order to investigate certain third-order bounds on the effective elastic properties of two-phase random media, i.e. bounds that employ statistical information, given by the two- and three-point correlation functions. In this way we were, first, able to unify the existing bounding procedures, due to Beran and Molyneux and McCoy, as certain Ritz-type procedures, corresponding to the choice of the respective perturbation kernels in the one-tuple term of the truncated functional series. Second, and more important, we were led to the problem of optimality of the bounds, due to the mentioned authors, in the sense whether the bounds are the most respective ones under the statistical information, used in their evaluation. The answer appears negative even for the classical example of a random dispersion of nonoverlapping spheres. However, the bounds in the latter case are optimal to the order c^2, similarly to the scalar conductivity case. The explicit evaluation of the said bounds to the order c^2 leads to the appearance of two statistical parameters, which linearly depend on the zero-density limit of the radial distribution function for the random set of sphere centers. The parameters are closely related to the coefficients of the leading c-terms of the Milton parameters ζ_2 and η_2 for the dispersion. This fact indicates once more the importance of the Milton parameters in the theory of two-phase random media. Similarly to [4], the obtained c^2-bounds allow to check on the applicability of certain heuristic theories in elasticity of composite materials for random dispersions of spheres. The most curious result of such a check here is that the well-known self-consistent theory, due to Hill and Budiansky, is not applicable to random dispersion even to the order c^2, because its predictions violate the respective bounds whatever be the random distribution of the spheres.

ACKNOWLEDGEMENT

The support of this work by the Bulgarian Ministry of Science and Education under Grant No 8 is acknowledged.

REFERENCES

1. M a r k o v, K. Z., K. D. Z v y a t k o v. On the optimal third-order boundson the effective conductivity of random dispersions of spheres. — Theor. Appl. Mechanics (Bulg. Acad. Sci.), **22**, 1991, No 3, 107 - 116.
2. B e r a n, M. J., J. M o l y n e u x. Use of classical variational principles to determine bounds for the effective bulk modulus in heterogeneous media. — Q. Appl. Math., **24**, 1966, 107 - 118.
3. M c C o y, J. J. On the displacement field in an elastic medium with random variations in material properties. — In: Recent Advances in Engeneering Sciences (Eringen A. C. ed.), Vol. **5**, New York, Gordon and Breach, 1970, 235 - 254.
4. M a r k o v, K. Z. Application of Volterra-Wiener series for bounding the overall conductivity of heterogeneous media. I. General procedure. II. Suspensions of equi-sized spheres. — SIAM J. Appl. Math. **47**, 1987, 831 - 850, 851 - 870.
5. M a r k o v, K. Z., K. D. Z v y a t k o v. On the bounds for the effective properties of random dispersions of spheres. — C. R. Acad. bulg. Sci., **40**, (No 3), 1987, 17 - 20.
6. B e r a n, M. J. Use of a variational approach to determine bounds for the effective bulk permittivity of a random medium. — Nuovo Cimento, **38**, 1965, 771 - 782.
7. S o k o l n i k ö f f, I. S. Mathematical Theory of Elasticity, New York, McGraw Hill, 1956.

8. M i l t o n, G. W. Bounds on the electromagnetic, elastic and other properties of two-component composites. — Phys. Rev. Lett., **46**, 1981, 542 – 545.
9. T o r q u a t o, S., G. S t e l l. Microstructure of two-phase random media. I. The n-point probability functions. — J. Chem. Phys., **77**, 1982, 2071 – 2077.
10. H a s h i n, Z., S. S h t r i k m a n n. A variational approach to the theory of the elastic behaviour of multiphase materials. — J. Mech. Phys. Solids, **11**, 1963, 127 – 140.
11. H i l l, R. Elastic properties of reinforced solids: some theoretical principles. — J. Mech. Phys. Solids, **11**, 1963, 357 – 372.
12. M i l t o n, G. W., N. P h a n-T h i e n. New bounds on the effective elastic moduli of two-components materials. — Proc. Roy. Soc. London, Ser. A, **380**, 1982, 305 – 331.
13. M i l t o n, G. W. Bounds on the elastic and transport properties of two-component composites. — J. Mech. Phys. Solids, **30**, 1982, 177 – 191.
14. E s h e l b y, J. D. The determination of the elastic fields of an ellipsoidal inclusion, and related problems. — Proc. Roy. Soc. London, Ser. A, **241**, 1957, 376 – 396.
15. T o r q u a t o, S. Bulk properties of two-phase disordered media. III. New bounds on the effective conductivity of dispersions of penetrable spheres. — J. Chem. Phys., **84**, 1986, 6345 – 6359.
16. S t r a t o n o v i c h, R. L. *Topics in the Theory of Random Noises*, Vol. 1, New York, Gordon and Breach 1967.
17. M a r k o v, K. Z., K. D. Z v y a t k o v. c^2-bounds on the elastic moduli of random dispersions of spheres. — C. R. Acad. bulg. Sci., **41**, (No 7), 1988, 21 – 24.
18. H i l l, R. A self-consistent mechanics of composite materials. — J. Mech. Phys. Solids, **13**, 1965, 213 – 222.
19. B u d i a n s k y, B. On the elastic moduli of some heterogeneous materials. — J. Mech. Phys. Solids, **13**, 1965, 223 – 227.
20. W i l l i s, J. R., J. A c t o n. The overall elastic moduli of a dilute suspension of spheres. — Q. J. Mech. Appl. Math., **29**, 1976, 163 – 177.

Received 2. VII. 1990

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

Книга 2 — Механика

Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"

FACULTE DE MATHEMATIQUES ET INFORMATIQUE

Livre 2 — Mécanique

Tome 83, 1989

NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS III. THE AXIOMS

GEORGI CHOBANOV, IVAN CHOBANOV

Expressum facit cessare tacitum

Георги Чобанов, Иван Чобанов. ДИНАМИЧЕСКИЕ АКСИОМЫ НЬЮТОНА И ЭЙЛЕРА. III. АКСИОМЫ.

Эта работа является третьей частью серии исследования под общим наименованием *Динамические аксиомы Ньютона и Эйлера*, первые две части которой опубликованы в 79-том томе этого *Ежегодника* 1985 г. (книга 2 — *Механика*). Целью этой серии является исследование роли динамических аксиом Ньютона и Эйлера в процессе логической консолидации математических основ динамики массовых точек и твердых тел, а также точного места, которое эти фундаментальные динамические постулаты занимают в системе аналитической механики. В этом смысле вопросная серия представляет принос к решению шестой проблеме Гильберта относительно аксиоматического построения аналитической механики. Специальное внимание обращено понятию инерциальности твердых систем отсчета как согласно Ньютона, так и Эйлера (предложения 11 – 13), особенно в случае динамики массовых точек и твердых тел с променливыми массами.

Georgi Chobanov, Ivan Chobanov. NEWTONIAN AND EULERIAN DYNAMICAL AXIOMS. III. THE AXIOMS.

This paper is the third part of a series of studies under the general title *Newtonian and Eulerian dynamical axioms*, the first two parts of which have been published in the 79th volume of this *Annual* for 1985 (book 2 — *Mechanics*). The aim of the series is to examine the role of the Newtonian and Eulerian dynamical axioms in the process of the logical

consolidation of the mathematical foundations of mass-point and rigid body dynamics, as well as the exact place these basic dynamical postulates take up in the edifice of the science analytical mechanics. In such a sense the series in question represents a contribution to the solution of Hilbert's sixth problem concerning the axiomatical construction of analytical mechanics. Special attention is paid to the notion of inertiality of rigid systems of reference according to both Newton and Euler (Pr 11 – Pr 13), particularly in the case of dynamics of mass-points and rigid bodies with variable masses.

This paper is the third part of a series of studies under the general title *Newtonian and Eulerian dynamical axioms*, the first two parts of which have been published in the 79th volume of this *Annual* for 1985 (book 2 — *Mechanics*). The aim of the series is to examine the role the Newtonian and Eulerian dynamical axioms play in the process of the logical consolidation of the mathematical foundations of mass-point and rigid body dynamics, as well as the exact place these basic dynamical postulates take up in the edifice of the science analytical mechanics. This may be accomplished by a thorough analysis of all the aspects of Newtonian and Eulerian dynamical axioms. In such a sense the series in question represents a contribution to the solution of Hilbert's sixth problem concerning the axiomatical construction of analytical mechanics.

The Newtonian and Eulerian dynamical axioms have a crucial role in the historical development of analytical dynamics. As a matter of fact, the mass-point dynamics has been borne in 1687 with the publication of Newton's famous *Philosophiae Naturalis Principia Mathematica*; and the rigid body dynamics — in 1775, when Euler wrote his *Nova methodus motum corporum rigidorum determinandi*, a work unfortunately still obscure even among professional mechanicians. That is why in the first part of the series a historical review has been proposed on the meanders that analytical mechanics was destined to wander about before the laws or principles of momentum and of moment of momentum of mass-points and rigid bodies have been discovered.

In the second part a review has been proposed on the manner these fundamental dynamical laws are represented (or sooner misrepresented) in the traditional literary sources on analytical dynamics.

The present third part is dealing with the axioms themselves. It contains strict mathematical formulations of these axioms along with several preliminary definitions of mechanical entities, therein involved, and some immediate but important corollaries.

Sch 1. For the sake of brevity the symbols Sgn, sgn:, Ax, Df, Pr, Dm, and Sch replace the words *notation, denotes by definition, axiom, definition, proposition, proof,* and *scholium* respectively, and the letters, R and C are reserved for the fields of all real and all complex numbers respectively.

Sch 2. The bibliography of all three parts of the series has a unified numeration.

Sch 3. Numbers in *brevier* refer to the *Appendix* in the end of the article.

Sch 4. Quotations from the *Appendix* are made in the following manner: relation (17) and proposition 19 therein are cited simply as (17) and Pr 19 respectively in the *Appendix* itself, but as App(17) and AppPr 19 respectively elsewhere.

Sch 5. Similarly, relation (17) and proposition 19 from the *main text* of this paper are cited simply as (17) and Pr 19 respectively in the main text itself, but as M(17) and M Pr 19 respectively in the *Appendix*.

The whole of mass-point dynamics is based upon, and is developed from, the following two postulates.

Ax 1 N (*fisrt Newtonian dynamical axioms,* alias *law* or ***principle of momentum of mass-point***). There exists such a rigid system of reference S that, all derivatives being taken with respect to S, for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the derivative with respect to the time of the momentum of P equals the basis of $\underrightarrow{F}$.

Df 1 N. Any system of reference, satisfying Ax 1 N, is called *inertial according to Newton.*

Ax 2 N (*second Newtonian dynamical axiom,* alias *law* or ***principle of moment of momentum*** (*kinetical moment*) ***of a mass-point***). If S is an inertial according to Newton system of reference and all derivatives are taken with respect to S, then for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the derivative with respect to the time of the moment of momentum of P equals the moment of $\underrightarrow{F}$, both moments being taken with respect to the origin of S.

The whole of rigid body dynamics is based upon, and is developed from, the following two postulates.

Ax 1 E (*first Eulerian dynamical axiom,* alias *law* or principle of momentum of rigid body). There exists such a rigid system of reference S that, all derivatives being taken with respect to S, for any rigid body B and for any system of forces $\underrightarrow{F}$ acting on B, the derivative with respect to the time of the momentum of B equals the basis of $\underrightarrow{F}$.

Df 1 E. Any system of reference satisfying Ax 1 E is called *inertial according to Euler.*

Ax 2 E (*second Eulerian dynamical axiom,* alias *law* or ***principle of moment of momentum*** (*kinetical moment*) ***of rigid body***). If S is an inertial according to Euler system of reference and all derivatives are taken with respect to S, then for any rigid body B and for any system of forces $\underrightarrow{F}$ acting on B, the derivative with respect to the time of the moment of momentum of B equals the moment of $\underrightarrow{F}$, both moments being taken with respect to the origin of S.

Sch 6. These formulations of the Newtonian and Eulerian dynamical axioms may be found nowhere in the current literature on analytical dynamics of mass-points and rigid bodies. Instead, amorphous redactions of imitations of Ax 1 N and possibly of Ax 1 E are proposed to the reader, the role of the inertial systems of reference according to Newton, as well as to Euler, being as a rule completely economized if not surpressed. As regards Ax 2 N and Ax 2 E, in the traditional literature on analytical dynamics these dynamical suppositions or hypotheses are taken down from their logical pedestal of dynamical axioms to the unenviable level of theorems, being labelled "the theorem of kinetical moment" of mass-points and of rigid bodies respectively. Moreover, even Ax 1 E is called, by some authors at least, "the theorem of momentum" of rigid bodies, with the claim that it is derivable from Ax 1 N. This is a most unpardonable logical error rooted in a deep ignorance of the real state of affairs in analytical mechanics, at least as far as its logical foundations are concerned. It is a topic we shall discuss at length subsequently.

Sch 7. For the time being we confine ourselves to the most categorical declaration that Ax 1 N, Ax 2 N, as well as Ax 1 E, Ax 2 E, are unprovable mathematical statements, as unprovable at least, as for instance Euclid's fifth postulate or Pascal's principle of mathematical induction are.

Sch 8. One of the aspects of these realities lies in the fact that the Newtonian and the Eulerian axioms involve mechanical terms which are unsusceptible to explicit mathematical definitions. Since this mathematical phenomenon is one of the most important, let us sumbit it to a closer analysis.

The meaning of both Newtonian and Eulerian dynamical axioms is out of reach unless and until the meaning of any term these verbal propositions involve is made clear. These terms are: *system of reference, rigid* system of reference, *derivative* of a vector function with respect to a system of reference, *mass-point* and *rigid body, momentum* and *moment of momentum* (*kinetical moment*) of a mass-point and a rigid body, *system of forces, basis* and *moment* of a system of forces with respect to a given point (pole), *origin* of a system of reference, *time,* and *acting* (a system of forces is "acting" on a mass-point and a rigid body). If all these terms were susceptible to explicit mathematical definitions, then the Newtonian and Eulerian dynamical axioms would turn out to be (true or false) mathematical theorems.

And if not?

The answer of this question, as regards more elementary mathematical theories than analytical mechanics (as, for instance, arithmetic and Euclidean geometry), was known to nobody until the end of the last century[1]. According to the proclaimed in 1899 Hilbert's *axiomatical principle*, a system of axioms for a mathematical theory must unconditionally include a certain number of void of explicit definitions terms of this theory. Now all the terms, numbered above and involved in Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E, are susceptible to strict explicit mathematical definitions with the only exception of the last two ones, namely *time* and *acting*. Those are primary notions of the mathematical theory called analytical mechanics, and they are defined implicitly namely by the aid of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E (along with other mechanical axioms which will not be formulated manifestly here for the time being). At that, the term time is a *primary notion-object*[2] and the term acting is a *primary notion-relation*[3].

Sch 9. Ax 1 N and Ax 1 E are *existance statements.* Any of them asserts that there exists one at least system of reference with certain properties, and Df 1 N, Df 1 E give special appelations of these kinds of systems of reference.

There is a definite lack of information in Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E, however. Indeed, if a particular system of reference S is given, then neither Ax 1 N, Ax 2 N nor Ax 1 E, Ax 2 E give any possibility to decide whether S is inertial according to Newton or according to Euler respectively. This is a question that will be discussed in detail below.

Sch 10. There is also a lack of distinctness about the relation between inertialities according to Newton and according to Euler. In other words, on the basis of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E only, one cannot answer the question whether there exists one at least inertial according to Newton system of reference which is, or is not, inertial according to Euler too. Alias, Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E are tolerant to any of these alternatives.

Sch 11. A mere glance at the Newtonian and Eulerian dynamical axioms displays at once that the notions of *mass-point* and of *rigid body* play a central role among all other notions these axioms involve. Their role is comparable to that the notion of integral plays in mathematical analysis. As well as it is impossible to build-up a logically irreproachable mathematical analysis without a strict mathematical definition of the term *intégral*, it is not lesser impossible to construct a logically unimpeachable analytical mechanics without strict mathematical definitions of the terms *mass-point* and *rigid body*[4,5].

Sch 12. A last general remark concerning the Newtonian and Eulerian dynamical axioms affects the striking similitude, the complete analogy between Ax 1 N, Ax 2 N, on the one hand, and Ax 1 E, Ax 2 E respectively, on the other hand. No great perspicacity is needed, indeed, to see that it is quite sufficient to substitude the term *rigid body* for the term *mass-point* in Ax 1 N, Ax 2 N in order to obtain automatically Ax 1 E, Ax 2 E respectively.

This formal resemblance between the Newtonian and the Eulerian dynamical axioms quite naturally brings forward the question: was there as much *marifet* needed, factually, as to ascribe Euler's name to the laws of momentum and of moment of momentum of rigid bodies? After all, once disposing with Ax 1 N and Ax 2 N, is it not a trivial whim to substitute in them the word *mass-point* by *rigid body* in order to obtain Ax 1 E and Ax 2 E respectively? Is there a great merit in such a procedure in order to perpetuate someone's name?[6]

Such an attitude toward the Eulerian dynamical axioms may be evinced only by someone who is entirely ignorant of the very essence of rigid body dynamics. *Cum grano salis,* to support such an outlook is as unwise as to uphold that a woman may be created out of a man by a mere substitution of the pronoun *she* for *he*.

Sch 13. First of all, neither Newton nor Euler have had the slightest idea of the above formulations of their dynamical laws[7]. Those are formulations that only modern mathematics could propose: it is hardly accidental that, as already underlined, they are nowhere to be seen even in the current mechanical literature. Euler did not dispose of Ax 1 N and Ax 2 N in order to substitute in them *rigid body* for *mass-point* and to obtain, in such a parrot way, Ax 1 E and Ax 2 E respectively.

Sch 14. The most that Euler took from Newton's *Principia* was Lex II. Besides, Euler did never have the slightest idea that Ax 1 E and Ax 2 E are beyond proof. A son of his epoch, he believed he had proved them. He never understood that his "demonstrationes" are, at the best, only plausible inferences. The nature of his reasonings is bordering on physical intuition. In his arguments, let us emphasize that once more, Newton's Lex II has played a most essential role.

All these circumstances being *bien entendu,* let us at last give a formal mathematical redaction of the Newtonian and Eulerian dynamical axioms, together with some immediate corollaries.

V denoting the *real standard vector space*[8], a *mass-point* P is defined as an ordered pair (r, m) of a vector function[9]

(1) $$\boldsymbol{r} : R \longrightarrow V$$

(the *radius-vector* of P) and a scalar function

(2) $$m : R \longrightarrow R \qquad (0 < m(t),\ t \in R)$$

(the *mass* of P). Under these notations it is written $P(\boldsymbol{r}, m)$.

Let $Oxyz$ be a *right-hand orientated orthonormal Cartesian system of reference*[10] and let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be the *unit vectors* of the axes Ox, Oy, Oz respectively. If $P(\boldsymbol{r}, m)$ is a mass-point, its radius-vector (1) being taken with respect to O, then the derivative

(3) $$\boldsymbol{v} \text{ sgn} : \frac{d\boldsymbol{r}}{dt} = \dot{\boldsymbol{r}}$$

of $\boldsymbol{r}$ with respect[11] to $Oxyz$ is called the *velocity* of P with respect to $Oxyz$.

The quantity

(4) $$\boldsymbol{k} \text{ sgn} : m\boldsymbol{v}$$

is called the *momentum* of P with respect to $Oxyz$, and the quantity

(5) $$\boldsymbol{l} \text{ sgn} : \boldsymbol{r} \times m\boldsymbol{v}$$

is called the *moment of momentum* (*kinetical moment*) of P with respect to $Oxyz$.

Let now P be under the action of the system

(6) $$\underrightarrow{F} \text{ sgn} : \{\overrightarrow{F}_\nu\}_{\nu=1}^n$$

of forces

(7) $$\overrightarrow{F}_\nu \text{ sgn} : (\boldsymbol{F}_\nu, \boldsymbol{M}_\nu) \qquad (\nu = 1, \dots, n),$$

where by definition

(8) $$\boldsymbol{F}_\nu = \boldsymbol{M}_\nu = \boldsymbol{O} \qquad (1 \leqq \nu \leqq n)$$

or otherwise

$$(9) \qquad \boldsymbol{F}_\nu \neq \boldsymbol{O},\ \boldsymbol{F}_\nu \boldsymbol{M}_\nu = O, \qquad (1 \leqq \nu \leqq n),$$

all moments $\boldsymbol{M}_\nu$ ($\nu = 1, \ldots, n$) being taken with respact to O.

For the sake of brevity let

$$(10) \qquad \boldsymbol{F} \text{ sgn} : \sum_{\nu=1}^{n} \boldsymbol{F}_\nu, \qquad \boldsymbol{M} \text{ sgn} : \sum_{\nu=1}^{n} \boldsymbol{M}_\nu$$

be the basis and the moment (with respect to O) of (6) respectively.

If the system of reference $Oxyz$ is, by hypothesis, inertial according to Newton, then the mathematical expressions of Newton's dynamical axioms Ax 1 N and Ax 2 N are

$$(11) \qquad \frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{F}$$

and

$$(12) \qquad \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{M}$$

respectively, the derivatives in the left-hand sides of (11) and (12) being taken with respect to $Oxyz$.

Using the abbreviated notations (4), (5), instead of (11) and (12) one can write

$$(13) \qquad \dot{\boldsymbol{k}} - \boldsymbol{F} = \boldsymbol{O}$$

and

$$(14) \qquad \dot{\boldsymbol{l}} - \boldsymbol{M} = \boldsymbol{O}$$

respectively.

While the notion of *mass-point* is a most simple one, the notion of ***rigid body*** is, on the contrary, a most complicated one. We are devoid of the opportunity of entering here in any details in this connection and we are compelled to refer the reader to the articles [44, 87 – 89]. Still some explications are impreventable with a view to better comprehension of the exposition.

Let B be a rigid body and P be any of its points. If $\boldsymbol{r} = \overline{OP}$, then (3) defines the *velocity* of P with respect to $Oxyz$. The definition of B requires the prescription of a function

$$(15) \qquad \varkappa : V \longrightarrow [0, \infty)$$

(*density* of B at r). If $d\mu$ denotes an element of arc, area, or volume of B, according to the dimensions of B (1-*dimensional*, 2-*dimensional*, or 3-*dimensional* rigid body respectively[12]), then the differential

$$dm = \varkappa(\boldsymbol{r})d\mu \tag{16}$$

is called the *element of mass* (*mass-element*, *elementary mass*) of B at $\boldsymbol{r}$.

Extremely important for rigid body dynamics are the following kinetical quantities:

$$m \text{ sgn} : \int dm \tag{17}$$

(mass of B),

$$\boldsymbol{r}_G \text{ sgn} : \frac{1}{m}\int \boldsymbol{r}dm \tag{18}$$

(radius-vector with respect to O of the mass-center G of B),

$$\boldsymbol{K} \text{ sgn} : \int \boldsymbol{v}dm \tag{19}$$

(momentum of B with respect to $Oxyz$), and

$$\boldsymbol{L} \text{ sgn} : \int \boldsymbol{r} \times \boldsymbol{v}dm \tag{20}$$

(moment of momentum, alias kinetical moment, of B with respect to $Oxyz$).

It is seen that while the mass of a rigid body is invariant with respect to the chosen system of reference, all the other three quantities (18) – (20) are not. At that, all integrals in the right-hand sides of (17) – (20) are taken over the occupied by the rigid body[13] space.

If the rigid body B is under the action of the system (6) of forces (7) with (8), (9) and the notations (10) are accepted, and if the system of reference $Oxyz$ is, by hypothesis, inertial according to Euler, then the mathematical expressions of Euler's dynamical axioms Ax 1 E and Ax 2 E are

$$\frac{d}{dt}\int \boldsymbol{v}dm = \boldsymbol{F} \tag{21}$$

and

$$\frac{d}{dt}\int \boldsymbol{r} \times \boldsymbol{v} dm = \boldsymbol{M} \tag{22}$$

respectively, the derivatives in the left-hand sides of (21) and (22) being taken with respect to $Oxyz$.

Using the abbreviated notations (19), (20), instead of (21) and (22) one can write

$$\dot{\boldsymbol{K}} - \boldsymbol{F} = \boldsymbol{O} \tag{23}$$

and

$$\dot{\boldsymbol{L}} - \boldsymbol{M} = \boldsymbol{O} \tag{24}$$

respectively.

Sch 15. The complete analogy between (23), (24), on the one hand, and (13), (14) respectively, on the other hand, is obvious. This similitude is only formal though, as a mere glance at the definitions (4), (5), on the one hand, and (19), (20) respectively, on the other hand, at once displays. This juxtaposition throws a new light on the raised in Sch 12 — Sch 14 question concerning Euler's attributions in rigid body dynamics, which are now seen in its true colours.

The difference between the momenta and the moments of momenta of mass points, on the one hand, and of rigid bodies, on the other hand, are tremendous indeed. One should not be deceived by the fact that such radically different mathematical objects as (4) and (19), as well as (5) and (20), are called with the same names. If, as underlined, the analogy between (13) and (23), as well as between (14) and (24), is perfect, yet one is at a loss to see the nuclei, or the germs, or the embrios of $\boldsymbol{K}$ and $\boldsymbol{L}$ in $\boldsymbol{k}$ and $\boldsymbol{l}$ respectively, to say nothing about some similarity whatever. The blind Euler managed to see $\boldsymbol{K}$ and $\boldsymbol{L}$ through $\boldsymbol{k}$ and $\boldsymbol{l}$ respectively, as well as (21) and (22) through (11) and (12). How did he succeed in doing so?

The only answer we can give is: Frankly, we don't know.

We have our suspicions, though. They will be exposed somewhat later.

The following three propositions are almost obvious.

Pr 1 N. (11), (12)

$$\boldsymbol{F} = \boldsymbol{O} \tag{25}$$

imply

$$\boldsymbol{M} = \boldsymbol{O}. \tag{26}$$

Dm. (11) implies

$$\boldsymbol{r} \times \frac{d}{dt}(m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F}. \tag{27}$$

On the other hand, obviously

$$\mathbf{r} \times \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}). \tag{28}$$

Now (27), (28) imply

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{r} \times \mathbf{F} \tag{29}$$

and (12), (29), (25) imply (26).

Pr 2 N. (11), (12) imply

$$\mathbf{F}\mathbf{M} = 0. \tag{30}$$

Dm. (12), (29) imply

$$\mathbf{r} \times \mathbf{F} = \mathbf{M} \tag{31}$$

and (31) implies (30).

Pr 3 N. (11), (12) imply

$$\mathbf{r}\mathbf{M} = 0. \tag{32}$$

Dm. (31).

Pr 4 N. If P is a mass-point and $\underrightarrow{F}$ is a system of forces acting on P, then

$$\text{rank } \underrightarrow{F} \neq 1 \tag{33}$$

and

$$\text{rank } \underrightarrow{F} \neq 3. \tag{34}$$

Dm. If a system of forces $\underrightarrow{F}$ is given, then it gives rise to a mapping

$$\mu : V \longrightarrow V \tag{35}$$

defined by

$$\mu(\mathbf{r}) = \text{mom}_{\mathbf{r}} \underrightarrow{F}, \tag{36}$$

where by definition

$$\text{(37)} \qquad \operatorname{mom}_{\boldsymbol{r}} \underset{\longrightarrow}{F} \qquad \operatorname{sgn}: \ \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}$$

is the *moment* of $\underset{\longrightarrow}{F}$ *with respect to* $\boldsymbol{r}$ (the $\boldsymbol{r}$*-moment of* $\underset{\longrightarrow}{F}$), $\boldsymbol{F}$ and $\boldsymbol{M}$ being, as until now, the basis and the moment (with respect to $\boldsymbol{o}$) of $\underset{\longrightarrow}{F}$ respectively. The mapping (35), defined by (36), is called the *momental field* of $\underset{\longrightarrow}{F}$. Now the *rank* of $\underset{\longrightarrow}{F}$ (symbolically — rank $\underset{\longrightarrow}{F}$) is defined as the maximal number of the linearly independent elements of the image $\mu(V)$ of V. According to the *rank-theorem* [90].

$$\text{(38)} \qquad \operatorname{rank} \underset{\longrightarrow}{F} = \begin{cases} 0 & \text{iff } \boldsymbol{F} = \boldsymbol{O},\ \boldsymbol{M} = \boldsymbol{O}, \\ 1 & \text{iff } \boldsymbol{F} = \boldsymbol{O},\ \boldsymbol{M} \neq \boldsymbol{O}, \\ 2 & \text{iff } \boldsymbol{F} \neq \boldsymbol{O},\ \boldsymbol{F}\boldsymbol{M} \neq \boldsymbol{O}, \\ 3 & \text{iff } \boldsymbol{F}\boldsymbol{M} = 0. \end{cases}$$

Now (38), Pr 1 N, Pr 2 N imply (33), (34).

Sch 16. Pr 4 N is very instructive. It manifests prohibitions in mass-point dynamics. According to it, a mass-point P and a system of forces $\underset{\longrightarrow}{F}$ acting on P being given, then necessarily rank $\underset{\longrightarrow}{F} = 0$ or rank $\underset{\longrightarrow}{F} = 2$.

Pr 5 N. P being a mass-point under the action of the system of forces $\underset{\longrightarrow}{F}$, the latter is equivalent to the zero-force or to a single non-zero force with a directrix passing through P.

Dm. Pr 4 N, [90] Pr 2, (31).

Sch 17. A direct, though not purely mathematical, corollary from Pr 4 N consists in the conclusion that *the rigid body dynamics cannot be derived from, or be reduced to, the mass-point one.* Indeed, as particular problems of rigid body dynamics display at once, the systems of acting on rigid bodies forces may be quite arbitrary; in particular, their ranks may equal 1 or 3. In other words, the systems of forces that are warrantable to competition as regards their actions on mass-points form an inessential part of the set of all systems of admissible to actions on rigid bodies forces. *Quod erat demonstrandum.*

Let us analyse the possibilities of the following alternative for Ax 2 N postulate.

Ax 2 N bis. If S is an inertial according to Newton system of reference, then for any mass-point $P(\boldsymbol{r}, m)$ and for any system of forces $\underset{\longrightarrow}{F}(\boldsymbol{F}, \boldsymbol{M})$ acting on P, the relation (31) holds, both $\boldsymbol{r}$ and $\boldsymbol{M}$ being taken with respect to the origin of S.

The following proposition is almost obvious.

Pr 6 N. The system of axioms Ax 1 N, Ax 2 N and Ax 1 N, Ax 2 N bis are equivalent.

Dm. Ax 1 N and Ax 2 N imply Ax 2 N bis (Pr 2 N). Inversely, Ax 1 N and Ax 2 N bis, alias (11) and (31), imply (12), i.e. Ax 2 N by virtue of (28).

