ГОДИШНИК

НА

СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

КНИГА 1 — МАТЕМАТИКА

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ИНТЕГРАЛНИ ПРЕДСТАВЯНИЯ НА ХОЛОМОРФНИ ФУНКЦИИ ПОСРЕДСТВОМ ФУНКЦИИТЕ НА ВЕБЕР–ЕРМИТ

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Петр Русев. **ИНТЕГРАЛЬНЫЕ ПРЕДСТАВЛЕНИЯ ГОЛОМОРФНЫХ ФУНКЦИЙ С ПОМОЩЬЮ ФУНКЦИЙ ВЕБЕРА-ЭРМИТА**

Пусть $D_\nu(z)$ обозначает функция Вебера-Эрмита с индексом ν. Рассматриваются интегральные преобразования

$$(*)\qquad A(z) = \int_0^\infty a(t)H(z,t)\,dt,$$

где $H(z,t) = 2^{t/2}\exp(z^2/2)D_t(z\sqrt{2})$. Дано необходимое условие типа „роста" для представления функций $A(z)$, голоморфной в полосе $|\mathrm{Im}\, z| < \tau_0$ $(0 < \tau_0 \leqq +\infty)$, в виде $(*)$. Выяснена связь с классическим преобразованием Фурие и как следствие доказана единственность интегрального представления вида $(*)$.

Peter Rusev. **INTEGRAL REPRESENTATION OF HOLOMORPHIC FUNCTIONS BY MEANS OF WEBER-HERMITE FUNCTIONS**

Integral representations of the kind

$$(*)\qquad A(z) = \int_0^\infty a(t)H(z,t)\,dt$$

are considered where $H(z,t) = 2^{t/2}\exp(z^2/2)D_t(z\sqrt{2})$. A necessary "growth" condition is given for a function $A(z)$, holomorphic in the stripe $|\mathrm{Im}\, z| < \tau_0$ $(0 < \tau_0 \leqq +\infty)$, to be represented in the form $(*)$. The connection with the classical Fourier transformation is clarified and as a consequence the uniqueness property of the transformation $(*)$ is proved.

1. Всяко (аналитично) решение на диференциалното уравнение

$$(1.1)\qquad y'' + (\nu + 1/2 - z^2/4)y = 0, \quad \nu \in \mathbf{C},$$

се нарича функция на параболичния цилиндър или още функция на Вебер–Ермит ([1], II, 8.2). В следващото изложение последното наименование се употребява за частното решение на уравнението (1.1), което има вида ([1], II, 8.2, (4))

$$(1.2)\qquad \begin{aligned} D_\nu(z) =& 2^{\nu/2}\exp(-z^2/4)\frac{\Gamma(1/2)}{\Gamma((1-\nu)/2)}\Phi(-\nu/2,1/2;z^2/2) \\ &+(z/\sqrt{2})\frac{\Gamma(-1/2)}{\Gamma(-\nu/2)}\Phi((1-\nu)/2,3/2;z^2/2), \end{aligned}$$

като с $\Phi(a,c;z)$ е означена изродената хипергеометрична функция на Кумер ([1], I, 6.1).

Ако ν е фиксирано, $D_\nu(z)$ като функция на z е холоморфна в цялата комплексна равнина, т. е. е цяла функция на z. В частност, ако $\nu = n$ е цяло неотрицателно число, то

$$(1.3)\qquad \exp(z^2/4)D_n(z) = 2^{-n/2}H_n(z/\sqrt{2}),$$

където H_n е n-тият полином на Ермит ([1], II, 8.2, (9)). От горното съотношение следва, че

$$(1.4)\qquad H_n(z) = 2^{n/2}\exp(z^2/2)D_n(z\sqrt{2}),\quad n = 0, 1, 2, \dots$$

При фиксирани z и $c \neq 0, -1, -2, \dots$, $\Phi(a,c;z)$ като функция на a е холоморфна в цялата комплексна равнина. От (1.2) следва тогава, че $D_\nu(z)$ при фиксирано z е цяла функция на ν.

2. Като имаме предвид (1.4), дефинираме за $z \in \mathbb{C}$ и $t \in [0,+\infty)$

$$(2.1)\qquad H(z,t) = 2^{t/2}\exp(z^2/2)D_t(z\sqrt{2}).$$

От интегралното представяне ([1], II, 8.3, (4))

$$(2.2)\qquad D_\nu(z) = \sqrt{2/\pi}\exp(z^2/4)\int_0^\infty \exp(-u^2/2)u^\nu\cos(zu-\nu\pi/2)\,du,$$

което е валидно при $\operatorname{Re}\nu > -1$, получаваме, че

$$(2.3)\qquad \sqrt{\pi/2}\,2^{-t/2}\exp(-z^2)H(z,t) = \int_0^\infty \exp(-u^2/2)u^t\cos(zu\sqrt{2}-\pi t/2)\,du.$$

Лема 2.1. *Каквото и да е* $0 \leq \tau < +\infty$, *за всяко* $z = x+iy$ *с* $|y| \leq \tau$ *и за всяко* $t \in [0,+\infty)$ *е изпълнено неравенство от вида*

$$(2.4)\qquad \exp(-x^2)|H(z,t)| \leq \operatorname{const}(\tau)(2t/e)^{t/2}\exp(\tau\sqrt{2t}).$$

Доказателство. От (2.3) следва, че щом $|y| \leq \tau$,

$$\exp(-x^2)|H(z,t)| \leq \operatorname{const}(\tau)2^{t/2}\int_0^\infty \exp(-u^2/2+\tau u\sqrt{2})u^t\,du.$$

Като имаме предвид интегралното представяне ([1], II, 8.3, (3))

$$D_\nu(z) = \frac{\exp(-z^2/4)}{\Gamma(-\nu)} \int_0^\infty \exp(-u^2/2 - zu)u^{-\nu-1}\, du, \tag{2.5}$$

което е валидно при $\operatorname{Re}\nu < 0$, намираме, че

$$\exp(-x^2)|H(z,t)| \leq \mathrm{const}(\tau)2^{t/2}\Gamma(t+1)D_{-t-1}(-\tau\sqrt{2}). \tag{2.6}$$

За функцията $D_\nu(z)$ е в сила следното представяне ([1], II, 8.4, (5)):

$$D_\nu(z) = \frac{1}{\sqrt{2}} \exp\left\{(\nu/2)\log(-\nu) - \nu/2 - \sqrt{-\nu}\, z\right\}\{1 + \delta_\nu(z)\}, \tag{2.7}$$

$$\nu \in \mathbf{C} \setminus [0, +\infty),$$

където $\delta_\nu(z)$ е цяла функция на z, $\delta_\nu(z) = O\left(|\nu|^{-1/2}\right)$ при $|\nu| \to +\infty$ и $|\arg(-\nu)| \leq \pi/2$ равномерно върху всяко ограничено подмножество на $\mathbf{C}$.

От (2.7) следва тогава, че при $t \to +\infty$ е в сила

$$\begin{aligned} &D_{-t-1}(-\tau\sqrt{2}) \\ &= \frac{1}{\sqrt{2}} \exp\left\{-\frac{t+1}{2}\log(t+1) + \frac{t+1}{2} + \sqrt{t+1}\,\tau\sqrt{2}\right\}\left\{1 + O(t^{-1/2})\right\}. \end{aligned} \tag{2.8}$$

Формулата на Стирлинг дава, че

$$\Gamma(t+1) = \sqrt{2\pi}\exp\{(t+1/2)\log(t+1) - t - 1\}\{1 + O(t^{-1})\}, \quad t \to +\infty. \tag{2.9}$$

Тъй като $\sqrt{2t+2} - \sqrt{2t} < 1$ и $\log(t+1) - \log t < t^{-1}$ за $t > 0$, от (2.8) и (2.9) следва, че съществува такова $\lambda = \lambda(\tau) > 0$, че за $t \in [\lambda, +\infty)$ е изпълнено неравенство от вида

$$\Gamma(t+1)D_{-t-1}(-\tau\sqrt{2}) \leq \mathrm{const}(\tau)\exp\{(t/2)\log t - t/2 + \tau\sqrt{2t}\},$$

или все едно

$$\Gamma(t+1)D_{-t-1}(-\tau\sqrt{2}) \leq \mathrm{const}(\tau)(t/e)^{t/2}\exp(\tau\sqrt{2t}). \tag{2.10}$$

Функцията $(t/e)^{-t/2}\exp(-\tau\sqrt{2t})\Gamma(t+1)D_{-t-1}(-\tau\sqrt{2})$ е непрекъсната (като функция на t) в интервала $[0, +\infty)$. Следователно в интервала $[0, \lambda]$ също е изпълнено неравенство от вида (2.10), т. е. можем да считаме, че такова неравенство е валидно за всяко $t \in [0, +\infty)$. Тогава от (2.6) и (2.10) следва (2.4).

Да означим с $E(\tau_0)$ $(0 < \tau_0 \leq +\infty)$ множеството на комплексните функции $a(t)$ $(0 \leq t < +\infty)$, които са локално интегрируеми и за които е изпълнено

$$\limsup_{t \to +\infty} (2t)^{-1/2} \log\left|(2t/e)^{t/2}a(t)\right| \leq -\tau_0. \tag{2.11}$$

Една функция $a \in E(\tau_0)$ тогава и само тогава, когато каквото и да е $-\infty < \tau < \tau_0$, съществува такова $r = r(\tau) > 0$, че за $t > r$ е изпълнено неравенството

$$|a(t)| \leq (2t/e)^{-t/2}\exp(-\tau\sqrt{2t}). \tag{2.12}$$

Теорема 2.1. *Ако* $0 < \tau_0 \leqq +\infty$ *и функцията* $a \in E(\tau_0)$*, интегралът*

$$A(z) = \int_0^\infty a(t)H(z,t)\,dt \tag{2.13}$$

е абсолютно равномерно сходящ върху всяко компактно подмножество на ивицата $S(\tau_0) : |\operatorname{Im} z| < \tau_0$ *и следователно дефинира холоморфна функция в нея. Освен това, каквото и да е* $0 \leqq \tau < \tau_0$*, изпълнено е неравенство от вида*

$$|A(z)| \leqq \operatorname{const}(\tau)\exp(x^2) \tag{2.14}$$

за всяко $z \in \overline{S(\tau)} : |\operatorname{Im} z| \leqq \tau$.

Доказателство. Нека K е компактно подмножество на $S(\tau_0)$. Тогава съществуват такива $0 \leqq \tau < \tau_0$ и $0 < R < +\infty$, че $|x| \leqq R$ и $|y| \leqq \tau_0$ за всяко $z = x + iy \in K$.

Да допуснем, че $\tau_0 < +\infty$ и нека $\delta = (\tau_0 - \tau)/2$. За $t \geqq r(\tau + \delta) = r((\tau + \tau_0)/2)$ е изпълнено неравенството

$$|a(t)| \leqq (2t/e)^{-t/2}\exp\{-(\tau+\delta)\sqrt{2t}\}. \tag{2.15}$$

Тогава съгласно лема 2.1 за $z \in K$ и $t \geqq r(\tau+\delta)$ е изпълнено неравенство от вида

$$|a(t)H(z,t)| \leqq \operatorname{const}(\tau, R)\exp(-\delta\sqrt{2t}\,)$$

и следователно интегралът

$$\int_{r(\tau+\delta)}^\infty |a(t)H(z,t)|\,dt$$

се мажорира върху K от интеграла

$$\int_{r(\tau+\delta)}^\infty \exp(-\delta\sqrt{2t})\,dt.$$

Това означава, че интегралът в (2.13) е абсолютно равномерно сходящ върху компактното множество $K \subset S(\tau_0)$.

Горните изводи бяха направени при предположението, че $\tau_0 < +\infty$. Ако $\tau_0 = +\infty$, избираме $\delta = 1$.

От лема 2.1 следва, че щом $|y| \leqq \tau$, то

$$|A(z)| \leqq \int_0^\infty |a(t)H(z,t)|\,dt \leqq \operatorname{const}(\tau)\exp(x^2)\int_0^\infty |a(t)|(2t/e)^{t/2}\exp(\tau\sqrt{2t}\,)\,dt.$$

Но интегралът от дясната страна на горното неравенство е сходящ, което следва от (2.15) и локалната интегрируемост на $a(t)$.

Забележка. От теорема 2.1 или по-точно от неравенството (2.14) следва, че не всяка функция, холоморфна в ивица от вида $S(\tau_0)$, се представя в нея във вида (2.13) с функция от класа $E(\tau_0)$.

3. Представянето (2.3), по-точно дясната му страна, наподобява преобразование на Фурие. Естествено е да се очаква, че за функция, която е холоморфна в ивицата $S(\tau_0)$ $(0 < \tau_0 \leqq +\infty)$ и има в нея интегрално представяне от вида (2.13), е налице и подходящо интегрално представяне на Фурие.

За да се убедим в това, дефинираме класа $F(\tau_0)$ от функции f, които са преобразования на Лаплас на функциите от класа $E(\tau_0)$, т. е.

$$f(\zeta) = \int_0^\infty a(t)\exp(\zeta t)\,dt, \quad a \in E(\tau_0). \tag{3.1}$$

Лема 3.1. *Нека* $0 < \tau_0 \leqq +\infty$ *и* $a \in E(\tau_0)$. *Интегралът в* (3.1) *дефинира цяла функция и освен това, каквото и да е* $0 < \tau < \tau_0$, *изпълнено е неравенство от вида* $(\zeta = \xi + i\eta)$

$$|f(\zeta)| \leqq \operatorname{const}(\tau)\exp(\exp(2\xi)/4 - \tau\exp\xi). \tag{3.2}$$

Доказателството на тази лема е напълно аналогично на това на лема 3.2 от [2, с. 110].

Теорема 3.1. *Ако* $a(t) \in E(\tau_0)$ $(0 < \tau_0 \leqq +\infty)$, *за функцията* $A(z)$, *дефинирана чрез* (2.13) *в ивицата* $S(\tau_0)$, *е в сила представянето*

$$2\sqrt{\pi}\exp(-z^2)A(z) \tag{3.3}$$

$$= \int_0^\infty \exp(-u^2/4)\{f(\ln u + i\pi/2)\exp(-izu) + f(\ln u - i\pi/2)\exp(izu)\}\,du,$$

където f *е функция от* $F(\tau_0)$.

Доказателство. Да обърнем внимание, че интегралът в (3.3) е абсолютно равномерно сходящ върху всяка (затворена) ивица $\overline{S(\tau)}$ с $0 \leqq \tau < \tau_0$. Наистина, ако $\tau + \delta < \tau_0$, от (3.2) следва, че за всяко $u \in (0, +\infty)$ е изпълнено неравенство от вида

$$|f(\ln u \pm i\pi/2)| \leqq \operatorname{const}(\tau + \delta)\exp(u^2/4 - (\tau + \delta)u)$$

и следователно за $z = x + iy$ с $|y| \leqq \tau$ и $u \in (0, +\infty)$ е изпълнено

$$\exp(-u^2/4)|f(\ln u \pm i\pi/2)\exp(\pm izu)| \leqq \operatorname{const}(\tau + \delta)\exp(-\delta u).$$

Да заместим в дясната страна на (3.3) функцията f с (3.1). Получаваме двукратния интеграл

$$2\int_0^\infty \exp(-u^2/4)\,du \int_0^\infty a(t)u^t\cos(zu - \pi t/2)\,dt. \tag{3.4}$$

Нека $0 \leqq \tau < \tau_0$ е фиксирано и δ е така избрано, че $\tau + \delta < \tau_0$ (ако $\tau_0 < +\infty$, можем да считаме, че $\delta = (\tau_0 - \tau)/2$, а ако $\tau_0 = +\infty$, избираме $\delta = 1$). Тогава за $t \geqq r = r(\tau + \delta)$ е изпълнено неравенството (2.15). Следователно, за да се убедим, че двукратният интеграл в (3.4) е абсолютно сходящ при $|\operatorname{Im} z| \leqq \tau$, достатъчно е да покажем, че е сходящ двукратният интеграл

$$\int_0^\infty \exp(-u^2/4 + \tau u)\,du \int_{r(\tau+\delta)}^\infty \tilde{a}(t)u^t\,dt,$$

където
$$\tilde a(t) = (2t/e)^{-t/2}\exp(-(\tau+\delta)\sqrt{2t}),$$
или, което е едно и също, че е сходящ интегралът

$$\int_0^\infty \exp(-u^2/4+\tau u)\,du \int_0^\infty \tilde a(t) u^t\,dt. \tag{3.5}$$

Дефинираме функцията $\tilde f(\zeta)$ чрез

$$\tilde f(\zeta) = \int_0^\infty \tilde a(t)\exp(\zeta t)\,dt.$$

Понеже $\tilde a \in E(\tau+\delta)$, то $\tilde f \in F(\tau+\delta)$ и е изпълнено неравенство от вида ($\zeta = \xi + i\eta$)

$$|\tilde f(\zeta)| \le \mathrm{const}(\tau)\exp\{\exp(2\xi)/4 - (\tau+\delta/2)\exp\xi\}.$$

От него следва, че за $u \in (0,+\infty)$ е изпълнено

$$\int_0^\infty \tilde a(t) u^t\,du = \tilde f(\ln u) \le \mathrm{const}(\delta)\exp(u^2/4 - (\tau+\delta/2)u).$$

Но тогава получаваме, че

$$\int_0^\infty \exp(-u^2/4+\tau u)\,du \int_0^\infty \tilde a(t) u^t\,dt \le \mathrm{const}(\delta)\int_0^\infty \exp(-(\delta/2)u)\,du < +\infty,$$

т. е. двукратният интеграл (3.5) е наистина сходящ.

Като разменим реда на интегриранията в (3.4), получаваме двукратния интеграл

$$2\int_0^\infty a(t)\,dt \int_0^\infty \exp(-u^2/4)u^t\cos(zu-\pi t/2)\,du$$
$$= 2^{3/2}\int_0^\infty 2^{t/2}a(t)\,dt \int_0^\infty \exp(-u^2/2)u^t\cos(zu\sqrt{2}-\pi t/2)\,du.$$

Но съгласно (2.3) и (2.13) това е точно лявата страна на (3.3).

Да дефинираме функциите p и q посредством равенствата

$$p(\zeta) = \frac{1}{2}\left(f(\zeta+i\pi/2)+f(\zeta-i\pi/2)\right) \tag{3.6}$$

и

$$q(\zeta) = \frac{1}{2i}\left(f(\zeta+i\pi/2)-f(\zeta-i\pi/2)\right). \tag{3.7}$$

Тогава от (3.3) следва, че за $z \in S(\tau_0)$ е в сила представянето

$$(3.8) \qquad \sqrt{\pi}\exp(-z^2)A(z) = \int_0^\infty \exp(-u^2/4)\{p(\ln u)\cos zu + q(\ln u)\sin zu\}\, du.$$

Също така, като имаме предвид (3.1), (3.6) и (3.7), намираме, че

$$(3.9) \qquad p(\zeta) = \int_0^\infty a(t)\cos(\pi t/2)\exp(\zeta t)\, dt$$

и

$$(3.10) \qquad q(\zeta) = \int_0^\infty a(t)\sin(\pi t/2)\exp(\zeta t)\, dt.$$

От представянето (3.8) може да бъде направен важен извод, който изразява свойството единственост на интегралното представяне (2.13). По-точно в сила е следното твърдение:

Теорема 3.2. *Ако $a(t) \in E(\tau_0)$ $(0 < \tau_0 \leq +\infty)$ и функцията A, дефинирана чрез (2.13), е тъждествено нула в $S(\tau_0)$, то $a \sim 0$ в интервала $[0, +\infty)$, т. е. $a(t) = 0$ почти навсякъде в този интервал.*

Доказателство. За всяко $z \in S(\tau_0)$ наред с равенството

$$\int_0^\infty \exp(-u^2/4)\{p(\ln u)\cos zu + q(\ln u)\sin zu\}\, du = 0$$

е изпълнено и

$$\int_0^\infty \exp(-u^2/4)\{p(\ln u)\cos zu - q(\ln u)\sin zu\}\, du = 0.$$

От горните две равенства следва в частност, че за всяко $x \in (-\infty, +\infty)$ е в сила равенството

$$(3.11) \qquad \int_0^\infty \exp(-u^2/4)p(\ln u)\cos xu\, du = 0.$$

Ако функцията $a(t) \in E(\tau_0)$, функциите $a(t)\cos(\pi t/2)$ и $a(t)\sin(\pi t/2)$ са също от $E(\tau_0)$, и следователно функциите p и q са от класа $F(\tau_0)$. Но тогава от лема 3.1 следва, че всяка от функциите $\exp(-u^2/4)p(\ln u)$ и $\exp(-u^2/4)q(\ln u)$ е от пространството $L(0, +\infty)$. Съгласно теоремата за единственост на cos–преобразованието на Фурие в това пространство от (3.11) следва, че $p(\ln u) = 0$ за всяко $u \in (0, +\infty)$, т. е. $p(\xi) = 0$ за $\xi \in (-\infty, +\infty)$. Теоремата за идентичност на холоморфните функции дава, че $p(\zeta) \equiv 0$. Но съгласно свойството единственост на преобразованието на Лаплас от (3.9) следва, че $a \sim 0$ в интервала $(0, +\infty)$.