Sch 18. It has been mentioned in App 7 that Newton "thought wrongly that" Ax 2 N "is an immediate corollary from Lex II", i.e. from Ax 1 N. Now Pr 6 N displays that this idea of Newton is a half-truth: Ax 2 N is a corollary from Ax 1 N and Ax 2 N bis *coniunctim* rather than from Ax 1 N alone. It is clear that on the logical background of that epoch Newton's error is easy to be explained.

Sch 19. It must not be left unobserved that Ax 2 N bis, in its simplest form at least, is a much more natural and intuitively clear proposition than Ax 2 N. Indeed, Ax 2 N speaks nothing to the physical experience. On the contrary, if the mass-point $P(\boldsymbol{r}, m)$ is acted on by a single non-zero force

$$\overrightarrow{F} = (\boldsymbol{F}, \boldsymbol{M}), \tag{39}$$

then Ax 2 N bis simply states that its directrix must unconditionally pass through P. This supposition is as natural as to seem obvious. No wonder the authors of text-books on analytical dynamics never bothered to formulate it explicitly.

Sch 20. The relation (31) is certainly satisfied in the case

$$\boldsymbol{r} \times \boldsymbol{F}_\nu = \boldsymbol{M}_\nu \qquad (\nu = 1, \dots, n), \tag{40}$$

as it is immediately seen by virtue of (10). Since any of the relations (40) represents (provided the radius-vector $\boldsymbol{r}$ of P is fluent) the equation of the directrix of the force $\overrightarrow{F}_\nu$ $(\nu = 1, \dots, n)$ respectively, these relations give expression to the requirement that any of these directrices must pass through the mass-point P. This is what all physicists bear in mind when asserting that there "would be but a single law of motion"[14]. Although *physically* quite natural, *mathematically* this requirement follows from no other hypothesis of mass-point dynamics and is, consequently, a new dynamical supposition.

Being unprovable, it may be raised to the rank of a new dynamical axiom, namely:

Ax 2 N bis bis. If a system of forces is acting on a mass-point P, then P lies on the directrix of any of these forces.

Sch 21. Naturally, it is possible to build a mass-point dynamics founded on Ax 1 N and Ax 2 N bis bis instead of Ax 1 N and Ax 2 N. The range of action of this hypothetical dynamics is, however, considerably narrower than the Newtonian one. It is true that Ax 1 N and Ax 2 N bis bis imply Ax 2 N, but the inverse is not true: Ax 1 N and Ax 2 N may be satisfied while Ax 2 N bis bis may not.

Sch 22. In such a way we are faced with the alternative: on the one hand, to develop the Newtonian mass-point dynamics on the basis of Ax 1 N, Ax 2 N; on the other hand, to develop an entirely identical dynamics on the basis of Ax 1 N, Ax 2 N bis. Both ways are completely equal in rights, as regards the Newtonian mass-point dynamics solely. If, however, the last is regarded together with the Eulerian rigid body dynamics, then the first way is preferable from an aesthetic point of view at least. Indeed, as it has been mentioned above, there is a complete parallelism between Ax 1 N, Ax 2 N, on the one hand, and Ax 1 E, Ax 2E, on the other hand. This parallelism, however, vanishes into thin air, if one chooses Ax 2 N bis instead of Ax 2 N, since Ax 2 E bis analogous to Ax 2 N bis simply does not exist: in the Eulerian rigid body dynamics there is no true proposition (axiom or theorem) similar to Ax 2 N bis.

In other words, in rigid body dynamical the analogue of Ax 2 N bis, if any, is simply and purely false: whereas (31) implies (30), in rigid body dynamics, as underlined in Sch 17, there is no obligatory relation connecting the basis and the moment of a system of forces acting on a rigid body — both these quantities may be absolutely arbitrary in a particular dynamical problem concerning rigid bodies.

Sch 23. Let us now display how "proofs" of the Eulerian dynamical axioms Ax 1 E and Ax 2 E are fabricated by most authors of text-books, treatises and monographs on analytical dynamics. To this end let $P_\nu(\boldsymbol{r}_\nu, m_\nu)$ $(\nu = 1, \dots, n)$ be mass-points on which the systems of forces $F_\nu(\boldsymbol{F}_\nu, \boldsymbol{M}_\nu)$ $(\nu = 1, \dots, n)$ respectively are acting, the moments $\overrightarrow{\boldsymbol{M}}_\nu$ $(\nu = 1, \dots, n)$ being taken with respect to O. Then Ax 1 N and Ax 2 N imply

$$\frac{d}{dt}(m_\nu \boldsymbol{v}_\nu) = \boldsymbol{F}_\nu \qquad (\nu = 1, \dots, n) \tag{41}$$

and

$$\frac{d}{dt}(\boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu) = \boldsymbol{M}_\nu \qquad (\nu = 1, \dots, n) \tag{42}$$

respectively, provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$ $(\nu = 1, \dots, n)$, the derivatives being taken with respect to the inertial according to Newton system of reference $Oxyz$. Adding (41) and (42) together and adopting the notations (10) one obtains

$$\frac{d}{dt}\sum_{\nu=1}^{n} m_\nu \boldsymbol{v}_\nu = \boldsymbol{F} \tag{43}$$

and

$$\frac{d}{dt}\sum_{\nu=1}^{n} \boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu = \boldsymbol{M} \tag{44}$$

respectively. The quantities

$$\boldsymbol{k}_n \ \mathrm{sgn} : \ \sum_{\nu=1}^{n} m_\nu \boldsymbol{v}_\nu \tag{45}$$

and

(46) $$l_n \text{ sgn} : \sum_{\nu=1}^{n} \boldsymbol{r}_\nu \times m_\nu \boldsymbol{v}_\nu$$

are by definition the momentum and the moment of momentum (kinetical moment) respectively of the system S_n of mass-points P_ν ($\nu = 1, \ldots, n$). Now (43) — (46) imply

(47) $$\dot{\boldsymbol{k}}_n - \boldsymbol{F} = \boldsymbol{O}$$

and

(48) $$\dot{\boldsymbol{l}}_n - \boldsymbol{M} = \boldsymbol{O}$$

respectively.[15]

The formal analogy between the laws (47), (48), on the one hand, and (23), (24) respectively, on the other hand, is obvious. Not so obvious is the sinNlitide between the relations (43), (44), on the one hand, and (21), (22) respectively, on the other hand. Some words in this connection are therefore not pointless.

Let us imagine, to this end, that a partition of a rigid body B is accomplished by the aid of three series of mutually perpendicular planes into a system of parallelepipeds, $n = m^3$ in number, in such a manner that *all* dimensions of any of them tend to zero with increasing m. Let p_ν denote the ν-th of these parallelepipeds, m_ν — its mass, and P_ν — any point inside of p_ν ($\nu = 1, \ldots, n$).

If now one condescends to follow the logical process of those authors of textbooks on analytical dynamics which are pretending to "prove" the Eulerian dynamical axioms Ax 1 E and Ax 2 E, then one could fancy that the rigid body B is "substituted" by a system of mass-points $P_\nu(\boldsymbol{r}_\nu, m_\nu)$, provided $\boldsymbol{r}_\nu = OP_\nu$ ($\nu = 1, \ldots, n$). According to these authors, this peculiar imitation of B is as much more adequate as greater n is, with the tendency to transmute into a complete identity with the infinitely increasing n. According to this current of thoughts the discrete $\boldsymbol{r}_\nu$, $\boldsymbol{v}_\nu$, and m_ν ($\nu = 1, \ldots, n$) in the right-hand sides of (45) and (46) are transformed into the general, deprived of individuality, $\boldsymbol{r}$, $\boldsymbol{v}$, and dm respectively in the right-hand sides of (19) and (20) respectively. According to the same ideas, the dynamical equations (43) and (44) are transmuted into the dynamical equations (21) and (22) respectively: the Eulerian dynamical axioms are proved!

There are, however, two at least points in connection with this *Hokus Pokus* that badly need explications.

First of all, let us remind the way the relations (43) and (44) have been obtained. We arrived at them by adding together the equations (41) and (42) respectively, in other words the latter are inescapable for our gains. If one does not dispose of (41) and (42) in any particular case, then one simply and purely has no right, both logical and ethical, to appeal to (43) at all. Now in the above reasonings one is entirely denuded of the possibility to write down (41) and (42), since there is no information about the forces in the right hand sides of these equations. In other words, one knows *nothing*, but *nothing indeed* (*nichts, rien de rien, nada, niente, ничто*) about both the bases $\boldsymbol{F}_\nu$ and the moments $\boldsymbol{M}_\nu$ of those hypothetical forces (7) which are acting (in the heads of eminent authors at least) on the mass-points P_ν of p_ν ($\nu = 1, \dots, n$). All that is known is the system of forces acting on the rigid body B itself. As far as the problem is concerned how are these latter forses decomposed (mentally at least, if not actually, alias physically) or distributed in order to act on those mass-points which, in their *Mannigfaltigkeit*, are intended to replace B in the above reasonings — its only answer is *ignoramus et ignorabimus.*

In such a manner, the whole mental procedure, described above, remains hanging in the air. It is rotten through and through. *In my end is my begining*, the proverb says. As regards the efforts to prove the Eulerian dynamical axioms, their end is in their beginning. Reasonings of the sort just exposed should not be written in black and white. The addle eggs must be cast out of the nest.

As regards the passage from m_ν ($\nu = 1, \dots, n$) to dm, mentioned above, the things stand topsyturvy.

Physically the notion of *density* of a rigid body B at any of its points P is defined as follows. Let

$$(49) \qquad \Delta_1, \ \Delta_2, \dots, \Delta_n, \dots$$

be an infinite sequence of parts of B, any of which involves P and is deposited in the preceding one, and let their dimensions in every direction tend to zero with increasing n. Let $\boldsymbol{v}_\nu$ and m_ν denote the volume and the mass of Δ_ν ($\nu = 1, 2, \dots, n$) respectively. Then the fraction

$$(50) \qquad \varkappa_\nu = \frac{m_\nu}{v_\nu} \qquad (\nu = 1, \ 2, \dots)$$

is called the *mean density* of Δ_ν. Its limit

$$(51) \qquad \varkappa = \lim_{\nu \to \infty} \varkappa_\nu$$

is called the *density* of B *at* P. (The physicists take for gospel truth the hypothesis that $\varkappa$ is a function of P only, *independently* of the particular sequence (49) by means of which it is defined.)

Mathematically, however, the whole procedure is unrecognizably reversed. The *density* of a rigid body B at any of its points P is a beforehand given function (15) of the radius-vector $\boldsymbol{r}$ of P that takes part in the very definition of B. (Naturally, certain additional conditions about this function must be hypothesized, in the first place its integrability in a certain sense, in view of the definition (16) of the elementary mass dm and its participation in all the integrals (17) — (20), etc.) This function being known, the mass of the part Δ_ν of B is defined by the integral

$$(52) \qquad m_\nu .= \int_{\Delta_\nu} dm = \int_{\Delta_\nu} \varkappa(\boldsymbol{r}) d\mu$$

according to (17), taken over Δ_ν ($\nu = 1, 2, \ldots$), the meaning of $d\mu$ being explained above.

This analysis displays once more the complete bankruptcy of the mechanical texts pretending to give "proofs" of the Eulerian dynamical laws of momentum and of kinetical moment.

Sch 24. And yet, this approach has been broadly used in mechanics in times gone by. So, for instance, the *mass* of a system S_n of masspoints $P_\nu(\boldsymbol{r}_\nu, m_\nu)$ ($\nu = 1, \ldots, n$) is defined by

$$(53) \qquad m \text{ sgn} : \sum_{\nu=1}^{n} m_\nu;$$

the radius-vector $\boldsymbol{r}_G$ of its *mass-center* G by

$$(54) \qquad \boldsymbol{r}_G \text{ sgn} : \frac{1}{m} \sum_{\nu=1}^{n} m_\nu \boldsymbol{r}_\nu;$$

and its *kinetic energy* by

$$(55) \qquad T \text{ sgn} : \frac{1}{2} \sum_{\nu=1}^{n} m_\nu v_\nu^2,$$

provided $\boldsymbol{v}_\nu = \dot{\boldsymbol{r}}_\nu$ ($\nu = 1, \ldots, n$). Completely similar to (53) — (55), the *mass* and the *mass-center* of a rigid body B are, as already mentioned, defined by (17) and (18) respectively, and its *kinetic energy* by

$$(56) \qquad T \text{ sgn} : \frac{1}{2} \int v^2 dm.$$

Now, not a word could be said justly against the definitions (17), (18), and (56), no matter that, treating the continual case, they are suggested by the definitions (53) - (55) respectively, treating the discrete one. The role of (53) — (55) for the formulation of the *definitions* (17), (18), and (56) respectively, however, is purely suggestive, inductive, heuristic, by no means logical. As regards the theorems, none of them can be proved for continua on the basis of facts known in the discrete case only, i.e. by analogy.

Sch 25. The question, raised at the end of Sch 15, namely — how did Euler arrive at the idea of his dynamical laws, may now be concretisized: did he use the logical process described in Sch 23, — or in other words, did he attain to Ax 1 E and Ax 2 E through (43) and (44) respectively? Although it is rather tempting to answer this last question in the affirmative, the justice requires some cautiousness. It is true that the equations (43), (44) are Euler's discoveries. It is also true that the heuristic approaches, described in Sch 24, have been widely used in Euler's days for definitional goals. At that, the associative abilities of Euler's mind were proverbial; as regards the formal analogies, he was a universally recognized master (let us remind the established by him relation between the exponential and the trigonometrical functions, or his summations of divergent series — to cite a few examples out of a legion). At last, it is true that in the epoch immediately foregoing the French revolution the mental picture of a rigid body as composed of a large number of mass-points has been most popular. And yet, the mathematical creation is a phenomenon that belongs to psychology rather than to mathematics itself. Let us remind that someone had said: he, who states categorically something that lies outside pure mathematics, is at least imprudent.

It has been underlined in Sch 9 that, S being a particular rigid system of reference, neither Ax 1 N, Ax 2 N, nor Ax 1 E, Ax 2 E, give any possibility to decide whether S is inertial or not according to Newton or to Euler respectively. We are now in a position to subject this problem to a detailed analysis.

The following propositions play an auxiliary role in solving this problem.

Pr 7. If α and β are rigid systems of reference, the function

$$\boldsymbol{p}: R \longrightarrow V \tag{57}$$

is differentiable, and $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then

$$\frac{d_\alpha \boldsymbol{p}}{dt} = \frac{d_\beta \boldsymbol{p}}{dt} + \overline{\omega}_{\alpha\beta} \times \boldsymbol{p} \qquad (t \in R). \tag{58}$$

Dm. Let by definition

$$\beta \text{ sgn}: \{\overrightarrow{b}_\nu\}_{\nu=1}^{3}, \tag{59}$$

where

$$\overrightarrow{b}_\nu \text{ sgn}: (\boldsymbol{b}_\nu, \boldsymbol{B}_\nu) \qquad (\nu = 1, 2, 3), \tag{60}$$

$$\boldsymbol{b}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{61}$$

$$\boldsymbol{B}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{62}$$

$$b_1(t) \times b_2(t) \cdot b_3(t) \neq 0 \qquad (t \in R), \tag{63}$$

$$b_\mu(t)B_\nu(t) + b_\nu(t)B_\mu(t) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ t \in R), \tag{64}$$

$$\frac{d}{dt}(b_\mu(t)b_\nu(t)) = 0 \qquad (\mu,\ \nu = 1,\ 2,\ 3;\ t \in R) \tag{65}$$

(Sch 10). Then $\overline{\omega}_{\alpha\beta}$ is defined as the only solution of the system of vector equations

$$\frac{d_\alpha b_\nu}{dt} = \overline{\omega}_{\alpha\beta} \times b_\nu \qquad (\nu = 1,\ 2,\ 3;\ t \in R), \tag{66}$$

namely

$$\overline{\omega}_{\alpha\beta} = \frac{1}{2}\sum_{\nu=1}^{3} b_\nu^{-1} \times \frac{d_\alpha b_\nu}{dt} \qquad (t \in R) \tag{67}$$

[91], the reciprocal vectors b_ν^{-1} of b_ν ($\nu = 1,\ 2,\ 3$) being defined as in Sch 10. It is proved that $\overline{\omega}_{\alpha\beta}$ satisfies also the system

$$\frac{d_\alpha b_\nu^{-1}}{dt} = \overline{\omega}_{\alpha\beta} \times b_\nu^{-1} \qquad (\nu = 1,\ 2,\ 3;\ t \in R). \tag{68}$$

The definition App (17) implies

$$\frac{d_\beta p}{dt} = \sum_{\nu=1}^{3}\left(\frac{d}{dt}(pb_\nu^{-1})\right) b_\nu \qquad (t \in R). \tag{69}$$

It is proved that if the functions

$$p_\nu :\ R \longrightarrow V \qquad (\nu = 1,\ 2) \tag{70}$$

are differentiable, then for any rigid system of reference α

$$\frac{d(p_1 p_2)}{dt} = \frac{d_\alpha p_1}{dt}p_2 + p_1\frac{d_\alpha p_2}{dt} \qquad (t \in R). \tag{71}$$

The relation (71) implies

$$\frac{d}{dt}(pb_\nu^{-1}) = \frac{d_\alpha p}{dt}b_\nu^{-1} + p\frac{d_\alpha b_\nu^{-1}}{dt} \qquad (t \in R) \tag{72}$$

($\nu = 1,\ 2,\ 3$), and (72), (68) imply

$$\frac{d}{dt}(\boldsymbol{p}\boldsymbol{b}_\nu^{-1}) = \frac{d_\alpha \boldsymbol{p}}{dt}\boldsymbol{b}_\nu^{-1} + \boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \cdot \overline{\boldsymbol{b}_\nu}^{-1} \qquad (t \in R) \tag{73}$$

($\nu = 1,\ 2,\ 3$). Now (69), (73) imply

$$\frac{d_\beta \boldsymbol{p}}{dt} = \sum_{\nu=1}^{3}\left(\frac{d_\alpha \boldsymbol{p}}{dt}\boldsymbol{b}_\nu^{-1}\right)\boldsymbol{b}_\nu + \sum_{\nu=1}^{3}(\boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \cdot \boldsymbol{b}_\nu^{-1})\boldsymbol{b}_\nu \tag{74}$$

($t \in R$), and (74), App (16) with $\frac{d_\alpha \boldsymbol{p}}{dt}$ and $\boldsymbol{p} \times \overline{\omega}_{\alpha\beta}$ respectively instead of $\boldsymbol{p}$ imply

$$\frac{d_\beta \boldsymbol{p}}{dt} = \frac{d_\alpha \boldsymbol{p}}{dt} + \boldsymbol{p} \times \overline{\omega}_{\alpha\beta} \qquad (t \in R), \tag{75}$$

whence (58).

Pr 8. If α and β are rigid systems of reference and $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then

$$\frac{d_\alpha \boldsymbol{p}}{dt} = \frac{d_\beta \boldsymbol{p}}{dt} \qquad (t \in R) \tag{76}$$

for any differentiable function (57) if, and only if,

$$\overline{\omega}_{\alpha\beta} = \boldsymbol{o} \qquad (t \in R). \tag{77}$$

Dm. Pr 7.

Sch 26. The relation (58) is usually called the *connection between the local derivatives of a vector function with respect to two systems of reference.*

Sch 27. Let us note in passing that

$$\overline{\omega}_{\beta\alpha} = -\overline{\omega}_{\alpha\beta}. \tag{78}$$

Indeed, Pr 7 implies

$$\frac{d_\beta \boldsymbol{p}}{dt} = \frac{d_\alpha \boldsymbol{p}}{dt} + \overline{\omega}_{\beta\alpha} \times \boldsymbol{p} \qquad (t \in R). \tag{79}$$

Now (58) and (79) imply

$$(\overline{\omega}_{\alpha\beta} + \overline{\omega}_{\beta\alpha}) \times \boldsymbol{p} = \boldsymbol{o}. \tag{80}$$

Since (80) is satisfied for any differentiable function (57), it implies (78).

Pr 9. If α and β are rigid systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, P is any moving point and

$$\text{(81)} \qquad \boldsymbol{r} = \boldsymbol{AP}, \qquad \boldsymbol{r}_B = \boldsymbol{AB}, \qquad \overline{\rho} = \boldsymbol{BP},$$

then

$$\text{(82)} \qquad \frac{d_\alpha \boldsymbol{r}}{dt} = \frac{d_\alpha \boldsymbol{r}_B}{dt} + \overline{\omega}_{\alpha\beta} \times \overline{\rho} + \frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R).$$

Dm. (81) and the obvious identity $\boldsymbol{AP} = \boldsymbol{AB} + \boldsymbol{BP}$ imply

$$\text{(83)} \qquad \boldsymbol{r} = \boldsymbol{r}_B + \overline{\rho},$$

whence (82).

Sch 28. If α is a rigid system of reference, n is a natural number, and the function (57) is $n+1$ times differentiable, then by definition

$$\text{(84)} \qquad \frac{d_\alpha^{n+1}\boldsymbol{p}}{dt^{n+1}} \text{ sgn : } \frac{d_\alpha}{dt}\frac{d_\alpha^n \boldsymbol{p}}{dt}.$$

The left-hand side of (84) is called the $(n+1)$*th derivative* of $\boldsymbol{p}$ *with respect to* α or the *local* (*with respect to* α) $(n+1)$*th derivative* of $\boldsymbol{p}$.

Sch 29. If $\boldsymbol{r}$ denotes the radius-vector of a moving point with respect to the origin of the rigid system of reference α, then the first and the second local derivatives of $\boldsymbol{r}$ with respect to α are usually called respectively the *local velocity* and the *local acceleration* of P *with respect to* α. If there is no danger of collision of notations, they are traditionally denoted by $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively.

Pr 10. If α and β are rigid systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α,

$$\text{(85)} \qquad \overline{\varepsilon}_{\alpha\beta} \text{ sgn : } \frac{d_\alpha \overline{\omega}_{\alpha\beta}}{dt}$$

is by definition the *instantaneous angular acceleration* of β with respect to α, P is a mass-point with two times differentiable mass

$$\text{(86)} \qquad m : R \longrightarrow R$$

and (81) hold, then

$$(87) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho})$$

$$+\overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta\overline{\rho}}{dt} + \frac{d_\beta(m\overline{\rho})}{dt}\right) + \frac{d_\beta}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) \qquad (t \in R).$$

Dm. (82) implies

$$(88) \qquad m\frac{d_\alpha \boldsymbol{r}}{dt} = m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \overline{\omega}_{\alpha\beta} \times m\overline{\rho} + m\frac{d_\beta\overline{\rho}}{dt}$$

$(t \in R)$, whence

$$(89) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho}$$

$$+\overline{\omega}_{\alpha\beta} \times \frac{d_\alpha(m\overline{\rho})}{dt} + \frac{d_\alpha}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) \qquad (t \in R)$$

in view of (85). On the other hand, Pr 7 implies

$$(90) \qquad \frac{d_\alpha(m\overline{\rho})}{dt} = \frac{d_\beta(m\overline{\rho})}{dt} + \overline{\omega}_{\alpha\beta} \times m\overline{\rho} \qquad (t \in R),$$

$$(91) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) = \frac{d_\beta}{dt}\left(m\frac{d_\beta\overline{\rho}}{dt}\right) + \overline{\omega}_{\alpha\beta} \times m\frac{d_\beta\overline{\rho}}{dt} \qquad (t \in R).$$

Now (89) — (91) imply (87).

Sch 30. Now we are capable of proving some important propositions shedding some light on the problem of inertiality according to Newton and Euler of rigid systems of reference. With a view to a better comprehension of these propositions they are somewhat dismembered.

Pr 11 N. If α and β are inertial according to Newton systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α and $\boldsymbol{r}_B = \boldsymbol{AB}$, then

$$(92) \qquad \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) = \boldsymbol{o} \qquad (t \in R)$$

for any function (86) and

$$(93) \qquad \overline{\omega}_{\alpha\beta} = \boldsymbol{o} \qquad (t \in R).$$

Dm. α and β being inertial according to Newton by hypothesis, Df 1 N and Ax 1 N imply

$$\frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \boldsymbol{F} \qquad (t \in R), \tag{94}$$

$$\frac{d_\beta}{dt}\left(m\frac{d_\beta \boldsymbol{r}}{dt}\right) = \boldsymbol{F} \qquad (t \in R), \tag{95}$$

provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (94), (95), and (87) imply

$$\frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) + \overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho}) + \overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta (m\overline{\rho})}{dt}\right) \qquad (t \in R). \tag{96}$$

Since Ax 1 N holds for any mass-point and for any system of forces acting on it, the corollary (96) from (94), (95) holds for any m and $\overline{\rho}$. Let us first choose

$$\overline{\rho} = \boldsymbol{o} \qquad (t \in R). \tag{97}$$

Then (96) implies (92), and (96), (92) imply

$$\overline{\varepsilon}_{\alpha\beta} \times m\overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times m\overline{\rho}) + \overline{\omega}_{\alpha\beta} \times \left(m\frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta (m\overline{\rho})}{dt}\right) = \boldsymbol{o} \tag{98}$$

$(t \in R)$. Thereupon let us choose

$$m = 1 \qquad (t \in R), \tag{99}$$

$$\frac{d_\beta \overline{\rho}}{dt} = \boldsymbol{o} \qquad (t \in R). \tag{100}$$

Then (98) — (100) imply

$$\overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho}) = \boldsymbol{o} \qquad (t \in R). \tag{101}$$

Scalar multiplication of (101) with $\overline{\rho}$ implies

$$(\overline{\omega}_{\alpha\beta} \times \overline{\rho})^2 = 0 \qquad (t \in R), \tag{102}$$

whence

$$\overline{\omega}_{\alpha\beta} \times \overline{\rho} = o \qquad (t \in R). \tag{103}$$

In particular , if one puts in (103) successively $\overline{\rho} = \boldsymbol{b}_1$ and $\overline{\rho} = \boldsymbol{b}_2$, then one obtains the system of vector equations

$$\overline{\omega}_{\alpha\beta} \times \boldsymbol{b}_\nu = o \qquad (\nu = 1,\ 2;\ t \in R). \tag{104}$$

Since by hypothesis $\boldsymbol{b}_1 \times \boldsymbol{b}_2 \neq o$ according to (63), the system (104) has exactly one solution, namely (93).

Pr 12 N. If α is an inertial according to Newton system of reference and β is a rigid system of reference, with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then (92) for any function (86) and (93) imply that β is inertial according to Newton too.

Dm. α being inertial according to Newton by hypothesis, Df 1 N and Ax 1 N imply (94) provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (87), (92) — (94) imply (95), i.e. β is inertial according to Newton (Df 1 N, Ax 1 N).

Pr 13 N. If α and β are rigid systems of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then necessary and sufficient conditions in order that α and β are simultaneously inertial according to Newton are (92) for any function (86) and (93).

Dm. Pr 11 N, Pr 12 N.

Sch 31. Before proceeding further, let us make some remarks in connection with Pr 13 N.

In the formulation of the Newtonian dynamical axioms no hypotheses have been made concerning the mathematical nature of the masses of the mass-points. Following the Newtonian tradition, however, for a long period of time the classical mechanics has worked under the acceptance (not explicity formulated, it is true) that the masses are absolute constants, especially, that they are invariant with respect to the time. In any case, such has been the state of affairs in mass-point and rigid body dynamics until the end of the last century.

In 1904, however, the Russian nechanician Meshtcherski proposed the differential equation

$$m\frac{dv}{dt} = \boldsymbol{F} + \overline{\boldsymbol{\Phi}}_1 + \overline{\boldsymbol{\Phi}}_2 \qquad (t \in R) \tag{105}$$

for the motion of mass-points with variable masses. At that, by definition

$$\overline{\Phi}_{\nu} \ \text{sgn} : \ \frac{dm_{\nu}}{dt} v_{\nu} \qquad (\nu = 1,\ 2) \tag{106}$$

are additional forces generated by the alterations of the masses, where $\frac{dm_1}{dt}$ is the rate of change of the outgo of m and $\frac{dm_2}{dt}$ — that of its income; v_1 is the relative velocity of the "particles separating from the mass-point" (according to Meshtcherski's mechanical ideas) and v_2 the relative velocity of the "particles added to the mass-point"; $\overline{\Phi}_1$ is called the "reactive traction" and $\overline{\Phi}_2$ — the "arresting force". It is supposed that the equation (105) is related to an inertial according to Newton system of reference. After Meshtcherski's work a new branch of analytical dynamics germinated: variable mass dynamics (although mainly mass-point problems have been discussed).

Sch 32. As it will be shown soon, the necessary and sufficient conditions, formulated in Pr 13 N, for the simultaneous inertiality according to Newton of two rigid systems of reference do not coincide in the cases of constant masses, on the one hand, and of variable masses, on the other hand. In view of the importance of this circumstance for the Newtonian mass-point dynamics in general, we shall subject it to a close analysis.

To this end we shall formulate two additional dynamical hypotheses which are mutually exclusive, i.e. inconsistent *coniunctim*. Afterwards we shall re-redact Pr 13 N separately for any of these cases.

Hpth NC. If m is the mass of any mass-point, then

$$\frac{dm}{dt} = 0 \qquad (t \in R). \tag{107}$$

Hpth NV. There exists one at least mass-point, the mass of which satisfies

$$\frac{dm}{dt} \neq 0 \qquad (t \in R). \tag{108}$$

Pr 14 NC. Under the conditions and notations of Pr 13 N, the supposition Hpth NC implies that the relation (92) is equivalent with

$$\frac{d_{\alpha}^2 \boldsymbol{r}_B}{dt^2} = o \qquad (t \in R). \tag{109}$$

Dm. Clear.

Pr 14 NV. Under the conditions and notations of Pr 13 N, the supposition Hpth NV implies that the relation (92) is equivalent with

$$\frac{d_{\alpha} \boldsymbol{r}_B}{dt} \doteq o \qquad (t \in R). \tag{110}$$

Dm. Since (92) must hold for any function (86), let (99) hold. Then (92) implies (109). Now (86) is equivalent with

$$\frac{dm}{dt}\frac{d_\alpha \bar{\boldsymbol{r}}_B}{dt} + m\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} = \boldsymbol{o} \qquad (t \in R) \tag{111}$$

and (111), (109), (108) imply (110).

Sch 33. If (109), (93) hold, then it is said that the motion of β with respect to α is a *rectilinear uniform translation*. On the other hand, if (110), (93) hold, then obviously β is *at rest* with respect to α. Indeed, (110) implies that the origin B of β does not move with respect to α, whereas (93) and (66) imply that the axis vectors $\boldsymbol{b}_\nu$ ($\nu = 1, 2, 3$) of β do not move with respect to α.

Using this terminology, we may re-redact Pr 13 N, splitting it into two propositions, the one corresponding to Hpth NC and the other — to Hpth NV.

Pr 15 NC. If there exists no mass-point with variable mass, then a necessary and sufficient condition in order that two rigid systems of reference are simultaneously inertial according to Newton is that they move with a rectilinear uniform translation with respect to each other.

Dm. Pr 13 N, Pr 14 NC, Sch 33.

Pr 15 NV. If there exists one at least mass-point with a variable mass, then a necessary and sufficient condition in order that two rigid systems of reference are similtaneously inertial according to Newton is that they are at rest with respect to each other.

Dm. Pr 13 N, Pr 14 NV, Sch 33.

Sch 34. Let us note that the necessary and sufficient conditions of Pr 14 NV are obviously considerably more restrictive than those of Pr 14 NC. In such a manner, as regards these two propositions, we are faced with a problem that, poetically at least, may be compared with the Gordian knot.

Pr 14 NC represents a fundamental credo of the classical Newtonian mass-point dynamics. Moreover, appropriate versions of this theorem belong to the basic acceptance of the Eulerian rigid body dynamics too, as well as of the theory of elasticity and of fluid mechanics, in other words, of the classical rational mechanics as a whole. In the mathematical reference book [91], for instance, chosen at random by the way, one reads:

"Всякая система отсчета, к-рая движется относительно И. с. о. прямолинейно и равномерно, является И. с. о."[16] (p. 562).

Now Pr 14 NV seems to destroy this dynamical credo. Indeed, according to it, if α is an inertial according to Newton system of reference, then the rigid system β is inertial according to Newton if, and only if, it does not move with respect to α (Pr 15 NV).

Sch 35. Physical arguments are *personae non grata* in mathematics. And yet, they may serve as a compass or, if one likes it better, as Ariadne's thread, even for pure mathematicians. For doubtlessly Newton and Euler have been striving at the shaping of a rational mechanics, applicable in the real world they were living in. In this connection let us underline that Pr 14 NC has successfully sustained the trials of practical examinations for two clear centuries.

Sch 36. Things standing as they ar, Pr 14 NV will persist in being *anguis in herba*, a logical trap for analytical dynamics until its contradictions with Pr 14 NC are abolished. It is obvious that in the eyes of a Newtonian purist the very idea that two rigid systems of reference are similtaneously inertial according to Newton only in case of mutual rest would seem a little short of heresy. One must not forget, however, that *nothing in mathematics is heresy enough to be worthy of the name*, the greatest virtue of a genuine mathematician consisting in his only ideology to have no ideology.

In its long adventuresome life mathematics has overlived quite a lot of mental shocks in order to be impressed of any. The fiest one has been when mathematics was *in cunabula*: $\sqrt{2}$ turned out to be no broken number! Then the collapse of the hopes for trisecting angles, doubling cubes, and squaring circles. And the fifth postulate — a far cry from what it has been imagined! To say nothing about tangentless curves, Mengenlehre-antinomies, the choice axiom, or the crash of Hilbert's axiomatical expectations... There is hardly something on God's earth to disturb mathematician's peace of mind nowadays.

Exits out of the logical pitfall that Pr 14 NC has driven mass-point dynamics in may be sought in several directions.

Sch 37. The first line of conduct may be capitulatory one: the avowal that the acceptance of (110) and (93) in the capacity of necessary and sufficient conditions of inertiality is OK. This is the easiest and at the same time the silliest solution.

In the second place, one could hypothesize the impossibility of (108) in the frames of Ax 1 N and Ax 2 N. This is equivalent to the acceptance of Hpth NC along with Ax 1 N and Ax 2 N, alias with the postulate that no Newtonian mass-point dynamics with variable masses exists.

Third, and last as we can see, one could come at the idea that a slight reformulation of Ax 1 N may render a helpful assistance.

Since there is an extremism in the air in the cases of the first two possibilities, we shall fix our attention on the last of these alternatives.

Sch 38. *Entre paranthèses*, the second of the above three opportunities is not as radical as it may seem at first glance. Indeed, the following questions may quite naturally arise. Is up to now classical mechanics to such an extent and so closely intimate with variable mass-point problems — sensible at that, not concocted, though the meaning of the last requirement is somewhat vague — that it could by no means divorce them? Is the classical example of Meshtcherski's dynamical equation (105) as blameless indeed, as it may seem at first sight? Is it not an underhand constant mass-point problem disguised as variable, as a matter of faet? For $\frac{dm}{dt}$ does not take part in this equation, as it should, and nothing in it suggests that m is variable with the time. Let us quote an excerpt from Ax 1 N: "for any mass-point P and for any system of forces $\underrightarrow{F}$, acting on it". In other words, the genuine mathematical equivalent of Ax 1 N in the variable mass-point case should be

(112) $$\frac{dm}{dt}v + m\frac{dv}{dt} = \boldsymbol{F},$$

the function (86) being prescribed for any particular mass-point problem and $\boldsymbol{F}$ — involving all the forces acting on P. It is true that $\frac{dm_1}{dt}$ and $\frac{dm_2}{dt}$ are at hand in (105), but m_1 and m_2 have nothing to do with m. In the same time it is also true that additional forces $\overline{\Phi}_1$ and $\overline{\Phi}_2$ are supplemented to $\boldsymbol{F}$ in the right-hand side of (105), unwarranted by (112). And so on, and so forth, etcetera... All these questions badly need a thorough mathematical analysis. Instead of it, in the mechanical literature one finds only texts written *currente calamo*.

Sch 39. In the constant mass-point case (107) the equation (112) reduces to

(113) $$m\boldsymbol{w} = \boldsymbol{F},$$

provided $\boldsymbol{w} = \dot{\boldsymbol{v}}$. In other words, if (107) holds, then (11) and (113) coincide.

The question now arises, whether Pr 14 NC could be saved in the variable mass-point case (108) too if (113) is substituted for the first Newtonian axiom (11)? In other words, let us try the possibilities of the following variant of Ax 1 N:

Ax 1 N bis. There exixts such a system of reference S that, all derivatives being taken with respect to S, for any mass-point P and for any system of forces $\underrightarrow{F}$ acting on P, the product of the mass and the acceleration of P equals the basis of $\underrightarrow{F}$.

It is easy to prove now that Ax 1 N bis implies a theorem analogous to Pr 15 NC, making, however, no use of the hypothesis Hpth NC. Beforehand the following definition must be recognized.

Df 1 N bis. Any system of reference satisfying Ax 1 N bis is called *inertial according to Newton.*

Pr 11 N bis. Ax 1 N bis being accepted, if α and β are inertial according to Newton systems of reference with origins A and B respectively, $\overline{\omega}_{\alpha\beta}$ is the instantaneous velocity of β with respect to α and $\boldsymbol{r}_B = \boldsymbol{AB}$, then (109) and (93) hold.

Dm. The demonstration imitates that of Pr 11 N. The relation (82) implies

(114) $$\frac{d_\alpha^2 \boldsymbol{r}}{dt} = \frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} + \overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta^2 \overline{\rho}}{dt^2} \qquad (t \in R),$$

whence

(115) $$m\frac{d_\alpha^2 \boldsymbol{r}}{dt^2} = m\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} + m\overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + m\overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2m\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} + m\frac{d_\beta^2 \overline{\rho}}{dt^2} \qquad (t \in R).$$

The systems of reference α and β being by hypothesis inertial according to Newton to the effect of Df 1 N bis, the latter together with Ax 1 N bis imply

(116) $$m\frac{d_\alpha^2 \boldsymbol{r}}{dt^2} = \boldsymbol{F} \qquad (t \in R),$$

(117) $$m\frac{d_\beta^2 \overline{\rho}}{dt^2} = \boldsymbol{F} \qquad (t \in R)$$

provided (81), m denoting the mass of any mass-point P and F — the basis of any system of forces acting on P. Then (115) — (117) imply

(118) $$\frac{d_\alpha^2 \boldsymbol{r}_B}{dt^2} + \overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho})$$

$$+ 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} + \frac{d_\beta^2 \overline{\rho}}{dt^2} \qquad (t \in R)$$

after canceling m.

Disposing of the equation (118) applying to any $\overline{\rho}$, in the particular case (97) it implies (109), and (109) and (118) imply

(119) $$\overline{\varepsilon}_{\alpha\beta} \times \overline{\rho} + \overline{\omega}_{\alpha\beta} \times (\overline{\omega}_{\alpha\beta} \times \overline{\rho}) + 2\overline{\omega}_{\alpha\beta} \times \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\beta^2 \overline{\rho}}{dt^2} = \boldsymbol{o}$$

$(t \in R)$. Thereupon (119) and the choice (100) imply (101). Afterwards (93) is proved in the same way as in the proof of Pr 11 N.

Pr 12 N bis. Ax 1 N bis being accepted, if α is an inertial according to Newton system of reference and β is a rigid system of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then (109) and (93) imply that β is inertial according to Newton too.

Dm. α being inertial according to Newton by hypothesis, Df 1 N bis and Ax 1 N bis imply (116) provided (81), m denoting the mass of any mass-point P and $\boldsymbol{F}$ — the basis of any system of forces acting on P. Then (115), (116), (109), (93) imply (117), i.e. β is inertial according to Newton (Df 1 N bis, Ax 1 N bis).

Pr 13 N bis. Ax 1 N bis being accepted, if α and β are rigid systems of reference with origins A and B respectively, $\boldsymbol{r}_B = \boldsymbol{AB}$, $\overline{\omega}_{\alpha\beta}$ is the instantaneous angular velocity of β with respect to α, then necessary and sufficient conditions in order that α and β are simultaneously inertial according to Newton are (109) and (93).

Dm. Pr 11 N bis, Pr 12 N bis.

Pr 15 N bis. Ax 1 N bis being accepted, a necessary and sufficient condition in order that two rigid systems of reference are simultaneously inertial according to Newton is that they move with a rectilinear uniform translation with respect to each other.

Dm. Pr 13 N bis, Sch 33.

Sch 40. Coming back to the general case let us note that by virtue of Df 1 N the only criterion for the inertiality according to Newton of a rigid system of reference is the answer to the question whether is satisfies Ax 1 N or not: if yes, then it is inertial; if not, it isn't. That is why no use of Ax 2 N has been made in the proof of the formulated in Pr 13 N criterion.

And yet, a pending question remains in connection with Ax 2 N and it is: is the axiom stable with respect to the established by Pr 13 N inertiality criterion? Since the meaning of this formulation is somewhat vague, let us make it more precise.

Suppose that α and β are rigid systems of reference for which the conditions (92), (93) hold and let α be inertial according to Newton. Then α *eo ipso* satisfies Ax 2 N. On the other hand β is inertial according to Newton in view of Pr 12 N, hence it also must *eo ipso* satisfy Ax 2 N. Now the question arises: it *must*, but *does it* indeed?

In other words, if $P(\boldsymbol{r}, m)$ is any mass-point and $\underrightarrow{F}\,(\boldsymbol{F}, \boldsymbol{M})$ is any system of forces acting on it, then it is certain that

$$\frac{d_\alpha}{dt}\left(\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt}\right) = \boldsymbol{M} \qquad (t \in R) \tag{120}$$

(under the notations already repeatedly used) by virtue of Ax 2 N, α being inertial according to Newton by hypothesis and $\boldsymbol{M}$ being taken with respect to the origin A of α. On the other hand, Pr 12 N warrants that β is inertial according to Newton too and, consequently, the equation

$$\frac{d_\beta}{dt}\left(\overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt}\right) = \boldsymbol{M}_B \qquad (t \in R) \tag{121}$$

(where

$$\boldsymbol{M}_B = \boldsymbol{M} + \boldsymbol{F} \times \boldsymbol{r}_B \tag{122}$$

is the moment of $\underrightarrow{F}$ with respect to the origin B of β) must also be satisfied by virtue of Ax 2 N. The meaning of the question, brought up above, is: *is it satisfed indeed*, in other words, *can* (121) *be proved*?

It is obvious that this question must unconditionally be answered *in the affirmative.*

Technically this problem is equivalent with the question: do (120), (92), (93), (122) imply (121)?

There are two ways leading to the answer.

The first of them is the direct one. Let (120), (92) hold for any function (86), as well as (93) and (122). Because of (93) the identity (88) becomes

$$m\frac{d_\alpha \boldsymbol{r}}{dt} = m\frac{d_\alpha \boldsymbol{r}_B}{dt} + m\frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R) \tag{123}$$

and (123), (83) imply

$$\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt} = (\boldsymbol{r}_B + \overline{\rho}) \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + (\boldsymbol{r}_B + \overline{\rho}) \times m\frac{d_\beta \overline{\rho}}{dt} \tag{124}$$

$(t \in R)$, i.e.

$$\begin{aligned} \boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt} &= \boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}}{dt} + \overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt} \\ &+ \overline{\rho} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \boldsymbol{r}_B \times m\frac{d_\beta \overline{\rho}}{dt} \qquad (t \in R). \end{aligned} \tag{125}$$

On the other hand, (93) and Pr 8 imply (76) for any differentiable function (57).

The equation (125) therefore implies

$$\begin{aligned} \frac{d_\alpha}{dt}\left(\boldsymbol{r} \times m\frac{d_\alpha \boldsymbol{r}}{dt}\right) &= \frac{d_\alpha}{dt}\left(\boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) \\ &+ \frac{d_\beta}{dt}\left(\overline{\rho} \times m\frac{d_\beta \overline{\rho}}{dt}\right) + \frac{d_\beta \overline{\rho}}{dt} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} \\ +\overline{\rho} \times \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) &+ \frac{d_\alpha \boldsymbol{r}_B}{dt} \times m\frac{d_\beta \overline{\rho}}{dt} + \boldsymbol{r}_B \times \frac{d_\beta}{dt}\left(m\frac{d_\beta \overline{\rho}}{dt}\right) \end{aligned} \tag{126}$$

$(t \in R)$. But obviously

$$\frac{d_\alpha}{dt}\left(\boldsymbol{r}_B \times m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) = \boldsymbol{r}_B \times \frac{d_\alpha}{dt}\left(m\frac{d_\alpha \boldsymbol{r}_B}{dt}\right), \tag{127}$$

$$\frac{d_\beta \overline{\rho}}{dt} \times m\frac{d_\alpha \boldsymbol{r}_B}{dt} + \frac{d_\alpha \boldsymbol{r}_B}{dt} \times m\frac{d_\beta \overline{\rho}}{dt} = \boldsymbol{o}, \tag{128}$$

$$\frac{d_\beta}{dt}\left(m\frac{d_\beta\bar{\rho}}{dt}\right) = \boldsymbol{F} \tag{129}$$

($t \in R$), the latter equation by virtue of Pr 13 N, i.e. (95). Now (126) — (129), (92), (120) imply

$$\boldsymbol{M} = \frac{d_\beta}{dt}\left(\bar{\rho} \times m\frac{d_\beta\bar{\rho}}{dt}\right) + \boldsymbol{r}_B \times \boldsymbol{F} \qquad (t \in R) \tag{130}$$

and (130), (122) imply (121).

The second way to the proof of (121) is extraordinary tricky. According to Pr 6, (120) may be written in the equivalent form

$$\boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M} \tag{131}$$

and (131), (83), (122) imply

$$\bar{\rho} \times \boldsymbol{F} = \boldsymbol{M}_B. \tag{132}$$

On the other hand, β is inertial according to Newton by virtue of Pr 13 N, hence (95) holds. Now (132), (95) imply

$$\bar{\rho} \times \frac{d_\beta}{dt}\left(m\frac{d_\beta\bar{\rho}}{dt}\right) = \boldsymbol{M}_B \qquad (t \in R), \tag{133}$$

whence (121).

Sch 41. *Acta est fabula.* For the time being this is almost all one could say apropos of ***inertiality according to Newton.*** Now it is high time to proceed to the discussing of analogous problems concerning ***inertiality according to Euler.***

Comparing (11), (12), on the one hand, with (21), (22) respectively, on the other hand, one should observe that these latter problems promise to be considerably harder. This is true. In the same time it is also true that up to now we have accumulated a certain experience in such matters.

Let the Cartesian system of reference $Oxyz$ be inertial according to Euler and let $\Omega\xi\eta\zeta$ be a Cartesian system of reference invariably connected with the rigid body B, its origin coinciding with the mass-center G of B. As it is well known from rigid body kinematics, then the following identity takes place

$$\boldsymbol{v} = \boldsymbol{v}_G + \bar{\omega} \times \bar{\rho}, \tag{134}$$

provided $\boldsymbol{r} = \boldsymbol{OP}$ for any point P of B, $\boldsymbol{r}_G = \boldsymbol{OG}$, $\boldsymbol{v} = \dot{\boldsymbol{r}}$, $\boldsymbol{v}_G = \dot{\boldsymbol{r}}_G$,

$$\boldsymbol{r} = \boldsymbol{r}_G + \bar{\rho}, \tag{135}$$

$\overline{\omega}$ denoting the instantaneous angular velocity of $\Omega\xi\eta\zeta$ with respect to $Oxyz$, all derivatives being taken with respect to $Oxyz$.

The identity (135) implies

$$\int \overline{\rho} dm = o \tag{136}$$

in view of the definition (18). Then (134), (136), (17) imply

$$\int \boldsymbol{v} dm = m\boldsymbol{v}_G \tag{137}$$

and (137), (21) imply

$$\frac{d}{dt}(m\boldsymbol{v}_G) = \boldsymbol{F} \qquad (t \in R), \tag{138}$$

$\boldsymbol{F}$ denoting the basis of any system of forces acting on B.

In other words, the equation (138) may be chosen in the capacity of a mathematical formulation of Euler's law of momentum, alias of Ax 1 E. It expresses the famous theorem of Euler according to which *the mass-center of any rigid body is moving like a mass-point with mass equal to the mass of the body and acted on by all forces acting on the body.*

Sch 42. The reader should not let himself be misled by the resemblance between (138) and (11). It is only formal. In other words, directly contrary to the wide-spread belief of all physicists, mechanicians, and mathematicians, the mass-center of a rigid body is no mass-point.

Indeed, if it was, then Ax 2 N would imply

$$\frac{d}{dt}(\boldsymbol{r}_G \times m\boldsymbol{v}_G) = \boldsymbol{M} \qquad (t \in R), \tag{139}$$

whence (30) by virtue of Pr 2 N, which is an absurdity.

By the way, another absurdity is obtainable in the following manner. First, (134) and (135) imply

$$\int \boldsymbol{r} \times \boldsymbol{v} dm = \int (\boldsymbol{r}_G + \overline{\rho}) \times (\boldsymbol{v}_G + \overline{\omega} \times \overline{\rho}) dm \tag{140}$$

$$= \int \boldsymbol{r}_G \times \boldsymbol{v}_G dm + \int \boldsymbol{r}_G \times (\overline{\omega} \times \overline{\rho}) dm + \int \overline{\rho} \times \boldsymbol{v}_G dm + \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho}) dm$$

($t \in R$). On the other hand, (17) and (136) imply

(141) $$\int \boldsymbol{r}_G \times \boldsymbol{v}_G dm = \boldsymbol{r}_G \times \boldsymbol{v}_G \int dm = \boldsymbol{r}_G \times m\boldsymbol{v}_G,$$

(142) $$\int \overline{\boldsymbol{r}}_G \times (\overline{\omega} \times \overline{\rho})dm = \boldsymbol{r}_G \times (\overline{\omega} \times \int \overline{\rho} dm) = \boldsymbol{o},$$

(143) $$\int \overline{\rho} \times \boldsymbol{v}_G dm = \int \overline{\rho} dm \times \boldsymbol{v}_G = \boldsymbol{o}$$

($t \in R$) and (140) — (143) imply

(144) $$\int \boldsymbol{r} \times \boldsymbol{v} dm = \boldsymbol{r}_G \times m\boldsymbol{v}_G + \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho})dm$$

($t \in R$). Now (144) and (22) imply

(145) $$\frac{d}{dt}(\boldsymbol{r}_G \times m\boldsymbol{v}_G) + \frac{d}{dt} \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho})dm = \boldsymbol{M}$$

($t \in R$) and (145), (135) imply the absurdity

(146) $$\frac{d}{dt} \int \overline{\rho} \times (\overline{\omega} \times \overline{\rho})dm = \boldsymbol{o} \qquad (t \in R).$$

Sch 43. The analogy between (138) and Ax 1 N being entirely formal, it is all the same enough for our goal, namely to use it in order to economize all the reasoning and reconings spent in connection with Pr 11 N — Pr 15 N bis, to say nothing about the enigmatical Hpth NC and Hpth NV.

The reader has certainly become aware of the trade dodge, long ago notorious as *Steiner's tea-kettle principle*: reduce unknown to known. In our case it is practicable in the following manner: Pr 11 N — Pr 15 N bis being demonstrable on the basis of Ax 1 N only and (138) imitating Ax 1 N up to the least, to prove anew their rigid body analogies would certainly mean useless efforts and needless time-wasting. The formal analogy between (138) and (11) secures the validity of these theorems in the rigid body case too (with the obvious *mutatis mutandis*, of course) without any specific proofs whatever.

We shall save the reader the bitter cup of reiteration of all those formulations. They are obvious and reducible to substitutions of the terms *rigid body* and *inertial according to Euler* for the terms *mass-point* and *inertial according to Newton* respectively in these propositions. Naturally, all the fuss in connection with variable masses jumps out again and is settled in the same way. No, the rigid body case deserves no special attention after the pains we have taken in connection with the mass-point dynamics. Instead, we shall turn our interest to another topic.

Sch 44. The economy we have realized by avoiding explicit formulations of the rigid body analogues of Pr 11 N — Pr 15 N bis imposes the following convention. If some of them has to be quoted, we shall cite it under the same number as in the corresponding mass-point case, substituting E for N. In such a manner Pr 14 NV, say, becomes Pr 14 EV, etc.

Sch 45. Summing up we may now state that, on the basis of the criteria of Pr 11 N — Pr 15 N bis (Pr 11 E — Pr 15 E bis), it is enough and to spare to know that a particular system of reference α is inertial according to Newton (Euler) in order to decide, for any rigid system of reference β, whether it is inertial according to Newton (Euler) or not. The point now consists in this peculiar system α.

Physically the choice of an inertial (no matter whether according to Newton or to Euler) system of reference is a matter of experiment. There are quite a lot of physical phenomena (declination toward east of a body falling freely in the northern hemisphere, the effect of Ber[17], etc.) indicating that no invariably connected with the Earth system of reference may be qualified as inertial. On the other hand, there are not a few physically quite trustworthy grounds to state that any rigid system of reference, the origin of which coincides with the mass-center of the Sun while its axes are directed toward immovable (far distant) stars, may be accepted in the capacity of an inertial one.

Mathematically, however, α being any particular rigid system of reference, there is no reason either to incriminate it as non-inertial or to make a fetish of it as inertial. In other words, any such system may be qualified as inertial, as well as non-inertial. Alias, all rigid systems of reference are allowed to competition as regards the title "inertial". Mathematically this qualification is a matter of definition, of definition only, and of nothing but definition.

Nevertheless there is a but there. In mathematical affairs there is an authoritative rule, the principle of economy. In other words, the most desirable case is the most simple one. But what in our case does actually most simple mean? The answer is given in the following two propositions.

Pr 16. If

$$e_\nu \in V \qquad (\nu = 1,\ 2), \tag{147}$$

$$e_\mu e_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2), \tag{148}$$

$$e_3 \operatorname{sgn} :\ e_1 \times e_2, \tag{149}$$

then

$$e_\mu e_\nu = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2,\ 3), \tag{150}$$

$$e_1 \times e_2 \cdot e_3 > 0. \tag{151}$$

Dm. Clear.
Pr 17. If (147) — (149),

$$\overrightarrow{e}_\nu \text{ sgn}:\ (e_\nu,\ o) \qquad (\nu = 1,\ 2,\ 3), \tag{152}$$

$$\varepsilon \text{ sgn}:\ \{\overrightarrow{e}_\nu\}_{\nu=1}, \tag{153}$$

then ε is a rigid right-hand orientated orthonormal Cartesian system of reference and o is its origin.

Dm. App (1) — (6), App (12) — (14), App (9), Pr 16.

We now manifest the following dynamical axiom.

Ax 3 N. The defined by (153) system of reference ε, provided (147) —(149), (152), is inertial according to Newton.

Sch 46. On the basis of Pr 11 N — Pr 15 N bis and Ax 3N one is capable of determining, for any rigid system of reference, whether it is inertial according to Newton or not. Especially:

Pr 18 N. If

$$a_\nu \in V \qquad (\nu = 1,\ 2,\ 3), \tag{154}$$

$$a_1 \times a_2 \cdot a_3 \neq 0, \tag{155}$$

$$A_\nu \in V \qquad (\nu = 1,\ 2,\ 3), \tag{156}$$

$$a_\mu A_\nu + a_\nu A_\mu = 0 \qquad (\mu, \nu = 1,\ 2,\ 3), \tag{157}$$

App (1) — (2), then α is inertial according to Newton.

Dm. Ax 3 N, Pr 14 NC, Pr 14 NV.

Sch 47. In such a way, the problem about inertiality according to Newton is settled. As regards Euler, we are faced with the following alternative:

1. There exists no system of reference inertial according to both Newton and Euler.
2. There exists one at least system of reference inertial according to both Newton and Euler.

Tertium non datur.

For the time being we are ignorant which of these two possibilities is true. As a matter of fact, any of them could be true, or could be untrue. Indeed, if α is any particular rigid system of reference, then we can decide, on the basis of Ax 3 N, whether it is inertial according to Newton or not. As regards its inertiality according to Euler, for the time being at least, we know as much as nothing. Before answering the question which of the above possibilities is true, let us make a little thinking.

If the first situation is realized and S_N and S_E are systems of reference inertial according to Newton and Euler respectively, then no proposition of the Newtonian mass-point dynamics does hold in S_E and no proposition of the Eulerian rigid body dynamics does hold in S_N (under the supposition that these propositions are not invariant with respect to the systems of reference). In other words, in the first case there exist two entirely distinct dynamical theories which are completely alienate from one another. In particular, no dynamical problem, simultaneously treating a mass-point and a rigid body *coniunctum*, can be solved by the direct application of both Ax 1 N, Ax 2 N and Ax 1 E; Ax 2 E (although there are indirect methods to this end). It is obvious that such a perspective does not seem a very attractive one.

Besides, in this first case a rather complicated problem arises in connection with the *time*-notion. As already explained in Sch 8, this notion is incapable of an explicit mathematical definition, being a primary notion-object of analytical mechanics, definable implicitly by means of Ax 1 N, Ax 2 N and Ax 1 E, Ax 2 E namely.

As a matter of fact, there are two rather than one time-notions, the one defined by the aid of Ax 1 N, Ax 2 N, and the other — by means of Ax 1 E, Ax 2 E. Correspondingly one should speak of *Newtonian time* and of *Eulerian time* and nobody knows apriory do they have in general something in common at all. This circumstance once again makes the first of the above possibilities entirely unacceptable.

In the second case any rigid system of reference is either inertial or non-inertial both according to Newton and Euler. Indeed, let the system of reference S be inertial both according to Newton and Euler and let the system Σ be inertial according to Newton. Then by virtue of Pr 11 N — Pr 15 N bis, the motion of Σ with respect to S is necessarily a rectilinear uniform translation or possibly a rest respectively. This condition, however, is sufficient for the inertiality of Σ according to Euler by virtue of the corresponding criterion among Pr 11 E — Pr 12 E bis. And *vice versa*, if Σ is inertial according to Euler, then by virtue of Pr 11 E — Pr 15 E bis its motion with respect to S is necessarily a rectilinear uniform translation or possibly a rest respectively. This condition is, however, sufficient for the inertiality of Σ according to Newton by virtue of the corresponding criterion among Pr 11 N — Pr 15 N bis. Hence, the Newtonian and Eulerian dynamical axioms hold for exactly the same sets of rigid systems of reference. In other words, in the second of the above cases there will exist a general dynamics, the Newton — Eulerian mass-point and rigid body dynamics. That is why this possibility is beyond comparison more tempting than the first one.

These considerations justify the acceptance of the following axiom.

Ax 3 E. The defined by (153) system of reference ε, provided (147) — (149), (152), is inertial according to Euler.

The following proposition is an immediate corollary from Ax 3 N, Ax 3 E and from the argumentation adduced above.

Pr 19 NE. Any system of reference which is inertial according to Newton is inertial according to Euler and *vice versa.*

Pr 19 NE and Pr 18 N imply:

Pr 18 E. If (154) — (157) and App (1) — (2), then the system of reference α is inertial according to Euler.

Pr 19 NE justifies the advisability of the following definition:

Df 2 NE. A system of reference is called *inertial* if it is inertial according to Newton.

Pr 20 E. A system of reference is inertial if, and only if, it is inertial according to Euler.

Dm. Pr 19 NE, Df 2 NE.

Sch 48. A question of intransient interest for analytical dynamics is the formulation and use of both Newtonian and Eulerian dynamical axioms and of their corollaries for non-inertial rigid and non-rigid systems of reference. This is, however, a topic we shall not discuss here.

Sch 49. A problem similar to that formulated and solved in Sch 40, comes into being in rigid body dynamics too. Making a long story short, it may be formulated in the following manner. If B is any rigid body and $\underrightarrow{F}(\boldsymbol{F}, \boldsymbol{M})$ is any system of forces acting on it, then it is certain that

$$\frac{d_\alpha}{dt}\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \boldsymbol{M} \qquad (t \in R) \tag{158}$$

(under the notations already repeatedly used) by virtue of Ax 2 E, α being inertial according to Euler by hypothesis and $\boldsymbol{M}$ being taken with respect to the origin A of α. On the other hand, Pr 12 E warrants that if (92), (93) hold, then β is inertial according to Euler and consequently the equation

$$\frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm = \boldsymbol{M}_G \qquad (t \in R), \tag{159}$$

provided (122), must hold. The problem mentioned above now is: does it hold indeed.

In order to solve it let us note that because of (93) the identity (82) implies

$$\frac{d_\alpha \boldsymbol{r}}{dt} dm = \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R) \tag{160}$$

and (160), (83) imply

$$\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \int (\boldsymbol{r}_B + \overline{\rho}) \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm \tag{161}$$

$$+ \int (\boldsymbol{r}_B + \overline{\rho}) \times \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R),$$

i.e.

$$\int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm \tag{162}$$

$$+ \int \overline{\rho} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \boldsymbol{r}_B \times \frac{d_\beta \overline{\rho}}{dt} dm \qquad (t \in R).$$

On the other hand, (93) and Pr 8 imply (76) for any differentiable function (57).