Друг интересен извод от представянето (3.8) е следното твърдение:

Теорема 3.3. *Четна (нечетна) функция A, която е холоморфна в ивицата* $S(\tau_0)$ $(0 < \tau_0 \leqq +\infty)$ *и се представя в нея във вида* (2.13) *с функция* $a(t) \in E(\tau_0)$, *е тъждествено нула.*

Доказателство. Ако функцията A е четна, от (3.8) следва, че

$$\int_0^\infty \exp(-u^2/4)q(\ln u)\sin zu\,du = 0$$

за $z \in S(\tau_0)$. Но тогава както в доказателството на теорема 3.2 заключаваме, че $q(\ln u) = 0$ за всяко $u \in (0,+\infty)$ и от (3.10) следва, че $a \sim 0$ в интервала $(0,+\infty)$. Ако A е нечетна, идваме пак до равенството (3.11).

Да означим с $F^*(\tau_0)$ $(0 < \tau_0 \leqq +\infty)$ класа на целите функции $f(\zeta)$ със следното свойство: за всяко $0 < \tau < \tau_0$ съществува такова $\delta = \delta(\tau) > 0$, че за всяко $\zeta = \xi + i\eta$ е изпълнено неравенство от вида

$$|f(\zeta)| \leqq \mathrm{const}(\tau)(1+|\zeta|)^{-1-\delta(\tau)}\exp(\exp(2\xi)/4 - \tau\exp\xi). \tag{3.12}$$

Лема 3.2. *Ако цялата функция* $f(\zeta)$ *принадлежи на класа* $F^*(\tau_0)$ $(0 < \tau_0 \leqq +\infty)$, *тя се представя във вида* (3.1) *с функция* $a(t)$ *от класа* $E(\tau_0)$.

Доказателство. За $0 \leqq t < +\infty$ дефинираме

$$a(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} f(\zeta)\exp(-t\zeta)\,d\zeta. \tag{3.13}$$

От (3.12) следва, че интегралът в (3.13) е абсолютно сходящ върху всяка права линия $\mathrm{Re}\,\zeta = \sigma$ $(-\infty < \sigma < +\infty)$ и стойността му не зависи от σ. Освен това във всяка полуравнина $\mathrm{Re}\,\zeta \leqq \sigma$ е изпълнено неравенство от вида

$$|f(\zeta)| \leqq \mathrm{const}(\tau,\sigma)(1+|\zeta|)^{-1-\delta(\tau)}.$$

Съгласно формулата за обръщане на преобразованието на Лаплас в комплексна област за всяко $\zeta \in \mathbb{C}$ е налице (3.1).

Остава да се убедим, че дефинираната с (3.13) функция $a(t)$ е от класа $E(\tau_0)$. За целта заместваме σ с $(\ln 2t)/2$ $(0 < t < +\infty)$ и като вземем предвид (3.12), намираме, че

$$|a(t)| \leqq \mathrm{const}(\tau)(2t/e)^{-t/2}\exp(-\tau\sqrt{2t})\int_{-\infty}^{\infty}\frac{d\eta}{(1+|\eta|)^{1+\delta(\tau)}}$$

и следователно $a \in E(\tau_0)$.

Забележка. От лема 3.1 следва, че класът $F^*(\tau_0)$ се съдържа в класа $F(\tau_0)$, като последният е по-широк от $F^*(\tau_0)$. Наистина нека $0 < \tau_0 < +\infty$ и да дефинираме

$$f_0(\zeta) = \int_0^\infty a_0(t)\exp(\zeta t)\,dt$$

с $a_0(t) = (2t/e)^{-t/2}\exp(-\tau_0\sqrt{2t})$. Функцията $a_0 \in E(\tau_0)$ и следователно

$f_0 \in F(\tau_0)$. Понеже $a_0'(t) \in L(0,+\infty)$, от равенството

$$f_0(\zeta) = \frac{1}{\zeta}\left\{-1 - \int_0^\infty a_0'(t)\exp(\zeta t)\,dt\right\}$$

и лемата на Риман следва, че

$$\lim_{|\eta|\to+\infty} i\eta f(i\eta) = -1,$$

което показва, че функцията f_0 не принадлежи на класа $F^*(\tau_0)$.

Сега да формулираме твърдение, което може да се счита за обратно на теорема 3.1, а именно:

Теорема 3.4. *Ако цялата функция $f \in F^*(\tau_0)$ $(0 < \tau_0 \leq +\infty)$, комплексната функция A, дефинирана с (3.3), е холоморфна в ивицата $S(\tau_0)$ и се представя в нея във вида (2.13) с функция $a \in E(\tau_0)$.*

4. Дефинираме функциите $G^{\pm}(z,t)$ за $z \in \mathbb{C}$ и $t \in [0,+\infty)$ чрез

(4.1) $$G^{\pm}(z,t) = 2^{t/2}\Gamma(t+1)D_{-t-1}(\mp iz\sqrt{2}).$$

Лема 4.1. *Каквото и да е $0 \leq \tau < +\infty$, във всяка от полуравнините $\pm\mathrm{Im}\, z \geq \tau$ е изпълнено неравенство от вида*

(4.2) $$|G^{\pm}(z,t)| \leq \mathrm{const}(\tau)(2t/e)^{t/2}\exp(z^2/2 - \tau\sqrt{2t}).$$

Твърдението на лема 4.1, т. е. валидността на горното неравенство, се доказва както и тази на неравенството (2.4).

Теорема 4.1. *Ако $0 \leq \tau_0 < +\infty$ и функцията $b(t) \in E(-\tau_0)$, интегралът*

(4.3) $$B^{\pm}(z) = \int_0^\infty b(t)G^{\pm}(z,t)\,dt$$

е абсолютно равномерно сходящ върху всяка полуравнина $\pm\mathrm{Im}\, z \geq \tau > \tau_0$ и следователно дефинира холоморфна функция на z при $\pm\mathrm{Im}\, z > \tau_0$. Освен това във всяка полуравнина $\pm\mathrm{Im}\, z \geq \tau > \tau_0$ е изпълнено неравенство от вида

(4.4) $$|B^{\pm}(z)| \leq \mathrm{const}(\tau)\exp(z^2/2).$$

Горното твърдение е следствие от лема 4.1 и от факта, че $b(t) \in E(-\tau_0)$ означава, че за произволно $\tau \in (\tau_0, +\infty)$ е изпълнено неравенство от вида

(4.5) $$|b(t)| \leq (2t/e)^{-t/2}\exp(\tau\sqrt{2t}), \quad t \geq r = r(\tau) > 0.$$

Лема 4.2. *За функциите $G^{\pm}(z,t)$ е валидно интегралното представяне*

(4.6) $$\sqrt{2}\exp(-z^2/2)G^{\pm}(z,t) = \int_0^\infty \exp(-u^2/4)u^t\exp(\pm izu)\,du.$$

Горните равенства са непосредствено следствие от дефиницията на функциите $G^{\pm}(z,t)$ чрез (4.1) и интегралното представяне (2.5).

Теорема 4.2. *Ако функцията* $b \in E(-\tau_0)$ $(0 \leqq \tau_0 < +\infty)$, *за функциите* $B^{\pm}(z)$, *дефинирани чрез* (4.3) *в полуравнините* $\pm\operatorname{Im} z > \tau_0$, *е в сила представянето*

$$\sqrt{2}\exp(-z^2/2)B^{\pm}(z) = \int_0^{\infty} \exp(-u^2/4)g(\ln u)\exp(\pm izu)\,du, \tag{4.7}$$

където g *е функция от* $F(-\tau_0)$.

Доказателство. Дефинираме функцията $g(\zeta)$ чрез

$$g(\zeta) = \int_0^{\infty} b(t)\exp(\zeta t)\,dt. \tag{4.8}$$

За такава функция е в сила твърдение, аналогично на лема 3.1, а именно: g е цяла функция и освен това каквото и да е $\tau_0 < \tau < +\infty$, в сила е неравенство от вида $(\zeta = \xi + i\eta \in \mathbb{C})$

$$|g(\zeta)| \leqq \operatorname{const}(\tau)\exp(\exp(2\xi)/4 + \tau\exp\xi). \tag{4.9}$$

От наличието на такова неравенство за всяко $\tau_0 < \tau < +\infty$ следва, че интегралът отдясно на (4.7) е абсолютно равномерно сходящ върху всяка полуравнина $\pm\operatorname{Im} z \geqq \tau > \tau_0$.

Както в доказателството на теорема 3.1 се убеждаваме, че абсолютно сходящ върху такава полуравнина е и всеки от двукратните интеграли, който се получава, като заместим g в (4.7) с (4.8), а именно:

$$\int_0^{\infty} \exp(-u^2/4 \pm izu)\,du \int_0^{\infty} b(t)u^t\,dt. \tag{4.10}$$

След размяната на реда на интегриранията в (4.10), дясната страна на (4.7) става

$$\int_0^{\infty} b(t)\,dt \int_0^{\infty} \exp(-u^2/4)u^t\exp(\pm izu)\,du.$$

Но съгласно интегралното представяне (4.6) това е точно лявата страна на (4.7).

Следствие. *Ако* $b(t) \in E(-\tau_0)$ $(0 \leqq \tau_0 < +\infty)$ *и* $B^{\pm}(z) \equiv 0$ *в някоя от полуравнините* $\pm\operatorname{Im} z > \tau_0$, *то* $b \sim 0$ *в* $[0, +\infty)$.

Теорема 4.3. *Ако цялата функция* $g(\zeta) \in F^*(-\tau_0)$ $(0 \leqq \tau_0 < +\infty)$, *то комплексните функции* $B^{\pm}(z)$, *дефинирани с* (4.7), *са холоморфни във всяка от полуравнините* $\pm\operatorname{Im} z > \tau_0$ *и се представят в тях във вида* (4.3) *с функция* $b(t) \in E(-\tau_0)$.

Доказателството на горното твърдение е аналогично на това на теорема 3.4 и се опира на факта, че щом $g \in F^*(-\tau_0)$, функцията $b(t)$, дефинирана чрез

$$b(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} g(\zeta)\exp(-t\zeta)\,d\zeta,$$

принадлежи на класа $E(-\tau_0)$.

5. При фиксирано $0 < \tau_0 < +\infty$ дефинираме функциите

$$G^{(1)}(\tau_0; z, t) = \frac{1}{2}\left\{G^+(z + i\tau_0, t) + G^-(z - i\tau_0, t)\right\} \tag{5.1}$$

и

$$G^{(2)}(\tau_0; z, t) = \frac{1}{2i}\left\{G^+(z + i\tau_0, t) - G^-(z - i\tau_0, t)\right\}. \tag{5.2}$$

От лема 4.1 и по-точно от неравенствата (4.2) следва, че каквото и да е $0 \leqq \tau \leqq \tau_0$, в ивицата $\overline{S(\tau)}$: $|\mathrm{Im}\, z| \leqq \tau$ са изпълнени неравенства от вида

$$|G^{(k)}(\tau_0; z, t)| \leq \mathrm{const}(\tau)(2t/e)^{t/2} \exp\{x^2/2 - (\tau_0 - \tau)\sqrt{2t}\}, \quad k = 1, 2. \tag{5.3}$$

С помощта на горните неравенства се доказва, че ако функциите $b_k(t) \in E(0)$ $(k = 1, 2)$, интегралът

$$B(z) = \int_0^\infty \left\{b_1(t) G^{(1)}(\tau_0; z, t) + b_2(t) G^{(2)}(\tau_0; z, t)\right\} dt \tag{5.4}$$

е абсолютно равномерно сходящ във всяка (затворена) ивица $\overline{S(\tau)}$ с $0 \leq \tau < \tau_0$ и освен това във всяка такава ивица за холоморфната функция, която той дефинира, е изпълнено неравенство от вида

$$|B(z)| \leq \mathrm{const}(\tau) \exp(x^2/2). \tag{5.5}$$

За функциите (5.1) и (5.2) са валидни съответно интегралните представяния

$$\sqrt{2} \exp(-z^2/2) G^{(1)}(\tau_0; z, t) = \int_0^\infty \exp(-u^2/4 - \tau_0 u) u^t \cos zu\, du \tag{5.6}$$

и

$$\sqrt{2} \exp(-z^2/2) G^{(2)}(\tau_0; z, t) = \int_0^\infty \exp(-u^2/4 - \tau_0 u) u^t \sin zu\, du, \tag{5.7}$$

които са непосредствени следствия от дефиницията им и от интегралното представяне (4.6).

Теорема 5.1. *Ако функциите $b_k(t) \in E(0)$ $(k = 1, 2)$, за функцията $B(z)$, дефинирана чрез* (5.4) *в ивицата $S(\tau_0)$, е в сила представянето*

$$\sqrt{2} \exp(-z^2/2) B(z) = \int_0^\infty \exp(-u^2/4 - \tau_0 u) \left\{g_1(\ln u) \cos zu + g_2(\ln u) \sin zu\right\} du \tag{5.8}$$

с функции $g_k(\zeta) \in F(0)$ $(k = 1, 2)$.

Валидно е и твърдение, аналогично на теорема 4.3, а именно:

Теорема 5.2. *Ако целите функции* $g_k(\zeta) \in F^*(0)$ $(k = 1, 2)$, *то комплексната функция* $B(z)$, *дефинирана с* (5.8), *е холоморфна в ивицата* $S(\tau_0)$ *и се представя в нея във вида* (5.4) *с функции* $b_k(t) \in \tilde{E}(0)$ $(k = 1, 2)$.

ЛИТЕРАТУРА

1. Бейтман, Г., А. Эрдейи. Высшие трансцендентные функции. I, М., 1973, II, М., 1974.
2. Русев, П. Интегрално представяне на аналитични функции посредством изродени хипергеометрични функции. Год. Соф. унив., Фак. мат. и мех., 72, 1978, 99–120.

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RAMSEY MULTIPLICITY $M(3,6)$ IS 2

Nikolay G. Khadzhiivanov, Ivan Zh. Pashov

Николай Г. Хаджииванов, Иван Пашов. РАМСЕЕВСКАЯ КРАТНОСТЬ $M(3,6)$ ЕСТЬ 2

Обозначим через $c_p(G)$ число p-клик графа G, а через $\overline{G}$ — его дополнение. Кери сконструировал 17-вершинный граф G, для которого $c_3(G)+c_6(\overline{G})=0$ и доказал, что неравенство $c_3(G)+c_6(\overline{G})\geqq 1$ имеет место для любого 18-вершинного графа G. Ив. Пашов указал 18-вершинный граф G, для которого $c_3(G)+c_6(\overline{G})=2$. В настоящей статье доказано, что $c_3(G)+c_6(\overline{G})\geqq 2$ для любого 18-вершинного графа G.

Nikolay G. Khadzhiivanov, Ivan Zh. Pashov. RAMSEY MULTIPLICITY $M(3,6)$ IS 2

Let $c_p(G)$ be the number of the p-cliques of a graph G and $\overline{G}$ be the complement of G. Kéry showed a 17-vertex graph G with $c_3(G)+c_6(\overline{G})=0$ and proved for every 18-vertex graph G the inequality $c_3(G)+c_6(\overline{G})\geqq 1$. I. Pashov constructed an 18-vertex graph G with $c_3(G)+c_6(\overline{G})=2$. In this paper is proved that $c_3(G)+c_6(\overline{G})\geqq 2$ for every 18-vertex graph G.

1. INTRODUCTION

A set of p vertices of a graph will be called p-clique (p-anticlique) if every pair of vertices in the set are (are not) adjacent.

F. Ramsey [1] proved that for every two natural numbers p, q there exists a natural number n such that each n-vertex graph has a p-clique or a q-anticlique. The minimum n with this property is denoted by $R(p,q)$ and is called Ramsey number. Obviously we have $R(p,q)=R(q,p)$, $R(1,q)=1$, $R(2,q)=q$. Thus, $R(p,q)$ is of interest when $3\leq p\leq q$. There are known only seven such Ramsey numbers: $R(3,3)=6$, $R(3,4)=9$, $R(3,5)=14$, $R(4,4)=18$ (R. Greenwood and

A. Gleason [2]); $R(3,6) = 18$ (G. Kéry [3]); $R(3,7) = 23$ (J. Graver and J. Yackel [4]); $R(3,9) = 36$ (C. Grinstead and S. Roberts [5]).

We will denote with $c_p(G)$ the number of p-cliques of a graph G and with $\bar{c}_q(G)$ — the number of its q-anticliques; it is clear that $\bar{c}_q(G) = c_q(\overline{G})$.

The number $c_p(G) + \bar{c}_q(G)$ we will call (p,q)-multiplicity of the graph G. The minimum (p,q)-multiplicity in the set of the n-vertex graphs we will denote by $M(n;p,q)$.

It is clear that $M(n;p,q) \geq 1$ when $n \geq R(p,q)$ and $M(n;p,q) = 0$ when $n < R(p,q)$.

The number $M(n;3,3)$ has been determined for every n by A. Goodman [6].

The number $M(R(p,q);p,q)$ is called Ramsey multiplicity and is denoted by $M(p,q)$. Of course, this number is of interest only when $3 \leq p \leq q$ because $M(p,q) = M(q,p)$ and $M(1,q) = M(2,q) = 1$.

There are known only three Ramsey multiplicities: $M(3,3) = 2$ (A. Goodman [6]); $M(3,4) = 1$ (N. Khadzhiivanov and N. Nenov [7]); $M(3,5) = 4$ (I. Pashov [8]).

It is known that every 18-vertex graph has a 3-clique or a 6-anticlique (G. Kéry [3]). In this paper we will prove that there is not a graph without 3-cliques with just one 6-anticlique (see Theorem 1) and, similarly — there is not a graph without 6-anticliques with just one 3-clique (see Theorem 2). Hence $M(3,6) \geq 2$.

On the other hand, the second author showed in [9] a 18-vertex graph without 6-anticliques and with exactly two 3-cliques. Thus $M(3,6) = 2$.

2. NOTATIONS AND PRELIMINARY RESULTS

We will write $G \in (n; \alpha \times p, \beta \times \overline{q})$ for an n-vertex graph G with $c_p(G) = \alpha$ and $\bar{c}_q(G) = \beta$.

The set of all neighbours of the vertex v in the considered graph will be denoted by $A(v)$; then $|A(v)| = d(v)$ is the degree of v. The same notation $A(v)$ will be used for the subgraph of G generated by the set $A(v)$. With $N(v)$ we will denote the set of the not adjacent to v (except v) vertices of the graph (and the correspondent generated subgraph). For the set of vertices not adjacent to either u and v (except u and v) and for the correspondent generated subgraph we will used the notation $N(u,v)$.

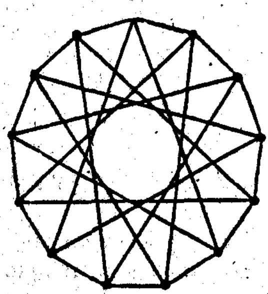

Fig. 1. The unique graph from $(13;0 \times 3, 0 \times \overline{5})$.

In the proofs of the theorems we will also use except a part of the above mentioned results the following propositions.

Proposition 1 (N. Nenov, I. Pashov and N. Khadzhiivanov [10]). *There is no graph* $G \in (13; 0 \times 3, 1 \times \overline{5})$.

Proposition 2 (G.Kéry [3]). *There is a unique graph* $G \in (13; 0 \times 3, 0 \times \overline{5})$; *it is presented in* Fig. 1.

Proposition 3 (N. Khadzhiivanov, N. Nenov and I. Pashov [11]). *There is a unique graph* $G \in (13; 1 \times 3, 0 \times \overline{5})$; *this graph is shown in* Fig. 2.

Proposition 4 (N. Nenov and N. Khadzhiivanov [12]). *There is a unique graph* $G \in (12; 0 \times 3,$

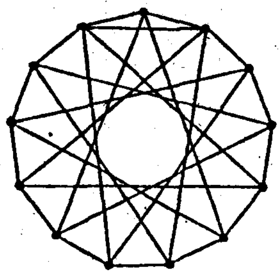

Fig. 2. The unique graph from $(13;1 \times 3, 0 \times \overline{5})$.

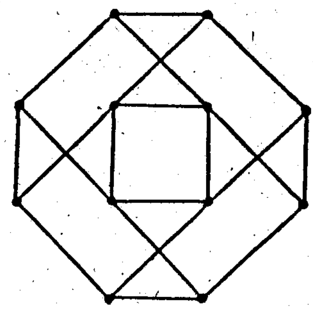

Fig. 3 The unique 20-edge graph from $(12;0 \times 3, 0 \times \overline{5})$.

$0 \times \overline{5})$ *with* 20 *edges*; *it is presented in* Fig. 3.

3. EVERY 18-VERTEX GRAPH WITHOUT 3-CLIQUES HAS AT LEAST TWO 6-ANTICLIQUES

Lemma 1. *Let* $G \in (18;0 \times 3, 1 \times \overline{6})$ *and let* A *be the unique 6-anticlique in* G. *If* u *and* v *are not adjacent vertices of* G *and* $\{u,v\} \not\subset A$, *then* $|A(u) \cap A(v)| \leq d(u) + d(v) - 8$.

Proof. Clearly, $N(u,v)$ does not contain 3-cliques and 4-anticliques because if M is a 4-anticlique in $N(u,v)$, then $M \cup \{u,v\}$ is a 6-anticlique in G and is different from A. From $R(3,4) = 9$, we have $|N(u,v)| \leq 8$. But $|N(u,v)| = 16 - |A(u) \cup A(v)| = 16 - d(u) - d(v) + |A(u) \cap A(v)|$. Hence $|A(u) \cap A(v)| \leq d(u) + d(v) - 8$.

Lemma 2. *If* $G \in (18;0 \times 3, 1 \times \overline{6})$, *then* $d(v) \leq 6$ *for every vertex* v *and the equality is achieved for at most one vertex.*

Proof. For every vertex v the set $A(v)$ is an anticlique because G has no 3-cliques. Since G has only one 6-anticlique — A, we have $d(v) = |A(v)| \leq 6$ and $A(v) = A$ when $d(v) = 6$.

If $d(u) = d(v) = 6$, then $A(u) = A(v) = A$. Hence u and v are not adjacent (G has no 3-cliques) and $|A(u) \cap A(v)| = 6$. This contradicts Lemma 1).

Lemma 3. *If* $G \in (18;0 \times 3, 1 \times \overline{6})$, *then* $d(v) \geq 4$ *for every vertex* v.

Proof. Assume $d(v) \leq 3$, i.e. $|N(v)| \geq 14$. Remember $M(3,5) > 1$, but $c_3(N(v)) = 0$, so $N(v)$ has at least two 5-anticliques K_1 and K_2. Then $K_1 \cup \{v\}$ and $K_2 \cup \{v\}$ are two different 6-anticliques in G. This contradiction completes the proof of the lemma.

Lemma 4. *If* $G \in (18;0 \times 3, 1 \times \overline{6})$ *and* v *is a vertex of* G *with* $d(v) = 4$, *then* $N(v) \in (13;0 \times 3, 0 \times \overline{5})$.

Proof. Obviously $|N(v)| = 13$, $c_3(N(v)) = 0$ and any 5-anticlique in $N(v)$ forms with v a 6-anticlique in G. Hence $\overline{c}_5(N(v)) \leq 1$.

The assumption $\overline{c}_5(N(v)) = 1$ implies $N(v) \in (13;0 \times 3, 1 \times \overline{5})$ which contradicts Proposition 1. Hence $\overline{c}_5(N(v)) = 0$, i.e. $N(v) \in (13;0 \times 3, 0 \times \overline{5})$.

Lemma 5. *If* $G \in (18;0 \times 3, 1 \times \overline{6})$ *and* A *is the unique 6-anticlique in* G, *then* $d(v) \geq 5$ *for every* $v \in A$.

Proof. Assume $v \in A$ and $d(v) = 4$ (see Lemma 3). According to Lemma 4, $N(v) \in (13; 0 \times 3, 0 \times \overline{5})$, which is impossible because $N(v)$ contains the 5-anticlique $A \backslash \{v\}$.

Lemma 6. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A is the unique 6-anticlique in G, then $d(v) \geqq 5$ for each $v \notin A$.*

Proof. Let us assume $v_0 \notin A$ and $d(v_0) = 4$ (see lemma 3). By Lemma 4, $N(v_0) \in (13; 0 \times 3, 0 \times \overline{5})$. Hence (see Proposition 2) $N(v_0)$ is isomorphic to the 4-regular graph in Fig. 1.

Denote by m the number of edges $[u, v]$ with $u \in A(v_0)$, $v \in N(v_0)$. Obviously, $m = \sum_{v \in N(v_0)} (d(v) - 4)$. This equality and Lemma 2 yields $m \leqq 12.1 + 1.2 = 14$.

On the other hand, it is clear that $m = \sum_{u \in A(v_0)} (d(v) - 1)$. Hence

$$\sum_{u \in A(v_0)} d(v) \leqq 18. \tag{1}$$

The set $A(v_0)$ contains at least two vertices from A because otherwise $N(v_0)$ will have a 5-anticlique and v_0 will be in a 6-anticlique, different from A. Let $\{u_1, u_2\} \subset A(v_0) \cap A$ and u_3, u_4 be the other two vertices in $A(v_0)$. By Lemma 5, $\sum_{u \in A(v_0)} d(u) \geqq 10 + d(u_3) + d(u_4)$. This inequality and (1) yield $d(u_3) + d(u_4) \leqq 8$. Then Lemma 3 gives $d(u_3) = d(u_4) = 4$.

Apply Lemma 1 for vertices u_3 and u_4 ($u_3 \notin A$, by Lemma 5). Thus we conclude $|A(u_3) \cap A(u_4)| \leqq 0$, which is absurdity because $v_0 \in A(u_3) \cap A(u_4)$. The received contradiction completes the proof of the Lemma.

Lemma 7. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$, then G is 5-regular.*

Proof. By Lemmas 5 and 6 we know that $d(v) \geqq 5$ for every vertex v of G and by Lemma 2 that $d(v) \leqq 6$ with equality at most for one vertex v. Hence G has at least 17 vertices of degree 5. If the rest vertex has degree 6, then the number of the vertices of an odd degree will be odd, which is impossible. Consequently, all vertices of G have degree 5.

Lemma 8. *Let $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A be the unique 6-anticlique in G. If $u \notin A$, then $|A(u) \cap A| \geqq 2$. If $u \notin A$, $v \notin A$ and $A(u) \cup A(v) \supset A$, then u and v are not adjacent.*

Proof. Any k vertices from A which are not adjacent to $u \notin A$ form with u a $(k+1)$-anticlique different from A. Hence, $k \leqq 4$, i.e. $|A(u) \cap A| \geqq 2$ for every $u \notin A$.

Now let $u \notin A$, $v \notin A$ and $A(u) \cup A(v) \supset A$. Assume that u and v are adjacent. We may regard that $|A \cap A(v)| \leqq |A \cap A(u)|$. Because of $c_3(G) = 0$, $A(u) \cap A(v) = \varnothing$. Hence $|A \cap A(v)| \leqq 3$, i.e. v has at least two neighbours out of A (see Lemma 7). Consequently, there is a neighbour w of v, $w \neq u$, $w \notin A$. The adjacent vertices w, v cannot have common neighbours. From the relations $|A(w) \cap A| \geqq 2$ and $A(u) \cup A(v) \supset A$ follows $|A(w) \cap A \cap A(u)| \geqq 2$. The vertices w and u have also the vertex $v \notin A$ as a common neighbour. Thus $|A(u) \cap A(w)| \geqq 3$.

On the other hand, applying Lemma 1 for the vertices u and w we receive $|A(u) \cap A(w)| \leqq d(u) + d(w) - 8 = 5 + 5 - 8 = 2$.

The obtained contradiction completes the proof of the Lemma.

Lemma 9. *If $G \in (18; 0 \times 3, 1 \times \overline{6})$ and A is the unique 6-anticlique in G, then $|A(v) \cap A| \leqq 3$ for every vertex $v \notin A$.*

Proof. Assume $|A(v)\cap A| \geqq 4$.

It is easy to see that $|A(v)\cap A| = 4$. Really, otherwise all five neighbours of v are in A and if w is the rest vertex of A, then $A(w)\cup v$ is a 6-anticlique, different from A, which is impossible.

Thus, $|A(v)\cap A| = 4$ and there is a vertex $u \notin A$ which is adjacent to v. By Lemma 8, $|A(u)\cap A| \geqq 2$ and because of $A(u)\cap A(v) = \emptyset$ we have $A(u)\cup A(v) \supset A$. Again Lemma 8 yields that u and v are not adjacent. This contradiction completes the proof of the Lemma.

Theorem 1. *Every 18-vertex graph without 3-cliques contains at least two 6-anticliques.*

Proof. Let G be an 18-vertex graph and $c_3(G) = 0$. Assume $\bar{c}_6(G) \leqq 1$. By $R(3,6) = 18$ we have $\bar{c}_6(G) \neq 0$. Hence $\bar{c}_6(G) = 1$, i.e. $G \in (18; 0\times 3, 1\times \bar{6})$. Denote by A the unique 6-anticlique in G.

Let B be the set of vertices v for which $|A(v)\cap A| = 3$ and $|B| = n$. Let V be the set of the vertices of G and $C = V\setminus(A\cup B)$. Obviously $B\cap A = \emptyset$ and $|C| = 12 - n$. By Lemmas 8 and 9 follows $|A(v)\cap A| = 2$ for each $v \in C$.

Let l be the number of the edges $[u,v]$ with $u \in A$ and $v \notin A$. Every vertex in A is an end point for five such edges; hence $l = 6.5 = 30$.

On the other hand, clearly, $l = \sum_{v\in B} |A(v)\cap A| + \sum_{v\in C} |A(v)\cap A| = n.3 + (12 - n).2 = 24 + n$. Hence $24 + n = 30$ and $n = 6$.

We will prove that B is an anticlique. Assume the opposite and let u, v be adjacent vertices from B. Since $A(u)\cap A(v) = \emptyset$, $|A(u)\cap A| = 3$ and $|A(v)\cap A| = 3$ we have $A(u)\cup A(v) \supset A$. From Lemma 8 follows that u and v are not adjacent.

The received contradiction shows that B is an 6-anticlique. But $B\cap A = \emptyset$, thus we obtain a contradiction with equality $\bar{c}_6(G) = 1$.

The Theorem is proved.

4. EVERY 18-VERTEX GRAPH WITHOUT 6-ANTICLIQUES HAS AT LEAST TWO 3-CLIQUES

Lemma 10. *If $G \in (18; 1\times 3, 0\times \bar{6})$ and T is the unique 3-clique in G, then $d(v) \leqq 6$ for every vertex v. The equality may be attained only for a vertex from T.*

Proof. If $v \notin T$, then $A(v)$ is an anticlique; hence $d(v) = |A(v)| \leqq 5$.

When $v \in T$, the subgraph $A(v)$ has only one edge; thus $d(v) = |A(v)| \leqq 6$.

Lemma 11. *If $G \in (18; 1\times 3, 0\times \bar{6})$, then $d(v) \geqq 4$ for every vertex v.*

Proof. If $d(v) \leqq 3$, then $|N(v)| \geqq 14$. Since $\bar{c}_5(N(v)) = 0$ and $M(3,5) > 1$ (see the Introduction), the subgraph $N(v)$ will have at least two 3-cliques, which is impossible.

Lemma 12. *Let $G \in (18; 1\times 3, 0\times \bar{6})$ and T is the 3-clique of G. If the vertex v_0 be adjacent to at least one vertex from T and $d(v_0) = 4$, then every vertex from $N(v_0)\setminus T$ is adjacent to at most one vertex from $A(v_0)$.*

Proof. Obviously $|N(v_0)| = 13$ and $\bar{c}_5(N(v_0)) = 0$. Since v_0 is adjacent to a vertex from T, we have $c_3(N(v_0)) = 0$.

Proposition 2 implies that the graph $N(v_0)$ is isomorphic to the graph in Fig. 1. We will use only the fact that $N(v_0)$ must be a 4-regular graph (see Fig. 1).

If $v \notin T$, by Lemma 10 we will have $d(v) \leq 5$. Then every $v \in N(v_0)\setminus T$, having 4 neighbours in $N(v_0)$, is adjacent to at most one vertex out of $N(v_0)$, i.e. to at most one vertex from $A(v_0)$.

Lemma 13. *If $G \in (18; 1 \times 3, 0 \times \overline{6})$ and $T = \{v_1, v_2, v_3\}$ is the 3-clique of G, then $d(v_i) \geq 5$, $i = 1, 2, 3$.*

Proof. Assume $d(v_1) < 5$. Then (see Lemma 11) $d(v_1) = 4$. Let $A(v_1)\setminus T = \{v_4, v_5\}$.

We will prove that $d(v_4) = d(v_5) = 4$.

Assume $d(v_4) \neq 4$. By Lemma 11, $d(v_4) \geq 5$. Since $A(v_4) \cap A(v_1) = \varnothing$, we have $|A(v_4) \cap N(v_1)| \geq 4$. The set $A(v_4)$ is an anticlique because $v_4 \notin T$. Hence $A(v_4) \cap N(v_1)$ is an anticlique too.

Applying Lemma 12 for the vertex v_1 we have that every vertex v from $N(v_1)\setminus T = N(v_1)$ is adjacent to at most one vertex from $A(v_1)$; in particular, the same is valid for each vertex $v \in A(v_4) \cap N(v_1)$. But such a v is adjacent to $v_4 \in A(v_1)$; hence v is not adjacent to the vertices v_2 and v_5 from $A(v_1)$.

Thus we receive that the set $(A(v_4) \cap N(v_1)) \cup \{v_2, v_5\}$ is an anticlique, which is a contradiction because this set has at least 6 vertices.

In this way we have that $d(v_4) = d(v_5) = 4$. The non-adjacent vertices v_4 and v_5 have a common neighbour (the vertex v_1). Hence $|N(v_4, v_5)| \geq 9$. Moreover, $c_3(N(v_4, v_5)) = 0$, because $v_1 \notin N(v_4, v_5)$. Note also that $\overline{c}_4(N(v_4, v_5)) = 0$. The obtained results for the graph $N(v_4, v_5)$ contradict the equality $R(3, 4) = 9$.

Lemma 14. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and $T = \{v_1, v_2, v_3\}$ be the 3-clique of G. If $v_0 \notin T$ and v_0 is adjacent to a vertex from T, then $d(v_0) = 5$.*

Proof. Assume $d(v_0) < 5$. Then (see Lemma 11) $d(v_0) = 4$. Let v_0 be adjacent to v_1 from T and the rest three neighbours of v_0 be v_4, v_5, and v_6.

We will prove that $d(v_4) = 4$ and $A(v_4) \cap T \neq \varnothing$.

Let $A = A(v_4) \cap (N(v_0)\setminus T)$. Since the set $A(v_4)$ is an anticlique, then A is an anticlique, too. By Lemma 12 we know that every vertex from $N(v_0)\setminus T$ is adjacent to at most one vertex from $A(v_0)$. The vertices of A are adjacent to $v_4 \in A(v_0)$ and belong to $N(v_0)\setminus T$. Hence each of these vertices is not adjacent to the other vertices v_1, v_5, v_6 from $A(v_0)$.

Consequently, $A \cup \{v_1, v_5, v_6\}$ is an anticlique, so that $|A| \leq 2$. Therefore v_4 is adjacent to at most two vertices from $N(v_0)\setminus T$. Besides v_4 is adjacent to v_0 and not adjacent to v_1, v_5, v_6. According to Lemma 11, $d(v_4) \geq 4$. Hence v_4 must be adjacent to at least one of the vertices v_2, v_3. On the other hand, v_4 can not be adjacent to both v_2, v_3.

Consequently, $d(v_4) = 4$ and $A(v_4) \cap T \neq \varnothing$.

Similarly, $d(v_5) = 4$. The vertices v_4 and v_5 have a common neighbour (the vertex v_0); hence $|N(v_4, v_5)| \geq 9$. It is clear that $\overline{c}_4(N(v_4, v_5)) = 0$ because v_4 and v_5 are not adjacent and G has no 6-anticliques. Then from $R(3, 4) = 9$ follows that $N(v_4, v_5)$ has a 3-clique. But this is impossible because $A(v_4) \cap T \neq \varnothing$.

The obtained contradiction shows that $d(v_0) \geq 5$. From Lemma 10 it follows $d(v_0) = 5$.

Lemma 15. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and T be the 3-clique in G. If v_0 is not adjacent to vertices from T, then $d(v_0) = 5$.*

Proof. By Lemma 10 we know that $d(v_0) \leq 5$. Assume $d(v_0) < 5$. Then Lemma 11 yields $d(v_0) = 4$.

Obviously, $|N(v_0)| = 13$, $c_3(N(v_0)) = 1$ and $\overline{c}_5(N(v_0)) = 0$. According to Proposition 3, the graph $N(v_0)$ is isomorphic to the graph in Fig. 2. Every vertex

v from $N(v_0)\backslash T$ has degree 4 in $N(v_0)$ (see Fig. 2). Consequently, v is adjacent to at most one vertex out of $N(v_0)$ (because $d(v) \leqq 5$, see Lemma 10). We will essentially use the just proved proposition in the end of the proof.

Every vertex u from $A(v_0)$ has at most one neighbour in T; hence it has at least two neighbours in $N(v_0)\backslash T$ (according to Lemma 11, $d(u) \geqq 4$). Moreover, if u is not adjacent to the vertices from T or $d(u) \geqq 5$, then u will have at least three neighbours in $N(v_0)\backslash T$.

We will prove that $A(v_0)$ contains at least three vertices, each with at least three neighbours in $N(v_0)\backslash T$.

This is clear, if three vertices from $A(v_0)$ have degree $\geqq 5$. Thus we will assume that there are $u_1, u_2 \in A(v_0)$ both of degree 4 (see Lemma 11). Then $|N(u_1, u_2)| \geqq 9$ (u_1 and u_2 have a common neighbour). The vertices u_1 and u_2 are not adjacent ($v_0 \notin T$), so that $\bar{c}_4(N(u_1, u_2)) = 0$. The equality $R(3,4) = 9$ implies that $N(u_1, u_2)$ has a 3-clique, i.e. $N(u_1, u_2) \supset T$. So u_2 is not adjacent to the vertices from T. Then, as we now, u_2 has three neighbours in $N(v_0)\backslash T$.

Thus $A(v_0)$ contains at least three vertices, each with at least three neighbours in $N(v_0)\backslash T$. The fourth vertex from $A(v_0)$ has at least two neighbours in $N(v_0)\backslash T$. But $|N(v_0)\backslash T| = 10$. Consequently, there is a vertex from $N(v_0)\backslash T$ with at least two neighbours in $A(v_0)$, which is impossible (see the beginning of the proof).

Lemma 16. *Let G be not regular graph and $G \in (18; 1 \times 3, 0 \times \bar{6})$. Then two vertices, say v_2 and v_3, from the unique 3-clique $T = \{v_1, v_2, v_3\}$ have degree 6 and all the rest vertices of G have degree 5. Moreover $N(v_1)$ contains a simple 8-cycle B which is an induced subgraph of G and has the properties:*

1. *Every vertex from $A = N(v_1)\backslash B$ is adjacent exactly to one vertex from $A(v_1)$.*
2. *Every vertex from B is adjacent exactly to two vertices from $A(v_1)$.*

P r o o f. By Lemmas 14 and 15 we have $d(v) = 5$ for every $v \notin T$ and Lemmas 10 and 13 give $5 \leqq d(v_i) \leqq 6$, $i = 1, 2, 3$. Since the number of the vertices of odd degree must be even, T contains one or three vertices of degree 5. But G is not regular. Hence T contains only one vertex of degree 5 — assume $d(v_1) = 5$ and $d(v_2) = d(v_3) = 6$.