The equation (162) therefore implies

$$\frac{d_\alpha}{dt} \int \boldsymbol{r} \times \frac{d_\alpha \boldsymbol{r}}{dt} dm = \frac{d_\alpha}{dt} \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm \tag{163}$$

$$+ \frac{d_\beta}{dt} \int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt} dm + \int \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm$$

$$+ \int \overline{\rho} \times \frac{d_\alpha}{dt} \left(\frac{d_\alpha \boldsymbol{r}_B}{dt} dm \right) + \int \frac{d_\alpha \boldsymbol{r}_B}{dt} \times \frac{d_\beta \overline{\rho}}{dt} dm + \int \boldsymbol{r}_B \times \frac{d_\beta}{dt} \left(\frac{d_\beta \overline{\rho}}{dt} dm \right)$$

$(t \in R)$. Obviously

$$\frac{d_\alpha}{dt} \int \boldsymbol{r}_B \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm = \boldsymbol{r}_B \times \frac{d_\alpha}{dt} \left(m \frac{d_\alpha \boldsymbol{r}_B}{dt} \right), \tag{164}$$

$$\int \frac{d_\beta \overline{\rho}}{dt} \times \frac{d_\alpha \boldsymbol{r}_B}{dt} dm + \int \frac{d_\alpha \boldsymbol{r}_B}{dt} \times \frac{d_\beta \overline{\rho}}{dt} dm = o, \tag{165}$$

$$(166) \qquad \int \frac{d_\beta}{dt}\left(\frac{d_\beta \overline{\rho}}{dt}dm\right) = \frac{d_\beta}{dt}\int \frac{d_\beta \overline{\rho}}{dt}dm = \boldsymbol{F}$$

$(t \in R)$, the latter equation by virtue of Pr 13 E. Now (163) — (166), (92), (158) imply

$$(167) \qquad \boldsymbol{M} = \frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt}dm + \boldsymbol{r}_B \times \boldsymbol{F} + \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}\right) dm$$

$(t \in R)$ and (167), (122) imply

$$(168) \qquad \frac{d_\beta}{dt}\int \overline{\rho} \times \frac{d_\beta \overline{\rho}}{dt}dm = \boldsymbol{M}_G + \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right)$$

$(t \in R)$. It is immediately seen that (168) would imply (159) if, and only if, the relation

$$(169) \qquad \int \overline{\rho} \times \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right) = \boldsymbol{o} \qquad (t \in R).$$

Now (169) would surely hold if

$$(170) \qquad \frac{d}{dt}(dm) = 0 \qquad (t \in R),$$

i.e. in the constant mass case. Indeed, then (17) implies (107) and (107), (92), (170) imply

$$(171) \qquad \frac{d_\alpha}{dt}\left(\frac{d_\alpha \boldsymbol{r}_B}{dt}dm\right) = \frac{d_\alpha^2 \boldsymbol{r}_B}{dt}dm \qquad (t \in R)$$

and (171), (109) imply (169).

In such a manner, the affirmative answer of this problem depends on the mass-constancy problem. In view of the logical difficulties of the latter we leave things here as they are.

Finally let us note again that the considerations in this article, as already mentioned, are intended to revive the mathematician's interest in Hilbert's sixth problem concerning the axiomatical construction of rational mechanics in general, and of analytical mechanics in particular, as well as to constribute, humble as it is, to its headway. In our mind, such efforts are not useless on the background of not a few *argumenta ad ignorantiam*, one has the chance to see printed in black and white in the literary sources on this domain. The maxim *hoc volo; sic jubeo, sit pro ratione voluntas*, often carried out in everyday life, sounds ridiculously in the mathematical routine. *Volens nolens*, the mechanicians must become reconciled with the fact that rational mechanics "in its relation to experience, intuition, abstraction, and everyday life does not differ in essence from other branches of mathematics"[18], that the axiomatical consolidation of its logical foundations is hence forthcoming.

APPENDIX

1. As a matter of fact, this problem has been looked on as a *circulus vitiosus* or, more picturesquely, as a dog striving to bite its tail. Indeed, a geometrical notion Z, for instance, is defined by means of one or several geometrical notions Y, X, etc. Going back, one arrives in the long run at several geometrical notions A, B, C, etc., which are so fundamental, so elementar, and so simple, that there are no other geometrical notions by means of which these A, B, C, etc. could be defined explicitly. In such a manner, at first sight at least, the circuit seems to close and the geometers' honourable intentions for an irreproachable consolidation of the logical foundations of their science seem to be a complete failure.

A way out of this dead-lock has been discovered by Hilbert. In his non-pareil work [13] marking a new mathematical era simultaneously with the change of two centuries, he proclaimed a new mathematical principle ordained to break up the ancient mental stereotypes as only the theory of relativity did. According to Hilbert's *axiomatical principle*, in the process of logical consolidation of the foundations of any mathematical theory T certain *primary notions-objects* $A_1, \ldots, A_a$ and certain *primary notions-relations* $B_1, \ldots, B_b$ of T must be discovered, or selected, or proclaimed, which are *unsusceptible to explicit definitions* by the aid of any other notions of T. These *primary notions* of T must be *defined implicitly* by the aid of a *system of axioms* of T, i.e. a set of statements Ax 1, ..., Ax N, involving $A_1, \ldots, A_a$ and $B_1, \ldots, B_b$ and stating elementary properties of $A_1, \ldots, A_a$, suggested by the intuition, or by the naive ideas primarily incarnated in $A_1, \ldots, A_a$, or on the basis of God knows what reasons. The question about the authenticity, or reliability, or trustworthiness, etc. of Ax 1, ..., Ax N does not come into being at all: according to the axiomatical principle of Hilbert this question is pointless, i.e. unsubstantiated, devoid of sense, empty of matter. The Ax 1, ..., Ax N of T are true by definition, or by hypothesis, or by decree, etc., inasmuch as two adamant conditions are satisfied. First, the system of statements Ax 1, ..., Ax N must be unconditionally *consistent*, i.e. free from inner contradictions. Second, its logical corollaries must form a system *identical* to T rather than to some of its far away cousins: any theorem of T must be demonstrable on the basis of Ax 1, ..., Ax N. (A system of axioms for the Euclidean geometry is proposed in the *Appendix* of the article [85, p. 160 – 161], while a system of axioms for arithmetic of natural numbers is given ibidem, p. 161 – 162.)

2. As, for instance, the notion *point*, *line*, and *plane* in geometry.

3. As for instance, the notion *incident* (*zusammengehört, liegt*, see [85, p. 160]) in geometry.

4. The rigid body dynamics is the main subject in analytical mechanics. As it is well known, traditionally the latter is divided into three parts: kinematics, statics and dynamics. In their turn, any of them is divided into two parts: kinematics of points and rigid bodies, statics of mass-points and rigid bodies, dynamics of mass-points and rigid bodies. The first ones of all these, namely kinematics of points, statics of mass-points, and dynamics of mass-points respectively, belong to the most trivial parts of analytical mechanics.

On the other hand, statics of rigid bodies is a trivial part of analytical mechanics of rigid bodies too, having to deal, first, with the most restricted case when the rank of the system of forces acting on the rigid body (both active and passive forces, alias forces determined by the conditions of the particular statical problem under consideration and reactions of the geometrical constraints imposed on the rigid body respectively) is equal to zero, and, second, with algebraic mathematical conditions if equilibrium, rather than with systems of differential equations of motion as in the dynamical case.

As regards kinematics of rigid bodies, its role in the system of analytical mechanics may be assessed as an auxiliary one. Indeed, its predetermination is to supply the analytical mechanics with the necessary geometry. As a matter of fact, rigid body kinematics could be qualified as the geometry of motion. Its main aim is to define and describe such fundamental for analytical mechanics mathematical entities, as for instance the notions of *affine* and *rigid Cartesian systems of reference*, *motion* of such systems, *local derivatives* of vector functions with respect to these systems, *affine* and *rigid kinematical bodies* along with their basic attributes, as for instance *partial* and *total instantaneous angular velocities*, as well as the proofs of *Eulerian theorems* concerning the *relations between linear* and *angular velocities* and of *Euler's kinematical equations* involving the Eulerian angles and their time-derivatives, and so on, and so forth, etcetera, to say nothing about the definition of the most important for the whole of rigid body dynamics notion of *kinetical rigid body*, with its basic attributes: *mass, mass-center, momentum,* and *kinetical moment.*

In such a manner, in the long run, analytical dynamics of rigid bodies remains the specific part of analytical mechanics in general — its genuine core, as a matter of fact.

5. Before proceeding farther, let us say some more words concerning the rigid body notion in its dynamical as well as kinematical aspect.

As already emphasized in the previous parts of this article, the rigid body concept is traditionally looked upon by the authors of writings on analytical mechanics as an apriori notion of this science. This attitude is due to the fact they have not yet overcome the mentality of the puberty period in the history of mechanics, pretending rational mechanics to be physics in its substance. It is not. The erroneous belief that it is resulted in the deplorable state of affairs as far as the axiomatical consolidation of its logical fundaments are concerned and has postponed the execution of Hilbert's program towards its axiomatical construction [85, p. 158 – 159, 166] *ad calendas graecas.*

Rational mechanics in general, analytical mechanics in particular, are mental, not experimental, as well as geometry is mental, not instrumental, "and in its relations to experience, intuition, abstraction, and everyday life it does not differ in essence from" [1, p. 336] the theory of numbers, say. "In this audience, I am sure, mathematics itself needs no defence. It is unnecessary to persuade you that mathematics is trying to be physics or trying to be engeneering. It should also be unnecessary to point out that mathematics, however abstract and however precise, is a science of *experience*, for experience is not confined to the gross senses: Also the human mind can experience, and we need not be so naive as to see in an oscilloscope an instrument more precise than the brain of a man" [*ibid.*].

Analytical mechanics is pure mathematics *par excellence*, and this is borne by the fact that it is now, *in our days*, as much a deductive science — neither more nor less — as arithmetic and geometry for instance are. It is true that *long ago*, in its embryonic inductive state, analytical mechanics belonged to physics. (One should not forget that Newton christened his first-borne child namely *Philosophiae Naturalis Principia Mathematica*, and that *Philosophia Naturalis* meant exactly *physics* in his days.) In the same time it is also true that the degree of this appurtenance, of these affiliations, has not been higher than those of arithmetic and geometry. For, once upon a time, there has been a period when arithmetic and geometry were parts of physics too: in their experimental and instrumental age respectively, when commutativity of multiplication has been established by check-ups, and volumes of solids have been determinated by the aid of sand and water. Let us not forget the historical truth that even Leibniz knew by a physical experiment rather than by proof that 2 times 2 makes 4.

6. Such objections are not made up, or fabricated, or concocted. They correspond to, they reflect scientific reality. They have been not once nor twice brought forward before us even by highly educated professional mathematicians who, however, as far as analytical mechanics is concerned, behave (a not infrequent phenomenon) as haughtily as only dilletanti could (improvising mechanics, as a matter of fact, entirely forgetting that they have settled their accounts with rational mechanics as late as they have left their student's desks).

7. Newton's dynamical ideas culminated in his famous postulate:

Lex II. *Mutationem motus proportionalem esse vi motrici impresse et fieri secundum lineam rectam qua vis illa imprimitur* (alias, the alteration of motion is ever proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed).

Now, a mere glance at Lex II and its modern version Ax 1 N at once displays an essential flaw in Newton's formulation: the total absence of the notion of system of reference in it, to say nothing of derivatives with respect to such systems. But Lex II is not unconditionally true: it is true for some systems of reference (inertial according to Newton) and untrue for other ones.

As regards Ax 2 N, it is completely wanting among Newton's *axiomata sive leges motus* [30, p. 129]. Undoubtedly, Newton knew and used it. Nevertheless, he thought wrongfully that it is an immediate corollary from Lex II. It is not. The erroneous belief that it is represents a prejudice shared even by modern authors of text-books, treatises and monographs on analytical mechanics. Its analysis is put off until later.

8. For an axiomatical definition of V see, for instance, [86].

9. For the sake of simplicity the definitional domain of (1) is hypothesized here to be R rather than some appropriate subset of R.

10. An *affine Cartesian system of reference* α is defined as the set

$$\alpha \text{ sgn} : \{\overrightarrow{a}_\nu\}_{\nu=1}^{3}, \tag{1}$$

where

$$\overrightarrow{a}_\nu \text{ sgn} : (\boldsymbol{a}_\nu, \boldsymbol{A}_\nu) \qquad (\nu = 1, 2, 3), \tag{2}$$

provided

$$\boldsymbol{a}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3), \tag{3}$$

$$\boldsymbol{A}_\nu : R \longrightarrow V \qquad (\nu = 1, 2, 3) \tag{4}$$

are given vector functions with

$$\boldsymbol{a}_1(t) \times \boldsymbol{a}_2(t) \cdot \boldsymbol{a}_3(t) \neq 0 \qquad (t \in R), \tag{5}$$

$$\boldsymbol{a}_\mu(t)\boldsymbol{A}_\nu(t) + \boldsymbol{a}_\nu(t)\boldsymbol{A}_\mu(t) = 0 \qquad (t \in R) \tag{6}$$

($\mu, \nu = 1, 2, 3$). The arrows (2) are called the *axes* of α and the vectors $\boldsymbol{a}_\nu$ are called the *axis vectors* of $\overrightarrow{a}_\nu$ ($\nu = 1, 2, 3$) respectively.

By virtue of (5), (6) the system of vector equations

$$\boldsymbol{a} \times \boldsymbol{a}_\nu = \boldsymbol{A}_\nu \qquad (\nu = 1, 2, 3) \tag{7}$$

has exactly one solution

$$\boldsymbol{a} : R \longrightarrow V, \tag{8}$$

namely

$$\boldsymbol{a} = \frac{1}{2}\sum_{\nu=1}^{3} \boldsymbol{a}_\nu^{-1} \times \boldsymbol{A}_\nu, \tag{9}$$

provided

(10) $$\mathbf{a}_\nu^{-1} \operatorname{sgn} : \frac{\mathbf{a}_{\nu+1} \times \mathbf{a}_{\nu+2}}{\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{a}_3} \qquad (\nu = 1,\ 2,\ 3)$$

with

(11) $$\mathbf{a}_{\nu+3} \operatorname{sgn} : \ \mathbf{a}_\nu \qquad (\nu = 1,\ 2)$$

are the *reciprocal vectors* of the reper (3). The function (8), defined by (9), is called the *origin* of α. It is the intersecting point of the three axes of α.

The system of reference (1) is called *rigid* if

(12) $$\frac{d}{dt}(\mathbf{a}_\mu(t)\mathbf{a}_\nu(t)) = 0 \qquad (t \in R)$$

$(\mu, \nu = 1,\ 2,\ 3)$. It is called *orthonormal* if

(13) $$\mathbf{a}_\mu(t)\mathbf{a}_\nu(t) = \begin{cases} 1 & (\mu = \nu) \\ 0 & (\mu \neq \nu) \end{cases} \qquad (\mu, \nu = 1,\ 2,\ 3)$$

$(t \in R)$ and *right-hand oriented* if

(14) $$\mathbf{a}_1(t) \times \mathbf{a}_2(t) \cdot \mathbf{a}_3(t) > 0 \qquad (t \in R).$$

11. The system of reference α being defined as above and the functions (3), (4) being differentiable, let

(15) $$\mathbf{p} : \ R \longrightarrow V$$

be any differentiable function. Then

(16) $$\mathbf{p} = \sum_{\nu=1}^{3} (\mathbf{p}\mathbf{a}_\nu^{-1})\mathbf{a}_\nu$$

and the function

(17) $$\frac{d_\alpha \mathbf{p}}{dt} \operatorname{sgn} : \ \sum_{\nu=1}^{3} \left(\frac{d}{dt}(\mathbf{p}\mathbf{a}_\nu^{-1})\right) \mathbf{a}_\nu$$

is called the *derivative of* $\mathbf{p}$ *with respect to* α or the *local* (*with respect to* α) *derivative of* $\mathbf{p}$. If the context permits no collision of notations, a simpler symbolics is used.

So for instance the derivatives with respect to $Oxyz$ are usually denoted by $\frac{d}{dt}$ instead of $\frac{d_\alpha}{dt}$. In the special case when t represents time, dots are traditionally used, for instance

$$(18) \qquad \dot{p} = \frac{dp}{dt}.$$

12. Examples of 1-dimensional rigid bodies are the so-called rings, wires, rods, etc.; examples of 2-dimensional rigid bodies are the so-called discs, lamellae, plates, slabs, etc.; with the exception of these extravagant samples, all "normal" rigid bodies are 3-dimensional.Naturally, the dimensions of rigid bodies are described strictly in the mathematical definition of the rigid body concept in any particular case. As regards this part of the exposition, an appeal is made to the reader's own experience in analytical mechanics.

13. Over the whole space V, as a matter of fact, in the mathematical definition of the rigid body concept. At that, it is supposed that $\varkappa = 0$ outside the "geometrical borders", or the "delineations" of the particular rigid body under consideration.

14. "It is clear enough that in statics the equilibrium of moments is not insured by the equilibrium of forces, nor *vice versa.* In dynamics, the principle of moment of momentum developed late, and much of the earlier work concerning it gives the impression that the two principles were somehow hoped to be equivalent, so that there would be but a single law of motion. This illusion is fostered in the teaching of mechanics by physicists today ... The law of moment of momentum is subtle, often misunderstood even today" [1, p. 128 – 129].

15. Strange though it may seem, the simple corollaries (47) and (48) from Ax 1 N and Ax 2 N respectively concerning S_n, nowadays known as Newton's laws of ***momentum*** and of ***kinetical moment*** respectively for a system of finite number of discrete mass-points, are nowhere to be found in Newton's ***Principia.*** They have been discovered by Euler about half a century after the publication of Newton's work.

16. Any system of reference, which is moving rectilinearly and uniformly without rotation with respect to an inertial system of reference, is an inertial system of reference itself.

17. Карл Максимович Бэр (1792 — 1876), Russian natural scientist. In 1857 he explained the erosion of the right (left) banks of rivers flowing in the directions of the meridians in the northern (southern) hemisphere by means of the Earth revolution.

18. We beg the reader's pardon for citing for the second time this position of Truesdell. We shall, however, never get tired in repeating it over and over again, as long as some mechanician's mental constitutions make it so timely and topical.

85. I. C h o b a n o v. Newtonian and Eulerian dynamical axioms. II. The literary tradition. — *Год. Соф. унив., Фак. мат. мех.*, **79**, 1985, кн. 2 — Механика, 141 - 168.
86. I. C h o b a n o v. Complex standard vector spaces. — *Год. Соф. унив., Фак. мат. мех.*, **75**, 1981, кн. 2 — Механика, 3 - 26.
87. Г. П а с к а л е в, И. Ч о б а н о в. Върху понятието "кинематично твърдо тяло. I. — *Год. Соф. унив., Физ. — мат. Фак.*, **54**, 1959 — 1960, кн. 1 — Математика, 173 - 230.
88. I. C h o b a n o v. Affine und Euklidische Körper in Hilbertschen Räumen. — *Год. Соф. унив., Мат. Фак.*, **64**, 1969 — 1970, 293 - 330.
89. G. C h o b a n o v, I. C h o b a n o v. Affine and rigid bodies in Hermitean spaces. — *Год. Соф. унив., Фак. мат. мех.*, **71**, 1976 — 1977, ч. I, 187 - 268.
90. G. C h o b a n o v, I. C h o b a n o v. The rank-theorem for systems of arrows in complex standard vector spaces. — *Год. Соф. унив., Фак. мат. мех.*, **79**, 1985, кн. 1 — Математика, 183 - 200.
91. G. C h o b a n o v, I. C h o b a n o v. On the instantaneous angular velocity. — *Год. Соф. унив., Фак. мат. мех.*, **71**, 1976 — 1977, ч. I, 217 - 245.
92. Математическая энциклопедия. Т. 2. М., 1979.

Received 21.V.1990

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 83, 1989

ХИДРОДИНАМИКА НА ТВЪРДИ И ТЕЧНИ ЧАСТИЦИ В ГРАДИЕНТНИ ТЕЧЕНИЯ

ЗАПРЯН ЗАПРЯНОВ

Запрян Запрянов. ГИДРОДИНАМИКА ТВЕРДЫХ И ДЕФОРМИРУЕМЫХ ЧАСТИЦ В СДВИГОВЫХ ТЕЧЕНИЯХ.

В настоящей работе делается обзор некоторых основных проблем в гидромеханике твердых и деформируемых частиц (капли и пузыри), которые обтекаются сдвиговым вязким потоком. Обсуждаются поступательные и ротационные движения твердых сферических и несферических частиц, гидродинамическое взаимодействие частиц в многофазных системах, а также равномерные и неравномерные обтекания недеформируемых и деформируемых частиц и т.д. В дополнении приводится решение задачи обтекания сферической капли осесимметричным градиентным вязким потоком. В конце работы формулируются некоторые нерешенные до сих пор задачи в этой области.

Zapryan Zapryanov. FLUID MECHANICS OF RIGID OR FLUID PARTICLES IN SHEAR FLOWS.

The main purpose of this paper is to survey some basic problems in the field of fluid mechanics concerning rigid and fluid (drops and bubbles) particles in viscous shear flows. We discuss translation and rotation of spherical and nonspherical rigid or fluid particles, particle-particle interactions in multiphase systems, homogeneous and inhomogeneous shear flows, flow past undeformable and deformable fluid particles and so on. In addition to that we provide the solution of the problem of a spherical drop suspended in an axisymmetric shear flow. At the end of the paper some unsettled problems in the field of shear flows are pointed out.

Важен клас течения, които се изследват много интензивно през последните 15 – 20 години, са градиентните флуидни течения. Освен фундаментално те имат и важно приложно значение. Градиентни флуидни течения възникват в химическата и биологическата промишленост при движението на стените на различни апарати във флуидна среда. Примери за такива течения са течението на Кует между две равнинни или сферични стени, едната от които се движи успоредно на другата; течението на Поазьой в цилиндрична тръба, породено от зададен градиент на налягането; течението около критична точка, което се среща например при доближаването на частици до колектор; разтягащите и удължаващите течения при производството на влакнести или нишкови полимерни материали и др.

Резултатите от изследванията на градиентните течения се използват при моделиране на различни дисперсни системи (суспензии, емулсии и др.), движението на еритроцитите, движението на тела в следи (зад други тела) или гранични слоеве и др.

Простите градиентни течения се характеризират от един макроскопичен параметър S, наречен коефициент на изменение на профила на скоростта. Сложните градиентни течения могат да се определят от два, три и т. н. параметри. Коефициентът на изменение на профила на скоростта S при простите градиентни течения има измерение, реципрочно на времето. Той участвува в израза за числото на Рейнолдс

$$Re = \frac{a^2 S}{\nu},$$

което е основен динамичен параметър в хидродинамиката на градиентните течения (a е характерен размер на течението, а ν — кинематичният вискозитет на флуида).

Първите фундаментални резултати в изследването на градиентните течения са получени от Айнщайн [1], който пресмятайки ефективния вискозитет на суспензия от еднакви сферични частици (1906 и 1911 г.), решава задачата за определяне на хидродинамичното взаимодействие на просто градиентно течение и неутрално суспендирана в него сферична частица, т. е. частица, на която не действува силата на тежестта или други външни сили. В този случай на частицата въздействува само градиентното течение, което създава ненулеви напрежения върху повърхността ѝ.

Теорията на Айнщайн бе обобщена от Джефри [2] за суспензия от елипсоидални частици, при които важна роля играе ориентацията им относно равнината от градиентното течение, имаща скорост, равна на нула. Поради различната геометрия на елипсоида в сравнение със сферата изследванията на Джефри намериха приложения в реологията на анизотропни дисперсни и полимерни системи, течните кристали, теченията с двойно пречупване и др.

Изследванията на Джефри се отнасят за просто градиентно течение, зададено в равнината Oxy със скорости $u = Sy$, $v = 0$, $w = 0$. Той установява, че напрежението, индуцирано от течението върху повърхността на ротационния елипсоид, може еквивалентно да се замени с действието на две двоици — едната се стреми да застави частицата да се установи в потока, така че оста ѝ да бъде успоредна на оста Oz, а другата — оста ѝ да лежи в равнината Oxy.

Ако ротационният елипсоид има уравнение

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

където $b = c$, то под действието на двете двоици върховете му описват в сферични координати $(r,\ \theta,\ \varphi)$ траектории с уравнение

$$\mathrm{tg}^2\theta = \frac{a^2 b^2}{k^2(a^2\cos^2\varphi + b^2\sin\varphi)}.$$

Тук k е константа, която зависи от ориентацията на частицата, като при $k = 0$ голямата ос a се върти в равнината Oxy, а при $k = \infty$ същата ос е успоредна на Oz. За всяка друга междинна стойност на k ротационният елипсоид се върти около оста a с ъглова скорост

$$\omega = \frac{S}{2}\cos\theta.$$

От тази формула следва, че при $k = 0$ получаваме $\omega = 0$, а при $k = \infty$ — $\omega = \frac{S}{2}$.

Изследванията на Джефри, извършени посредством уравненията на Стокс, показват, че при обтичането на ротационен елипсоид от просто градиентно течение не се забелязва тенденция оста му да се разположи в някакво предпочитано положение относно несмутеното движение на флуида. Това е недостатък на модела, с който си служи Джефри. Моментът на елипсоида, изчислен в стоксово приближение, е равен на нула независимо от ориентацията му по отношение на посоката на обтичащия поток. Подобен недостатък на уравненията на Стокс се наблюдава и при ексцентрично разположена вътре във вертикален кръгов цилиндър сферична частица, която пада под действието на теглото си. При движението си сферата не изпитва сила, която да я принуди наред с движението си надолу да мигрира в радиално направление, но това противоречи на експерименталните наблюдения.

През 1923 г. Факсен [3, 4] разработва метод за решаване на задачи от обтичане на сфера, намираща се между две успоредни равнини. Той изхожда от фундаменталното решение на уравнението на Лаплас за налягането

$$\Delta p = 0,$$

което има вида

$$p = -\frac{1}{4\pi r}.$$

Представяйки това решение в интегрална форма, той получава

$$\frac{1}{r} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{\exp[i(\alpha x + \beta y) - k|z|]\}\frac{d\alpha d\beta}{k},$$

където $i = \sqrt{-1}$, α и β са интеграционни променливи и $k = \sqrt{\alpha^2 + \beta^2}$. Факсен използува тази функция и нейните частни производни относно декартовите координати за изразяване на общото решение на уравненията на Стокс за сфера в безкраен флуид в декартови координати

$$\begin{aligned}
u &= \frac{1}{2}\pi\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{\exp[i(\alpha x + \beta y) - x|z|]\}\cdot\left\{\frac{i\alpha}{k}g_2 + \frac{2g_1}{k}\right.\\
&- \left.\frac{g_1\alpha^2}{k^3}(k|z|+1) + \frac{\alpha^2 z}{k}g_3\right\}d\alpha d\beta;\\
v &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{\exp[i(\alpha x + \beta y)]\}\, i\beta\cdot\left\{\frac{g_2}{k} + \frac{i}{k^3}g_1(k|z|+1)\right.\\
&- \left.\frac{i\alpha x}{k}g_3\right\}d\alpha d\beta;\\
w &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{\exp[i(\alpha x + \beta y) - k|z|]\}\cdot\left\{-\frac{z}{|z|}g_2 - \frac{zi\alpha}{k}g_1\right.\\
&+ \left.\frac{i\alpha}{k}g_3 k|z| + 1\right\}d\alpha d\beta;\\
p &= \frac{\mu}{\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\{\exp[i(\alpha x + \beta y) - k|z|]\}\left\{-g_1 + \frac{z}{|z|}kg_3\right\}d\alpha d\beta,
\end{aligned}$$

където g_1, g_2, g_3 са произволни функции на α и β. Тъй като Факсен използува метода на отражението, това представяне е удобно за удовлетворяване на граничните условия поотделно върху сферата и върху равнините. От тях се определят функциите g_1, g_2, g_3.

За съпротивлението на сфера с радиус a, движеща се със скорост U, успоредно на една равнина, във вискозен флуид след второто отражение Факсен получава

$$W_x = \frac{-6\pi\mu a U}{1 - \frac{9}{16}\frac{a}{l} + \frac{1}{8}\left(\frac{a}{l}\right)^3 - \frac{45}{256}\left(\frac{a}{l}\right)^4 - \frac{1}{16}\left(\frac{a}{l}\right)^5},$$

където l е разстоянието от центъра на сферата до равнината. Ако сферата може да се върти, ъгловата ѝ скорост ще бъде

$$\omega = \frac{3U}{32a}\left(\frac{a}{l}\right)^4\left(1 - \frac{3}{8}\frac{a}{l}\right).$$

Системни изследвания на обтичането на сфера от градиентно вискозни течения между две успоредни равнини извършва Вакия [5, 6]. Използувайки развития от Факсен метод, Вакия разглежда два случая: а) течение на Поазьой между две неподвижни равнини; б) течение на Кует, при което едната равнина е неподвижна, а другата се движи.

Да означим с $2L$ разстоянието между двете равнини и в двата случая и да предположим, че центърът на сферата е разположен на разстояние $l = \frac{L}{2}$ от едната стена. Вакия взема началото на координатната система да съвпада с центъра на сферата и разглежда течение между двете равнини от вида

$$u_\infty = U - \frac{2U}{3l} z - \frac{U^*}{3l^2} z^2, \qquad v_\infty = w_\infty = 0,$$

където U е скоростта на флуида върху равнината $z = 0$. За силата на съпротивлението успоредно на равнините той получава

$$W_x = \frac{6\pi\mu a U \left[1 - \frac{1}{9}\left(\frac{a}{l}\right)^2\right]}{1 - 0,6526\frac{a}{l} + 0,3160\left(\frac{a}{l}\right)^3 - 0,242\left(\frac{a}{l}\right)^4},$$

а за момента

$$T_y = \frac{8}{3}\pi\mu a^2 U \cdot \frac{a}{l}\left[1 + 0,0758\left(\frac{a}{l}\right) + 0,049\left(\frac{a}{l}\right)^3\right],$$

където a е радиусът на сферата.

Предполагайки, че по-близката до сферата равнина е неподвижна, а по-далечната се движи със скорост U, във втория случай Вакия разглежда следното несмутено течение между равнините:

$$u_\infty = \frac{U}{4} + \frac{U}{4l} z, \qquad v_\infty = w_\infty = 0.$$

Като използува координатна система, неподвижно свързана със сферата, за съпротивлението той получава

$$F_x = \frac{3/2\pi\mu a U}{1 - 0,6526\left(\frac{a}{l}\right) + 0,4003\left(\frac{a}{l}\right)^3 - 0,297\left(\frac{a}{l}\right)^4},$$

а за момента —

$$T_y = 4\pi\mu a^2 U \frac{a}{l}\left[1 + 0,0506\frac{a}{l} + 0,033\left(\frac{a}{l}\right)^2\right].$$

Интересни теоретични изследвания на градиентни течения извършва Бредертън [7], който прилага метода за срастване на асимптотичните разлагания за решаване на задачата за неутрално суспендирана цилиндрична частица в просто градиентно течение. Освен това Бредертън [8] разглежда и други частици с по-сложна форма в градиентни течения и установява съществуването и на други движения (моди) в допълнение към изучените от Джефри и Факсен.