Let e be the number of the edges of the 12-vertex graph $N(v_1)$ and m be the number of the edges of the type $[u, v]$ with $u \in N(v_1)$, $v \in A(v_1)$. Clearly

$$(2) \qquad 12.5 = \sum_{u \in N(v_1)} d(u) = 2e + m.$$

Since every $v \in A(v_1)$ has 4 neighbours in $N(v_1)$, we have $m = 5.4 = 20$ and $e = 20$.

Thus $N(v_1)$ has 20 edges. Obviously $N(v_1) \in (12; 0 \times 3, 0 \times \bar{5})$. According to Proposition 4, $N(v_1)$ is isomorphic to the graph in Fig.3. Let B be the set of the vertices of $N(v_1)$ of degree 3 in $N(v_1)$. Clearly B is a simple 8-cycle which is an induced subgraph of G. Every vertex from the set $A = N(v_1)\backslash B$ has degree 4 in $N(v_1)$. The vertices of $N(v_1)$ have degree 5 in G. Therefore the properties 1 and 2 are obviously fulfilled and the lemma is proved.

The next lemma will be proved in assumptions and notations from Lemma 16. We know that $d(v) = 5$ when $v \notin T$. The neighbours of the vertex v_1 (which also has degree 5) we will denote by v_2, v_3, v_4, v_5, v_6. By Lemma 16, $N(v_1) = A \cup B$, where $A \cap B = \varnothing$, $|A| = 4$, B is a simple 8-cycle and A and B have the properties 1 and 2.

We will denote by B_k, $2 \leqq k \leqq 6$, the set of the vertices from B which are adjacent to v_k. Clearly B_k are anticliques and $|B_k| \leqq 4$.

Lemma 17. *The assumptions of Lemma 16 and the last notations imply the properties:*

1. $B_2 \cap B_3 = \emptyset$.
2. $|B_i| \geqq 3$ *when* $i \in \{2,3\}$.
3. $B_i \cap B_j \neq \emptyset$ *when* $i \in \{2,3\}$ *and* $j \in \{4,5,6\}$.
4. $B \neq B_2 \cup B_3$.
5. $|B_i| = 3$ *when* $i \in \{2,3\}$.
6. $|B_i \cap B_j| = 1$ *when* $i \in \{2,3\}$ *and* $j \in \{4,5,6\}$.
7. *There is* $j_0 \in \{4,5,6\}$ *with* $|B_{j_0}| = 4$.
8. *Let* j_0 *(see the previous property) be, for instance,* 6. *Then* $B\backslash(B_2 \cup B_3) \subset B_6$ *and one of the vertices from* $B\backslash(B_2 \cup B_3)$ *is in* $B_4\backslash B_5$ *and the other in* $B_5\backslash B_4$.

Proof. 1. If $v \in B_2 \cap B_3$, then $\{v, v_2, v_3\}$ is a 3-clique different from T, which is impossible.

2. The set $A(v_2) \cap N(v_1)$ is a 4-anticlique, since $d(v_2) = 6$ and G has no other 3-clique except T.

Assume $|B_2| \leqq 2$. Then $|A(v_2) \cap A| \geqq 2$. Let $a, b \in A(v_2) \cap A$. According to the property 1 in Lemma 16, a and b are not adjacent to the vertices from $A(v_1)\backslash\{v_2\}$. The set $A(v_1)\backslash\{v_2\}$ is a 4-anticlique because $d(v_1) = 5$ and G has not a 3-clique, different from T. Since a and b are not adjacent, $\{a,b\} \cup (A(v_1)\backslash\{v_2\})$ is a 6-anticlique, which is a contradiction.

3. It is sufficient to prove $A(v_2) \cap A(v_j) \cap N(v_1) \neq \emptyset$ because if $v \in A(v_2) \cap A(v_j) \cap N(v_1)$, then property 1 from Lemma 16 implies $v \in B$.

Assume $A(v_2) \cap A(v_j) \cap N(v_1) = \emptyset$. The set $A(v_2)\backslash\{v_1\}$ is a 5-anticlique because $d(v_2) = 6$ and G has no other 3-clique except T. We have $A(v_2)\backslash\{v_1\} = \{v_3\} \cup (A(v_2) \cap N(v_1))$. Our assumption yields that v_j is not adjacent to the vertices from $A(v_2) \cap N(v_1)$. On the other hand, v_j is obviously not adjacent to v_3. Hence $\{v_j\} \cup (A(v_2)\backslash\{v_1\})$ is a 6-anticlique, which is a contradiction.

4. Assume the opposite, i.e. $B = B_2 \cup B_3$. Then $|B_2| = |B_3| = 4$, since $|B_k| \leqq 4$. Hence v_2 and v_3 have no neighbours in A. Then every vertex from A is adjacent exactly to one vertex from $A(v_1) \backslash \{v_2, v_3\} = \{v_4, v_5, v_6\}$. Therefore, there are vertices a and b from A with a common neighbour from $\{v_4, v_5, v_6\}$; we may regard that a and b are adjacent to v_4.

According to property 3, v_4 and v_i, $i = 2, 3$, have a common neighbour in B; we will denote it by b_i. Obviously $b_2 \neq b_3$ because otherwise $B_2 \cap B_3 \neq \emptyset$. The set $\{a, b, b_2, b_3\}$ is a 4-anticlique because $A(v_4)$ contains this set and $v_4 \notin T$. From 1 in Lemma 16 it follows that a and b are not adjacent to v_5 and v_6 and from 2 in the same lemma — b_2 and b_3 are not adjacent to v_5 and v_6.

We come to the contradiction that $\{a, b, b_2, b_3, v_5, v_6\}$ is a 6-anticlique.

5. Assume the opposite. Then we may regard (see property 2) that $|B_2| = 4$. Let $c \in B\backslash(B_2 \cup B_3)$ (see property 4). Since B_2 is a 4-anticlique in the simple 8-cycle B and $c \notin B_2$, then c is adjacent to two vertices u_1 and u_2 from B_2. According to property 2 in lemma 16, the vertex c is adjacent exactly to two vertices from $A(v_1)\backslash\{v_2, v_3\} = \{v_4, v_5, v_6\}$; we may assume that c is adjacent to v_4 and v_5. The vertices u_1 and u_2 are not adjacent to v_4 and v_5 because otherwise will be a 3-clique which contains the vertex c. The vertices u_1 and u_2 are in B_2 and $B_2 \cap B_3 = \emptyset$. Hence u_1 and u_2 are not adjacent to v_3. From property 2 in Lemma 16 it follows that u_1 has another (different from v_2) neighbour from $A(v_1)$ and we have the same for the vertex u_2. But u_1 and u_2 are not adjacent to v_3, v_4 and v_5. Hence u_1 and u_2 are adjacent to v_6.

Thus $A(u_1) \cap A(u_2) \supset \{v_2, v_6, c\}$. Moreover $d(u_1) = d(u_2) = 5$. Hence $|N(u_1, u_2)| \geqq 9$. The vertices u_1 and u_2 are not adjacent (otherwise $\{c, u_1, u_2\}$ will be a 3-clique). Hence $\bar{c}_4(N(u_1, u_2)) = 0$. Then from $R(3, 4) = 9$ we have $c_3(N(u_1, u_2) > 0$, i.e. $N(u_1, u_2) \supset T$. But this is impossible because $v_2 \notin N(u_1, u_2)$.

6. The sets $B_2 \cap B_4$, $B_2 \cap B_5$ and $B_2 \cap B_6$ are disjoint (see property 2 in Lemma 16) and not empty (see property 3), and $|B_2| = 3$ (see property 5). Hence $|B_2 \cap B_j| = 1$, $j = 4, 5, 6$.

7. Since $|A(v_i) \cap N(v_1)| = 4$ and $|A(v_i) \cap B| = 3$ (see property 5) we have $|A(v_i) \cap A| = 1$, $i = 2, 3$. Let $a_i = A(v_i) \cap A$. The vertices a_i have no neighbours in $\{v_4, v_5, v_6\}$ (see property 1 in Lemma 16). Each of the rest two vertices from A has one neighbour in $A(v_1)$. Hence there is a certain vertex from $\{v_4, v_5, v_6\}$ which has no neighbours in A, so all the four neighbours from $N(v_1)$ of this vertex are in B.

8. From properties 1 and 5 it follows $|B \backslash (B_2 \cup B_3)| = 2$. Let $B \backslash (B_2 \cup B_3) = \{a, b\}$. Every set B_4, B_5, B_6 has exactly one element in B_2 and B_3 (see property 6). According to the condition $|B_6| = 4$. Hence $\{a, b\} \subset B_6$.

The vertex a has another (different from v_6) neighbour in $A(v_1)$. Since a is not adjacent to v_2 and v_3, then a is adjacent to v_4 or v_5; we will assume that a is adjacent to v_4. The vertex b is also adjacent to a certain vertex from $\{v_4, v_5\}$. But b can not be adjacent to v_4. Indeed, otherwise $A(v_4) \cap A(v_6) \supset \{a, b, v_1\}$ and since $d(v_4) = d(v_6) = 5$, we obtain $|N(v_4, v_6)| \geqq 9$. This is impossible because $R(3, 4) = 9$, $\bar{c}_4(N(v_4, v_6)) = 0$ and $c_3(N(v_4, v_6)) = 0$ ($T \not\subset N(v_4, v_6)$, because $v_1 \notin N(v_4, v_6)$).

Thus, b is not adjacent to v_4. Hence b is adjacent to v_5, i.e. $b \in B_5 \backslash B_4$. Similarly, $a \in B_4 \backslash B_5$.

The lemma is proved.

Let us summarize the results from Lemma 17. Replace $B_i \cap B_j = b_{i,j}$ when $i \in \{2, 3\}$, $j \in \{4, 5, 6\}$ (see property 6 in Lemma 17). Since $|B_2| = |B_3| = 3$ (see property 5 in Lemma 17) and every vertex from B belongs exactly to two B_k (see property 2 in Lemma 16), then $B_2 = \{b_{2,4}, b_{2,5}, b_{2,6}\}$ and $B_3 = \{b_{3,4}, b_{3,5}, b_{3,6}\}$ where all these $b_{i,j}$ are different.

According to properties 7 and 8 in Lemma 17, the 4-element set B_6 contains (except the vertices $b_{2,6}$ and $b_{3,6}$) the two vertices of $B \backslash (B_2 \cup B_3)$. One of these two vertices belongs to $B_4 \backslash B_5$ — we will denote it by $b_{4,6}$. The other belongs to $B_5 \backslash B_4$ — denote it by $b_{5,6}$. So $B_6 = \{b_{2,6}, b_{3,6}, b_{4,6}, b_{5,6}\}$.

Thus $B = \{b_{2,4}, b_{2,5}, b_{2,6}, b_{3,4}, b_{3,5}, b_{3,6}, b_{4,6}, b_{5,6}\}$ and $B_4 = \{b_{2,4}, b_{3,4}, b_{4,6}\}$, $B_5 = \{b_{2,5}, b_{3,5}, b_{5,6}\}$.

Lemma 18. *If $G \in (18; 1 \times 3, 0 \times \bar{6})$, then G is 5-regular.*

P r o o f. Assume the opposite. We will use Lemmas 16 and 17 and the introduced notations. So the simple 8-cycle B contains the anticliques B_k, $k = 2, 3, 4, 5, 6$. Hence the two neighbours from B of $b_{i,6}$ are $b_{m,n}$ where $m \in \{2, 3, 4, 5\} \backslash \{i\}$ and $n \in \{4, 5\} \backslash \{i\}$. Thus B is the simple 8-cycle $(b_{2,4}, b_{3,6}, b_{2,5}, b_{4,6}, b_{3,5}, b_{2,6}, b_{3,4}, b_{5,6})$. Denote by a_i, $i = 2, 3$, the unique neighbour of v_i in A. The vertex a_i have no other (different from v_i) neighbour in $A(v_1)$ (see property 1 in Lemma 16).

We will prove that a_i, $i = 2, 3$, is adjacent to $b_{4,6}$. First of all, the set $M = \{a_2, b_{2,4}, b_{2,6}, v_3, v_5\}$ is a 5-anticlique. Indeed, the set $\{a_2, b_{2,4}, b_{2,6}, v_3\}$ contains in the anticlique $A(v_2) \backslash \{v_1\}$ and v_5 is not adjacent to all vertices from this set.

The set $\{b_{4,6}, b_{2,4}, b_{2,6}, v_3, v_5\}$ is a 5-anticlique, too. Indeed, as we already know, $\{b_{2,4}, b_{2,6}, v_3, v_5\}$ is an anticlique. On the other hand, $b_{4,6}$ is not adjacent to $b_{2,4}$ (otherwise $\{b_{4,6}, b_{2,4}, v_4\}$ will be a 3-clique); $b_{4,6}$ is not adjacent to $b_{2,6}$ (by virtue of similar reasons) and $b_{4,6}$ is not adjacent to v_3 and v_5.

If we assume that $b_{4,6}$ is not adjacent to a_2, then $\{b_{4,6}\} \cup M$ will be a 6-anticlique, which is impossible.

We proved that a_2 is adjacent to $b_{4,6}$. In the same way we can prove that a_3 is adjacent to $b_{4,6}$. For this purpose it is sufficient to note that $\{a_3, b_{3,4}, b_{3,6}, v_2, v_5\}$ and $\{b_{4,6}, b_{3,4}, b_{3,6}, v_2, v_5\}$ are anticliques.

Thereby we arrive at the conclusion that $A(b_{4,6}) \supset \{a_2, a_3, b_{2,5}, b_{3,5}, v_4, v_6\}$, which is a contradiction because $d(b_{4,6}) = 5$ (see Lemma 16).

The obtained contradiction completes the proof of the lemma.

Lemma 19. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$ and T be the unique 3-clique in G. As we know by Lemma 18, G is a 5-regular graph. Denote by v_0 a vertex of G, $v_0 \notin T$ and let v_0 be adjacent to a vertex from T. Then $N(v_0)$ has a simple 8-cycle B which is an induced subgraph in G and has the properties:*

1. *Every vertex from $A = N(v_0) \backslash B$ is adjacent exactly to one vertex from $A(v_0)$.*

2. *Every vertex from B is adjacent exactly to two vertices from $A(v_0)$.*

The proof of this lemma is similar to the proof of Lemma 16 and we will not present it.

In relation with Lemma 19 we will introduce some notations.

Denote by v_i, $1 \leqq i \leqq 5$, the neighbours of v_0. Let $v_1 \in T$ and $\Delta_i = A(v_i) \cap N(v_0)$. Clearly, $|\Delta_i| = 4$, Δ_1 contains a unique edge and the other Δ_i are anticliques.

Lemma 20. *Let $G \in (18; 1 \times 3, 0 \times \overline{6})$. In the notations from (and after) Lemma 19 we may affirm that:*

1. *$|\Delta_i \cap \Delta_j| \leqq 1$ when $i < j$.*
2. *$\Delta_k \subset B$ for a certain k.*
3. *$|\Delta_i \cap B| = 3$ when $i \neq k$.*
4. *$k = 1$.*

P r o o f. 1. Assume the opposite and let $\{a, b\} \subset \Delta_i \cap \Delta_j$. First we will prove that $i > 1$. Assume $i = 1$. Then $T \not\subset N(v_1, v_j)$; hence $c_3(N(v_1, v_j)) = 0$. The vertices v_1 and v_j are not adjacent and therefore $\bar{c}_4(N(v_1, v_j)) = 0$. We have $A(v_1) \cap A(v_j) \supset \{v_0, a, b\}$ and $d(v_1) = d(v_j) = 5$, so that $|N(v_1, v_j)| \geqq 9$. The obtained result contradicts the equality $R(3,4) = 9$.

Thus, $i > 1$. Hence Δ_i and Δ_j are 4-anticliques.

It is easy to see that $\Delta_i \cup \Delta_j \not\subset B$. Indeed, if $\Delta_i \cup \Delta_j \subset B$, then since B is a simple 8-cycle and the 4-anticliques Δ_i, Δ_j have a common vertex, certainly $\Delta_i = \Delta_j$. Thus, the vertices of Δ_i have no other (different from v_i and v_j) neighbours in $A(v_0)$ (see Lemma 19), thereby $\Delta_i \cup (A(v_0) \backslash \{v_i, v_j\})$ is a 7-anticlique in G, which is an absurdity.

Therefore we may assume that $\Delta_i \not\subset B$. Then $\Delta_i \cap A \neq \emptyset$ and let $c \in \Delta_i \cap A$.

We will prove that $M = \{a, b, c\} \cup (A(v_0) \backslash \{v_i, v_j\})$ is an anticlique. Indeed, $\{a, b, c\} \subset A(v_i)$; hence $\{a, b, c\}$ is an anticlique. On the other hand, c is not adjacent to the vertices from $A(v_0) \backslash \{v_i\}$ (see property 1 in Lemma 19), and a and b are not adjacent to the vertices from $A(v_0) \backslash \{v_i, v_j\}$ (see property 2 in Lemma 19). Since $A(v_0)$ is an anticlique, then M is really a 6-anticlique.

The obtained contradiction shows that $|\Delta_i \cap \Delta_j| \leqq 1$.

2. Remind that every vertex from A is adjacent to exactly one vertex from $A(v_0)$. Since $|A| = 4$ and $|A(v_0)| = 5$, certainly there is a vertex v_k which has no neighbours in A. Hence $\Delta_k \subset B$.

3. First we will prove that if $i \neq k$, then $\Delta_i \not\subset B$. Assume the opposite, i.e. that there is an $i \neq k$ with $\Delta_i \subset B$. Since $\Delta_k \cup \Delta_i \subset B$, then the vertices from A are not adjacent to the vertices v_k and v_i, so that every vertex from A is adjacent to exactly one vertex from $A(v_0)\backslash\{v_k, v_i\}$. We have $|A| = 4$ and $|A(v_0)\backslash\{v_k, v_i\}| = 3$. Hence, there are two vertices a and b in A with a common neighbour $v_n \in A(v_0)\backslash\{v_k, v_i\}$.

The vertices a and b must be adjacent because otherwise $\{a, b\} \cup (A(v_0)\backslash\{v_n\})$ will be a 6-anticlique (see property 1 in Lemma 19). Thus $\{a, b, v_n\}$ is a 3-clique. But v_1 is the unique vertex from $A(v_0)$ which is a vertex from a 3-clique, so $n = 1$.

Consequently $k \neq 1$ and $i \neq 1$. Then Δ_k and Δ_i are 4-anticliques. From $|\Delta_k \cap \Delta_i| \leqq 1$ (see property 1) it follows that Δ_k and Δ_i are different 4-anticliques in the simple 8-cycle B, so that $\Delta_k \cap \Delta_i = \varnothing$.

This shows that $B = \Delta_k \cup \Delta_i$. From property 1 it follows that Δ_m $(m \neq k, i)$ has at most two elements in B, so that $|\Delta_m \cap A| \geqq 2$. The subscript m takes three values and $|A| = 4 < 6$. Consequently, there are such m_1, m_2 that $\Delta_{m_1} \cap \Delta_{m_2} \cap A \neq \varnothing$. This contradicts to property 1 in Lemma 19.

Thus $\Delta_i \not\subset B$ when $i \neq k$, i.e. $\Delta_i \cap A \neq \varnothing$. Since $\Delta_{i_1} \cap \Delta_{i_2} \cap A = \varnothing$ when $i_1 \neq i_2$, then the four sets $\Delta_i \cap A$, $i \neq k$, are 1-element sets, i.e. $|\Delta_i \cap B| = 3$ when $i \neq k$.

4. Assume the opposite. We may suppose that $k = 2$ (see property 2). Then Δ_2 is a 4-anticlique in B. Number the vertices of B consecutively: $B = (c_1, c_2, \ldots, c_7, c_8)$. By property 1, $|\Delta_i \cap \Delta_j| \leqq 1$, $i \neq j$; say $i < j$. If the set $B \cap \Delta_i \cap \Delta_j$ is not empty, then we denote by $b_{i,j}$ the unique element of this set. In this way (see property 2 in Lemma 19) every vertex of B has a single notation.

Without loss of generality we may assume that the 4-anticlique Δ_2 consists of vertices $c_1 = b_{2,3}$, $c_3 = b_{2,4}$, $c_5 = b_{2,5}$ and $c_7 = b_{1,2}$. Since no one of the vertices v_2, v_3, v_4, v_5 is a vertex from a 3-clique, then $c_2 = b_{1,5}$ and $c_4 = b_{1,3}$.

According to property 3, $\Delta_1 \cap B = \{c_2, c_4, c_7\}$. Then c_6, $c_8 \notin \Delta_1$; hence $c_6 = b_{3,4}$ and $c_8 = b_{4,5}$.

From property 3 it follows $|\Delta_i \cap A| = 1$ when $i \neq 2$. Let $a_i = \Delta_i \cap A$. It is clear that the vertices a_i are different (see property 1 in Lemma 19).

Obviously, the sets $\{b_{2,4}, b_{3,4}, b_{2,3}, v_1, v_5\}$ and $M = \{a_3, b_{3,4}, b_{2,3}, v_1, v_5\}$ are 5-anticliques (v_0 and v_3 do not belong to 3-cliques). The vertex a_3 is adjacent to $b_{2,4}$ because otherwise $M \cup \{b_{2,4}\}$ will be a 6-anticlique. The vertex a_5 is adjacent to $b_{2,4}$, too. Indeed, the sets $\{b_{2,4}, b_{2,5}, b_{4,5}, v_1, v_3\}$ and $\{a_5, b_{2,5}, b_{4,5}, v_1, v_3\}$ are anticliques; so if a_5 is not adjacent to $b_{2,4}$, we have a 6-anticlique in G.

Thus $A(c_3) \supset \{c_2, c_4, v_2, v_4, a_3, a_5\}$, which contradicts the equality $d(c_3) = 5$.

The proof of the lemma is completed.

Theorem 2. *Every 18-vertex graph without 6-anticliques contains at least two 3-cliques.*

P r o o f. Let G be an 18-vertex graph and $\bar{c}_6(G) = 0$. Assume $c_3(G) \leqq 1$. From the equality $R(3, 6) = 18$ it follows $c_3(G) = 1$; hence $G \in (18; 1 \times 3, 0 \times \bar{6})$.

We will use the notations introduced in Lemmas 19 and 20.

We know that $\Delta_1 \subset B$ (see properties 2 and 4 in Lemma 20) and Δ_1 contains a unique edge. We may suppose that this edge is $[c_1, c_2]$ (the set B induces a simple 8-cycle in G). Then $T = \{v_1, c_1, c_2\}$ is the unique 3-clique of G and $c_3, c_8 \notin \Delta_1$. At least one of the vertices c_4, c_7 is in Δ_1 (because $|\Delta_1| = 4$ and T is the unique 3-clique of G). By the symmetry we may suppose that $c_4 \in \Delta_1$. Then $c_6 \in \Delta_1$ or $c_7 \in \Delta_1$.

We will prove that $c_7 \in \Delta_1$. Assume the opposite, i.e. $c_6 \in \Delta_1$.

Without loss of generality we assume $c_1 = b_{1,2}$, $c_2 = b_{1,3}$, $c_4 = b_{1,4}$, $c_6 = b_{1,5}$. It is easy to see that $c_3 = b_{2,5}$ and $c_5 = b_{2,3}$. Then $\Delta_2 \cap B = \{c_1, c_3, c_5\}$ (see property 3 in Lemma 20) and $c_7 \notin \Delta_2$. Moreover $c_7 \notin \Delta_1$, $c_7 \notin \Delta_5$ (since $c_6 \in \Delta_5$). This shows that $c_7 = b_{3,4}$. It is easy to see that $c_8 \notin \Delta_i$, $i = 1, 2, 3, 4$ and hence, c_8 can not have two neighbours in $A(v_0)$.

The obtained contradiction shows that $c_7 \in \Delta_1$. Therefore $\Delta_1 = \{c_1, c_2, c_4, c_7\}$. We may suppose that $c_1 = b_{1,2}$, $c_2 = b_{1,3}$, $c_4 = b_{1,4}$, $c_7 = b_{1,5}$. Then $c_3 = b_{2,5}$ and $c_8 = b_{3,4}$. Consequently, the 3-anticlique $\Delta_5 \cap B = \{c_3, c_7, c_5\}$. On the other hand $c_5 \notin (\Delta_1 \cup \Delta_4)$; hence $c_5 = b_{2,5}$ or $c_5 = b_{3,5}$. But we have already $c_3 = b_{2,5}$; hence $c_5 = b_{3,5}$. Finally, we have $c_6 \notin (\Delta_1 \cup \Delta_3 \cup \Delta_5)$ and hence $c_6 = b_{2,4}$.

Let $a_i = \Delta_i \cap A$, $i = 2, 3, 4, 5$ (see property 3 in Lemma 20). The vertices a_i are different (see property 1 in Lemma 19).

It is clear that $M = \{c_4, c_2, c_8, v_2, v_5\}$ and $\{a_3, c_2, c_8, v_2, v_5\}$ are anticliques. If we assume that a_3 and c_4 are not adjacent, then $\{a_3\} \cup M$ will be a 6-anticlique.

Consequently, a_3 is adjacent to c_4. In order to prove in this way that a_2 is adjacent to c_4 , it is sufficient to keep in mind that $\{c_4, c_6, c_1, v_3, v_5\}$ and $\{a_2, c_6, c_1, v_3, v_5\}$ are anticliques. Finally, $A(c_4) \supset \{c_3, c_5, a_2, a_3, v_1, v_4\}$, which contradicts the equality $d(c_4) = 5$.

The theorem is proved.

REFERENCES

1. R a m s e y, F. P. On a problem of formal logic. *Proc. London Math. Soc.*, **30** (1930), 264–286.
2. G r e e n w o o d, R. E., A. M. G l e a s o n. Combinatorial relations and chromatic graphs. *Canad. J. Math.*, **7** (1955), 1–7.
3. K é r y, G. Ramsey egy gràfelméleti tételéröl. *Mat. Lapok*, **15** (1964), 204–224 (Hungarian, Russian summary).
4. G r a v e r, J. E., J. Y a c k e l. Some graph theoretic results associated with Ramsey's theorem. *J. Combin. Theory*, **4** (1968), 125–175.
5. G r i n s t e a d, C. M., S. M. R o b e r t s. On the Ramsey numbers R(3,8) and R(3,9). *J. Combin. Theory Ser. B*, **33** (1982), 27–51.
6. G o o d m a n, A. W. On sets of acquaintances and strangers at any party. *Amer. Math. Monthly*, **66** (1959), 778–783.
7. N e n o v, N. D., N. G. K h a d s h i i v a n o v. Some two-colorings of edges of a complete graph with nine vertices (Russian, English summary), *Annuaire Univ. Sofia Fac. Math. Mec.*, **71** (1976/77), 95–115.
8. P a s h o v, I. Z h. Ramsey multiplicity in 2-colorings of the edges of a complete graph with fourteen vertices (Russian). *Serdica*, **10** (1984), 192–197.
9. P a s h o v, I. Z h. An 18-vertex graph with two 3-cliques and without 6-anticliques. *C. R. Acad. Bulgare Sci.*, **41** (1988), 35–36.
10. N e n o v, N. D., I. Z h. P a s h o v, N. G. K h a d z h i i v a n o v. On some extremal 2-colorings of the edges of the complete graph with thirteen vertices (Russian, English summary). *Godishnik Viss. Ped. Inst. Shumen, Prirod.-Mat. fak.*, VIIB (1983), 53–64.
11. K h a d z h i i v a n o v, N. G., N. D. N e n o v, I. Z h. P a s h o v. Uniqueness of the 13-vertex graph with independence number 4 and with one triangle (Russian). *C. R. Akad. Bulgare Sci.*, **35** (1982), 1363–1366.
12. N e n o v, N. D., N. G. K h a d z h i i v a n o v. Description of the graphs with 13 vertices having a unique triangle and a unique 5-anticlique (Russian, English summary). *Annuaire Univ. Sofia Fac. Math. Mec.*, **76** (1982), 91–107.

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ON THE EFFECTIVE ENUMERATIONS OF PARTIAL STRUCTURES*

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Ангел В. Дичев. ОБ ЭФФЕКТИВНЫХ НУМЕРАЦИЯХ ЧАСТИЧНЫХ АЛГЕБРАИЧЕСКИХ СИСТЕМ

Проблема существования эффективной нумерации данной частичной алгебраической системы, у которой носитель счетен, является одной из важнейших проблем теории конструктивных моделей.

В этой статье дана характеризация алгебраических систем с унарными функциями и предикатами, допускающих эффективные нумерации. Показано, что существуют только два типа структур, допускающих эффективные нумерации, независимо от конкретного вида функции и предикатов алгебраической системы.

В конце статьи дан пример структуры не допускающей эффективной нумерации, в которой каждое множество, определимое посредством SC с добавлением конечного числа констант, является рекурсивно перечислимым.

Angel V. Ditchev. ON THE EFFECTIVE ENUMERATIONS OF PARTIAL STRUCTURES

The question of the existence of effective enumeration of a given partial structure with denumerable domain is one of the most important problems in the theory of recursively enumerable models.

In this paper a characterization of the unary structures which admit effective enumerations is obtained. In the rest cases the question is open. It is shown also that there exist two sorts of structures only which admit effective enumerations independently of the concrete sort of the functions and predicates in the structure.

An example is given of a structure in which every subset of N definable by means of SC with finitely many constants is recursively enumerable, but the structure does not admit an effective enumeration.

The question of the existence of effective enumerations (recursively enumerable presentations) of a given partial structure with denumerable domain is one of the

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important problems in the theory of recursively enumerable (in the sequel r.e. for the sake of brevity) models.

In section 2 of this paper, a characterization of the unary structures which admit effective enumerations is obtained. In the rest cases the question is open.

Besides it is well known that every structure without predicates and with total functions admits an effective enumeration; every structure without functions and with finitely many unary predicates admits an effective enumerations, as well. It is natural to ask whether these are all partial structures which admit effective enumerations independently of the concrete sort of the functions and predicates in the structures. It turns out that there is only one sort structures more which admit effective enumerations. These are the structures without predicates and with only one partial function. In all another cases the structures admitting effective enumerations depend on the concrete sort of the functions and predicates.

In [4] Soskova and Soskov have defined a notion of effective enumeration and shown that a partial structure admits an effective enumeration in their sense iff every subset of $\mathbb{N}$ definable in this structure by means of REDS with finitely many constants is r.e. Soskov has proved also that if a partial structure admits an effective enumeration then every definable in this structure subset of $\mathbb{N}$ by means of SC (search computability, cf. [3]) with finitely many constants is r.e. In connection with this, I. Soskov stated the conjecture that if in a partial structure every subset of $\mathbb{N}$ definable by means of SC with finitely many constants is r.e., then the structure admits an effective enumeration. Soskov's conjecture is an attempt to characterize via search computability the structures which admit an effective enumeration. Let us remark that there is an example in [1] of a structure which has r.e. $\exists$-theory, but does not admit effective enumeration. In it, however, not all definable subsets of $\mathbb{N}$ are r.e.

In section 3 we give an example of a simple structure, in which every subset of $\mathbb{N}$ definable by means of SC with finitely many constants is r.e., but the structure does not admit an effective enumeration.

1. PRELIMINARIES

In this paper $\mathbb{N}$ denotes the set of all natural numbers $\{0, 1, \dots\}$, and $\mathbb{N}_n$ denotes the set $\{k : k \in \mathbb{N} \,\&\, k < n\}$.

If f is a partial function $Dom(f)$ denotes the domain, and $Ran(f)$ denotes the range of values of the function f. We use W_e to denote the e-th recursively enumerable (r.e.) set. We denote by $\langle .,. \rangle$ an effective coding of all ordered pairs of natural numbers with recursive functions $\lambda x.(x)_0$ and $\lambda x.(x)_1$ such that $(\langle x_0, x_1 \rangle)_0 = x_0$ and $(\langle x_0, x_1 \rangle)_1 = x_1$.

Let $U \subseteq \mathbb{N}^2$ and $\mathfrak{F}$ be a family of subsets of $\mathbb{N}$. The set U is said to be *universal* for the family $\mathfrak{F}$ iff for any n the set $\{x : (n, x) \in U\}$ is in the family $\mathfrak{F}$, and, conversely, for any set A in $\mathfrak{F}$ there exists such an n that $A = \{x : (n, x) \in U\}$.

Let us recall some definitions from [4] and [5].

Let $\mathfrak{A} = (B; \theta_1, \dots, \theta_n; \Sigma_1, \dots, \Sigma_k)$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1, \dots, \theta_n$ are partial functions of many arguments on B, $\Sigma_1, \dots, \Sigma_k$ are partial predicates of many arguments on B and $n \geq 0$ and $k \geq 0$. The relational type of $\mathfrak{A}$ is the ordered pair $(\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_k \rangle)$ where each θ_i is a_i-ary and each Σ_j is b_j-ary.

If every θ_i $(1 \leq i \leq n)$ and every Σ_j $(1 \leq j \leq k)$ are totally defined, then we say that the structure $\mathfrak{A}$ is a total one.

Effective enumeration (or r.e. presentation) of the structure $\mathfrak{A}$ is every ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = (\mathbf{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k)$ is a partial structure of the same relational type as $\mathfrak{A}$ and α is a partial, surjective mapping of $\mathbf{N}$ onto B such that the following conditions hold:

(i) $Dom(\alpha)$ is recursively enumerable and $\varphi_1, \dots, \varphi_n$, $\sigma_1, \dots, \sigma_k$ are partial recursive;

(ii) $\alpha(\varphi_i(x_1, \dots, x_{a_i})) \cong \theta_i(\alpha(x_1), \dots, \alpha(x_{a_i}))$ for every natural x_1, ..., x_{a_i}, $1 \leq i \leq n$;

(iii) $\sigma_j(x_1, \dots, x_{b_j}) \cong \Sigma_j(\alpha(x_1), \dots, \alpha(x_{b_j}))$ for every natural x_1, ..., x_{b_j}, $1 \leq j \leq k$.

We say that the structure $\mathfrak{A}$ admits an effective enumeration iff there exists an effective enumeration of the structure $\mathfrak{A}$.

Remark 1. If the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ then it admits an effective enumeration $\langle \alpha, \mathfrak{B}^* \rangle$ which satisfies the additional condition

(iv) $Dom(\alpha)$ is closed with respect to the partial operations $\varphi_1^*, \dots, \varphi_n^*$.

Indeed, let the structure $\mathfrak{A}$ admit an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ and φ_i^* be the restriction of the function φ_i to the set $\varphi_i^{-1}(Dom(\alpha))$, $i = 1, \dots, n$, and $\sigma_j^* = \alpha_j$, $j = 1, \dots, k$. Then it is easy to verify that $\langle \alpha, \mathfrak{B}^* \rangle$ is an effective enumeration which satisfies (iv).

Remark 2. If the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$, then it admits a totally defined effective enumeration $\langle \alpha, \mathfrak{B}^* \rangle$.

Indeed, let us suppose that the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$. We may assume that $\langle \alpha, \mathfrak{B} \rangle$ satisfies condition (iv). Let f be a total recursive function such that $Ran(f) = Dom(\alpha)$. We define the enumeration $\langle \alpha^*, \mathfrak{B}^* \rangle$ as follows:

$\alpha^* = \lambda x.\alpha(f(x))$;

$\varphi_i^* = \lambda x_1 \dots \lambda x_{a_i}.f^{-1}(\varphi_i(f(x_1), \dots, f(x_{a_i})))$, $\quad i = 1, \dots, n$;

$\sigma_j^* = \lambda x_1 \dots \lambda x_{b_j}.f^{-1}(\sigma_j(f(x_1), \dots, f(x_{b_j})))$, $\quad j = 1, \dots, k$.

Again, it is easy to verify that $\langle \alpha^*, \mathfrak{B}^* \rangle$ is a totally defined effective enumeration.

We shall identify the partial predicates with the partial mappings which obtain values in $\{0, 1\}$, taking 0 for true and 1 for false.

Let $\mathfrak{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathfrak{L}$ consists of n functional symbols $f_1, \dots, f_n$ and k predicate symbols $T_1, \dots, T_k$ where each f_i is a_i-ary and each T_j is b_j-ary. Let T_0 be a new unary predicate symbol which is intended to represent the unary total predicate $\Sigma_0 = \lambda s.0$.

Let $\{X_1, X_2, \dots\}$ be a denumerable set of variables. We use capital letters X, Y, Z to denote variables.

If τ is a term in the language $\mathfrak{L}$, then we write $\tau(X_1, \dots, X_a)$ to denote that all of the variables in τ are among X_1, ..., X_a. If $\tau(X_1, \dots, X_a)$ is a term and $s_1, \dots, s_a$ are arbitrary elements of B, then by $\tau_{\mathfrak{A}}(X_1/s_1, \dots, X_a/s_a)$ we denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements $s_1, \dots, s_a$.

Termal predicates in the language $\mathfrak{L}$ are defined by the following inductive clauses:

If $T \in \{T_0, \dots, T_k\}$, T is b-ary and τ^1, ..., τ^b are terms, then $T(\tau^1, \dots, \tau^b)$ and $\neg T(\tau^1, \dots, \tau^b)$ are atomic termal predicates;

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(X_1, \ldots, X_a)$ be a termal predicate with variables among $X_1, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of B. The value $\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Π over $s_1, \ldots, s_a$ in $\mathfrak{A}$ is defined by the inductive clauses:

If $\Pi = T_j(\tau^1, \ldots, \tau^{b_j})$, $0 \leq j \leq k$, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \Sigma_j(\tau^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), \ldots, \tau^{b_j}_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a));$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong (\Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) = 0 \supset 1, 0).$$

Here each expressions of the form "$\Pi^1 = 0 \supset \Pi^2, \Pi^3$" should be read as "if $\Pi^1 = 0$ then Π^2, else Π^3".

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong (\Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) = 0 \supset \Pi^2_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), 1).$$

If Π is a termal predicate and n is a natural number, then $\exists Y_1 \ldots \exists Y_r(\Pi \supset n)$ is called *conditional expression.*

Let $Q = \exists Y_1 \ldots \exists Y_r(\Pi \supset n)$ be a conditional expression with free variables among $X_1, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of B. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Q is defined by the equivalence

$$Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong l \iff \text{there exist } p_1, \ldots, p_r \text{ in } B \text{ such that } \Pi_{\mathfrak{A}}(Y_1/p_1, \ldots, Y_r/p_r, X_1/s_1, \ldots, X_a/s_a) \cong 0 \quad \text{and} \quad l = n.$$

We assume fixed an effective coding of the atomic predicates, the termal predicates and the conditional expressions of the language $\mathfrak{L}$.

Let $A \subseteq \mathbb{N}$. The set A is said to be definable in the structure $\mathfrak{A}$ iff for some r.e. set of conditional expressions $\{Q^v\}_{v \in V}$ with free variables among $X_1, \ldots, X_a$ and for some fixed elements $t_1, \ldots, t_a$ of B the following equivalence is true

$$l \in A \iff \exists v(v \in V \,\&\, Q^v_{\mathfrak{A}}(X_1/t_1, \ldots, X_a/t_a) \cong l).$$

If s is an element of B, then $T_{\mathfrak{A}}[s]$ (the type of s) is the set of natural numbers

$\{v : \Pi_v$ is an atomic termal predicate with a single variable X_1 occuring in Π_v and $\Pi^v_{\mathfrak{A}}(X_1/s) \cong 0\}$.

2. THE CONNECTION BETWEEN THE EFFECTIVE ENUMERATION OF A GIVEN STRUCTURE AND THE EXISTENCE OF AN R.E. SET UNIVERSAL FOR THE FAMILY OF THE TYPES OF THE STRUCTURE

Suppose a partial structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ be given, where all functions and predicates are unary and B is a denumerable set. Then the following theorem holds

Theorem. *A partial structure $\mathfrak{A}$ admits an effective enumeration iff the family of all types of the elements of the structure $\mathfrak{A}$ has an universal r.e. set.*

Proof. Suppose that the partial structure $\mathfrak{A}$ admits an effective enumeration $(\alpha, \mathfrak{B})$, where $\mathfrak{B} = (\mathbb{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k)$. Clearly, we can consider that α is totally defined over $\mathbb{N}$. A routine construction shows that there exists a primitive recursive in $\{\varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k\}$ function Ψ, such that for each termal predicate Π with code v, $\Psi(v, x) = \Pi_{\mathfrak{A}}(\underline{X}_1/\alpha(x))$ for all x of $\mathbb{N}$. Then, it is obvious that the set

$$U = \{(x, v) : \Psi(v, x) = 0 \,\&\, v \text{ is a code of an atomic predicate }\}$$

is r.e. and universal for the family of all types of the elements of the structure $\mathfrak{A}$.