Изследвано е и влиянието на цилиндрична стена върху движението на твърди частици във вискозен флуид. Изборът на този вид граница (стена) е свързан с това, че цилиндричната повърхнина обгражда напълно както частицата, така и флуидния поток.

През 1907 г. Ладенбург [9] разглежда задачата за движение на твърда сфера с постоянна скорост U в направление на оста на кръгова цилиндрична тръба. Като използува приближен метод, той получава за съпротивлението, което изпитва сферата при движението ѝ в тръбата, формулата

$$\frac{W}{6\pi\mu aU} = 1 + 2,4\frac{a}{r_0}. \tag{1}$$

Тук a е радиусът на частицата, а r_0 — радиусът на сечението на кръговата тръба. Използувайки метода си, Факсен [10] решава същата задача и получава също, че частицата изпитва по-голямо съпротивление, отколкото при движението ѝ в неограничен вискозен флуид. С това той потвърждава извода, направен от Ладенбург, за задържащото влияние на цилиндричната стена върху движението на частицата. Като използува метода на Факсен, Вакия [11] решава задачата за движение на сфера по оста на кръгова цилиндрична тръба, в която има развито течение на Поазьой. За съпротивлението, което изпитва сферата, той получава

$$\frac{W}{6\pi\mu aU} = \frac{1 - \frac{2}{3}\frac{a}{r_0}}{1 - 2,104\frac{a}{r_0} + 2,09\frac{a}{r_0} - 1,11\left(\frac{a}{r_0}\right)^2}. \tag{2}$$

За сравнение с резултатите на Ладенбург той решава и задачата, когато сферата се движи по оста на тръбата с постоянна скорост U и получава по-точната в сравнение с (1) формула

$$\frac{W}{6\pi\mu aU} = 1 + 2,1\frac{a}{r_0}. \tag{3}$$

Извършвайки експерименти за бавно движение на твърда сферична частица, неутрално (свободно) суспендирана в течение на Поазьой в кръгова цилиндрична тръба, през 1961 и 1962 г. Сегре и Зилберберг [12, 13] наблюдават странична миграция на частицата, чийто център се установява в равновесно положение, отдалечено на около 0,6 радиуса от оста на цилиндъра, независимо от началното ѝ положение относно оста. Ако частицата е суспендирана близо до стената, тя мигрира навътре, а ако е суспендирана близо до оста, тя мигрира в посока към стената. Този ефект на странична миграция бе потвърден и в изследванията на други автори [14, 15, 16], които установиха относително неголяма чувствителност към големината на числото на Рейнолдс (пресметнато, като се използува дължината на диаметъра на тръбата) и отношението на диаметъра на тръбата $2r_0$ и диаметъра на частицата $2a$.

При движение на разредена суспензия от неутрално плуващи сферични частици във флуид в кръгла цилиндрична тръба Сегре и Зилберберг констатират наличието на напречни сили, стремящи се да преместят частиците, които са близо до оста, към стената и обратно — частиците, които са до стената — към центъра на тръбата. В резултат на действието на тези сили независимо от началното си положение частиците се концентрират в пръстеновиден слой, който е разположен приблизително в средата между оста и стената на тръбата.

Опитите на Сегре и Зилберберг са продължени от Смол и Ейхорн [17], Дей и Генети [18], Денсън и съавтори [19] и др. за твърди частици, които не са неутрално суспендирани в течението на Поазьой в кръгова тръба, а имат по-голяма или по-малка плътност от плътността на заобикалящия ги флуид. От изследванията на тези автори следва, че ако плътността на сферичната частица (намираща се в течение на Поазьой, насочено надолу във вертикална кръгова тръба) е по-голяма от плътността на заобикалящия я флуид, имаме наслагване на скоростта на сферата, предизвикана от силата на тежестта и носещата скорост на течението и миграцията на частицата l към стената на тръбата. Ако плътността ѝ е по-малка от плътността на флуида, споменатите две скорости имат противоположни посоки и миграцията на частицата е към оста на тръбата.

Теоретичното обяснение на резултатите от тези експерименти среща големи затруднения. Използувайки метода за срастване на асимптотичните разлагания, Рубинов и Келер [20] показват, че движещата се и едновременно с това въртяща се сфера в неподвижен вискозен флуид изпитва сила, която е перпендикулярна на посоката на движение, т.е. изпитва подемна сила

$$\overline{F}_L = \pi a^3 \rho \overline{\Omega} \times \overline{U} \left[1 + O(Re)\right].$$

Тук $\overline{U}$ е скоростта на транслационното движение, $\overline{\Omega}$ — ъгловата скорост, ρ — плътността на флуида и $Re = \dfrac{\rho U a}{\mu}$ — числото на Рейнолдс. Като използуват този резултат, Рубинов и Келер правят опит да обяснят теоретично експериментите на Сегре и Зилберберг, но получават, че страничната сила е насочена така, че свободно суспендираната в поазьоевото течение сферична частица винаги мигрира към оста на тръбата. По този начин дори с привличането на конвективните членове в уравненията на движение не може да се даде задоволително теоретично обяснение на посочения ефект. Бредертън [8] изследва теоретично миграцията на различни по форма твърди частици в течения на Поазьой и изказва предположението, че това влияние е свързано с инерционните членове в уравненията на Навие — Стокс. В експериментите на Сегре и Зилберберг (1961, 1962 г.) числото на Рейнолдс с характерен линеен размер — диаметъра на тръбата — $2r_0$, е около 30, макар че числото на Рейнолдс, пресметнато чрез диаметъра на сферичната частица, е по-малко от единица. Голдсмит и Мейсън [21] разглеждат свободно суспендирани сферични частици в течение на Поазьой, когато числото на Рейнолдс $Re = \dfrac{2r_0 \cdot v\rho}{\mu} \ll 1$, и установяват експериментално, че няма миграция при число на Рейнолдс $Re = \dfrac{2r_0 \cdot v\rho}{\mu}$, по-малко от единица. Те не наблюдават миграция и при свободно суспендирани твърди частици с форма на дискове и пръти с кръгово сечение.

За някои частици с по-особена форма обаче Бредертън установява, че и при $Re \ll 1$ съществува странична миграция. Същият резултат за наличие на миграция и в условията на бавно вискозно движение на деформирани капки е получен експериментално от Голдсмит и Мейсън [21]. Анализирайки критично обсъжданото явление, Сафмън [22] изказва предположение, че при някои условия могат да се окажат важни не само инерционните, но и ненютоновите ефекти.

Като използват бисферични координати, Голдман, Кокс и Бренер [23] получават точно решение за бавното транслационно движение на неутрално суспендирана твърда сферична частица в течение на Кует. Намереното решение е валидно за малки числа на Рейнолдс и произволно отношение на разстоянието на сферата до равнината (без случая на допиране на частицата до равнината) и радиуса ѝ.

Точно решение в стоксово приближение на задачата за обтичане от градиентен вискозен поток на сфера, допираща се до равнина, е дадено от Нил [24] през 1968 г. Обобщавайки метода на Дийн и О'Нил [25] за решаване на пространствени задачи за обтичане от равномерен вискозен поток на сфера близо до равнина или две сфери, Вакия [26] решава задачата за две сферични частици, суспендирани в градиентен вискозен поток, който е насочен в направление, перпендикулярно на оста на центровете им. Тъй като Вакия разглежда само частния случай (оста на центровете на сферите лежи в равнината на градиентното течение и има скорост, равна на нула), през 1971 г. Дейвис [27] разглежда общия случай, при който оста на двете сфери и равнината с нулева скорост в градиентното течение са в общо положение. Анализ на траекториите за движение на сфери с равни радиуси в градиентно вискозно течение е направен от Лин и др. [28]. По-сложният случай, когато наред с градиентния поток на сферите действува и допълнителна сила на привличане от типа на Лондон, е разгледан от Картис и Носкинг [29].

Системно изследване на хидродинамичното взаимодействие на две свободно движещи се сфери в градиентен вискозен поток е извършено от Бетчелор и Грийн [30]. От техните резултати като частен случай се получават резултатите на Дейвис [27] и на Лин и др. [28]. Точно решение на уравненията на Стокс за задачата за две допиращи се сфери, суспендирани в градиентно вискозно течение, е получено от Вакия [31] през 1971 г. Две години по-късно Нир и Акривос [32] разглеждат по-общия случай, когато допиращите се сфери имат различни радиуси. Те пресмятат силите и моментите, действуващи на двете частици, и използуват получените резултати при моделирането на разредени суспензии.

В [33] е решена класическата задача за обтичане на сфера от равномерен вискозен поток при малки и крайни числа на Рейнолдс. Тук ще разгледаме задачата за обтичане на сфера от прост градиентен поток в стоксово приближение. При липса на външни сили уравненията на Стокс в декартови координати $(x,\ y,\ z)$ имат вида

$$\mu \,\Delta\, u = \frac{\partial p}{\partial x}, \qquad \mu \,\Delta\, v = \frac{\partial p}{\partial y}, \qquad \mu \,\Delta\, w = \frac{\partial p}{\partial z}. \tag{4}$$

Чрез диференциране на уравненията (4) и използуване на уравнението на непрекъснатостта

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5}$$

получаваме, че налягането удовлетворява уравнението

$$\Delta\, p = 0. \tag{6}$$

Тогава решението $p(x,\ y,\ z)$ на (6) може да се запише в ред по сферичните функции P_n:

$$p = \sum_n P_n, \tag{7}$$

като отделните членове на това разлагане са независими. За да изразим компонентите на скоростта $u,\ v,\ w$ посредством P_n, полагаме

$$\begin{aligned} u &= Ar^2\frac{\partial P_n}{\partial x} + Br^{2n+3}\frac{\partial}{\partial x}\left(\frac{P_n}{r^{2n+1}}\right), \\ v &= Ar^2\frac{\partial P_n}{\partial y} + B^{2n+3}\frac{\partial}{\partial y}\left(\frac{P_n}{r^{2n+1}}\right), \\ w &= Ar^2\frac{\partial P_n}{\partial z} + B^{2n+3}\frac{\partial}{\partial z}\left(\frac{P_n}{r^{2n+1}}\right), \end{aligned} \tag{8}$$

където $r^2 = x^2 + y^2 + z^2$, а A и B са константи. Константата A се определя така, че изразът $Ar^2\dfrac{\partial P_n}{\partial x}$ да удовлетворява първото уравнение на (4). Тъй като

$$\Delta\left(r^2\frac{\partial P_n}{\partial x}\right) = r^2\Delta\left(\frac{\partial P_n}{\partial x}\right) + 4\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\frac{\partial P_n}{\partial x}$$

$$+\frac{\partial P_n}{\partial x}\Delta(r^2) = 2(2n+1)\frac{\partial P_n}{\partial x},$$

то за A получаваме

$$A = \frac{1}{2(2n+1)\mu}.$$

Лесно се проверява, че при този избор на константата A изразите $Ar^2\frac{\partial P_n}{\partial y}$ и $Ar^2\frac{\partial P_n}{\partial z}$ удовлетворяват съответно второто и третото уравнение на (4). Константата B определяме, като заместим (8) в уравнението на непрекъснатостта (5):

$$B = \frac{n}{(n+1)(2n+1)(2n+3)\mu}.$$

Общото решение на (4) и (5) се получава, като към общото решение на хомогенната система

$$\Delta u^h = 0, \qquad \Delta v^h = 0, \qquad \Delta w^h = 0, \tag{9}$$

$$\frac{\partial u^h}{\partial x} + \frac{\partial v^h}{\partial y} + \frac{\partial w^h}{\partial z} = 0 \tag{10}$$

прибавим едно частно решение на нехомогенната система (4), (5).

От (9) следва, че функциите u^h, v^h, w^h (аналогично на налягането) могат да се представят във вид на ред посредством сферични функции, т.е.

$$u^h = \sum_n u_n^h, \qquad v^h = \sum_n v_n^h, \qquad w^h = \sum_n w_n^h.$$

Тук u_n^h, v_n^h, w_n^h удовлетворяват уравнението на непрекъснатостта (10) за всяко n, тъй като са линейно независими.

Като диференцираме уравнението

$$\frac{\partial u_n^h}{\partial x} + \frac{\partial v_n^h}{\partial y} + \frac{\partial w_n^h}{\partial z} = 0$$

относно x и го прибавим към първото уравнение на (9), получаваме уравнение, което може да се запише във вида

$$\frac{\partial}{\partial y}\left(\frac{\partial v_n^h}{\partial x} - \frac{\partial u_n^h}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial u_n^h}{\partial z} - \frac{\partial w_n^h}{\partial x}\right). \tag{11}$$

По аналогичен начин получаваме и уравненията

$$\frac{\partial}{\partial z}\left(\frac{\partial w_n^h}{\partial y} - \frac{\partial v_n^h}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v_n^h}{\partial x} - \frac{\partial v_n^h}{\partial y}\right), \tag{12}$$

(13) $$\frac{\partial}{\partial x}\left(\frac{\partial u_n^h}{\partial z}-\frac{\partial w_n^h}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial w_n^h}{\partial y}-\frac{\partial v_n^h}{\partial z}\right).$$

Равенствата (11) – (13) показват, че съществува функция $\varkappa_n(x,\ y,\ z)$, за която са в сила равенствата

(14) $$\frac{\partial \varkappa_n}{\partial x}=\frac{\partial w_n^h}{\partial y}-\frac{\partial v_n^h}{\partial z};\qquad \frac{\partial \varkappa_n}{\partial y}=\frac{\partial u_n^h}{\partial z}-\frac{\partial w_n^h}{\partial x};$$

$$\frac{\partial \varkappa_n}{\partial z}=\frac{\partial v_n^h}{\partial x}-\frac{\partial u_n^h}{\partial y}.$$

Следователно
$$\frac{\partial^2 \varkappa_n}{\partial x^2}+\frac{\partial^2 \varkappa_n}{\partial y^2}+\frac{\partial^2 \varkappa_n}{\partial z^2}=0,$$

т.е. функцията $\varkappa_n(x,\ y,\ z)$ е също сферична функция от степен n.

Лесно се установява, че изразът $xu_n^h+yv_n^h+zw_n^h$ е сферична функция от $(n+1)$-ва степен. От (11) – (13) следват

(15) $$z\frac{\partial \varkappa_n}{\partial y}-y\frac{\partial \varkappa_n}{\partial z}=x\frac{\partial u_n^h}{\partial x}+y\frac{\partial u_n^h}{\partial y}+z\frac{\partial u_n^h}{\partial z}+u_n^h$$
$$-\frac{\partial}{\partial x}(xu_n^h+yv_n^h+zw_n^h)$$

и две други аналогични уравнения, които се получават от (15) чрез циклична замяна на x, y, z и u_n^h, v_n^h, w_n^h. След известни преобразувания от (9) – (10) получаваме

(16) $$\Delta\left[xu_n^h+yv_n^h+zw_n^h\right]=0.$$

Следователно може да запишем

(17) $$xu_n^h+yv_n^h+zw_n^h=\Phi_{n+1},$$

където Φ_{n+1} е сферична функция от $(n+1)$-ва степен. Като заместим (17) в (16), за неизвестната функция u_n^h намираме

(18) $$(n+1)u_n^h=\frac{\partial \Phi_{n+1}}{\partial x}+z\frac{\partial \varkappa_n}{\partial y}-y\frac{\partial \varkappa_n}{\partial z}.$$

Аналогично за v_n^h и w_n^h получаваме

(19) $$(n+1)v_n^h=\frac{\partial \Phi_{n+1}}{\partial y}+x\frac{\partial \varkappa_n}{\partial z}-z\frac{\partial \varkappa_n}{\partial x},$$

(20) $$w_n^h=\frac{\partial \Phi_{n+1}}{\partial z}+y\frac{\partial \varkappa_n}{\partial x}-x\frac{\partial \varkappa_n}{\partial y}.$$

Имайки предвид, че щом функциите Φ_{n+1} и $\varkappa_n$ са сферични, то и $(n+1)\Phi_{n+1}$, $(n+1)\varkappa_n$ също са сферични функции, може без ограничение на общността да изоставим в (18), (19) множителя $(n+1)$. Така получаваме, че решенията на хомогенната система (9), (10) могат да се запишат във вида

$$(21)\qquad \begin{aligned} u_n^h &= \sum\left(\frac{\partial\Phi_n}{\partial x}+z\frac{\partial\varkappa_n}{\partial y}-y\frac{\partial\varkappa_n}{\partial z}\right),\\ v_n^h &= \sum\left(\frac{\partial\Phi_n}{\partial y}+x\frac{\partial\varkappa_n}{\partial z}-z\frac{\partial\varkappa_n}{\partial x}\right),\\ w_n^h &= \sum\left(\frac{\partial\Phi_n}{\partial z}+y\frac{\partial\varkappa_n}{\partial x}-x\frac{\partial\varkappa_n}{\partial y}\right).\end{aligned}$$

Като вземем предвид изразите за A и B, за общото решение на хомогенната система (4), (5) получаваме

$$(22)\qquad \begin{aligned} u &= \frac{1}{\mu}\sum_n\left[\frac{r^2}{2(2n+1)}\frac{\partial P_n}{\partial x}+\frac{nr^{2n+3}}{(n+1)(2n+1)(2n+3)}\frac{\partial}{\partial x}\frac{P_n}{r^{2n+1}}\right]\\ &+ \sum_n\frac{\partial\Phi_n}{\partial x}+\sum_n\left(z\frac{\partial\varkappa_n}{\partial y}-y\frac{\partial\varkappa_n}{\partial z}\right),\end{aligned}$$

$$(23)\qquad \begin{aligned} v &= \frac{1}{\mu}\sum_n\left[\frac{r^2}{2(2n+1)}\frac{\partial P_n}{\partial y}+\frac{nr^{2n+3}}{(n+1)(2n+1)(2n+3)}\frac{\partial}{\partial y}\frac{P_n}{r^{2n+1}}\right]\\ &+ \sum_n\frac{\partial\Phi_n}{\partial y}+\sum_n\left(x\frac{\partial\varkappa_n}{\partial z}-z\frac{\partial\varkappa_n}{\partial x}\right),\end{aligned}$$

$$(24)\qquad \begin{aligned} w &= \frac{1}{\mu}\sum_n\left[\frac{r^2}{2(2n+1)}\frac{\partial P_n}{\partial z}+\frac{nr^{2n+3}}{(n+1)(2n+1)(2n+3)}\frac{\partial}{\partial z}\frac{P_n}{r^{2n+1}}\right]\\ &+ \sum_n\frac{\partial\Phi_n}{\partial z}+\sum_n\left(y\frac{\partial\varkappa_n}{\partial x}-x-\frac{\partial\varkappa_n}{\partial y}\right).\end{aligned}$$

От (21) – (24) следва, че полученото решение за компонентите на скоростта се изразява чрез функциите P_n, Φ_n и $\varkappa_n$. Трите суми в дясната страна на уравненията (21) – (24) характеризират съответно влиянието на разпределението на налягането, потенциалния и вихровия характер на обтичането на твърдата частица. Във векторен вид уравненията (21) – (24) се записват по-компактно така:

$$(25)\qquad \vec{v}=\sum_{n=0}^{\infty}\left[(\nabla\varkappa_n\times\vec{r})+\nabla\Phi_n+\frac{n+3}{2(n+1)(2n+3)}r^2\nabla P_n\right.$$
$$\left.-\frac{n}{(n+1)(2n+3)}rP_n\right].$$

За да приложи полученото общо решение (25) за решаване на задачата за обтичане на твърда сфера от градиентен поток, Айнщайн избира координатна система така, че началото ѝ да съвпада с центъра на частицата и оста x да е насочена в направлението на скоростта ($\vec{v}_\infty$) в безкрайност. Тогава граничните условия се записват така:

$$(26)\qquad u = v = w = 0 \qquad \text{при} \qquad r = a;$$

$$\vec{v} = \vec{v}_\infty(\alpha y,\ 0,\ 0) \qquad \text{при} \qquad r \to \infty,$$

където a е радиусът на сферата и α е единствената ненулева компонента на тензора на скоростта на деформацията за разглежданото течение. При този избор на координатната система членовете в (25), съдържащи $\varkappa_n$, отпадат, защото осите ѝ съвпадат с главните оси на деформацията. Като отчита това, след известни преобразувания Айнщайн [1] намира, че на нула не са равни само функциите

$$\Phi_2 = \frac{1}{4}\alpha(x^2 - y^2), \qquad \Phi_{-3} = B_{-3}a^5\frac{x^2 - y^2}{r^2},$$

$$P_{-3} = \mu A_{-3}\frac{x^2 - y^2}{r^5},$$

където A_{-3} и B_{-3} са константи. След известни преобразувания от (21) – (24) следва

$$(27)\qquad \begin{aligned} u &= \frac{1}{2}A_{-3}a^3x\frac{x^2 - y^2}{r^5} + B_{-3}a^5\left[-\frac{5x}{r^4}(x^2 - y^2) + \frac{2x}{r^2}\right] + \frac{\alpha x}{2}; \\ v &= \frac{1}{2}A_{-3}a^3y\frac{x^2 - y^2}{r^5} + B_{-3}a^5\left[-\frac{5y}{r^4}(x^2 - y^2) - \frac{2y}{r^2}\right] + \frac{\alpha y}{2}; \\ w &= \frac{1}{2}A_{-3}a^3z\frac{x^2 - y^2}{r^5} + B_{-3}a^5\left[-\frac{5z}{r^4}(x^2 - y^2)\right]. \end{aligned}$$

От граничните условия (26) намираме

$$A_{-3} = -\frac{5}{2}\alpha, \qquad B_{-3} = -\frac{1}{4}\alpha a^{-3}.$$

Замествайки в (27), за компонентите на скоростта окончателно се получава

$$(28)\qquad \begin{aligned} u &= \frac{\alpha x}{2}\left[-\frac{5}{2}\left(\frac{a^3}{r^3} - \frac{a^2}{r^2}\right)\frac{x^2 - y^2}{r^2} - \frac{a^2}{r^2} + 1\right]; \\ v &= \frac{\alpha y}{2}\left[-\frac{5}{2}\left(\frac{a^3}{r^3} - \frac{a^2}{r^2}\right)\frac{x^2 - y^2}{r^2} + \frac{a^2}{r^2} - 1\right]; \\ w &= \frac{5\alpha z}{4}\left(\frac{a^2}{r^2} - \frac{a^3}{r^3}\right)\frac{x^2 - y^2}{r^2}. \end{aligned}$$

Известно е, че силата, действуваща на сфера, обтичана от равномерен безкраен вискозен флуид, се дава от формулата на Стокс

$$\vec{F} = 6\pi\mu a \vec{v}_\infty, \tag{29}$$

където $\vec{v}_\infty = \text{const}$. През 1924 г. Факсен [34] обобщава (29) за случая на твърда сфера, обтичана от стационарен прост градиентен вискозен поток. Той установява, че теоремата за силата, с която действува стационарен вискозен неравномерен поток върху твърда сфера, намираща се в него, се дава с формулата

$$\vec{F} = 6\pi\mu a \frac{1}{4\pi a^2} \int\limits_S \vec{v}(r) d\sigma = 6\pi\mu a (\vec{v}_\infty)^S. \tag{30}$$

Тук $\vec{v}_\infty(r)$ е скоростта на неравномерния поток в безкрайност, S е повърхността на сферичната частица, а

$$(\vec{v}_\infty)^S = \frac{1}{4\pi a^2} \int\limits_S \vec{v}_\infty(r) d\sigma \tag{31}$$

е средната скорост на повърхността S.

От (30) при $\vec{v}_\infty(r) = \text{const}$, т.е. при равномерно обтичане, се получава като частен случай формулата на Стокс (29).

Поведението на флуидните частици (капки и мехури), суспендирани в градиентни течения, се различава много от поведението им в равномерните в безкрайност течения. Докато при бавни неградиентни течения (т.е. равномерни течения при малки числа на Рейнолдс) сферичните капки и мехури запазват формата си, ако имат малки размери или голямо повърхностно напрежение, то при силно градиентни течения това не е така. Формата на флуидните частици при втория вид течения зависи не само от обемните или междуфазовите свойства на флуида, но и от градиента на профила на скоростта S на течението. При малки стойности на S формата на флуидните частици е близка до сферичната, а при много големи S настъпват извънредно силни деформации, при които флуидната частица става неустойчива и се разделя на две еднакви капки или няколко по-малки сателитни капки.

Ще отбележим, че макар уравненията на Стокс и съответните гранични условия да са линейни, тъй като формата на капките (мехурите) се записва с нелинейното уравнение

$$f\left(\frac{x'}{a}, \frac{y'}{a}, \frac{z'}{a}, C_a, \frac{\hat{\mu}}{\mu}\right) = 0,$$

разглежданият проблем е нелинеен. Именно поради тази нелинейност на проблема за разлика от обтичането на твърди частици (с известни граници) от градиентен вискозен поток при флуидните частици досега не са получени точни решения в случая, когато формата им не е сферична.

Първите фундаментални резултати в изследване поведението на капка в градиентно течение на Кует и хиперболично вискозно течение, като се използуват уравненията на Стокс, извършва Тейлор [35, 36] — съответно през 1932 и 1934 г. Удовлетворявайки граничните условия за непрекъснатост на скоростта и тангенциалното напрежение върху зададена сферична междуфазова граница, той пресмята скоростта и нормалните напрежения в областта на течението. Като използува уравнението за баланса на нормалните напрежения, Тейлор намира следната форма на флуидната частица:

$$r = a\left(1 + \frac{2D}{a^2}xy\right),$$

където D е параметър на деформацията, за който имаме

$$D = \frac{L - B}{L + B} = \frac{Sa\mu}{\sigma}\frac{19k + 16}{16k + 16} \ll 1.$$

Тук $B = a(1 - D)$, $L = a(1 + D)$, $k = \frac{\hat{\mu}}{\mu}$ ($\hat{\mu}$ и μ са вискозитетите на флуидите вътре и вън от капката).

Полученият резултат от Тейлор показва, че при малки деформации формата на капката от сферична преминава в елипсоидна. Най-голямата ос L и най-малката ос B на елипсоида [33] лежат в равнината Oxy, сключвайки ъгъл от 45° с осите Ox и Oy. Удължението на формата на капката е в посока от третия към първия квадрант, а свиването — от втория към четвъртия. Третата ос на елипсоида лежи на оста Oz и е равна на радиуса a на първоначалната форма на капката. Това уравнение е експериментално потвърдено за течение на Кует и хиперболичното градиентно течение от самия Тейлор [36] и Рамшид и Мейсън [37], а за течение на Поазьой — от Голдсмит и Мейсън [38].

Установено е, че при стойности на малкия параметър $\varepsilon > 0,2$ започват да настъпват значителни отклонения от предсказаната от теорията елипсоидна форма. При $\varepsilon = 0,5$ вискозните сили, които водят до промяна на формата на капката, превишават силите на повърхностното напрежение, стремящи се да я запазят, и тя се разрушава, т.е. губи устойчивост. При тези стойности на ε зависимостта на формата на капката от повърхностното напрежение става нелинейна и както посочват Рамшид и Мейсън [37], преди разрушаването тя има различен вид за различните видове градиентни течения.

Тейлор [36] изследва експериментално случая на силно вискозни капки при $\varepsilon \to \infty (\eta^{-1} \to 0)$. Когато ε е голямо, Тейлор разглежда в първо приближение формата на капката като функция само на отношението на вискозитетите k и отново намира, че капката приема формата на елипсоид с оси, успоредни на линиите на тока на несмутеното движение на флуида.

Като се проектира в равнината, съдържаща неравномерната част на градиентното течение, уравнението на деформираната повърхност на капката добива вида

$$\frac{r}{a} = 1 + \frac{5}{4}k^{-1}\cos 2\Phi + O(k^{-1}),$$

където Φ е ъгълът на градиента на скоростта на течението и линиите на тока.

Както показват експериментите на Тейлор [36] и Рамшид и Мейсън [37], при $k > 5$ капката се удължава във вид на дълга нишка при течение на Кует и се разрушава при хиперболично течение. Сложността на изследвания нелинеен проблем затруднява точното пресмятане на формата на флуидните частици в градиентни течения, поради което той досега не е решен. Голям принос в това направление е изследването на Кокс [38], който използва за малък параметър величината

$$\frac{1}{\eta} = \frac{\mu a S}{\sigma}, \qquad \text{или} \qquad \frac{1}{k} = \frac{\mu}{\bar{\mu}}.$$

Той изследва хиперболично течение, в което импулсивно от покой тръгва флуидна капка, чиято форма се приближава към равновесното си положение при неограничено растене на времето.

Ако началото на сферичната координатна система $(r,\ \theta,\ \varphi)$ съвпада с центъра на капката, за уравнението на формата на флуидната частица имаме $r = a[1 + F(\theta,\ \varphi)]$, където F се разлага посредством сферичните функции F_n във вида

$$F(\theta,\ \varphi) = \sum_{n=1}^{\infty} \varepsilon^n F_n(\theta,\ \varphi).$$

Кокс търси скоростта и налягането във вид на ред по степените на малкия параметър

$$\vec{v}(r,\ \theta,\ \varphi,\ \varepsilon) = \vec{v}_0(r,\ \theta,\ \varphi) + \varepsilon\vec{v}_1(r,\ \theta,\ \varphi) + \varepsilon^2\vec{v}_2(r,\ \theta,\ \varphi) + \ldots,$$

$$p(r,\ \theta,\ \varphi,\ \varepsilon) = p_0(r,\ \theta,\ \varphi) + \varepsilon p_1(r,\ \theta,\ \varphi) + \varepsilon^2 p_2(r,\ \theta,\ \varphi) + \ldots$$

Допускайки, че при малки деформации сферичната капка с радиус a приема формата на триосен елипсоид с оси B, L и a, за величината D той получава

$$D = \frac{L - B}{L + B} = \frac{5(19k + 16)}{4(\eta + 1)\sqrt{(19k)^2 + (20\eta)^2}} \ll 1.$$

Тя е в сила или при $\eta \gg 1$ и $k = O(1)$, или при $k \gg 1$ и $\eta = O(1)$. Тук се предполага, че малката ос (B) и голямата ос (L) лежат в равнината, определяща градиентното течение, докато третата ос (a) е перпендикулярна на тях.

Да изберем осите Ox и Oz на декартовата координатна система да са успоредни съответно на вихровите и токовите линии на несмутеното градиентно течение и най-малката и най-голямата ос на елипсоида да лежат в равнината Oyz. Тогава ориентацията на флуидната частица относно градиентното течение ще бъде определена от ъгъл α, заключен между оста Oy и най-голямата ос на елипсоида. За малки деформации Кокс получава

$$\alpha = \frac{\pi}{4} + \frac{1}{2}\mathrm{tg}^{-1}\left(\frac{19k}{20\eta}\right).$$

Тъй като $\frac{k}{\eta} = \frac{\hat{\mu}Sa}{\sigma}$, от тази формула следва, че $\alpha = 45^\circ$ при $S \to 0$ и капката е ориентирана под ъгъл 45° относно линиите на тока, а $\alpha = 90^\circ$ при $S \to \infty$ и голямата ос е насочена в направление на токовите линии на несмутеното течение.

Използвайки метода на Кокс, Хетсрони и Хабер [39, 40] изследват динамиката на деформируема капка, суспендирана в неограничено градиентно стоксово течение. Чрез метода на отражението Хетсрони и др. [41] решават приближено задачата за аксиално движение на слабо деформируема капка в кръгова тръба. Ако R_0 е радиусът на тръбата, b — разстоянието на капката до оста на тръбата, a — еквивалентният радиус на флуидната частица, ρ и $\hat{\rho}$ — плътностите на флуидите вън и вътре в нея и k — отношението на вискозитетите μ и $\hat{\mu}$, за скоростта на движение на капката в тръбата в направление на течението на Поазьой авторите на [42] получават

$$\begin{aligned} U &= \frac{2(\hat{\rho}-\rho)ga^2}{9\mu}\frac{1+k}{\frac{2}{3}+k}\left[1-\frac{2+3k}{3(1+k)}\frac{a}{R_0}f\left(\frac{b}{R_0}\right)\right] \\ &+ U_0\left[1-\left(\frac{b}{R_0}\right)^2-\frac{2k}{2+3k}\left(\frac{a}{R_0}\right)\right]+O\left(\frac{a}{R_0}\right)^3, \end{aligned}$$

където $f\left(\frac{b}{R_0}\right)$ е функция, табулирана от Фамуларо [42, таблица 1, с. 309]. Хабер и Хетсрони [40] пресмятат в течение на Поазьой скоростта на миграция на частицата в тръбата и траекторията, която описва тя. Според тези резултати неутрално суспендирана в течение на Поазьой капка ще мигрира радиално навън в посока към стената, докато експериментите на Голдсмит и Мейсън [21] показват, че тя се движи радиално навътре в посока към оста на тръбата. Това несъответствие на теоретичните и експерименталните резултати се дължи на допусната грешка при пресмятане на траекториите. Ако грешката се поправи, радиалната посока на движение на деформираната флуидна частица се променя, но големината на получената радиална скорост съществено се различава от получената експериментално.

По-добро съвпадение между теоретичните и експерименталните резултати получават Уол и Рабинов [43], които изследват по-обща задача, от която като частен случай се получава задачата на Хабер и Хетсрони [40].

Шафри и др. [44] изследват хидродинамичното взаимодействие между стена и близка до нея капка. Те разглеждат простия случай, когато флуидната частица е суспендирана близо до ограничаващата течението на Поазьой равнинна стена. Това течение се доближава до течението на Поазьой, в което радиусът на тръбата е голям и капката е близо до стената. Допуснатите грешки в [44] са коригирани от същите автори в [45]. За скоростта на миграцията на деформираната капка авторите на [44, 45] получават

$$U_m = Sa\left(\frac{\mu Sa}{\sigma}\right)\left(\frac{a}{h}\right)^2 f(k), \tag{32}$$

където

$$f(k) = \frac{(19k+16)(9k^2+17k+9)}{(k+1)^3}.$$

Формулата (32) е получена при условие, че величините $\frac{a}{h}$ и $\frac{\mu Sa}{\sigma}$ са малки, а k е от порядъка на единица. Формулата за капка близо до равнинна стена, ограничаваща течението на Поазьой, експериментално е потвърдена от Карние и др. [14].

През 1967 г. Шафри и Бренер [46] разработват метод за изследване с точност до втори порядък на деформациите на капка, суспендирана в течение на Кует. Получен е интересен резултат — като се отчете второто приближение при изследване деформациите на капката, нейната форма вече не е елипсоидна.

С цел да обобщи резултатите и за две флуидни частици в градиентен поток Грийнщейн [47] разглежда задачата за бавно движение на две сферични капки, разположени симетрично относно оста на кръгова тръба. Посоката на движение на частиците е перпендикулярна на правата, минаваща през центровете им, което усложнява значително решението на задачата. Прилагайки метода на отражението, Хетсрони и Хабер [48] също изследват хидродинамичното взаимодействие между две сферични капки, потопени в неограничено градиентно течение с произволна скорост. Тъй като посоката на движение на двете капки е в направление, перпендикулярно на правата, минаваща през центровете им, задачата е пространствена и полученото решение има сложна структура.

Котансо и Тизон [49] изследват експериментално и теоретично изплуването на мехур в тръба. За малки стойности на числото на Рейнолдс и $\frac{a}{h} < 0,2$ (a е радиусът на мехура, а h — най-малкото разстояние от центъра на частицата до стената на тръбата) те установяват, че мехурът остава почти сферичен. При $\frac{a}{h} > 0,6$ флуидната частица има цилиндрична форма, завършваща в двата края с "чадърообразна" форма. При $0,2 < \frac{a}{h} < 0,6$ мехурът има елипсоидна форма.

Глюкман и др. [50] разработват аналитико-числен колокационен метод, при който граничните условия се удовлетворяват само в определен неголям брой точки от повърхността на частицата, която се обтича. Използувайки този метод, Ганотос и др. [51] изследват движението на три твърди частици в неограничено пространствено течение. Главният проблем при прилагането на този метод е изборът на положението на колокационните точки. Ганотос и др. [52] прилагат този метод за изследване движението на твърда сфера в произволна посока между две успоредни равнини. Въз основа на същия метод Даган и др. [53] решават задачата за движение на твърда сферична частица близо до стена с крайна дължина.

През 1988 г. Шапира и Хабер [54] изследват бавно движение на деформируема капка, успоредно на две равнини. Тъй като върху флуидните частици точките на колокация не могат да се изберат предварително, когато те са деформируеми, Шапира и Хабер прилагат метода на отражението. Сходимостта на решението обаче не е добра, когато частицата е близо до една от равнините.

Много бавните вискозни течения между две успоредни вертикални равнини се описват от уравненията

$$\mu\frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x}, \qquad \mu\frac{\partial^2 v}{\partial z^2} = \frac{\partial p}{\partial y} + \rho g, \tag{33}$$

където началото на координатната система е взето в средата между пластините, оста Oy е насочена вертикално, оста Ox — хоризонтално и лежи в средата между двете равнини, а оста Oz — перпендикулярно на последните. Когато тези равнини са много близо една до друга и в течението между тях се намира цилиндрично препятствие, чрез образуващите успоредни равнини те получават т. нар. клетка на Хел-Шоу. Флуидът в клетката се движи под действие на градиента на налягането, приложен в двата ѝ края. Несмутеното равнинно течение на Поазьой се записва във вида

$$(u,\ v) = \frac{6z}{b}\left(1 - \frac{z}{b}\right)(U,\ V), \tag{34}$$

където b е разстоянието между равнините. Ако се замести в (33), се получават уравненията

$$\frac{12\mu}{b^2}U = -\frac{\partial p}{\partial x}, \qquad \frac{12\mu}{b^2}V = -\frac{\partial p}{\partial z} - \rho g, \tag{35}$$

които са идентични по вид с уравненията, описващи движението в пореста среда, т. е. с уравненията на Дарси. Затова клетката на Хел-Шоу се разглежда като модел на пори с пропускливост $\frac{b^2}{12}$ и решенията на задачите за движение на вискозни флуиди са пряко свързани с двумерните филтрационни течения. През 1898 г. Хел-Шоу [55] установява, че при постоянни стойности на z в равнината $(x,\ y)$ се получава двумерно скоростно поле, което е потенциално и върху твърдите стени удовлетворява само условието за непротичане. Затова линиите на тока при стационарно обтичане на препятствия в клетката на Хел-Шоу са идентични по форма с тези в хипотетично двумерно течение на идеален флуид около препятствие от същия вид. Този резултат позволява да се използува клетката на Хел-Шоу за демонстрационни визуализации на линиите на тока на идеални потенциални течения около препятствия с различна форма, като се въвежда оцветяване в няколко точки на входа на съоръжението.

Точни решения на задачата за стационарно изплуване на мехури в клетка на Хел-Шоу, съдържаща вискозен флуид, намират Тейлор и Сафман [56]. Те доказват, че от тези решения устойчиво е онова, което се отнася за мехур, чиято ширина е два пъти по-малка от разстоянието между двете успоредни равнини. Интересни резултати за хидродинамични проблеми, свързани с клетката на Хел-Шоу, се съдържат в работите [57] – [63].

Пълен обзор на изследванията до 1972 г. на хидродинамичното взаимодействие между твърди или флуидни частици и кръгови тръби, съдържащи вискозен флуид, е даден от Бренер в [64, 65]. Усъвършенствуване на метода на Кокс [38] и подобрение на резултатите са извършени от Франкъл и Акривос [66] и Бардес-Бисел и Акривос [67]. Теоретичните резултати добре се съгласуват с експерименталните резултати на Торза, Кокс и Мейсън [68].

Ралисън [69] извършва критичен анализ на получените резултати от изследванията на движението на флуидни частици в градиентни течения. Използувайки подхода, разработен в теорията на тънкото крило, той разглежда големи деформации на флуидни частици в градиентен поток. Ралисън и Акривос [70] разработват числен метод, който дава възможност да се пресмята формата на капката (мехура) в случаите, в които асимптотичната теория е неприложима.

В обзора си от 1984 г. Ралисън [69] анализира различни видове градиентни течения, като изхожда от разделянето на несмутената скорост $\vec{v}$ на потока на две компоненти: 1) свързана с тензора на деформацията e_{ij}; 2) свързана с тензора на завихрянето ω_{ij}:

$$\vec{v} \sim Ca(e_{ij} + \omega_{ij})\vec{x},$$

където $Ca = \dfrac{U\mu}{\sigma}$ е капилярното число. В зависимост от вида на тензорите e_{ij} и ω_{ij}, записани в декартови координати, най-простите градиентни течения, изследвани досега, са:

1. Просто градиентно течение

$$\vec{v} = \vec{v}(Sy,\ 0,\ 0); \qquad e_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \omega_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

2. Равнинно хиперболично течение

$$\vec{v} = \vec{v}(Sx,\ -Sy,\ 0); \qquad e_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \omega_{ij} = 0,$$

при което няма завихреност на потока. Този вид градиентно течение се получава например при четирицилиндровия уред на Тейлор.

3. Ортогонално реометрично течение

$$e_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \omega_{ij} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -2\varkappa \\ -1 & 0 & 0 \\ 2\varkappa & 0 & 0 \end{pmatrix};$$

Параметърът $\varkappa$ е свързан с промяната на относителната ориентация на вихъра спрямо главните оси на тензора на деформацията.

4. Чисто ососиметрично разтягане

$$\vec{v} = \vec{v}(Sx,\ -\frac{1}{2}Sy,\ -\frac{1}{2}Sz);$$

$$e_{ij} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}; \qquad \omega_{ij} = 0.$$

5. Равнинни градиентни течения

$$e_{ij} = \frac{1}{2}(1+\varkappa)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \omega_{ij} = \frac{1}{2}(1-\varkappa)\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Тук $-1 \leqq \varkappa \leqq 1$ е параметър, чиито стойности принадлежат на $[-1,\ 1]$. При $\varkappa = 0$ се получава като частен случай простото градиентно течение, а при $\varkappa = -1$ — равнинното хиперболично течение.

Изучаването на свойствата на кръвта при градиентното ѝ движение е в голяма степен свързано с изследване поведението на формените ѝ елементи — еритроцити, левкоцити и тромбоцити. Една от най-изследваните биомембранни структури, движещи се в градиентните течения, са еритроцитите. В хидромеханиката те се моделират като твърди частици, капки или микрокапсули. Като твърди частици еритроцитите са изследвани например в [71] – [76] и др. През 1969 г. Шмидт-Шонбейн и Уелс [77] предлагат движещите се в градиентен поток еритроцити да се моделират като капки. Този модел води началото си от изследванията на Голдсмит [78] на поведението на течни капки, намиращи се в движещ се в кръгова тръба вискозен флуид. През 1976 г. Голдсмит и Скейлак [79] наблюдават аксиална миграция на еритроцитите в кръгова тръба и деформирането им в елипсоидна форма със слабо изразен "димпъл". Моделирането на еритроцитите като флуидни частици е изследвано и от много други автори — [80] – [83] и др. През 1967 г. Роске [84] и Годард и Милер [85] изследват движението на хомогенни еластични сферични частици в градиентно течение.

По-реалистичен модел на еритроцитите се получава, като се вземат предвид техните еластични свойства. През 1968 г. Лайтхил [86] и през 1969 г. Фицджералд [87], прилагайки теорията на смазките, определят движението на еластични сфери в капилярни тръби.

През 1974 г. Ричардсън [88] разглежда елипсоидни микрокапсули, суспендирани в градиентно течение, като модел на еритроцитите в кръвта. Същия модел разглежда през 1980 г. Бадес-Бисел [89], като взема формата на микрокапсулата за сферична. Движенията на външния слой и на течността вътре в сферичната микрокапсула се описват с уравненията на Стокс. Материалът на мембраната е приет за несвиваем, еластичен и физически нелинеен от най-общ вид. Под действието на вискозните сили на течностите мембраната претърпява от двете страни крайни премествания и деформации. За решаване на така формулираната линейна задача е приложен методът на смущенията (пертурбациите) за малки отклонения от сферичната форма. Брун [90] разглежда също пертурбации на малки сферични микрокапсули във вискозно течение. Далеч от еритроцита течението се приема за хомогенно и зависещо от времето. Получени са в явен вид реологичните уравнения на разредена суспензия в сферични частици в случаите, когато те са твърди, еластични или представляват микрокапсули, запълнени с вискозен нютонов флуид. През 1985 г. Бадес-Бисел и Сгайер [91] обобщават полученото в [89] решение, като разглеждат микрокапсули, чиято деформируема мембрана има вискозно-еластична реология, съответствуваща на модела на Келвин—Фойхт. Те прилагат итерационна изчислителна процедура при малки отклонения на деформируемата мембрана от сферична форма. В [91] авторите разглеждат и частните случаи на чисто еластична и чисто вискозна мембрана. Изследван е и случаят, когато мембраната се моделира като безкрайно тънка черупка от тримерен еластичен материал при някои частни видове конститутивни уравнения.

Като допълнение на обзора на градиентни течения ще отбележим и изследването на картината на течението и силите на хидродинамичното взаимодействие при обтичане на две твърди сферични частици от ососиметричен вискозен поток с параболичен профил на скоростта, извършено от Калицова-Куртева и Запрянов [92].

Като използват резултатите от [28], Дарабанер и Мейсън [93] пресмятат траекториите на две частици при обтичането им от градиентен поток. Сравняването с експерименталните данни показва добро съвпадение. Тези резултати са използвани от други автори при определяне на ефективния вискозитет на суспензии от сферични частици с обемна концентрация от 1 до 2%. През 1977 г. Дейвис и О'Нил [94] решават задачата за обтичане на кръгов цилиндър от градиентен вискозен поток, ограничен от равнина. Когато разстоянието между цилиндъра и равнината клони към нула, от двете стени започват алтернативно да се генерират вихри. Тази структура на течението преминава в безкрайна редица от вихри, когато цилиндърът има обща точка с равнината [95]. Обширен клас от градиентни течения разглеждат Горен и О'Нил [96]. По-специално те изследват обтичането на твърда сферична частица близо до равнината от градиентен поток. В [97] се разглежда вискозният поток, индуциран от движението на кръгова тръба, в която се намира твърда сфера. Хидродинамичните ефекти, дължащи се на влиянието на стената, се пренебрегват, тъй като се предполага, че радиусът на сферата е много по-малък от радиуса на тръбата. Това дава възможност на авторите да разглеждат обтичане на сферична частица от т. нар. "неограничено течение на Поазьой".

В един от първите модели на движението на еритроцитите в кръвната плазма в капилярните кръвоносни съдове еритроцитите се моделират като твърди частици, кръвната плазма — като вискозна течност и кръвоносните съдове — като кръгови цилиндрични тръби. Една част от изследванията в това направление се основават на общото решение, получено от Хаберман и Сайре [98]. В [99] – [100] са разгледани и други възможности за моделиране на еритроцитите в кръвта, като особено е подчертан моделът, при който еритроцитите се моделират като течни капки, покрити с еластична мембрана.

Да разгледаме ососиметричен градиентен поток със скорост

$$\vec{v}^*_\infty = A_0\rho z\vec{i}_\rho - z^2\vec{i}_z, \tag{36}$$

където $\vec{i}_\rho$ и $\vec{i}_z$ са единични вектори в цилиндрична координатна система $(\rho,\ z,\ \varphi)$ и A е константа. Ако дефинираме функция на тока чрез равенствата

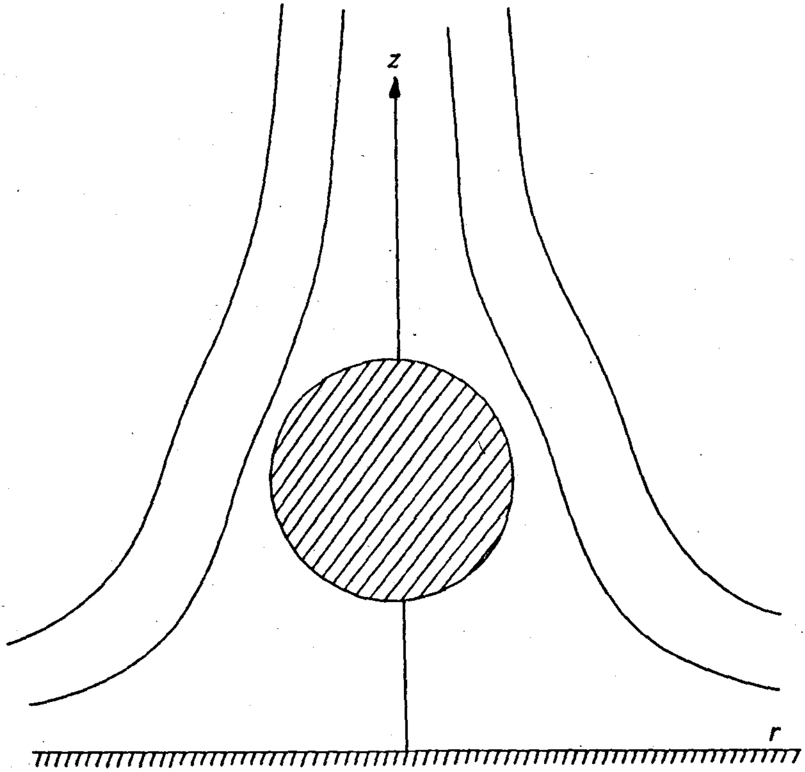
$$u^* = \frac{1}{\rho}\frac{\partial \Psi^*}{\partial z}, \qquad w^* = -\frac{1}{\rho}\frac{\partial \Psi^*}{\partial \rho}, \tag{37}$$

след интегриране получаваме

$$\Psi^* = \frac{1}{2}A_0\rho^2 z^2. \tag{38}$$

Течението, определено от тази функция на тока, е ососиметрично течение около критична точка. Нека сферична капка с радиус a се намира в течението, така че центърът ѝ да лежи на оста $\rho = 0$, и на разстояние $h > a$ от равнината $z = 0$ (фиг. 1) формата ѝ ще се запазва.

Да представим функцията на тока Ψ на смутеното течение във вида $\Psi = \Psi^* + \Psi_1$, където Ψ^* е несмутеното течение далеч от капката. Тогава Ψ_1 ще удовлетворява уравнението на Стокс за бавно движение на



Фиг. 1

вискозен флуид:

$$E^4\Psi_1 \equiv \left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right)\Psi_1 = 0, \tag{39}$$

тъй като функцията (38) удовлетворява това уравнение.

Да въведем бисферична координатна система, в която $\xi = 0$ да бъде уравнението на равнината, а $\xi = \alpha$ — уравнението на повърхността на капката. Тогава безразмерното разстояние $h = \mathrm{ch}\alpha$ от капката до равнината определя еднозначно α и фокалното безразмерно разстояние $c = \mathrm{sh}\alpha$. Ако означим с $\hat{\Psi}_1$ функцията на тока вътре в капката, уравнението (39) ще решаваме при следните гранични условия:

1) Върху твърдата равнина $\xi = 0$:

$$\Psi = 0, \qquad \Psi_1 = -\Psi^*; \tag{40}$$

$$\frac{\partial\Psi}{\partial\xi} = 0, \qquad \frac{\partial\Psi_1}{\partial\xi} = -\frac{\partial\Psi^*}{\partial\xi}; \tag{41}$$

2) Върху повърхността на междуфазовата граница $\xi = \alpha$:

$$\Psi_1 = \hat{\Psi}_1 = 0; \tag{42}$$

$$\frac{\partial \Psi_1}{\partial \xi} = \frac{\partial \hat{\Psi}_1}{\partial \xi}, \qquad \frac{\partial^2 \Psi_1}{\partial \xi^2} = \lambda \frac{\partial^2 \hat{\Psi}_1}{\partial \xi^2}; \tag{43}$$

3) В безкрайност, т.е. при $\xi^2 + \eta^2 \to 0$:

$$\Psi_1 \to 0. \tag{44}$$

Тук $\lambda = \dfrac{\hat{\mu}}{\mu}$, където $\hat{\mu}$ и μ са динамичните вискозитети на флуидите вътре и вън от капката.

За ососиметрично течение в бисферична координатна система операторът E^2 има вида

$$E^2 \equiv \frac{\text{ch}\xi - \beta}{c^2} \left\{ \frac{\partial}{\partial \xi} \left[(\text{ch}\xi - \beta) \frac{\partial}{\partial \xi} \right] + (1 - \beta^2) \frac{\partial}{\partial \beta} \left[(\text{ch}\xi - \beta) \frac{\partial}{\partial \beta} \right] \right.,$$

където $\beta = \cos \eta$.

Решението на уравнението (39) в бисферични координати вън и вътре в капката се записва така:

$$\Psi_1 = B(\text{ch}\xi - \beta)^{-3/2} \sum_{n=1}^{\infty} U_n(\xi) V_n(\beta), \tag{45}$$

$$\hat{\Psi}_1 = B(\text{ch}\xi - \beta)^{-3/2} \sum_{n=1}^{\infty} \hat{U}_n(\xi) V_n(\beta), \tag{46}$$

където

$$V_n(\beta) \equiv P_{n-1}(\beta) - P_{n+1}(\beta)$$

($P_n(\beta)$ са полиномите на Льожандър),

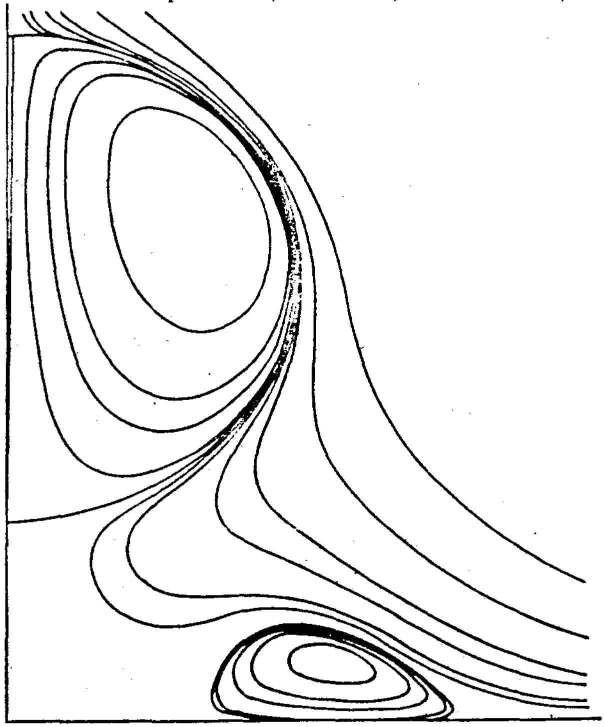
$$\begin{aligned} U_n(\xi) &= A_n \text{ch} j_n \xi + B_n \text{sh} j_n \xi + C_n \text{ch} j_n \xi + D_n \text{sh} k_n \xi, \\ \hat{U}_n(\xi) &= a_n e^{-j_n \xi} + b_n e^{-k_n \xi}, \end{aligned}$$

$$j_n = n - \frac{1}{2}, \qquad k_n = n + \frac{3}{2}, \qquad B = \frac{1}{2} A_0 c^4.$$

Коефициентите A_n, B_n, C_n, D_n, a_n и b_n се определят числено от граничните условия (40) – (44), като се вземе предвид, че

$$\begin{aligned} \Psi^* &= \frac{1}{2} A_0 c^4 \frac{\sin^2 \eta \psi \text{sh}^2 \xi}{(\text{ch}\xi - \beta)^4} \\ &= \frac{1}{2} A_0 c^4 (\text{ch}\xi - \beta)^{-3/2} \left\{ \frac{2\sqrt{2}}{3} \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \left[j_n e^{-\gamma_n \xi} - k_n e^{-k_n \xi} \right] \right. \\ &+ \left. \frac{\text{sh}\xi}{\text{ch}\xi} \left[e^{-j_n \xi} - e^{-k_n \xi} \right] \right\} V_n(\beta). \end{aligned}$$

Тъй като коефициентите A_n, B_n, C_n, D_n намаляват бързо с увеличаването на n, за пресмятането на функциите на тока на течението вън и вътре в капката е достатъчно във формулите (45), (46) да се вземат 15–16 члена. Линиите на тока зависят от два параметъра λ и h. На фиг. 2 те са показани при $\lambda = 0,5$ и $h = 1,8$. Вижда се, че зад капката близо до



Фиг. 2

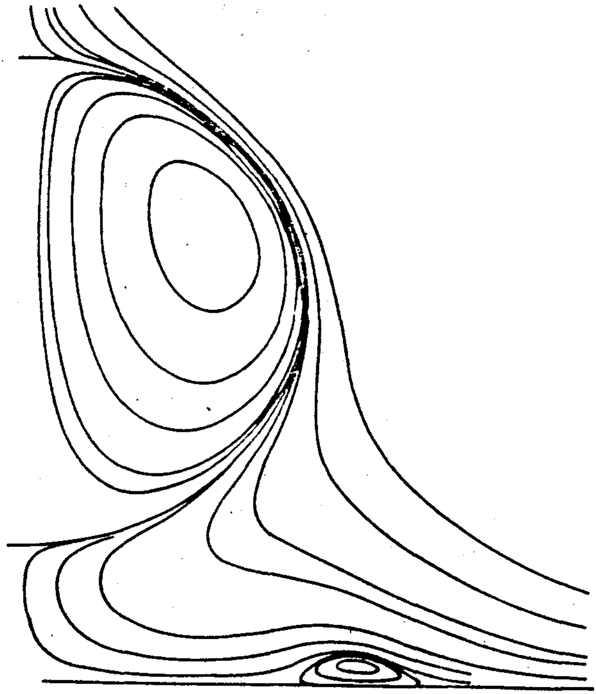
стената се появява вихър. При големи разстояния на флуидната частица от равнината този вихър изчезва. При $\lambda = 0,5$ за $h_{\text{кр}}$ се получава $h^2_{\text{кр}} \sim 7$ (по-точно $6,769 < h^2_{\text{кр}} < 7,470$). При $\lambda = 1,5$ и $h = 1,6$ също има два вихъра зад капката близо до стената (фиг. 3), които са симетрично разположени относно оста Oz. При $\lambda = 1,5$ за критичното разстояние получаваме $h^2_{\text{кр}} \sim 8$ (по-точно $7,473 < h^2_{\text{кр}} < 8,25$). Хидродинамичното силово взаимодействие между флуидната частица и твърдата стена се характеризира със силата, с която флуидът действува върху капката при наличие на стената. Поради осесиметричността на задачата

$$F_x = F_y = 0 \qquad \text{и} \qquad F_z = 6\pi\mu A_0 a^3 f_0,$$

където

$$f_0 = \frac{2\sqrt{2}}{3}\text{sh}^3\alpha \sum_{n=1}^{\infty} \frac{2n+1}{2n+3} B_n.$$

Зависимостта на f_0 от разстоянието h за различни стойности на λ е дадена на фиг. 4. При големи стойности на λ ($\lambda \to \infty$) флуидната частица

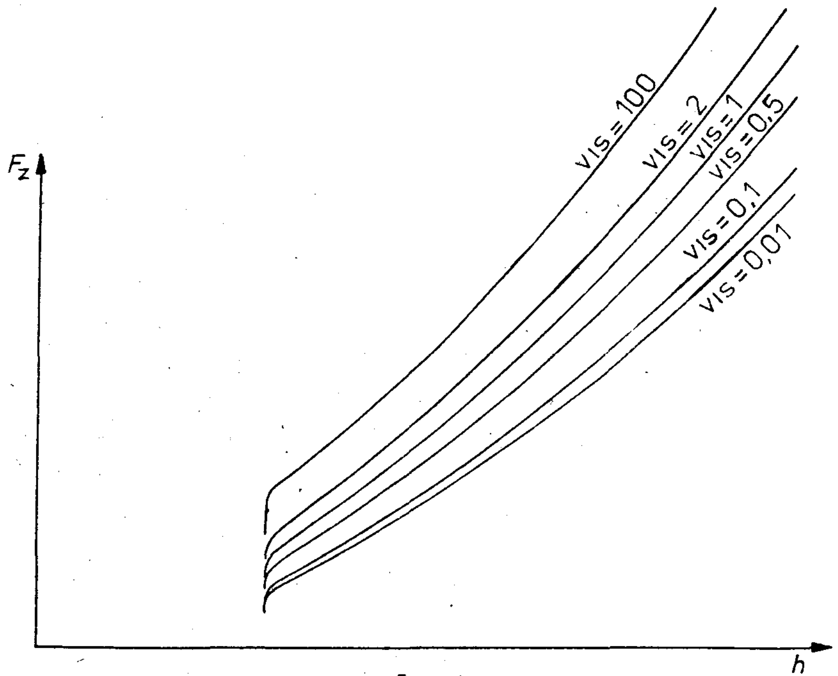


Фиг. 3

се "втвърдява" и зависимостта между f_0 и h за капка преминава в дадената от Горен и О'Нил [96] зависимост между f_0 и h за твърда сфера. Сравнението е дадено на табл. 1.

Изследването на обтичането на твърди или флуидни частици от градиентни вискозни течения е сравнително по-трудно от изследването на същите хидродинамични задачи при обтичане с равномерен поток. Когато частиците са свободно суспендирани в градиентни вискозни течения, те могат да описват много сложни траектории. Например в просто градиентно течение две сферични частици могат да се въртят безкрайно (неограничено) в затворени траектории (орбити), докато при удължаващите (разтягащите) течения техните траектории са отворени с изключение на случая, когато двете частици се допират.