Suppose now that all types of the elements of the structure $\mathfrak{A}$ are r.e. and that the family of all these types has an universal r.e. set U^1. Denote by U^2 the set $\{(x, v) : ((x)_0, v) \in U_1\}$ and by U^2_x the set $\{v : (x, v) \in U^2\}$. It is obvious that U^2 is r.e. and also universal for the family of all types of the elements of the structure $\mathfrak{A}$. For any natural x, U^2_x is a type of some element of B. Moreover, for every x there exist infinitely many y such that $U^2_x = U^2_y$.

Let φ_i be the function $\lambda x.\langle i, x\rangle$, $i = 1, \ldots, n$, $\mathbb{N}_0 = \mathbb{N} \setminus (Ran(\varphi_1) \cup \ldots \cup Ran(\varphi_n))$ and g be monotonically increasing function such that $Ran(g) = \mathbb{N}_0$. We denote by U the set $\{(g(x), v) : (x, v) \in U^2\}$ and by U_x the set $\{v : (x, v) \in U\}$. For any natural number x, let B_x be the set $\{s : s \in B \,\&\, T_{\mathfrak{A}}[s] = U_x\}$ and α^0 be arbitrary mapping of $\mathbb{N}_0$ onto B, satisfying the equalities $\alpha^0(\{y : U_x = U_y\}) = B_x$, $x \in \mathbb{N}$. Evidently, α^0 is surjective and $Dom(\alpha^0) = \{x : \exists x((x, v) \in U)\} = \mathbb{N}_0$.

We define the partial mapping α of $\mathbb{N}$ onto B by the inductive clauses:

If $x \in \mathbb{N}_0$, then $\alpha(x) = \alpha^0(x)$;

If $x = \langle i, y\rangle$, $1 \leq i \leq n$, $\alpha(y) = s$ and $\theta_i(s) = t$, then $\alpha(x) = t$.

The following simple lemmas are proved in [6].

Lemma 1. *For every $x \in \mathbb{N}$ and i, $1 \leq i \leq n$,*

$$\alpha(\varphi_i(x)) \cong \alpha(\langle i, x\rangle) \cong \theta_i(\alpha(x)).$$

Let us denote by $\mathfrak{B}^*$ the partial structure $(\mathbb{N}; \varphi_1, \ldots, \varphi_n)$.

Corollary. *Let $\tau(Y)$ be a term and $y \in \mathbb{N}$. Then,*

$$\alpha(\tau_{\mathfrak{B}^*}(Y/y)) \cong \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Lemma 2. *There exists an effective way to define for every x of $\mathbb{N}$ an element y of $\mathbb{N}_0$ and a term $\tau(\underline{X}_1)$ such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$.*

Lemma 3. *There exists an effective way to define for every x of $\mathbb{N}$ an element y of $\mathbb{N}_0$ and a term $\tau(\underline{X}_1)$ such that $\alpha(x) \cong \tau_{\mathfrak{A}}(\underline{X}_1/\alpha(y))$.*

We need also the following auxiliary lemma.

Lemma 4. *$Dom(\alpha)$ is recursively enumerable.*

P r o o f. Let for an arbitrary natural number x, $y = y(x) \in \mathbb{N}_0$ and the term $\tau(\underline{X}_1)$ be such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$. Let in addition $v = v(x)$ be the code of the atomic predicate $T_0(\tau(\underline{X}_1))$. Then it is clear that

$$x \in Dom(\alpha) \iff \tau_{\mathfrak{A}}(\underline{X}_1/\alpha(y)) \text{ is defined } \iff v \in T_{\mathfrak{A}}[\alpha_0(y)]$$
$$\iff (y(x), v(x)) \in U.$$

Therefore, $Dom(\alpha)$ is r.e.

Finally, let us define the partial predicates $\sigma_1, \ldots, \sigma_k$ on $\mathbb{N}$ using the conditional equalities $\sigma_j(x) \cong \Sigma_j(\alpha(x))$, $j = 1, \ldots, k$. Again, for arbitrary x let $y = y(x) \in \mathbb{N}$ and the term $\tau(\underline{X}_1)$ be such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$. Let in addition $v_1 = v_1(x)$ ($v_2 = v_2(x)$) be the code of the atomic predicate $T_j(\tau(\underline{X}_1))$ ($\neg T_j(\tau(\underline{X}_1))$). Then it is obvious that

$$\sigma_j(x) = 0 \iff \Sigma_j(\alpha(x)) \cong 0 \iff (y(x), v_1(x)) \in U;$$
$$\sigma_j(x) = 1 \iff \Sigma_j(\alpha(x)) \cong 1 \iff (y(x), v_2(x)) \in U;$$

Therefore, the predicates $\sigma_1, \ldots, \sigma_k$ are partial recursive. So, we have proved that $\langle \alpha, (\mathbb{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k) \rangle$ is an effective enumeration of the structure $\mathfrak{A}$.

Corollary 1. *Let $\mathfrak{A} = (B; \theta)$ be a partial structure, where B is a denumerable set and θ is a partial unary function such that $Dom(\theta) \subseteq B$ and $Ran(\theta) \subseteq B$. Then the structure $\mathfrak{A}$ admits an effective enumeration.*

Proof. If s is an arbitrary element of B then we use $[s]$ to denote the type $\{k : \theta^k(s) \text{ is defined}\}$ of the element s. So, if $s \in B$, then either $[s] = \mathbb{N}_n$ for some natural number n or $[s] = \mathbb{N}$.

Let us note that if for some element $s \in B$, $[s] = \mathbb{N}_n$ and $k < n$ then there exists such an element $t \in B$ that $[t] = \mathbb{N}_k$. Indeed, let t be the element $\theta^{n-k}(s)$. Therefore, for any structure $\mathfrak{A}$ we have the following possibilities for the families $\{[s] : s \in B\}$ of all types of the elements of B:

(i) $\{\mathbb{N}\}$;
(ii) $\{\mathbb{N}_k : k < n\}$ for some natural number n;
(iii) $\{\mathbb{N}\} \cup \{\mathbb{N}_k : k < n\}$ for some natural number n;
(iv) $\{\mathbb{N}_k : k \in \mathbb{N}\}$;
(v) $\{\mathbb{N}\} \cup \{\mathbb{N}_k : k \in \mathbb{N}\}$;

It is obvious that everyone of these families have an universal r.e. set. Thus, we obtain that the structure $\mathfrak{A}$ admits an effective enumeration.

If $\Pi^v = \Pi^{v_1} \& \ldots \& \Pi^{v_k}$ is a termal predicate where $\Pi^{v_1}, \ldots, \Pi^{v_k}$ are atomic predicates, then we use $\{\Pi^v\}$ to denote the set $\{v_1, \ldots, v_k\}$.

If all types of the structure $\mathfrak{A}$ are finite, then using Theorem 1 and Theorem 2 from [2] one can obtain more sophisticated necessary and sufficient conditions for the existence of an effective enumeration of the structure $\mathfrak{A}$. Namely, the following two theorems hold.

Corollary 2. *Let all function and predicates of the partial structure $\mathfrak{A}$ be unary and all types of the structure $\mathfrak{A}$ be finite. Then the structure $\mathfrak{A}$ admits an effective enumeration iff the following three conditions hold:*

(i) *The set $V = \{v : \exists s \in B(\Pi^v_{\mathfrak{A}}(s) = 0)\}$ is recursively enumerable;*

(ii) *The set $I = \{v : \{\Pi^v\} = T_{\mathfrak{A}}[s]$ for some $s \in B\}$ is a Σ^0_2 set in the arithmetical hierarchy;*

(iii) *There exists a partial recursive function f such that $V \subseteq Dom(f)$ and for every $v \in V$, $W_{f(v)}$ is a type of some element of the structure $\mathfrak{A}$ and $\{\Pi^v\} \subseteq W_{f(v)}$.*

Corollary 3. *Let all functions and predicates of the partial structure $\mathfrak{A}$ be unary, all types of the structure $\mathfrak{A}$ be finite and for any type t there exist at most finitely many types containing t. Then the structure $\mathfrak{A}$ admits an effective enumeration iff the following two conditions hold:*

(i) *The set $V = \{v : \exists s \in B(\Pi^v_{\mathfrak{A}}(s) = 0)\}$ is recursively enumerable;*

(ii) *The set $I = \{v : \{\Pi^v\} = T_{\mathfrak{A}}[s]$ for some $s \in B\}$ is a Σ^0_2 set of the arithmetical hierarchy.*

3. SOME COUNTER EXAMPLES

Example 1. *There exists a structure $\mathfrak{A} = (B; \theta_1; \Sigma_1)$ with unary θ_1 and Σ_1 which does not admit effective enumeration, but all definable in $\mathfrak{A}$ subsets of $\mathbb{N}$ are r.e.*

Let $\mathbf{Z}$ be the set of all integers and E be a set of natural numbers which is not Σ_2^0, $E = \{1 = p_0 < p_1 < \ldots\}$. Let in addition $B = \{b_{k,n} : k, n \in \mathbf{N}\} \cup \{a_m : m \in \mathbf{Z}\}$ where all $b_{k,n}$ and a_m are distinct for any $k, n \in \mathbf{N}$ and $m \in \mathbf{Z}$.

Define θ_1 and Σ_1 as follows:

$$\theta_1(b_{k,n}) = b_{k+1,n}, \quad k, n \in \mathbf{N};$$

$$\theta_1(a_m) = a_{m+1}, \quad m \in \mathbf{Z};$$

$$\Sigma_1(b_{k,n}) \cong 0 \iff k = 0 \vee k = p_n;$$

$$\Sigma_1(a_m) \cong 0 \iff m \leqq 0.$$

First, we show that every definable subset of $\mathbf{N}$ in the structure $\mathfrak{A}$ is r.e.

We use $f_1^k(Y)$ to denote the term $f_1(\ldots(f_1(Y))\ldots)$, where the symbol f_1 occurs k times in the term.

If s is an arbitrary element of B then we use $[s]$ to denote the type $\{k : \Sigma_1(\theta_1^k(s)) \cong 0\}$ of the element s; if Y is a variable and $Q = \exists Y_1 \ldots, \exists Y_r(\Pi \supset m)$ is a conditional expression, then we denote by $[Y; Q]$ the finite set

$$\{k : T_1(f_1^k(Y)) \text{ join in } \Pi \text{ as a conjunctive member }\}.$$

It is obvious that for any fixed element $s \in B$, any variable Y and conditional expression Q, there exists an effective way to verify whether or not $[Y; Q] = [s]$.

On the other hand, if $Q = \exists Y_1 \ldots \exists Y_r(\Pi \supset m)$ is a conditional expression, then for every conjunctive member $T_1(f_1^k(Y_i))$ of Π, $1 \leqq i \leqq r$, we can find an element s such that the conditional equality $\Sigma_1(\theta_1^k(s)) \cong 0$ is true. Therefore, in the structure $\mathfrak{A}$ the value $Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of every conditional expression Q does not depend on these conjunctive members which have no free variables.

Now, let A be a definable subset of $\mathbf{N}$, $\{Q^v\}_{v \in V}$ be an r.e. set of conditional expressions with free variables among $X_1, \ldots, X_a$, and $s_1, \ldots, s_a$ be elements of B, such that the following equivalence is true:

$$l \in A \iff \exists v(v \in V \;\&\; Q^v_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong l).$$

Therefore, if $Q^v = \exists Y_1 \ldots \exists Y_{r_v}(\Pi^v \supset m^v)$, then for arbitrary $v \in V$ the following equivalence hold:

$$m^v \in A \iff [X_1; Q^v] = [s_1] \;\&\; \ldots \;\&\; [X_a; Q^v] = s_a.$$

Therefore, there exists an effective way to verify whether or not $m^v \in A$, i.e. the set A is r.e.

Thus we have established that every definable subset of $\mathbf{N}$ is r.e. Now we shall see that the structure $\mathfrak{A}$ does not admit an effective enumeration.

Assume that $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = (\mathbf{N}; \varphi_1; \sigma_1)$ is a partial structure of the same relational type as $\mathfrak{A}$ and α is a partial, surjective mapping of $\mathbf{N}$ onto B such that the following conditions hold:

(i) $Dom(\alpha)$ is r.e. and φ_1, σ_1 are p.r.;

(ii) $\alpha(\varphi_1(x)) \cong \theta_1(\alpha(x))$ for every natural x;

(iii) $\sigma(x) \cong \Sigma_1(\alpha(x))$ for every natural x.

Then $\{\{k : \sigma_1(\varphi_1^k(n)) \cong 0\} : n \in \mathbf{N}\} = \{[s] : s \in B\}$. Therefore, the family of finite sets $\{[s] : s \in B\}$ has an universal set $U = \{(n, k) : \sigma_1(\varphi_1^k(n)) \cong 0\}$.

Let F be the set $\{v : \exists s(s \in B \;\&\; [s] = E_v)\}$. According to Theorem 1 the set F is Σ_2^0. Then, the set $F = \{v : v \in F \;\&\; E_v$ has exactly two elements $\}$ is Σ_2^0, as well. Thus, the set $E = \{x : x \neq 0 \;\&\; \exists v(v \in F_1 \;\&\; x \in E_v)\}$ is Σ_2^0, contrary to the choice of the set E. So we prove that $\mathfrak{A}$ does not admit effective enumeration.

Example 2. *There exists a structure $\mathfrak{A} = (B;\, \theta_1, \theta_2)$ with unary θ_1, θ_2 which does not admit effective enumeration.*

Let $B = \mathbb{N}$ and E be a set which is not r.e. and define the functions θ_1, θ_2 as follows:

$$\theta_1(k) = k + 1,\ k \in \mathbb{N};$$

$$\theta_2 \text{ be arbitrary function such that } Dom(\theta_2) = E.$$

It is easy to check that $E = \{k : \theta_2(\theta_1^k(0)) \text{ is defined }\}$. Therefore, the structure $\mathfrak{A}$ does not admit effective enumeration.

Example 3. *There exists a structure $\mathfrak{A} = (B;\, \Sigma_1)$ with a single binary predicate, which does not admit effective enumeration.*

Let E be a set of natural numbers which is not r.e., $p_0 < p_1 < \ldots$ be the sequence of all prime numbers, $E_1 = \{k : \exists l(l \in E \;\&\; p_l = k)\}$ and $E_2 = \{k : \exists l \exists m(m \in E_1 \;\&\; k = m.l)\}$.

It is obvious that the sets E_1 and E_2 are not r.e. too.

If $\mathfrak{A} = (B; \Sigma_1)$ is a structure where Σ_1 is a binary predicate, we say that there exists k-cycle ($k > 0$) for Σ_1 if for some elements $t_0, t_1, t_2, \ldots, t_{k-1}$ of B

$$\Sigma_1(t_0, t_1) = 0 \;\&\; \Sigma_1(t_1, t_2) = 0 \;\&\; \ldots \;\&\; \Sigma_1(t_{k-1}, t_0) = 0.$$

Let $\mathfrak{A} = (B; \Sigma_1)$ be a structure where B is infinite denumerable set and Σ_1 be a binary predicate totally defined over B such that there exists a k-cycle for Σ_1 iff $k \in E_2$. It is obvious that one can construct such a structure $\mathfrak{A}$.

Suppose that there exists an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$, where $\mathfrak{B} = (\mathbb{N}; \sigma_1)$. Then, there exists a k-cycle for Σ_1 iff for some natural numbers $x_0, x_1, x_2, \ldots, x_{k-1}$, $\sigma_1(x_0, x_1) = 0 \;\&\; \sigma_1(x_1, x_2) = 0 \;\&\; \ldots \;\&\; \sigma_1(x_{k-1}, x_0) = 0$, i.e. $k \in E_2$ iff $\exists x_0 \exists x_1 \ldots \exists x_{k-1}(\sigma_1(x_0, x_1) = 0 \& \sigma_1(x_1, x_2) = 0 \& \ldots \& \sigma_1(x_{k-1}, x_0) = 0)$. We obtain that E_2 is r.e. which contradicts the choice of the set E_2. Therefore, the structure $\mathfrak{A}$ does not admit effective enumeration.

From this example one can easily obtain that there exists a structure $\mathfrak{A} = (B; \theta_1)$ with binary θ_1 which does not admit effective enumeration.

It is well known the following

Proposition 1. *If the structure $\mathfrak{A} = (B;\, \theta_1, \ldots, \theta_n;\, \Sigma_1, \ldots, \Sigma_k)$ is a total one and* $[k = 0$ or ($n = 0$ & $\Sigma_1, \ldots, \Sigma_k$ are unary)] *then the structure $\mathfrak{A}$ admits an effective enumeration.*

From the above mentioned examples we obtain also the following

Proposition 2. a) *If the structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ is such that*

$$[(k = 0 \;\&\; n = 1 \;\&\; \theta_1 \text{ is unary }) \text{ or } (n \neq 0 \;\&\; \Sigma_1, \ldots, \Sigma_k \text{ are unary })]$$

then the structure $\mathfrak{A}$ admits an effective enumeration.

b) *If* $[(k \neq 0 \;\&\; n \neq 0)$ or $(k \neq 0$ & at least one of $\Sigma_1, \ldots, \Sigma_k$ is not unary) or $n \geq 2$ or ($n \neq 0$ & at least one of $\theta_1, \ldots, \theta_n$ is not unary)] *then there exists a structure $\mathfrak{A}$ which does not admit effective enumeration.*

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REFERENCES

1. Ershov, Y. L. Decision problems and constructivizable models (in Russian). Publishing house "Nauka", Moscow, 1980.
2. Ditchev, A. V. Effective enumerations of the families of finite sets of natural numbers (in print).
3. Moschovakis, Y. N. Abstract first order computability I. Trans. Amer. Math. Soc., 138, 1969, 427–464
4. Soskova, A. A. & I. N. Soskov. Effective enumerations of abstract structures. *In*: Heyting'88: Mathematical Logic, ed. P. Petkov, Plenum Press, New York and London, (to appear).
5. Soskov, I. N. Definability via enumerations. *J. Symb. Log.*, v. 54(2), 1989.
6. Soskov, I. N. Computability by means of effectively definable schemes and definability via enumerations. *Arch. for Math. Log.* (to appear).

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SCHEMES FOR RECURSION ELIMINATION IN BACKUS' FP-SYSTEMS

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Атанас Раденски, Милена Джамбазова. СХЕМЫ ДЛЯ УСТРАНЕНИЯ РЕКУРСИИ В БЭКУСОВСКИХ FP-СИСТЕМАХ

В статье рассмотрены трансформации с применением метода табюлации рекурсивных функций в среде FP-систем Бекуса. Корректность трансформации доказана методом индукций неподвижной точки. Рассмотрены следующие классы рекурсивных функций: рекурсия относительно одного целого аргумента; рекурсия относительно целого элемента списка; полная рекурсия; взаимно-рекурсивные функции.

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In this paper, the idea of tabulation is explored in order to develop transformations of recursive functions in Backus' FP-sytems into iterative ones. Transformation schemes are described in the paper and the correctness of some of them is proved by means of fixpoint induction. The following classes of recursion are considered: recursion with respect to one integer argument, recursion with respect to an integer element of a list, multiple recursion, mutual recursion.

1. INTRODUCTION

A well known approach to recursion elimination consists in the development of equivalent schemes which allow transformation of recursive programs into iterative ones [11, 5, 8]. Transformation of linear recursive schemes (schemes of De Backer and Scott) into iterative ones in purely syntactical studied [12,13]. Programs which support automated transformation of recursion into iteration are considered [4]. On the other hand, some authors develop methods which are usable "manually", by human programmers [1]. Recursion elimination is considered on an implementation level as well in [9]. Frequently, the low efficiency of evaluation of a function is due to the fact that the function is applied many times to the same argument during the evaluation. A well known example in this respect is the Fibonacci function:

```
function Fib(s : integer) : integer;
begin
  if (s = 0) or (s = 1) then Fib := 1
  else Fib := Fib(s-1) + Fib(s-2);
end;
```

The application of *Fib* to *s* causes one application of *Fib* to $s-1$, two different applications of *Fib* to $s-2$, and so on. Suppose, any value of the function *Fib* is memorized in a special table after its first computation. This value can be used without recomputation any time the function should be applied to the same argument. In addition, the order of computation can be changed from "top down" to "bottom up". In such a way the recursive function *Fib* can be replaced by an iterative one:

```
function Fib(s:integer) : integer;
var table : array [0..max] of integer;
    k : integer;
begin table[0] := 1; table[1] := 1;
      for k := 2 to s do
        table[k] := table[k-1] + table[k-2];
      Fib := table[s];
end;
```

Obviously, the most general variant of tabulation when all computed values are memorized may cause great memory overhead and is too inefficient. The more realistic case is when only some of the values are memorized for further use. It is clear that *table*[*k*] depends only on *table*[$k-1$] and *table*[$k-2$], i.e. only on the two preceding values. Using this dependence, the above function can be easily transformed into more efficient one, in which the table is of size 2. (Since the last function mentioned above is well known, it is not presented here.)

This example illustrates techniques for recursion elimination known as tabulation [3]. This technique consists in transforming recursive programs into iterative ones by means of additional tables. In this paper, the idea of tabulation is explored in order to develop transformations of recursive functions in Backus' FP-sytems [2] into iterative ones. As it will be shown, the iterative variant of the function itself implements some kind of tabulation. Actually, the function computes some values in some order, memorizes them during some steps of the computation and uses the memorized values in the further computations. A number of schemes for recursion elimination in FP-systems have been already developed. Kieburtz and Shultis [6] formulate in the language of FP-systems and prove the validity of many schemes well-known from outside the FP-systems. Reddy and Jayaraman [10] develop an approach to recursion elimination based on the representation of recursive FP-functions in the form of infinite conditions.

In this paper some transformaion schemes are described and the correctness of some of them is proved by means of fixpoint induction. All transformations considered reduce recursive functions to tail-recursive ones. The transformation of tail-recursion to iteration is straightforward [6]. The following classes of recursion are considered:

1) Recursion with respect to one integer argument;
2) Recursion with respect to an integer element of a list;
3) Multiple recursion;
4) Mutual recursion.

2. REMARKS ON THE PROOF METHOD AND ON THE NOTATION

Suppose, f and g are recursively defined functions: $Def\ f = F(f)$ and $Def\ g = G(g)$. In the paper, some equivalence relations in the form $L(f) = R(g)$ are considered. Throughout the paper such equality is proved by the following variant of fixpoint induction (Scot's μ-induction) [7]:

Base. $L(\bot) = R(\bot)$.

Let $f_0 \subseteq f_1 \subseteq f_2 \subseteq \ldots$ and $g_0 \subseteq g_1 \subseteq g_2 \subseteq \ldots$ are two monotonicaly increasing sequence of functions, such that $L(f_i) = R(g_i)$. Then $L(\bigcup_{i=0}^{\infty} f_i) = R(\bigcup_{i=0}^{\infty} g_i)$.

Induction. It is demonstrated that the assumption $L(f) = R(g)$ implies $L(F(f)) = R(G(g))$.

Second step of the base is always true in FP-systems because all function and functionals are continuous [6,7,14,15].

As far as notations are concerned, it is supposed that the reader is familiar with those ones suggested by Backus in [2].

Throughout the paper, E denotes an arbitrary function, which has integer values when is defined. In addition, q, q_0, q_1, $\ldots$, q_{n-1} denote arbitrary integers.

For the reader who is unfamiliar with the notation of FP-systems, suffice it to say that $\circ$ signifies the ordinary (left-associative) composition of functions; the construction $[f_1, f_2, \ldots, f_n]$ is n-tuple of of functions f_i, $(1 \leqq i \leqq n)$. $\bot$ is everywhere undefined function, $\bar{x}$ signify constant function.

Some of the laws of FP-algebra we use are:

1. $\mathrm{id} \circ f = f \circ \mathrm{id} = f$
2. $f \circ (p \longrightarrow g; h) = p \longrightarrow f \circ g; f \circ h$
3. $(p \longrightarrow g; h) \circ f = p \circ f \longrightarrow g \circ f; h \circ f$
4. $k \circ [f_1, \ldots, f_n] = f_k \quad (1 \leqq k \leqq n)$
5. $\bot \circ f = \bot$

3. RECURSION WITH RESPECT TO ONE INTEGER ARGUMENT

Let us consider functions whose definitions have the following form:

$$(\mathrm{R}_1) \qquad Def\ f = \mathrm{eq}_0 \longrightarrow \bar{q};\ E \circ [f \circ \mathrm{sub}_1, \mathrm{id}];$$

Example. The definition of the *factorial* function can be given in the following form:

$$Def\ factorial = \mathrm{eq}_0 \longrightarrow \bar{1}; x \circ [factorial \circ \mathrm{sub}_1, \mathrm{id}];$$

Suppose, f is defined according to the scheme (R_1). If $f(n-1)$ is known, then the computation of $f(n)$ is straightforward. $f(n)$ can be computed starting from $f(0) = q$ and computing $f(i+1)$ from $f(i)$ for $i = 0, 1, \ldots, n-1$. A simple table in the form $\langle f(i), i, n \rangle$ can be used. (In the beginning, the table is $\langle q, 0, n \rangle$.) For $i = 1, 2, \ldots, n-1$ the following is executed:

— $f(i+1)$ is computed by means of $f(i)$, which is in the table;
— $f(i+1)$ replaces $f(i)$ in the table;
— the second component i of the table (the "counter") is increased by 1;
— it is checked whether the counter i is already equal to the third component n.

This approach allows the function f to be replaced by a function f', defined according to the following scheme:

$$(\mathrm{I}_1)\qquad \begin{array}{l} Def\ f' = g \circ [\overline{q}, \overline{0}, \mathrm{id}]; \\ Def\ g = \mathrm{eq} \circ [2, 3] \longrightarrow 1;\ g \circ [E \circ [1, \mathrm{add}_1 \circ 2], \mathrm{add}_1 \circ 2, 3]; \end{array}$$

Example. In the case of the *factorial* function the above definitions look like this:

$$Def\ factorial' = g \circ [\overline{1}, \overline{0}, \mathrm{id}];$$

$$Def\ g = \mathrm{eq} \circ [2,3] \longrightarrow 1;\ g \circ [x \circ [1, \mathrm{add}_1 \circ 2], \mathrm{add}_1 \circ 2, 3];$$

The correctness of the transformation under consideration is established by the following theorem:

Theorem 1. *Let f be defined according to the scheme* (R_1) *and let f' be defined according to the scheme* (I_1). *Then $f = f'$.*

Proof. The equality $g \circ [q, 0, \mathrm{id}] = f$ is to be proved.

Base. $\perp \circ [\overline{q}, \overline{0}, \mathrm{id}] = \perp$.

Induction. Suppose $f = f'$. Substitute f' for f in the right side of the definition of f. It is necessary to prove that the resulting expression is equivalent to the right side of the definition of f':

(1) $\mathrm{eq}_0 \longrightarrow \overline{q}; E \circ [f' \circ \mathrm{sub}_1, \mathrm{id}]$

(Substitute the right side of the definition of f' for f'.)

$$= \mathrm{eq}_0 \longrightarrow \overline{q};\ E \circ [g \circ [\overline{q}, \overline{0}, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}]$$

$$= \mathrm{eq} \circ [2,3] \circ [\overline{q}, \overline{0}, \mathrm{id}] \longrightarrow 1 \circ [\overline{q}, \overline{0}, \mathrm{id}]; E \circ [g \circ [\overline{q}, \mathrm{sub}_1 \circ 1, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}]$$

In order to reduce the above expression, the following Lemma is used:

Lemma 1. *For arbitrary integers a, b:*

$$E \circ [g \circ [\overline{a}, \mathrm{sub}_1 \circ b, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}] = g \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}].$$

According to the above lemma, expression (1) becomes:

$$\mathrm{eq} \circ [2,3] \circ [\overline{q}, \overline{0}, \mathrm{id}] \longrightarrow 1 \circ [\overline{q}, \overline{0}, \mathrm{id}];\ g \circ [E \circ [\overline{q}, \overline{1}], \overline{1}, \mathrm{id}]$$

$$= (eq \circ [2,3] \longrightarrow 1;\ g \circ [E \circ [1, \mathrm{add}_1 \circ 2], \mathrm{add}_1 \circ 2, 3]) \circ [\overline{q}, \overline{0}, \mathrm{id}]$$

$$= g \circ [\overline{q}, \overline{0}, \mathrm{id}],$$

and Theorem 1 is proved.

As far as lemma 1 is concerned, it can be proved by fixpoint induction, too:

Base. $E \circ [\perp \circ [\overline{a}, \mathrm{sub}_1 \circ \overline{b}, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}] = \perp \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}]$.

Induction. Let denote $a' = E \circ [\overline{a}, \overline{b}]$, $b' = \mathrm{add}_1 \circ \overline{b}$. If a' and b' are integer constants, then according to the inductive hypothesis:

$$g \circ [E[\overline{a'}, \overline{b'}], \overline{b'}, \mathrm{id}] = E \circ [g \circ [\overline{a'}, \mathrm{sub}_1 \circ \overline{b'}, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}].$$

The same equality holds when $a' = \perp$ or $b' = \perp$, or the both.

Now, the right side of the equality under consideration can be unfolded and reduced as follows:

$$g \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}] =$$

(definition of g)

$$= \mathrm{eq} \circ [\overline{b}, \mathrm{id}] \longrightarrow E \circ [\overline{a}, \overline{b}]; g \circ [E \circ [E \circ [\overline{a}, \overline{b}], \mathrm{add}_1 \circ \overline{b}], \mathrm{add}_1 \circ \overline{b}, \mathrm{id}]$$

$$= \mathrm{eq} \circ [\overline{b}, \mathrm{id}] \longrightarrow E \circ [\overline{a}, \mathrm{id}]; g \circ [E \circ [\overline{a'}, \overline{b'}], \overline{b'}, \mathrm{id}]$$

(inductive hypothesis)

$$= \mathrm{eq} \circ [\overline{b}, \mathrm{id}] \longrightarrow E \circ [\overline{a}, \mathrm{id}]; E \circ [g \circ [\overline{a'}, \mathrm{sub}_1 \circ \overline{b'}, id] \circ \mathrm{sub}_1, \mathrm{id}]$$

$$= \mathrm{eq} \circ [\overline{b}, \mathrm{id}] \longrightarrow E \circ [\overline{a}, \mathrm{id}]; E \circ [g \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}]$$

(FP-algebra)

$$= E \circ [(\mathrm{eq} \circ [\overline{b}, \mathrm{id}] \longrightarrow \overline{a}; g \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}] \circ \mathrm{sub}_1), \mathrm{id}]$$

$$= E \circ [(\mathrm{eq} \circ [\overline{b-1}, \mathrm{id}] \longrightarrow \overline{a}; g \circ [E \circ [\overline{a}, \overline{b}], \overline{b}, \mathrm{id}]) \circ \mathrm{sub}_1, \mathrm{id}]$$

(definition of g, folding)

$$= E \circ [g \circ [\overline{a}, \mathrm{sub}_1 \circ \overline{b}, \mathrm{id}] \circ \mathrm{sub}_1, \mathrm{id}].$$

The last expression is just the left side of the equality. This is the end of the proof of lemma 1.

Now, the following more general class of definitions is considered:

$$(\mathrm{R}_2) \qquad Def\ f = \mathrm{eq}_0 \longrightarrow \overline{q_0}; \mathrm{eq}_1 \longrightarrow \overline{q_1}; \dots ; \mathrm{eq} \circ [\overline{k-1}, \mathrm{id}] \longrightarrow \overline{q_{k-1}};$$
$$E \circ [[f \circ \mathrm{sub}_1, f \circ \mathrm{sub}_1^2, \dots, f \circ \mathrm{sub}_1^k], \mathrm{id}].$$

Example. The Fibonacci function is defined by a particular instance of the scheme (R_2), with $k = 2$, $E = + \circ 1$:

$$Def\ Fib = \mathrm{eq}_0 \longrightarrow \overline{1}; \mathrm{eq}_1 \longrightarrow \overline{1}; + \circ [Fib \circ \mathrm{sub}_1, Fib \circ \mathrm{sub}_1^2].$$

In the scheme (R_2), $f(n)$ depends on $f(n-1), f(n-2), \dots, f(n-k)$. In that, the values of f for the first k natural numbers are known. It is convenient to use a table, which has the form

$$\langle\langle f(i), f(i-1), \dots, f(i-k+1)\rangle, i, n\rangle.$$

In the beginning of the computation, the table has the following contents:

$$\langle\langle q_{k-1}, q_{k-2}, \dots, q_0\rangle, k-1, n\rangle.$$

In order to compute $f(n)$, the values $f(k)$, $f(k+1)$, $f(k+2), \dots$ are computed one after the other. In that, $f(i+1)$ is computed simply by application of E to the first element of the table, i.e. the list $\langle f(i), f(i-1), \dots, f(i-k+1)\rangle$ of k preceding The value so obtained is appended to the left of the same list, and its rightmost component $f(i-k+1)$ is dropped. Finally, on each step of the computation the second component of the table (the counter) is increased by one. The computation stops when the counter becomes equal to n.

Using such kind of tabulation, f can be transformed into a tail-recursive function f', which is defined according to the following scheme:

$$(\mathrm{I}'_2)\quad \begin{aligned} &Def\ f = \mathrm{eq}_0 \longrightarrow \overline{q_0};\ \mathrm{eq}_1 \longrightarrow \overline{q_1};\ \dots;\ \mathrm{eq} \circ [\overline{k-1}, \mathrm{id}] \longrightarrow \overline{q_{k-1}};\\ &\qquad g \circ [[\overline{q_{k-1}}, \overline{q_{k-2}}, \dots, \overline{q_0}], \overline{k-1}, id],\\ &Def\ g = \mathrm{eq} \circ [2,3] \longrightarrow 1 \circ 1;\ g \circ [[E \circ [1, \mathrm{add}_1 \circ 2], 1 \circ 1, \dots, (k-1) \circ 1],\\ &\qquad \mathrm{add}_1 \circ 2, 3]. \end{aligned}$$

Example. The Fibonacci function can be defined by a tail-recursive definition, following the scheme (I'_2):

$$Def\ Fib' = \mathrm{eq}_0 \longrightarrow \overline{1};\ \mathrm{eq}_2 \longrightarrow \overline{1};\ g \circ [[\overline{1}, \overline{1}], \overline{1}, id],$$

$$Def\ g = \mathrm{eq} \circ [2,3] \longrightarrow 1 \circ 1;\ g \circ [[+ \circ 1, 1 \circ 1], \mathrm{add}_1 \circ 2, 3].$$

Theorem 2. *Let f be defined according to the scheme* (R_2) *and let f' be defined according to the scheme* (I'_2). *Then $f = f'$.*

4. RECURSION WITH RESPECT TO AN INTEGER ELEMENT OF A LIST

There are many functions which are applied to lists in the form

$$\langle n, a_2, a_3, \dots, a_r \rangle,$$

and which are defined recursively with respect to the first element n of the list (n is integer). Such functions can be defined according to the following scheme:

$$(\mathrm{R}_3)\qquad Def\ f = \mathrm{eq}_0 \circ 1 \longrightarrow \overline{q};\ E \circ [f \circ [\mathrm{sub}_1 \circ 1, 2, \dots, r], j].$$

Example. According to the above scheme, multiplication of natural numbers can be expressed by addition as follows:

$$Def\ mult = \mathrm{eq}_0 \circ 1 \longrightarrow \overline{0};\ + \circ [mult \circ [\mathrm{sub}_1 \circ 1, 2], 2].$$

Functions, defined according to the scheme (R_3), can be transformed to tail-recursive ones by means of a table in the form:

$$\langle f(i), i, \langle n, a_2, a_3, \dots, a_r \rangle \rangle.$$

(When the computation starts, the content of the table is $\langle q, 0, \langle n, a_2, a_3, \dots, a_r \rangle \rangle$.) The second component i of the table is the counter — the computation starts with a counter 0 and stops when the counter is equal to n.

A tail-recursive function f' which is equivalent to f can be defined according to the following scheme:

$$(\mathrm{I}_3)\quad \begin{aligned} &Def\ f' = g \circ [\overline{q}, \overline{0}, \mathrm{id}],\\ &Def\ g = \mathrm{eq} \circ [2, 1 \circ 3] \longrightarrow 1;\\ &\qquad g \circ [E \circ [1, j \circ [\mathrm{add}_1 \circ 2, 2 \circ 3, 3 \circ 3, \dots, r \circ 3]], \mathrm{add}_1 \circ 2, 3]. \end{aligned}$$

Example. The scheme (I_3) determines the following tail-recursive definition of the multiplication function:

$$Def\ mult' = g \circ [\overline{0}, \overline{0}, \mathrm{id}],$$

$$Def\ g = \mathrm{eq} \circ [2, 1 \circ 3] \longrightarrow 1;$$

$$g \circ [+ \circ [1, 2 \circ 3], \mathrm{add}_1 \circ 2, 3].$$

Theorem 3. *Let f be defined according to the scheme* (R_3) *and let f′ be defined according to the scheme* (I_3). *Then* $f \doteq f'$.

Proof. *Base.* $\bot \circ [\bar{q}, \bar{0}, \mathrm{id}] = \bot$.

Induction. The substitution of f' for f in the right side of the definition of f leads to the following:

$$\mathrm{eq}_0 \circ 1 \longrightarrow \bar{q};$$

$$E \circ [g \circ [\bar{q}, \bar{0}, \mathrm{id}] \circ [\mathrm{sub}_1 \circ 1, 2 \circ 3, 3 \circ 3, \ldots, r \circ 3], j]$$

(Here Lemma 2 — see below — is applied:)

$$= (\mathrm{eq} \circ [2, 1 \circ 3] \longrightarrow 1;$$

$$g \circ [E \circ [1, j \circ [\mathrm{add}_1 \circ 2, 2 \circ 3, 3 \circ 3, \ldots, r \circ 3]], \mathrm{add}_1 \circ 2, 3]) \circ [\bar{q}, \bar{0}, \mathrm{id}]$$

$$= g \circ [\bar{q}, \bar{0}, \mathrm{id}],$$

and this is the end of the proof of Theorem 3.

Lemma 2. *For any integers q and a the following equality holds:*

$$E \circ [g \circ [\bar{q}, \bar{a}, \mathrm{id}] \circ [\mathrm{sub}_1 \circ 1, 2 \circ 3, 3 \circ 3, \ldots, r \circ 3], j]$$

$$= g \circ [E \circ [\bar{q}, j \circ [\mathrm{add}_1 \circ \bar{a}, 2 \circ 3, 3 \circ 3, \ldots, r \circ 3]], \mathrm{add}_1 \circ \bar{a}, \mathrm{id}].$$

The proof again can be carried out by fixpoint induction.

Example. A function, which computes the power when applied to a list $\langle n, a \rangle$ is defined according to the scheme (R_3):

$$Def\ power = \mathrm{eq}_0 \circ 1 \longrightarrow \bar{1}; * \circ [power \circ [\mathrm{sub}_1 \circ 1, 2], 2].$$

Its tail-recursive variant, obtained according the scheme (I_3) looks like this:

$$Def\ power' = g \circ [\bar{1}, \bar{0}, \mathrm{id}],$$

$$Def\, g = \mathrm{eq} \circ [2, 1 \circ 3] \longrightarrow 1;\ g \circ [* \circ [1, 2 \circ 3], \mathrm{add}_1 \circ 2, 3].$$

We introduce two slightly modified versions of the schemes (R_3) and (I_3):

(R_3') $\quad Def\ f = \mathrm{eq}_0 \circ 1 \longrightarrow \bar{q};\ E \circ [f \circ \mathrm{apndl} \circ [\mathrm{sub}_1 \circ 1, \mathrm{tl}], j].$

$$Def\ f' = g \circ [\bar{q}, \bar{0}, \mathrm{id}],$$

(I_3') $\quad Def\ g = \mathrm{eq} \circ [2, 1 \circ 3] \longrightarrow 1;$

$$g \circ [E \circ [1, j \circ \mathrm{apndl} \circ [\mathrm{add}_1 \circ 2, \mathrm{tl} \circ 3]], \mathrm{add}_1 \circ 2, 3].$$

The last two schemes are useful, for instance, when computing sums and products of the following kind:

$$p(i, a_1, a_2, \ldots, a_r) = sigma(n, a_1, a_2, \ldots, a_r),$$

$$p(i, a_1, a_2, \ldots, a_r) = pi(n, a_1, a_2, \ldots, a_r).$$

Corresponding recursive functions are defined according to the scheme (R_3') as follows:

$$Def\ sigma = eq0 \circ 1 \longrightarrow \bar{0};$$

$$+ \circ [p, sigma \circ \mathrm{apndl} \circ [\mathrm{sub}_1 \circ 1, \mathrm{tl}]],$$

$$Def\ pi = \text{eq}_0 \circ 1 \longrightarrow \bar{1};$$
$$* \circ [p, pi \circ \text{apndl} \circ [\text{sub}_1 \circ 1, \text{tl}]].$$

A tail-recursive version of *sigma* is easily obtained according to the scheme (I_3'):

$$Def\ sigma' = g \circ [\bar{0}, \bar{0}, \text{id}],$$
$$Def\ g = \text{eq} \circ [2, 1 \circ 3] \longrightarrow 1;$$
$$g \circ [+ \circ [1, p \circ \text{apndl} \circ [\text{add}_1 \circ 2, \text{tl} \circ 3]], \text{add}_1 \circ 2, 3].$$

A tail-recursive version of the function *pi* can be obtained in an analogous way.