Простото несмутено градиентно течение е вихрово, а несмутените ососиметрични равнинни удължаващи течения са безвихрови. Поради съществуващия градиент на скоростта при градиентните течения свободно суспендираните в тях частици се движат с различни транслационни и ротационни скорости, което е причина за тяхното доближаване и дори сблъскване. Възникват въпроси като: 1) Може ли да се получи равновесен физичен контакт между две сблъскали се частици? 2) Как действуват



Фиг. 4

различните видове градиентни течения върху различните видове формирали се образувания (агрегати) от частици? 3) Поведението на агрегатите зависи ли от това, дали несмутеното градиентно течение е вихрово или не?

Таблица 1

h/a	f_0[96]	f_0
51,0	2660,0	2659,87
10,0	112,80	112,78
5,0	32,090	32,0912
2,0	7,533	7,532
1,5	5,146	5,1452
1,1	3,575	3,5748
1,05	3,400	3,3949
1,01	3,263	3,2606
1,005	3,246	3,21

В [93] е показано експериментално, че в суспензиите от сферични частици в просто градиентно течение едновременното сблъскване на n частици ($n \geqq 3$) може да доведе както до образуване, така и до разрушаване на двойки от частици, които имат затворени траектории.

При простото градиентно течение раздалечаването на частиците или образуването на дублети зависи от техните траектории. Отворените траектории водят до разпръскване (диспергиране) на частиците, докато затворените траектории не притежават това свойство.

Тъй като теоретично сферите не могат да установят физичен контакт [102], заключаваме, че в градиентните течения е невъзможно агрегиране на частици без действието на привличащи сили между тях. В зависимост от вида на градиентното течение то може да има или няма диспергиращи (разпръскващи) ефекти върху съществуващи агрегати. Ососиметричните и равнинните удължаващи течения лесно диспергират агрегати от почти допиращи се сферични частици.

Интересно свойство притежават суспензиите, чиито частици имат несферична форма — вместо да се ориентират в едно направление, частиците се въртят неравномерно по периодично затворени орбити. Такъв вид траектории са били получени най-напред от Джефри [2] при изследване на суспензии с частици, имащи формата на ротационни елипсоиди. Обобщени резултати от извършените експериментални изследвания на обтичането на твърди и флуидни частици са дадени в монографията на Клифт, Грейс и Вебер [103] през 1978 г.

Наред с решените съществуват и много нерешени проблеми от хидродинамичното взаимодействие при градиентни течения в стоксово приближение между деформируеми флуидни частици, между капки (мехури) и твърди или междуфазови граници и др. Най-неразработени са досега проблемите за определяне на хидродинамичното взаимодействие при градиентни течения между две и повече твърди или флуидни частици, когато се използуват пълните уравнения на Навие — Стокс.

ЛИТЕРАТУРА

1. A. Einstein. — Ann. Physik, **19**, 1906, p. 289; Ann. Physik, **34**, 1911, 591.
2. G. B. Jeffery. — Proc. Roy. Soc. (London), A 102, 1922, 161.
3. H. Faxen. — Arkiv Mat. Astron. Fys., **17**, N 27, 1923.
4. H. Faxen. — Arkiv Mat. Astron. Fys., **19**, N 22, 1925.
5. S. Wakiya. — J. Phys. Soc., **12**, 1957, 1130.
6. S. Wakiya. — Res. Rep. Fac. Eng. Niigata Univ. Japan, **7**, 1958, 1.
7. F. P. Bretherton. — J. Fluid Mech., **12**, 1962, 591.
8. F. P. Bretherton. — J. Fluid Mech., **14**, 1962, 284.
9. R. Ladenburg. — Ann. Physik, **23**, 1907, 447.
10. S. Wakiya. — Res. Rep. Fac. Eng. Niigata Univer., Japan, **5**, 1956, 1.
11. S. Wakiya. — J. Phys. Soc. Japan, **8**, 1953, 254.
12. G. Segre, A. Silberberg. — Nature, **189**, 1961, 209.

13. G. Segre, A. Silberberg. — J. Fluid Mech., **14**, 1962, 115, 136.
14. A. Karnis, H. Goldsmith, S. G. Mason. — Nature, **200**, 1963, 159.
15. H. L. Goldsmith, S. G. Mason. — Bibliotheca Anat., **4**, 1964, 462.
16. R. C. Jeffery. — J. Fluid Mech., **22**, 1965, 721.
17. R. Eichhorn, S. Small. — J. Fluid Mech., **20**, 1964, 513.
18. J. T. Day, W. E. Genetti. — B. S. thesis, Univ. Utah. Salt. Lake City, Utah., 1964.
19. C. D. Denson, E. B. Cristiansen, D. L. Salt. — A. I. Ch. E. J., **12**, 1966, 589.
20. S. I. Rubinow, J. B. Keller. — J. Fluid Mech., **11**, 1961, 447.
21. H. L. Goldsmith, S. G. Mason. — J. Colloid Sci., **17**, 1962, 448.
22. P. G. Saffman. — J. Fluid Mech., **22**, 1965, 385.
23. A. J. Goldman, H. Brenner. — Chem. Engng. Sci., **21**, 1966, 1151.
24. M. E. O'Neill. — Chem. Engn. Sci., **23**, 1963, 1293.
25. W. R. Dean, M. E. O'Neill. — Mathematika, **10**, 1963, 13.
26. S. Wakiya. — J. Phys. Soc. Japan, **22**, 1967, 1101.
27. M. H. Davis. Rept. N NCAR-TN/STR-64, National Center for Atmospheric Research Boulder, Colorado, 1971.
28. C. J. Lin, K. J. Lee, N. F. Sather. — J. Fluid Mech., **43**, 1970, 35.
29. S. G. Curtis, L. M. Hocking. — Trans. Faraday Soc., **66**, 1970, 1381.
30. G. K. Batchelor, J. T. Green. — J. Fluid Mech., **56**, 1972, 375.
31. S. Wakiya. — J. Phys. Soc. Japan, **31**, 1971, 1581.
32. A. Nir, A. Acrivos. — J. Fluid Mech., **59**, 1973, 209.
33. H. Lamb. Hydrodynamics, Dover, 1945.
34. H. Faxen. — Arkiv. Mat. Astron. Fys., **18**, N 29, 1924.
35. G. I. Taylor. — Proc. Roy. Soc. (London), A **138**, 1932, 41.
36. G. I. Taylor. — Proc. Roy. Soc. (London), A **146**, 1934, 501.
37. F. D. Rumscheidt, S. G. Mason. — J. Colloid Sci., **16**, 1961, 238.
38. R. G. Cox. — J. Fluid Mech., **37**, 1969, 601.
39. G. Hetsroni, S. Haber. — Rheol. Acta, **9**, 1970, 488.
40. S. Haber, G. Hetsroni. — J. Fluid Mech., **49**, 1971, 257.
41. G. Hetsroni, S. Haber, E. Wacholder. — J. Fluid Mech., **41**, 1970, 689.
42. J. Happel, H. Brenner. Low Reynolds Number Hydrodynamics, N. Y., Prentice-Hall, 1965.
43. P. R. Wohl, S. I. Rubinov. — J. Fluid Mech., **62**, 1974, 185.
44. C. E. Chaffey, H. Brenner, S. G. Mason. — Reologica Acta, **4**, 1965, 64.
45. C. E. Chaffey, H. Brenner, S. G. Mason. — Reologica Acta, **6**, 1967, 100.
46. C. H. Chaffey, H. Brenner. — J. Colloid and Interface Sci., **24**, 1967, 258.
47. T. Greenstein. — J. Phys. Soc. Japan, **32**, 1972, 1398.
48. G. Hetsroni, S. Haber. — Int. J. Multiphase Flow, **4**, 1978, 1.
49. M. Coutanceau, P. Thizon. — J. Fluid Mech., **107**, 1981, 339.
50. P. Gluckman, R. Preffer, S. Weinbaum. — J. Fluid Mech., **50**, 1971, 705.
51. P. Ganotos, R. Pfeffer, S. Weinbaum. — J. Fluid Mech., **84**, 1978, 79.

52. P. Ganotos, R. Pfeffer, S. Weinbaum. — J. Fluid Mech., **99**, 1980, 755.
53. Z. Dagan, R. Pfeffer, S. Weinbaum. — J. Fluid Mech., **122**, 1982, 273.
54. M. Shapira, S. Haber. — Int. J. Multiphase Flow, **14**, 1988, 483.
55. H. J. Hele Shaw. — Nature, **58**, 1988, 34.
56. J. W. McLean, P. G. Saffman. — J. Fluid Mech., **102**, 1981, 455.
57. F. P. Bretterton. — J. Fluid Mech., **10**, 1961, 166.
58. G. Tryggvason, H. Afer. — J. Fluid Mech., **136**, 1983, 1.
59. C. W. Park, G. M. Homsy. — J. Fluid Mech., **139**, 1985, 1583.
60. P. Tabeling, A. Libchaber. — Phys. Rev., A **33**, 1986, 794.
61. T. Maxworhty. — J. Fluid Mech., **173**, 1986, 95.
62. L. W. Schwartz, H. M. Princen, A. D. Kiss. — J. Fluid Mech., **172**, 1986, 259.
63. D. A. Reinelt. — J. Fluid Mech., **175**, 1987, 557.
64. H. Brenner. In Advances in chemical Engineering, Academic Press, 1966, 287.
65. H. Brenner. — Progress in Heat and Mass Transfer, **6**, 1971, 487.
66. N. A. Frankel, A. Acrivos. — J. Fluid Mech., **44**, 1970, 65.
67. D. Barthes-Biesel, A. Acrivos. — J. Fluid Mech., **61**, 1973, 1.
68. S. Torza, R. G. Cox, S. G. Mason. — J. Colloid Interface Sci., **38**, 1972.
69. J. M. Rallison. — Ann. Rev. Fluid Mech., **16**, 1984, 45.
70. J. M. Rallison, A. Acrivos. — J. Fluid Mech., **89**, 1978.
71. H. Goldsmith, G. Mason. The *microrheology of dispersions* in Rheology (Edited by Eirich F. R. v.4), Academic Press, N. Y., 1967.
72. H. Wang, R. Skalak. — J. Fluid Mech., **38**, 1969, 75.
73. T. Chen, R. Skalak. — Appl. Sci. Res., **22**, 1970, 403.
74. H. Brenner, P. Bungay. — Fedn. Proc., **30**, 1971, 1565.
75. P. Bungay, H. Brenner. — Int. J. Multiphase Flow, **1**, 1973, 25.
76. B. Skalak, P. Chen, S. Chien. — Biorheology, **9**, 1972, 67.
77. H. Schmid-Schonbein, R. Well. — Science, **165**, 1969, 288.
78. H. Goldsmith. — J. Gen. Physiol., **52**, 1968, 55.
79. H. Goldsmith, R. Skalak. — Hemodynamics, Ann. Rev. Fluid Mech., **7**, 1975, 213.
80. H. Gautheier, H. Goldsmith, S. G. Mason. — Biorheology, **9**, 1972, 205.
81. K. Kline. — Biorheology, **9**, 1972, 287.
82. W. Hyman, R. Skalak. — A. I. Ch. E. J., **18**, 1972, 287.
83. Отчет на договорна разработка 514/16623Г. Хидродинамично моделиране на емулсии и мембрани. Ръководител на договора З. Запрянов.
84. R. Roscoe. — J. Fluid Mech., **28**, 1967, 273.
85. J. Goddard, Ch. Miller. — J. Fluid Mech., **28**, 1967, 657.
86. M. Lighthil. — J. Fluid Mech., **34**, 1968, 113.
87. J. Fitz-Gerald. — Proc. Roy. Soc., B **174**, 1969, 193.
88. E. Richardson. — Proc. Roy. Soc., A **338**, 1974, 129.
89. D. Barthes-Biesel. — J. Fluid Mech., **100**, 1980, 831.
90. P. Brun. — Biorheology, **17**, 1980, 419.
91. D. Barthes-Biesel, H. Sgaier. — J. Fluid Mech., **160**, 1985, 119.
92. P. Kalitzova-Kurteva, Z. Zapryanov. Sixth congres of theoretical and applied Mechanics, Varna, 1989.
93. C. Darabaner, S. Mason. — Reol. Acta, **6**, 1967, 273.

94. A. Davis, M. O'Neill. — J. Fluid Mech., **81**, 1977, 551.
95. Schubert. — J. Fluid Mech., **27**, 1967, 647.
96. S. Goren, M. O'Neill. — Chem. Engn. Sci., **26**, 1971, 325.
97. A. Chwang, T. Yao-Tsu Wu. — J. Fluid Mech., **67**, 1975, 787.
98. W. Haberman, R. Sayre. — 1958, Davis Taylor Model Basin Report 1143, Washington, D.C.
99. P. Brunn. — J. Fluid Mech., **126**, 1983.
100. D. Bathes-Biesel, H. Soaier. — J. Fluid Mech., **160**, 1985, 119.
101. F. Gross, A. Popel. Mathematics of Microcirculation Phenomena, 1980, 17.
102. A. Arp, G. Mason. — J. Colloid and Interface Science, **61**, 1977, 21.
103. R. Clift, J. Grace, M. Weber. Bubbles, Drops and Particles Academic Press, New York, San Francisco, London, 1978.

Received 2. VII. 1990

FLUID MECHANICS OF RIGID OR FLUID PARTICLES IN SHEAR FLOWS

Z. Zapryanov

Summary

Beginning with the celebrated paper of Stokes* the theory of particle (rigid of fluid) motions in quiescent fluids has a long and auspicious history. This initial work has been extended in two main branches: uniform streaming flows and nonuniform (shear) flows.

In general, uniform flows are achieved by the action of an external force on the particles or of some tethering force when the fluid stream past stationary particles. Shear flows may arise because of the movement of apparatus boundaries as in the case of Coette flow or the action of an external pressure gradient as in Poiseuille flow through a circular tube.

Shearing flows may be generated as well by the extension of a thread of viscous fluid as in the polymer drawing processes or by the impaction of particles on collectors where one obtains a stagnation flow.

During the past two-three decades considerable progress has been made in the development of continuum theories of suspensions in which the suspended matter may consist of rigid or fluid particles. The understanding of the flow behaviour of particulate suspensions is important in many chemical technologies, sedimentation, engineering problems concerning with nuclear reactor cooling, performance of rigid-fuel rocket nozzles, aerosol spraying and others. Sedimentation, wherein particles fall under the action of gravity through a fluid, is commonly used in the petroleum industries as a way of separating particles from the fluid.

The hydrodynamic models of shear flows may be globally characterized by a shear rate S in units of reciprocal time and Reynolds number

$$Re = \frac{a^2 S}{\nu},$$

*G. G. Stokes. — Trans. Cambr. Phil. Soc., **9**, 1951, 8.

where a is the radius of the particle and ν is the kinematic viscosity of the fluid. For small Reynolds numbers the velocity $\overline{v}$ and the pressure p must satisfy the Stokes and continuity equations

$$(1) \qquad \nu\nabla^2\overline{v} = \frac{1}{\rho}\nabla p, \qquad \nabla\overline{v} = 0,$$

where ρ is the density of the fluid.

The mathematical modelling of heterogeneous systems (suspensions, emulsions, slurries etc.), as if it were homogeneous, has for many years been a challenge to both experimentalist and theoretician. As a result numerous models have been developed to show the link between the macroscopic behaviour of the system as a whole and the microscopic interactions between heterogeneities. Einstein [1] was the first who calculated the viscosity of a dilute suspension of rigid spheres in an incompressible Newtonian fluid. He considered creeping shear flow past a single sphere and multiplied the result obtained by the number of the spheres to give a total correction to indisturbed flow for a dilute suspension of noninteracting spheres. In this way he obtained his classical formula

$$(2) \qquad \frac{\mu_{ef}}{\mu} = 1 + \frac{5}{2}\varphi + O(\varphi^2),$$

where μ_{ef} is the effective viscosity of suspension, μ is the viscosity of Newtonian fluid, comprising the continuous phase, and φ is the volume fraction of spheres in suspension. It is worth to note that a modern derivation of Einstein's suspension viscosity has been presented by Landau and Lifshitz*.

In an oft-quoted paper Jeffery [2] studied the behaviour of an ellipsoidal particle in a shearing field on the basis of Stokes equation of motion. This work has since provided the starting point for a multitude of theoretical investigations related to antisotropism in laminar shear flows. He considered a suspension of neutrally buoyant ellipsoids of revolution, dispersed in a Couette flow. The orientational distribution function is a periodic function of time, the period being

$$T = \frac{2\pi}{S}\left(k + \frac{1}{k}\right),$$

in which $k = \dfrac{a_{\|}}{a_{\perp}}$ and $a_{\|}$, $a_{\perp}$ are the lengths of the semiaxes of the symmetry and transverse axes respectively.

The axis of the ellipsoid of revolution moves in one of a family of closed periodic orbits, the center of the particle moving with the velocity of the undisturbed fluid at that point. Jeffery [2] found the time-average viscosity of the suspension by utilizing additional dissipation arguments and intergrating this instantaneous, orientation-dependent quantity over one period. Accordingly, the energy dissipation depends upon the initial particle orientation and cannot be regarded as an intrinsinc property of the fluid-particle suspension. Since the concept of an infinitely dilute suspension is an idealized one, Mason et al.** have perfomed a sequence of experiments and established that after a sufficient number of individual particle rotations the distribution of orbital parameters approaches a unique, steady-state distribution, which is apparently independent of the initial orientational distribution. In this connection we shall note that two spheres approaching one

*L. D. Landau, E. M. Lifshitz. Fluid Mechanics, Addison-Wesley, 1959.

S. G. Mason, R. J. Manley. — Proc. Roy. Soc. (London), A **238, 1956, 117.

another along two neighbouring streamlines, moving at different velocities, may either "collide" to form an effectively permanent collision doublet or else they may merely approach to within some minimal separation distance before their receding. The existance of closed streamlines furnishes a rational hydrodynamic explanation of the existance of permanent, two-sphere doublets.

Bretherton [7] considered the steady, two-dimensional motion at low Reynolds number of an incompressible viscous fluid past a circular cylinder, the velocity at large distance being described by a uniform simple shear. In 1962 Bretherton [8] extended Jeffery's analysis to bodies of more general shape and showed the existance of modes of motion, incapable of being displayed by ellipsoidal particles. He showed that for some rigid particles of revolution there exists a definite "preferred" orientation and the rigid bodies do not undergo a periodic rotation of the Couette flow. He found that the "preffered" orientation of the particles is such that their symmetry axe is directed along the streamlines. If

$$\varepsilon = \frac{a-b}{a},$$

then

$$\frac{\mu_{ef}}{\mu} = 1 + m\varphi, \tag{3}$$

where $m = m(\varepsilon)$ is a known function.

When one liquid is at rest in another liquid of the same density, it assumes the form of a spherical drop. The physical and chemical conditions of emulsions of two fluids, which do not mix, have been the subject of many studies, but very little seems to be known about the mechanics of the stirring processes. In contrast to the rigid particles, the principle physical characteristic of a fluid particle (a drop or a bubble) is its ability to deform under the influence of shear. But if the drops (bubbles) are very small or the surface tension is large then the shape of the fluid particles will tend to keep spherical. However, in general, the fluid particle adopts a nonspherical shape and its precise shape, when suspended in a fluid undergoing simple shear, is governed by the ratio of the viscous shearing forces, μS, to the interfacial tension forces, $\frac{\sigma}{a}$, where σ is the surface tension and a — the radius of the underformed drop (bubble). The fluid particle shape is determined in part by the dimension parameter

$$k = \frac{\sigma}{\mu a S}$$

and in part by the viscosity ratio

$$\alpha = \frac{\text{continuous} - \text{phase viscosity}}{\text{fluid particle viscosity}} = \frac{\mu}{\hat{\mu}}.$$

It is important to note also that the droplet contour adopts a definite orientation, relative to the principle axes of shear, though there exists an internal circulation within it.

In his two extraordinary papers G. I. Taylor [35, 36] found that the emulsion behaves like a Newtonian fluid with a viscosity

$$\mu_{ef} = \mu\left[1 + \frac{5}{2}\frac{1+\frac{2}{5}\alpha}{1+\alpha}\varphi\right]. \tag{4}$$

For rigid particles $\alpha \to 0$ and one obtains Einstein's formula (2) but for gas bubble $\alpha \to \infty$ and

$$\mu_{ef} = \mu(1 + \varphi). \tag{5}$$

Not surprisingly, this viscosity is considerably less than that for rigid sphere suspension.

If the viscosity of moderately concentrated suspensions is presented by the power series expansion

$$\mu_{ef} = \mu \left(1 + \frac{5}{2}\varphi + k\varphi^2 + \cdots \right), \tag{6}$$

then the second-order coefficient is governed by "two-body" interactions. That is why the investigation of the interaction of just two particles alone in a large expanse of fluid is of special significance.

Problems, in which a viscous fluid interacts with a deformable fluid particle and a rigid wall, are of considerable interest in the study of liquids, such as blood, polymer solutions, and suspensions of liquid droplets. The understanding of the mechanics of the interactions between the deformable fluid particle and the rigid wall is important both for investigating phenomena of interest at the level of a single particle and for the bulk rheology of the suspension. One problem of this type for example, is the response of a red blood cell to a viscous shear field.

In the theory of particle capture by filtration or scrubbing processes one must know the hydrodynamic forces, exerted on particles suspended in the flow, and particularly the extent of the influence of walls on these forces when the particles are moving to the walls. The normal force, exerted by axisymmetric stagnation flow on small rigid sphere touching a rigid plane was culculated by Goren*. A method for estimating the hydrodynamic force, acting on a small particle of a dilute suspension in a slow streaming motion past a large spherical obstacle, was presented by Goren and O'Neill [96].

The steady flow in and around a deformable drop, moving in an unbounded viscous parabolic flow and subjected to an external body force, was calculated for creeping-flow regime. It was found, in addition to the drag force that, the drop experiences a force orthogonal to the undisturbed flow direction. Whol and Rubinov [43] also calculated the trajectories for a drop in Poiseuille flow on the basis of their derived force. They predicted that a neutrally buoyant drop would move radially inwards to the cylinder axis, in agreement with the observations of Goldsmith and Mason [21].

The radial migration of a single spherical particle across the streamlines of a Poiseuille flow in a tube cannot be explained on the basis of Stokes'equations, even in the presence of the bounding walls, i.e. a sphere experiences no transverse force at creeping flow regime. A transverse force does exist if inertial forces are taken into account (Rubinov and Keller [20]). In fact, the redial force producing axial migration of the deforming drops (and flexible solid particles) arises from the interaction between the drop deformation and the flow field around the drop, rather than from an inertial effect. In this way the transverse force on a particle should be computable solely on the basis of the Stokes' equations if the particle is flexible.

Let us consider an axisymmetric shear flow that at infinity has the form

$$\vec{v}^* = A_0(\rho z \vec{i}_\rho - z^2 \vec{i}_z), \tag{7}$$

*S. L. Goren. — Fluid Mech., 41, 1970, 619.

where $\vec{i}_\rho$, $\vec{i}_z$ are unit vectors in cylindrical coordinates $(\rho,\ z)$, A_0 is a constant and $\vec{v}^*$ is the velocity at infinity. If a stream function Ψ^* is defined in the usual manner

$$u^* = \frac{1}{\rho}\frac{\partial\Psi^*}{\partial z}, \qquad w^* = -\frac{1}{\rho}\frac{\partial\Psi^*}{\partial\rho}, \tag{8}$$

then one has

$$\Psi^* = \frac{1}{2}A_0\rho^2 z^2. \tag{9}$$

This indisturbed flow represents an axisymmetrical stagnation flow. Let us introduce a spherical drop of radius a into this flow, so that its centre lies on the axis $\rho = 0$ and at a distance from the plane $z = 0$. In this way the flow will be still symmetrical about the axis $\rho = 0$ and the stream function may be written as

$$\Psi = \Psi^* + \Psi_1,$$

where Ψ^* is the flow at infinity and $\hat{\Psi}$ must satisfy the Stokes' creeping flow equation

$$E^4\hat{\Psi} \equiv \left\{\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right\}^2 \hat{\Psi} = 0, \tag{10}$$

since Ψ^* satisfies this equation.

In order to determine the solution of equation (10) we introduce bispherical coordinates $(\xi,\ \eta)$ related to the cylindrical coordinates $(\rho,\ z)$ by the relations

$$\rho = \frac{c\sin\eta}{\mathrm{ch}\xi - \cos\eta}, \qquad z = \frac{c\ \mathrm{sh}\xi}{\mathrm{ch}\xi - \cos\eta},$$

where $-\infty < \xi < \infty$ and $0 \leqq \eta \leqq \pi$ and c is a positive constant. The coordinate surface $\xi = \alpha$ describes the drop surface and $\xi = 0$ represents the rigid plane.

The boundary conditions are:

(i) on the rigid plane $\xi = 0$

$$\Psi = 0,\ \Psi_1 = -\Psi^*; \tag{11}$$

$$\frac{\partial\Psi}{\partial\xi} = 0, \qquad \frac{\partial\Psi_1}{\partial\xi} = -\frac{\partial\Psi^*}{\partial\xi}; \tag{12}$$

(ii) on the drop interface $\xi = \alpha$

$$\Psi_1 = \hat{\Psi}_1 = 0, \tag{13}$$

$$\frac{\partial\Psi_1}{\partial\xi} = \frac{\partial\hat{\Psi}_1}{\partial\xi}, \tag{14}$$

$$\frac{\partial^2\Psi_1}{\partial\xi^2} = \lambda\frac{\partial^2\hat{\Psi}_1}{\partial\xi^2}; \tag{15}$$

(iii) at infinity $\xi^2 + \eta^2 \to 0$,

$$\Psi_1 \to 0, \tag{16}$$

where $\lambda = \frac{\mu^{(i)}}{\mu^{(e)}}$, $\mu^{(i)}$, $\mu^{(e)}$ are the dynamic viscosities inside and outside of the drop, respectively, and $\hat{\Psi}_1$ and Ψ_1 are the corresponding stream functions.

We have solved the problem of a stagnation (shear) flow past a spherical drop in proximity of a rigid wall at small Reynolds numbers (i.e. the problem (7) – (16)) and have found the structure of the flow and the force, exerted by creeping flow on the drop. Toshev and Zapryanov* have investigated numerically the problem of a stagnation flow past a rigid sphere in proximaty of a rigid wall at moderate Reynolds numbers. The above two problems are connected with the investigation of the interaction between the blood erythrocites and the walls of artificial organs. In the same connection Kalitzova-Kurteva and Zapryanov [92] have studied steady axisymmetric motion at low Reynolds numbers of an incompressible viscous fluid past two spherical particles, the velocity at large distance being described by an unbounded Poiseuille flow. If the radii of the particles are not smaller than that of the containing circular tube, one has to consider hydrodinamic effects due to the wall of the tube. They have obtained an exact solution for the flow fields of the above mentioned problem by an extension of the well known procedure, developed by Stimpson and Jeffery**.

Finally we will mention briefly some problems in the field of shear flows which are still not solved:

i) The problem of shear creeping flow past two deformable drops or bubbles.

ii) The problem of shear creeping flow past "two capsules" (two fluid drops, which are enclosed by an elastic membrane).

iii) The problem of shear flows past two particles (rigid or fluid), where inertia terms are included.

---

*E. Toshev, Z. Zapryanov. In preparation for publication.

**M. Stimpson, G. B. Jeffery. — Proc. Roy. Soc., London, 11, 1926, 110.

ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ "СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Книга 2 — Механика
Том 83, 1989

ANNUAIRE DE L'UNIVERSITE DE SOFIA "ST. KLIMENT OHRIDSKI"
FACULTE DE MATEMATIQUES ET INFORMATIQUE
Livre 2 — Mécanique
Tome 83, 1989

О НЕГОЛОНОМНЫХ СИСТЕМАХ С НЕЛИНЕЙНЫМИ СВЯЗЯМИ

СОНЯ ДЕНЕВА

Соня Денева. О НЕГОЛОНОМНЫХ СИСТЕМАХ С НЕЛИНЕЙНЫМИ СВЯЗЯМИ. В работе выведены уравнения движения нелинейных неголономных систем. Рассмотрены два примера применения этих уравнений.

Sonia Deneva. NONHOLONOMIC SYSTEMS WITH NONLINEAR CONNECTIONS. In this paper new equations of motion for nonholonomic nonlinear sistems are derived and two problems are solved as an illustration of the equations.

Механика нелинейных неголономных систем возникла в начале XX века благодаря работам П. Аппеля [1, 2, 3] и Е. Делассю [4, 5, 6], которые подробно исследовали такие системы. Первые сведения о нелинейной механике относятся к концу XIX-ого века. Одна из первых известных работ в этом направлении — статья И. В. Мещерского [7], в которой он показал, что в аналитической механике могут рассматриваться движения материальной точки при существовании конечны связей, но и дифференциальных связей произвольного порядка.

И. В. Мещерский очень подробно рассмотрел дифференциальные связи первого порядка, считая, что они могут осуществляться с помощью среды, воздействующей на точку, находящуюся в ней. Он вывел уравнения движения материальной точки с такой связью и дал два примера, в которых связи осуществляются при помощи шероховатой поверхности, а реакциями связей являются силы трения. В 1911 году примеры механических систем с нелинейными неголономными связями были данны П. Аппельем [1] и Е. Делассю [4]. В 1936 году А. Белимович [8] рассмотрел принципиально новый пример нелинейных неголономных систем, основанный на автоматическом регулировании скорости материальной точки. В последнее время в связи с развитием автоматики проявляется все больший интерес к нелинейной неголономной механике. Многие авторы как L. Castaddi [9], Г. С. Погасов [10], A. Melis [11], В. Новоселов [12] дают примеры таких систем, которые имеют более методологическое, чем практическое значение. Но можем быть уверенными, что в управляемых системах, в теории автоматического регулирования и в других вопросах современной техники механика нелинейных неголономных связей найдет широкое применение [13]. Отметим, что особое развитие теории нелинейных неголономных задач получила в работах И. Ценовой [14, 15] и Б. Долапчиева [16].

В настоящей работе сделан короткий обзор некоторых дифференциальных уравнений движения нелинейных неголономных систем. Рассмотрены уравнения Раута [12] для неголономных нелинейных систем первого и второго порядка. На основе принципов Журдена и Гаусса в независимых вариациях выведены уравнения движения без множителей Лагранжа. Эти уравнения совпадают с уравнениями типа Чаплыгина, данными в [15], но имеют более простую для применения форму. Вышеупомянутые уравнения применены к двумя примерам и сведены к квадратурам. При этом к первому примеру применены как уравнения Раута, так и уравнения без множителей Лагранжа.

1. Уравнения Раута для неголономных нелинейных систем первого и второго порядка.

Пусть движение механической системы определяется l параметрами, т. е.

$$r_j = r_j(t,\ q_i) \qquad (i = 1, \dots,\ l;\ j = 1, \dots,\ n). \tag{1}$$

Между параметрами системы существуют связи

$$f_\rho(t,\ q_i,\ \dot{q}_i) = 0 \qquad (\rho = 1, \dots,\ d), \tag{2}$$

которые вообще говоря нелинейными. Как следствия из (2) при варьировании только обобщенных скоростей получим

$$\sum_{i=1}^{l} \frac{\partial f_\rho}{\partial \dot{q}_i} \delta \dot{q}_i = 0 \qquad (\rho = 1, \dots,\ d). \tag{3}$$

Для вывода уравнения движения исходим из принципа Журдена, согласно которому

$$\sum_{j=1}^{n} (-m_j \ddot{r}_j + F_j) \delta r_j = 0, \tag{4}$$

где F_j силы, действующие на систему. Отметим, что принцип (4) записан в предположении, что

$$\sum_{j=1}^{n} R_j \delta \dot{r}_j = 0, \tag{5}$$

т. е. связи (2) идеалны по отношению принципа Журдена. Для конкретных примеров покажем, что реакции связей (2) удовлетворяют (5). Из (1) имеем

$$\dot{r}_j = \sum_{i=1}^{l} \frac{\partial r_j}{\partial q_i} \dot{q}_i + \frac{\partial r_j}{\partial t}, \tag{6}$$

откуда, варьируя скорости, получаем

$$\delta \dot{r}_j = \sum_{i=1}^{l} \frac{\partial r_j}{\partial q_i} \delta \dot{q}_i. \tag{7}$$

Подставляя (6) в (4), будем иметь

$$\sum_{i=1}^{l} \left\{ \sum_{j=1}^{l} \left(-m_j \ddot{r}_j \frac{\partial r_j}{\partial q_i} + F_j \frac{\partial r_j}{\partial q_i} \right) \right\} \delta \dot{q}_i = 0. \tag{8}$$

Доказано, что [14]

$$\sum_{j=1}^{n} m_j \ddot{r}_j \frac{\partial r_j}{\partial q_i} = \sum_{j=1}^{n} m_j \ddot{r}_j \frac{\partial \dot{r}_j}{\partial \dot{q}_i} = \frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i},$$

где

$$T = \sum_{j=1}^{n} \frac{m_j r_j^2}{2} \tag{9}$$

кинетическая энергия системы. Отсюда (7) принимает вид

$$\sum_{i=1}^{l} \left\{ -\frac{\partial \dot{T}}{\partial \dot{q}_i} + 2\frac{\partial T}{\partial q_i} + Q_i \right\} \delta \dot{q}_i, \tag{10}$$

где

$$Q_i = \sum_{j=1}^{n} F_j \frac{\partial r_j}{\partial q_i} \tag{11}$$

обобщенные силы, действующие на систему. Последние можно определить как коэффициенты перед вариациями обобщенных координат в выражении для виртуальной работы. Имеем

$$\delta A = \sum_{j=1}^{n} F_j \partial r_j = \sum_{i=1}^{l} \left(\sum_{j=1}^{n} F_j \frac{\partial r_j}{\partial q_i} \right) \delta q_i. \tag{12}$$

Из сравнения (11) и (12) следует вышеупомянутое утверждение. Если силы, действующие на точки систем F_j, консервативны, то выполняются соотношения

$$Q_i = \frac{\partial U}{\partial q_i}, \tag{13}$$

где U — силовая функция. Умножая (3) на множителы — λ_ρ, суммируя по ρ и прибавляя к (10), получаем

$$\sum_{i=1}^{l} \left\{ -\frac{\partial \dot{T}}{\partial \dot{q}_i} + 2\frac{\partial T}{\partial q_i} + Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \dot{q}_i} \right\} \delta \dot{q}_i = 0. \tag{14}$$

Из (3) видно, что d вариации $\delta \dot{q}_i$ зависимы и соответно $l-d$ независимы. Выбираем множители λ_ρ так, чтобы коэффициенты в (14) перед зависимыми вариациями координат $\partial \dot{q}_i (i = d+1, \ldots, l)$ обратились в нули. Тогда, так как остальные вариации $\delta \dot{q}_i (i = 1, \ldots, d)$ независимы, из (14) находим

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \dot{q}_i} \qquad (i = 1, \ldots, l). \tag{15}$$

Уравнения (15) в [14] выведены в форме Нильсона. Легко доказать на основе (9), что имеем

$$\frac{\partial \dot{T}}{\partial \dot{q}_i} - 2\frac{\partial T}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial \dot{q}_i},$$

откуда уравнения (15) принимают вид

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \dot{q}_i} \qquad (i = 1, \ldots, l). \tag{16}$$

К уравнениям (16) добавляем (2), так что имеем $l+d$ уравнения с $l+d$ неизвестными q_i, λ_ρ. Эти уравнения очевидно имеют место для нелинейных связей первого порядка.

Рассмотрим и нелинейные неголономные связи второго порядка. В этом случае будем иметь зависимости вида

$$f_\rho(t, q_i, \dot{q}_i, \ddot{q}_i) = 0 \qquad (\rho = 1, \ldots, d; \; i = 1, \ldots, l). \tag{17}$$

Варьируя только обобщенные ускорения, из (17) получаем

$$\sum_{i=1}^{l} \frac{\partial f_\rho}{\partial \ddot{q}_i} \delta \ddot{q}_i = 0 \qquad (\rho = 1, \dots, d). \tag{18}$$

Для вывода уравнения движения выходим из принципа Гаусса

$$\sum_{j=1}^{n} (-m_j \ddot{r}_j + F_j) \delta \ddot{r}_j = 0 \tag{19}$$

в предположении, что

$$\sum_{j=1}^{n} R_j \delta \ddot{r}_j = 0, \tag{20}$$

где R_j реакция связей (17). Иными словами, предполагаем, что связи (17) идеальны по отношению к принципу Гаусса. Из (6) получаем

$$\ddot{r}_j = \sum_{i=1}^{l} \frac{\partial r_j}{\partial q_i} \ddot{q}_i + \cdots, \tag{21}$$

где многоточие означает члены, не содержащие $\ddot{q}_i$. Из (21) варьируя только ускорения, получаем

$$\delta \ddot{r}_j = \sum_{i=1}^{l} \frac{\partial r_j}{\partial q_i} \delta \ddot{q}_i. \tag{22}$$

Подставляем (22) в (19), откуда находим

$$\sum_{i=1}^{l} \left\{ \sum_{j=1}^{n} \left(-m_j r_j \frac{\partial r_j}{\partial q_i} + F_j \frac{\partial r_j}{\partial q_i} \right) \right\} \delta \ddot{q}_i = 0. \tag{23}$$

Из [14] имеем

$$\sum_{j=1}^{n} -m_j \ddot{r}_j \frac{\partial r_j}{\partial q_i} = \frac{1}{2} \left(\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - 3 \frac{\partial T}{\partial q_i} \right).$$

Тогда (23) принимает вид

$$\sum_{i=1}^{l} \left(-\frac{1}{2} \frac{\partial \ddot{T}}{\partial \ddot{q}_i} + \frac{3}{2} \frac{\partial T}{\partial q_i} + Q_i \right) \delta \ddot{q}_i = 0. \tag{24}$$

Оперируя с вариациями (18), как и выше, получаем из (24) уравнения

$$\frac{1}{2} \frac{\partial \ddot{T}}{\partial \ddot{q}_i} - \frac{3}{2} \frac{\partial T}{\partial q_i} = Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \ddot{q}_i} \qquad (i = 1, \dots, l). \tag{25}$$

Очевидно (25) — расширение уравнений Ценова для нелинейных неголономных связей второго порядка. Не трудно показать, что

$$\frac{1}{2}\frac{\partial \ddot{T}}{\partial \ddot{q}_i} - \frac{3}{2}\frac{\partial T}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i},$$

откуда (25) принимает Лагранжевый вид

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \ddot{q}_i} \qquad (i = 1, \dots, l). \tag{26}$$

К (26) добавляем уравнения (17) и получаем систему из $l+d$ уравнений с $l+d$ неизвестными.

Отметим, что уравнения с множителями Лагранжа в форме Аппеля данны в [15], а именно

$$\frac{\partial S}{\partial \ddot{q}_i} = Q_i + \sum_{\rho=1}^{d} \lambda_\rho \frac{\partial f_\rho}{\partial \ddot{q}_i},$$

где S энергия ускорения.

2. Уравнения без множителей Лагранжа для нелинейных неголономных систем первого и второго порядка.

Пусть как и выше движение механической системы определяется l параметрами. Предполагается, что система (2) разрешима относительно части обобщенных скоростей, т. е. имеем

$$\dot{q}_\mu = F_\mu(t, q_i, \dot{q}_r) \qquad (\mu = k+1, \dots, l;\ r = 1, \dots, k), \tag{27}$$

где $k = l - d$ число независимых скоростей. Если ограничиться зависимостями (2), то как было показано выше, можно, исходя из принципа Журдена, получить уравнения Раута для нелинейных неголономных связей. Эти уравнения, однако, содержат множители Лагранжа, т.е. число неизвестных больше, чем число параметров. Здесь пользуясь связями (27), выведем уравнения движения в независимых скоростях. Будем исходит из принципа Журдена (4), и предположений (5). Используя заново зависимость

$$\sum_{j=1}^{n} m_j \ddot{r}_j \frac{\partial r_j}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \qquad (i = 1, \dots, l),$$

приведем принцип Журдена к виду

$$\sum_{i=1}^{l} \left\{ -\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i} + Q_i \right\} \delta \dot{q}_i = 0. \tag{28}$$

Варьируя (27) относительно обобщенных скоростей, получаем зависимости

$$\delta \dot{q}_\mu = \sum_{r=1}^{k} \frac{\partial F \mu}{\partial \dot{q}_r} \delta \dot{q}_r \qquad (\mu = k+1, \ldots , l), \tag{29}$$

где $\delta \dot{q}_r$ независимые вариации. Записываем (28) в виде

$$\sum_{r=1}^{k} \left\{ -\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) + \frac{\partial T}{\partial q_r} + Q_r \right\} \delta \dot{q}_r + \sum_{\mu=k+1}^{l} \left\{ -\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) + \frac{\partial T}{\partial q_\mu} + Q_\mu \right\} \delta q_\mu = 0. \tag{30}$$

Вносим (29) в (30), после чего получаем

$$\sum_{r=1}^{k} \left\{ -\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) + \frac{\partial T}{\partial q_r} + Q_r \right\} \delta \dot{q}_r + \sum_{r=1}^{k} \left\{ \sum_{\mu=k+1}^{l} \left\{ -\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) \frac{\partial F_\mu}{\partial \dot{q}_r} + \frac{\partial T}{\partial q_\mu} \frac{\partial F_\mu}{\partial \dot{q}_r} + Q_\mu \frac{\partial F_\mu}{\partial \dot{q}_r} \right\} \delta \dot{q}_r \right\} = 0 \tag{31}$$

$$(r = 1, \ldots , \; k = l - d).$$

Изза того, что $\delta \dot{q}_r$ зависимые вариации, из (31) следуют уравнения

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) + \sum_{\mu=k+1}^{l} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\mu} \right) \frac{\partial F_\mu}{\partial \dot{q}_r} - \frac{\partial T}{\partial q_r} - \sum_{\mu=k+1}^{l} \frac{\partial T}{\partial q_\mu} \frac{\partial F_\mu}{\partial \dot{q}_r} = Q_r + \sum_{\mu=k+1}^{l} Q_\mu \frac{\partial F_\mu}{\partial \dot{q}_r}, \tag{32}$$

$$(r = 1, \ldots , \; k).$$

Уравнения (32) и (27), число которых l, определяют вполне движение системы. Здесь отметим заново, что уравнения, подобные (32), данны [15], где названы уравнениями типа Чаплигина. В этой работе используется выражение для конетической энергии $\tilde{T}$, которое получается из T после замены зависимых скоростей. В этом смысле (32) можно третировать как новую форму уравнений аналитической механики для неголономных нелинейных систем.

Аналогично для нелинейных неголономных систем второго порядка, на которые наложены связи вида

$$\ddot{q}_\mu = F_\mu(t, \; q_i, \; \dot{q}_i, \; \ddot{q}_r) \qquad (\mu = k+1, \ldots , \; l; \; r = 1, \ldots , \; k). \tag{33}$$

Применяя принципа Гаусса для идеальных связей, получаем уравнения

$$(34)\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \ddot{q}_r}\right) + \sum_{\mu=k+1}^{l} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\mu}\right)\frac{\partial F_\mu}{\partial \ddot{q}_r} - \frac{\partial T}{\partial q_r}$$

$$- \sum_{\mu=k+1}^{l} \frac{\partial T}{\partial q_\mu}\frac{\partial F_\mu}{\partial \ddot{q}_r} = Q_r + \sum_{\mu=k+1}^{l} Q_\mu \frac{\partial F_\mu}{\partial \ddot{q}_r} \qquad (r = 1, \ldots, l-d).$$

Уравнения (34) совместно с (33) определяют движение системы.

В частном случае нелинейных неголономных систем первого порядка, для которых имеем (27), получается условие Пшеборского — Кочиной [17], [18] для вариаций координат

$$(35)\qquad \delta q_\mu = \sum_{r=1}^{k} \frac{\partial F_\mu}{\partial \dot{q}_r}\delta q_r \qquad (\mu = k+1, \ldots, l).$$

Отметим, что используя (35), можно получить уравнения (32) для нелинейных неголономных систем и с помощью принципа Даламбера — Лагранжа.

3. Пример системы с нелинейными неголономными связями первого порядка.

Тело (например космический аппарат) движется под действием силы тяжести и управляющей силы $\mathbf{R}$ так, что центр масс тела имеет постоянную по величине скорость $\boldsymbol{v}$. В качестве параметров движения выбираем координаты x, y, z центра масс G тела и углы Эйлера ϕ, θ, ψ для координатной системы, неизменно связанной с телом. Согласно теореме Кенига для кинетической энергии имеем

$$(36)\qquad T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2}(Ap^2 + Bq^2 + Cr^2),$$

где A, B, C главные моменты инерции тела и его масса. Соответственно p, q, r — компоненты угловой скорости сферического движения тела вокруг его центр масс. Согласно условию для скорости G имеем следующую нелинейную связь

$$(37)\qquad f = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{m}{2}v^2 = 0,$$

где v постоянная скорость.

Внешняя сила $\mathbf{P} = m\mathbf{g}$ консервативна и имеет силовую функцию

$$(38)\qquad V = -mgz.$$

Применяя уравнения Раута (16) и имея в виду (36), (37) и (38), получаем

$$(39)\qquad \ddot{x} = \lambda\dot{x}, \qquad \ddot{y} = \lambda\dot{y}, \qquad \ddot{z} = -g + \lambda\dot{z},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = 0; \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = 0; \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0.$$

Здесь λ — множитель Лагранжа, соответствующий (37). На основе кинематических формул

$$\begin{aligned} p &= \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi, \\ q &= \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi, \\ r &= \dot{\psi}\cos\theta + \dot{\phi} \end{aligned}$$

не трудно получить из (39) эйлеровые динамические уравнения

(40)
$$\begin{aligned} A\dot{p} &- (B-C)qr = 0, \\ B\dot{q} &- (C-A)rp = 0, \\ C\dot{r} &- (A-B)pq = 0. \end{aligned}$$

Очевидно (40) соответствуют случаю Эйлера движения твердого тела вокруг неподвижной точки. В случае, когда начальные угловые скорости $\dot{\phi}_0$, $\dot{\psi}_0$, $\dot{\theta}_0$ нули, уравнения (40) допускают только нулевое решение $p = q = r = 0$, т. е. тело совершает трансляционное движение. В противном случае (40) имеют известное решение в эллиптических функциях.

Согласно (37) и (39) движение тела определяется из уравнений

(41) $$\ddot{x} = \lambda\dot{x}, \qquad \ddot{y} = \lambda\dot{y}, \qquad \ddot{z} = -g + \lambda z, \qquad \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = v^2.$$

При условии, что $\dot{x} \neq 0$, $\dot{y} \neq 0$, из (41) находим

(42) $$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}} = \lambda.$$

Из (42) следуют зависимости

(43) $$\dot{y} = C_1\dot{x}, \qquad y = C_1 x + C_2,$$

где C_1, C_2 — интеграционные константы, зависящие от начальных условий. Из (41) и (43) следует

(44) $$\dot{z} = \varepsilon\sqrt{v^2 - (C_1^2+1)\dot{x}^2},$$

где $\varepsilon = \pm 1$ в зависимости от $\dot{z}_0 > 0$ или $\dot{z}_0 < 0$. Из (44) следует

(45) $$\ddot{z} = -\frac{\varepsilon(C_1^2+1)\dot{x}\ddot{x}}{\sqrt{v^2 - (C_1^2+1)\dot{x}^2}}.$$

Подставляя (45) и (44) в третьем уравнении (41), получаем

(46) $$\frac{v^2\ddot{x}}{\dot{x}\sqrt{v^2 - (C_1^2+1)\dot{x}^2}} = \varepsilon g.$$

После интегрирования (46) находим

$$\dot{x} = \frac{2C_3 v e^{-\frac{\delta g}{v}t}}{1 + C_1^2 + C_3^2 e^{-\frac{2\delta g t}{v}}}. \tag{47}$$

C_3 — интеграционная константа. Интегрируя еще раз (47), получаем

$$x(t) = C_4 - \frac{2\delta v^2}{g\sqrt{1+C_1^2}} \operatorname{arctg} \frac{C_3 e^{\frac{-\varepsilon g t}{v}}}{\sqrt{1+C_1^2}}. \tag{48}$$

Из (43) и (48) следует

$$y(t) = C_2 + C_1 C_4 - \frac{2\varepsilon v^2 C_1}{g\sqrt{1+C_1^2}}. \tag{49}$$

Из (44) и (47) после соответствующего интегрирования находим

$$z(t) = C_5 + \varepsilon v t + \frac{v^2}{g} \ln\left[1 + C_1^2 + C_3^2 e^{\frac{-2\delta g t}{v}}\right]. \tag{50}$$

Из формул (48) — (50) вытекает, что при достаточно больших значениях t и при $\dot{z}_0 > 0$ центр масс совершает приближенно праволинейное равномерное движение вдоль оси z. В общем случае центр масс движется с постоянной по величине скоростью.

Определим необходимую управляющую силу $\boldsymbol{R}$. На основании теоремы о движении центра масс имеем

$$m w_G = m g + \boldsymbol{R}. \tag{51}$$

Проектируя (51) по осям координатной системы, неподвижной по отношению к Земле, получаем

$$m\ddot{x} = R_x, \qquad m\ddot{y} = R_y, \qquad m\ddot{z} = -mg + R_z. \tag{52}$$

Из (42) и (47) следует

$$\lambda(t) = \frac{\varepsilon g}{v} \cdot \frac{1 + C_1^2 + C_3^2 e^{\frac{-2\varepsilon g}{v}t}}{1 + C_1^2 + C_3^2 e^{\frac{-2\varepsilon g}{v}t}}. \tag{53}$$

Равенства (52) и (53) определяют закон изменения реакции. Из (52) следует, что $\boldsymbol{R}$ коллинеарна скорости $\boldsymbol{v}$ центра масс G тела, т. е.

$$\boldsymbol{R} = m\lambda \boldsymbol{v}_G. \tag{54}$$

С помощью (54) легко проверить, что условие (5) идельности связи выполнено. Действительно (5) имеет в этом случае вид

$$\boldsymbol{R} \cdot \delta\dot{\boldsymbol{r}} = m\lambda \boldsymbol{v} \cdot \delta \boldsymbol{v} = m\lambda v \delta v = 0, \tag{55}$$

так как из за того, что $v = \text{const}$ следует $\delta v = 0$ и, следовательно, (55) выполнено.

К этому примеру применим уравнения (32) без множителей Лагранжа. В этом случае из уравнений (27) получаем

$$\dot{z} = \sqrt{v^2 - \dot{x}^2 - \dot{y}^2}, \tag{56}$$

т.е. $\dot{x}$ и $\dot{y}$ независимые скорости. Тогда уравнения (32) имеют вид

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right)\frac{\partial \dot{z}}{\partial \dot{x}} - \frac{\partial T}{\partial x} - \frac{\partial T}{\partial z}\frac{\partial \dot{z}}{\partial \dot{x}} = \frac{\partial U}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial \dot{z}}{\partial \dot{x}}, \tag{57}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right)\frac{\partial \dot{z}}{\partial \dot{y}} - \frac{\partial T}{\partial y} - \frac{\partial T}{\partial z}\frac{\partial \dot{z}}{\partial \dot{y}} = \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial \dot{z}}{\partial \dot{y}},$$

где U определено из (38), а T из (36). Делая соответствующие выкладки в (57), получаем

$$\ddot{x} - \ddot{z}\frac{\dot{x}}{\sqrt{v^2 - \dot{x}^2 - \dot{y}^2}} = \frac{g\dot{x}}{\sqrt{v^2 - \dot{x}^2 - \dot{y}^2}}, \tag{58}$$

$$\ddot{y} - \ddot{z}\frac{\dot{y}}{\sqrt{v^2 - \dot{x}^2 - \dot{y}^2}} = \frac{g\dot{y}}{\sqrt{v^2 - \dot{x}^2 - \dot{y}^2}}.$$

Как следствия из (58) будем иметь

$$\frac{\ddot{x}}{\dot{x}} = \frac{\ddot{y}}{\dot{y}},$$

т. е. (42), откуда следует (43). Поставляя в (58), находим

$$\frac{v^2\ddot{x}}{\sqrt{v^2 - (C_1^2 + 1)\dot{x}^2}} = g\dot{x},$$

т. е. заново получаем (46), откуда следуют теже квадратуры.

4. Пример системы с нелинейными неголономными связями второго порядка.

Рассмотрим движение твердого тела вокруг неподвижной точки, для которого выполнено условие обобщенной прецесии Гриоли [14]:

$$p\dot{q} - q\dot{p} + r(p^2 + q^2) - \mu(p^2 + q^2)^{3/2} = 0, \tag{59}$$

здесь снова p, q, r компоненты угловой скорости ω, μ — постоянная. Отметим, что для обыкновенной прецессии, т. е. когда ось собственного вращения тела и вертикаль образуют постояний угол θ, условие (59) выполняется тождественно при $\mu = \text{ctg}\theta$. Но для обобщенной прецессии оно ведет к зависимости

$$f = \ddot{\psi}\dot{\theta}\sin\theta - \ddot{\theta}\dot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}^2\cos\theta \tag{60}$$

$$+\dot\psi^3\sin^2\theta\cos\theta-\mu(\dot\psi^2\sin^2\theta+\dot\theta^2)^{3/2}=0.$$

Кинетическая энергия тела, как известно, имеет вид

$$T=\frac{1}{2}(Ap^2+Bq^2+Cr^2), \tag{61}$$

где A, B, C — главные моменты инерции тела. Силовая функция приложенной силы тяжести будет

$$U=-mgz_G=-mg(\xi_0\sin\theta\sin\phi+\eta_0\sin\theta\cos\phi+\zeta_0\cos\theta), \tag{62}$$

где ξ_0, η_0, ζ_0 координаты центра масс G тела по отношению осей связанны с телом. Для рассматриваемой системы применим уравнения Раута (26) с углами Эйлера ϕ, ψ, θ в качестве независимых параметров, т.е.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right)-\frac{\partial T}{\partial\phi}=\frac{\partial U}{\partial\phi}+\lambda\frac{\partial f}{\partial\ddot\phi}, \tag{63}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\psi}\right)-\frac{\partial T}{\partial\psi}=\frac{\partial U}{\partial\psi}+\lambda\frac{\partial f}{\partial\ddot\psi},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial U}{\partial\theta}+\lambda\frac{\partial f}{\partial\ddot\theta}.$$

На основе (61) и кинематических формул Эйлера будем иметь

$$\frac{\partial T}{\partial\dot\phi}=r,\qquad \frac{\partial T}{\partial\phi}=(A-B)pq, \tag{64}$$

$$\frac{\partial T}{\partial\dot\psi}=Ap\sin\theta\sin\phi+Bq\sin\theta\cos\phi+Cr\cos\theta,$$

$$\frac{\partial T}{\partial\psi}=0,\qquad \frac{\partial T}{\partial\dot\theta}=Ap\cos\phi-Bq\sin\phi,$$

$$\frac{\partial T}{\partial\theta}=Ap\dot\psi\cos\theta\sin\phi+Bq\dot\psi\cos\theta\cos\phi-Cr\dot\psi\sin\theta.$$

Из (62) и (60) получаем

$$\frac{\partial U}{\partial\phi}=mg(\eta_0\sin\theta\sin\phi-\xi_0\sin\theta\cos\phi), \tag{65}$$

$$\frac{\partial U}{\partial\psi}=0,\qquad \frac{\partial U}{\partial\theta}=-mg(\xi_0\cos\theta\sin\phi+\eta_0\cos\theta\cos\phi-\zeta_0\sin\theta),$$

$$\frac{\partial f}{\partial\ddot\phi}=0,\qquad \frac{\partial f}{\partial\ddot\psi}=\dot\theta\sin\theta,\qquad \frac{\partial f}{\partial\ddot\theta}=-\dot\psi\sin\theta.$$

Подставляя (64) и (65) в (63), получаем

$$C\dot r-(A-B)pq=mg(\eta_0\sin\theta\sin\theta-\xi_0\sin\theta\cos\phi), \tag{66}$$

$$A\dot{p}\sin\theta\sin\phi + B\dot{q}\sin\theta\cos\phi + C\dot{r}\cos\theta + Ap(\cos\theta\sin\phi\dot{\theta}$$
$$+\sin\theta\cos\phi\dot{\phi}) + Bq(cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi\dot{\phi}) - Cr\sin\theta\dot{\theta} = \lambda\dot{\theta}\sin\theta,$$
$$A\dot{p}\cos\phi - B\dot{q}\sin\phi + Ap(-\sin\phi\dot{\phi} - \dot{\psi}\cos\theta\sin\phi)$$
$$+Bq(-\dot{\phi}\cos\phi - \dot{\psi}\cos\theta\cos\phi) + Cr\dot{\psi}\sin\theta$$
$$= mg(-\xi_0\cos\theta\sin\phi - \eta_0\cos\theta\cos\phi + \zeta_0\sin\theta) - \lambda\dot{\psi}\sin\theta.$$

Исключая из второго и третьего уравнений (66) получим как следствие интеграл энергии тела

(67) $$Ap^2 + Bq^2 + C^2 + mg(\xi_0\sin\theta\sin\phi + \eta_0\sin\theta\cos\phi + \zeta_0\cos\theta) = h_1.$$

Следовательно система уравнений сферического движения при наличии обобщенной прецессии Гриоли имеет вид

(68) $$C\dot{r} - (A-B)pq = mg(\eta_0\sin\theta\sin\phi - \xi_0\sin\theta\cos\phi),$$
$$Ap^2 + Bq^2 + Cr^2 + mg(\xi_0\sin\theta\sin\phi + \eta_0\sin\theta\cos\phi + \zeta_0\cos\theta) = h_1,$$
$$\ddot{\psi}\dot{\theta}\sin\theta - \ddot{\theta}\dot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}^2\cos\theta$$
$$+\dot{\psi}^3\sin^2\theta\cos\theta - \mu(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2)^{\frac{3}{2}} = 0.$$

Как частный случай для уравнения (68) рассмотрим движение однородного шара с неподвижным геометрическим центром. В этом случае первые два уравнения (68) получаем в виде

(69) $$r = C_1,\ p^2 + q^2 = \dot{\psi}^2\sin^2\theta + \dot{\theta}^2 = h,$$

где h и C_1 постоянные интегрирования. Из (69) получаем

(70) $$\dot{\psi} = \frac{\sqrt{h-\dot{\theta}^2}}{\sin\theta}.$$

Подставляя (71) в последнее уравнение (68) и делая известные преобразования, получаем

(71) $$\ddot{\theta}\sin\theta - \cos\theta(h-\dot{\theta}^2) + \mu\sqrt{h}\sqrt{h-\dot{\theta}^2}\sin\theta = 0.$$

После замены

(72) $$u = \sqrt{h-\dot{\theta}^2}$$

запишем уравнение (71) в виде

(73) $$\frac{du}{d\theta} + \mathrm{ctg}\theta u = \mu\sqrt{h}.$$

Общее решение (73) имеет вид

(74) $$u = \frac{C_2 - \mu\sqrt{h}\cos\theta}{\sin\theta},$$

где C_2 постоянная интегрирования. На основе (72) и (74) получаем

(75) $$t + C_3 = \int \frac{\sin\theta d\theta}{\sqrt{h\sin^2\theta - (C_2 - \mu\sqrt{h}\cos\theta)^2}}.$$

Очевидно (75) можно довести до элементарных квадратур.

ЛИТЕРАТУРА

1. A p p e l, P. – C. R., t. 152, No 19, 1911, 1197–1199.
2. A p p e l, P. – Rendiconti del Circolo Mat. di Palermo, t. 33, 1912, 259–267.
3. A p p e l, P. – Memorial des sciences math., t. 1, 1925.
4. D e l a s s u s, E. – C. R., t. 152, No 25, 1911, 1739–1743.
5. D e l a s s u s, E. – C. R. , t. 154, No 16, 1912, 964–968.
6. D e l a s s u s, E. Annalies de l'Ecole Normale Super., t. 30, 1913, 483–520.
7. М е щ е р с к и й, И. В., Сообщения мат. общества при Харьковском университете, 1887.
8. B i l i m o v i t c h, A. Publication Math. de l'Université de Belgrade. t. 3, 1936.
9. C a s t o l d i, L. Rendiconti Jst. Lombardo di Scie Lettere, ser. 3, v. 79, 1946.
10. П о г а с о в, Г. С. Сб. статей ВЗПИ, в. 12, 1955, 83–90.
11. M e l i s, A. Rend. Semin. Fac. sci. Univ. Cagliari, v. 25, f. 3–4, 1956, 143–153.
12. Н о в о с е л о в, В. С. – Вестник ЛГУ, №19, вып. 4, 1957, 106–111.
13. Д о б р о н р а в о в, В. В. Сборник статей МВТУ, №104, 1961.
14. Ц е н о в, И. – Год. Соф. унив., т. XV–XVI, 1924.
15. Ц е н о в, И. – ЦАН СССР, 89, №1, 1953, 21–24.
16. Д о л а п ч и е в, Б. – Известия Болгарской АН, Математика, №2, 1960.
17. Д о б р о н р а в о в, В. В. Основы механики неголомных систем, Высшая школа, 1970.
18. Н о в о с е л о в, В. С. – Учение записки ЛГУ, №217, вып. 31, 1957.
19. P r z e b o r s k i. A Die allgemeinst den Gleichungen der klassischen Dynamik.– Math. Zeitschrift, 33, II, 2, 1932.
20. К о ч и н, И. Е. Об освобождаемости механических систем, ПММ, т. 10, 1946.

Received 21. V. 1990