5. FULL TABULATION

Let us consider a function f of an integer argument, such that:
— its value $f(0) = q$ is known;
— the value of $f(n)$ depends on n and $f(n-1)$, $f(n-2)$, ..., $f(0)$.
A function g can be considered, which tabulates all values of f:

$$g(n) = \langle f(n), f(n-1), \ldots, f(0) \rangle.$$

Hence, f can be defined by means of g as follows:

$$Def\ f = \text{eq}_0 \longrightarrow \bar{q};\ E \circ \text{apndl} \circ [\text{id}, g \circ \text{sub}_1],$$
$$Def\ g = \text{eq}_0 \longrightarrow [\bar{q}];\ \text{apndl} \circ [f, g \circ \text{sub}_1].$$

Since $f = 1 \circ g$, the computation of f is reduced to the computation of g. From the other hand, g can easily be replaced by a tail-recursive function:

$$Def\ g = g'[[\bar{q}], \bar{0}, \text{id}],$$
$$Def\ g' = \text{eq} \circ [2, 3] \longrightarrow 1;$$
$$g' \circ [\text{apndl} \circ [E \circ \text{apndl} \circ [\text{add}_1 \circ 2, 1], 1], \text{add}_1 \circ 2, 3].$$

The function g tabulates all values of f. Such kind of full tabulation is inefficient. Actually, it is not needed in practice. The idea behind it however can be used in order to implement some restricted forms of tabulation.

Let us consider again a function, whose value $f(n)$ depends only on n and $f(n-1)$, $f(n-2)$, ..., $f(n-k)$, as in scheme (R_2). Now we consider a slightly different scheme:

$$(R_4)\quad \begin{array}{l} Def\ f = \text{eq}_0 \longrightarrow \overline{q_0};\ \text{eq}_1 \longrightarrow \overline{q_1};\ \ldots\ \text{eq} \circ [\overline{k-1}, \text{id}] \longrightarrow \overline{q_{k-1}}; \\ \qquad E \circ [\text{id}, f \circ \text{sub}_1, f \circ \text{sub}_1^2, \ldots, f \circ \text{sub}_1^k]. \end{array}$$

An equivalent tail-recursive definition follows:

$$(I_4)\quad \begin{array}{l} Def\ f' = \text{eq}_0 \longrightarrow \overline{q_0};\ \text{eq}_1 \longrightarrow \overline{q_1};\ \ldots; \\ \qquad \text{eq} \circ [\overline{k-1}, \text{id}] \longrightarrow \overline{q_{k-1}};\ 1 \circ g; \\ Def\ g = \text{eq}_0 \longrightarrow [\overline{q_0}];\ \text{leq} \circ [\text{id}, \overline{k-1}] \longrightarrow \text{apndl} \circ [f', g \circ \text{sub}_1]; \\ \qquad \text{tlr} \circ \text{apndl} \circ [f', g \circ \text{sub}_1]. \end{array}$$

It can be demonstrated that the functions f and f', defined by the schemes (R_4) and (I_4), are equivalent.

Let us note that the function g can be easily defined a tail-recursive variant:

$$Def\ g = g' \circ [[\overline{q_{k-1}}, \overline{q_{k-2}}, \ldots, \overline{q_0}], \overline{k-1}, \mathrm{id}];$$
$$Def\ g' = \mathrm{eq} \circ [2,3] \longrightarrow 1;$$
$$g' \circ [\mathrm{apndl} \circ [E \circ [\mathrm{apndl} \circ [\mathrm{add}_1 \circ 2, 1]], \mathrm{tlr} \circ 1], \mathrm{add}_1 \circ 2, 3];$$

Examples. Let us consider again the factorial function:

$$Def\ factorial = \mathrm{eq}_0 \bar{1};\ x \circ [\mathrm{id}, factorial \circ \mathrm{sub}_1].$$

(The above definition slightly differs from the one considered earlier.) According to the above schemes, the same function can be defined with tail-recursion:

$$Def\ factorial' = \mathrm{eq}_0 \longrightarrow \bar{1};\ 1 \circ g;$$
$$Def\ g = g' \circ [[\bar{1}], \bar{0}, \mathrm{id}];$$
$$Def\ g' = \mathrm{eq} \circ [2,3] \longrightarrow 1;$$
$$g' \circ [\mathrm{apndl} \circ [x \circ [\mathrm{apndl} \circ [\mathrm{add}_1 \circ 2, 1]], \mathrm{tlr} \circ 1], \mathrm{add}_1 \circ 2, 3];$$

The same scheme can be applied to the Fibonacci function.

6. MUTUALLY RECURSIVE FUNCTIONS

Two mutually recursive functions f and g, defined as follows:

$$(R_5)\quad \begin{array}{l} Def\ f = \mathrm{eq}_0 \longrightarrow \overline{q_1};\ E_1 \circ [f \circ \mathrm{sub}_1, g \circ \mathrm{sub}_1, \mathrm{id}]; \\ Def\ g = \mathrm{eq}_0 \longrightarrow \overline{q_2};\ E_2 \circ [f \circ \mathrm{sub}_1, g \circ \mathrm{sub}_1, \mathrm{id}]; \end{array}$$

can be reduced to functions f' and g' according to the scheme below:

$$(R_5')\quad \begin{array}{l} Def\ f' = h \circ [\bar{1}, \mathrm{id}]; \qquad Def\ g' = h \circ [\bar{2}, \mathrm{id}]; \\ Def\ h = \mathrm{and} \circ [\mathrm{eq}_0 \circ 2, \mathrm{eq}_1 \circ 1] \longrightarrow \overline{q_1};\ \mathrm{and} \circ [\mathrm{eq}_0 \circ 2, \mathrm{eq}_2 \circ 1] \longrightarrow \overline{q_2}; \\ \qquad t \circ [1, h \circ a, h \circ b, \mathrm{id}]; \\ Def\ t = \mathrm{eq}_1 \circ 1 \longrightarrow E_1 \circ [2,3,4];\ E_2 \circ [2,3,4]; \\ Def\ a = [\bar{1}, \mathrm{sub}_1]; \qquad Def\ b = [\bar{2}, \mathrm{sub}_1]; \end{array}$$

The function h can be reduced to a tail-recursive one. To do so, tables in the form

$$\langle\langle h(1,i), h(2,i)\rangle, \langle 1,n\rangle, i\rangle, \qquad \langle\langle h(1,i), h(2,i)\rangle, \langle 2,n\rangle, i\rangle,$$

are used. (The first table is used when f is to be computed, and the second — respectively when g is to be computed.) The tables allow to obtain an iterative variant g' of the above defined function g, and consequently — iterative variants of f' and g':

$$Def\ f' = h' \circ [\bar{1}, \mathrm{id}]; \qquad Def\ g' = h' \circ [\bar{2}, \mathrm{id}];$$

$$Def\ h' = h'' \circ [[\overline{q_1}, \overline{q_2}], \mathrm{id}, \bar{0}];$$

$$Def\ h'' = \mathrm{and} \circ [\mathrm{eq} \circ [2 \circ 2, 3], \mathrm{eq}_1 \circ 1 \circ 2] \longrightarrow 1 \circ 1;$$
$$\mathrm{and} \circ [\mathrm{eq} \circ [2 \circ 2, 3], \mathrm{eq}_2 \circ 1 \circ 2] \longrightarrow 2 \circ 1;$$
$$h'' \circ [[E_1 \circ [1 \circ 1, 2 \circ 1, \mathrm{add}_1 \circ 3],$$
$$E_2 \circ [1 \circ 1, 2 \circ 1, \mathrm{add}_1 \circ 3]], 2, \mathrm{add}_1 \circ 3];$$

7. CONCLUSION

By means of tabulation, some more complicated recursive definitions can also be reduced to tail-recursive ones. Among them are, for instance, definitions in which recursion is with respect to several parts of the argument $\langle n, k \rangle$:

$$Def\ f = \mathrm{eq}_0 \circ 1 \longrightarrow \overline{q_1};\ \mathrm{eq}_0 \circ 2 \longrightarrow \overline{q_2};$$
$$E \circ [f \circ [\mathrm{sub}_1 \circ 1, 2], f \circ [1, \mathrm{sub}_1 \circ 2], \mathrm{id}].$$

Thus, tabulation techniques allow to obtain many interesting schemes for recursion removal in FP-systems.

REFERENCES

1. Arsac J., Y. Codratoff. Some techniques for recursion removal from recursive functions. *ACM Transactions on Programming Languages and Systems*, 4, 1982, 295–322.
2. Backus, J. Can programming be liberated from the von Neumann style? A functional style and its algebra of programs. *Communications of the ACM*, 21, 1978, 613–641.
3. Backus, J. W. The algebra of functional programs: functional level reasoning, linear equations and extended definitions. *In*: Formalization of programming concepts, ed. Diaz and Ramos, Springer–Verlag (1981).
4. Bird, R. Tabulation techniques for recursion removal. *Computing Surveys*, 12, 1980, 403–417.
5. Bustall, R., J. Darlington. A transformation system which automatically improves programs. *Acta Informatica*, 6, 1976, 41–60.
6. Hikita, T. On a class of recursive procedures and equivalent iterative ones. *Acta Informatica*, 12, 1979, 305–320.
7. Kieburtz, R., J. Shultis. Transformation of FP schemes. *In*: Proceedings of the conference on functional programming languages and computer architecture, Portmounth, New Hamphire, New York: ACM 1981 41–48.
8. Manna, Z. Mathematical theory of computation. New York, McGraw-Hill 1974.
9. Partsch, H., P. Pepper. A family of rules for recursion removal. *Information Processing Letters*, 5, 1976, 174–177.
10. Radensky, A. Computer, programming language, compiler. One nonconventional programming language , and its implementation. Sofia, Nauka i Izkustvo 1987 (in Bulgarian).
11. Reddy, U., B. Jayaraman. Theory of linear equations applied to program transformation. *In*: Proceedings of the eighth international joint conference on artificial intelligence, Karlsruhe (A. Bundy, ed.), Los Altos, California: WII, 1, 10–16.
12. Trachtenbrot, M. Transformations of recursive programs as a mean of efficient program construction. *In*: Program optimization and transformation. Proceedings of a national seminar (A. Ershov, ed.), Novosibirsk, VC SO AN SSSR 198?, part 2, 119–138.

13. W i l l i a m s, J. H. On the development of the algebra of functional program. *ACM Trans. on Prog. Lang. and Systems*, 4, 1982, 4, 733–757.
14. П а т е р с о н М. С., С. Е. Х ь ю и т. Сравнительная схематология. *В кн.*: Кибернетический сборник. Вып. 13(новая серия). М., „Мир", 1976, 62–72.
15. Г а р л э н д С., Д. Л а к х э м. Стандартные схемы, рекурсивные схемы и формальные языки. *В кн.*: Кибернетический сборник. Вып. 13(новая серия), М., „Мир", 73–119.

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РАВНОМЕРНАЯ АСИМПТОТИКА СПЕКТРАЛЬНОЙ ФУНКЦИИ ВОЗМУЩЕННОГО ГАРМОНИЧЕСКОГО ОСЦИЛЛЯТОРА И РАВНОСХОДИМОСТЬ РЯДОВ ПО СОБСТВЕННЫМ ФУНКЦИЯМ

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Георги Е. Караджов. РАВНОМЕРНАЯ АСИМПТОТИКА СПЕКТРАЛЬНОЙ ФУНКЦИИ ВОЗМУЩЕННОГО ГАРМОНИЧЕСКОГО ОСЦИЛЛЯТОРА И РАВНОСХОДИМОСТЬ РЯДОВ ПО СОБСТВЕННЫМ ФУНКЦИЯМ

Получена равномерная асимптотика спектральной функции $e(\lambda, x, y)$ при $\lambda \to +\infty$, $x, y \in \mathbb{R}$, для одного класса глобально эллиптических существенно самосопряженных псевдодифференциальных операторов с главным символом $\xi^2 + x^2$. На этой основе доказана теорема о равносходимости рядов Фурье по собственным функциям, обобщающая известную теорему Сеге [1] о равносходимости рядов Эрмита.

Georgi E. Karadzhov. UNIFORM ASYMPTOTICS OF THE SPECTRAL FUNCTION FOR PERTURBED HARMONIC OSCILLATOR AND EQUICONVERGENCE OF SERIES WITH RESPECT TO THE EIGENFUNCTIONS

One obtains an uniform asymptotics of the spectral function $e(\lambda, x, y)$ as $\lambda \to +\infty$, $x, y \in \mathbb{R}$ for a class of globally elliptic essentially self-adjoint pseudodifferential operators with a principal symbol $\xi^2 + x^2$. On this basis one proves an equiconvergence theorem for Fourier series with respect to the eigenfunctions, which generalises the well-known theorem of Szegő [1] on equiconvergency of Hermite series.

1. Пусть $A = a(x, D_x)$ — существенно самосопряженный в $L^2(\mathbb{R})$ псевдодифференциальный оператор с полным символом $a(x, \xi) = \xi^2 + x^2 + b(x, \xi)$. Будем предполагать, что функция $b \in C^\infty(\mathbb{R}^2)$ удовлетворяет оценке

(H₁) $$|\partial^\alpha b(x, \xi)| \leq C_\alpha (1 + |x| + |\xi|)^{-|\alpha|}, \qquad (x, \xi) \in \mathbb{R}^2,\ C_\alpha > 0,$$

либо асимптотике

(H₂) $$b(x, \xi) \sim \sum_{k \geq 0} b_k(x, \xi) \quad \text{и} \quad b_0(x, -\xi) = -b_0(x, \xi),\ x \neq 0,$$

где $b_k \in C^\infty(\mathbb{R}^2 \setminus 0)$ имеет свойство однородности:

$$b_k(\sqrt{\lambda}x, \sqrt{\lambda}\xi) = \lambda^{-k} b_k(x,\xi), \quad \lambda > 0,\ k \geqq 0. \tag{1}$$

Так как оператор A — глобально эллиптический, то его спектр состоит только из собственных чисел конечной кратности $\lambda_1 \leqq \lambda_2 \leqq \ldots, \lambda_j \to +\infty$, поэтому его спектральная функция $e(\lambda, x, y)$ определяется равенством

$$e(\lambda, x, y) = \sum_{\lambda_j \leqq \lambda} \bar{\varphi}_j(x)\varphi_j(y), \tag{2}$$

где $\{\varphi_j\}$ — полная ортонормированная система собственных функций [2]. Отметим, что $\varphi_j \in C^\infty(\mathbb{R})$.

Рассмотрим функцию $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ и ее ряд Фурье

$$f(y) \sim \sum_{n=0}^{\infty} a_n \varphi_n(y), \quad a_n = \int_{-\infty}^{+\infty} f(x)\bar{\varphi}_n(x)\, dx \tag{3}$$

по собственным функциям $\{\varphi_n\}$. Согласно (2) n-ая частичная сумма $s_n(f,y) = \sum\limits_{\lambda_k \leqq n} a_k\varphi_k(y)$ представляется в виде

$$s_n(f,y) = \int_{-\infty}^{+\infty} f(x)e(n, x, y)\, dx. \tag{4}$$

Цель работы найти равномерную асимптотику спектральной функции $e(\lambda, x, y)$ при $\lambda \to +\infty$, $x, y \in \mathbb{R}$, и на основе формулы (4) доказать следующую теорему о равносходимости.

Теорема о равносходимости. *Пусть оператор A удовлетворяет предположению* (H_1) *и функция $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ обладает свойствами*:

$$\int_{|x|>1} |x|^{-1}|f(x)|\, dx < \infty; \tag{S_1}$$

$$\int a(\lambda, x)(1 - x^2/\lambda)^{-1/4}|f(x)|\, dx = o(\lambda^{1/2}), \quad \lambda \to +\infty, \tag{S_2}$$

где $a(\lambda, x)$ — характеристическая функция множества $\{x : (1-\delta)\lambda < x^2 < \lambda - \lambda^{1/3+\delta}\}$ и

$$\int b(\lambda, x)|f(x)|\, dx = o(\lambda^{1/3}), \quad \lambda \to +\infty, \tag{S_3}$$

где $b(\lambda, x)$ — характеристическая функция множества $\{x : \lambda - \lambda^{1/3+\delta} < x^2 < \lambda + \lambda^{1/3+\delta}\}$ для некоторого $\delta > 0$.

Тогда имеет место следующее утверждение о равносходимости:

$$s_n(f,y) - \frac{1}{\pi}\int_{y-\delta}^{y+\delta} f(x)\frac{\sin\sqrt{n}(x-y)}{x-y}\, dx = o(1), \quad n \to +\infty, \tag{5}$$

где $o(1)$ — *локально равномерно по отношению к параметру* $y \in \mathbb{R}$.

Замечание 1. Если $b = 0$, то ряд Фурье (3) для функции f соответствует ряду Фурье по полиномам Эрмита для функции $f(x)e^{x^2/2}$. В этом случае теорема о равносходимости доказана в [5] и обобщает соответствующий результат Сеге [1, стр. 245].

Сформулируем полученные результаты об асимптотике спектральной функции $e(\lambda, x, y)$. Более удобно выписывать асимптотику функции

$$(6) \qquad E(\lambda, x, y) = e(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y), \quad \lambda \to +\infty.$$

Ввиду ее симметричности по переменным x, y будем предполагать, что $y^2 \leqq x^2$. Тогда достаточно рассмотреть следующие случаи:

1) $x^2 \leqq 1-\delta$, $\delta > 0$; 2) $1-\delta \leqq x^2 \leqq 1$;

3) $1 \leqq x^2 \leqq 1 + \lambda^{-2/3+\sigma}$, $\sigma > 0$; 4) $x^2 \geqq 1 + \lambda^{-2/3+\sigma}$.

Теорема 1 (случай $x^2 \leqq 1-\delta$, $\delta > 0$). *Пусть оператор A удовлетворяет предположению* (H_1). *Тогда для любого малого* $\delta > 0$ *существует* $\varepsilon > 0$ *так, чтобы:*

(i) *Если* $|x-y| \leqq \varepsilon$, *то имеет место равномерная асимптотика*

$$(7) \quad E(\lambda, x, y) = (\lambda\omega)^{-1/2}(2\pi t)^{-1}\sin\lambda\psi(t, \xi, x, y) + O(\lambda^{-1/2}), \quad \lambda \to +\infty,$$

где $\cos 2t = xy + \omega$, $\omega = \left[(1-x^2)(1-y^2)\right]^{1/2}$, $\xi = (1-y^2)^{1/2}$ *и*

$$(8) \qquad \psi(t, \xi, x, y) = t + \varphi(t, \xi, x) - \xi y,$$

$$(9) \qquad \varphi(t, \xi, x) = -\frac{1}{2}(x^2 + \xi^2)\,\mathrm{tg}\,2t + x\xi(\cos 2t)^{-1};$$

(ii) *Если* $|x-y| > \varepsilon$, *то выполнена равномерная оценка*

$$(10) \qquad E(\lambda, x, y) = O(\lambda^{-1/2}), \quad \lambda \to +\infty.$$

Следствие 1. *Пусть оператор A удовлетворяет предположению* (H_1). *Если* $c < x^2 < (1-\delta)\lambda$, *где* $\delta > 0$ *мало и* $y^2 < c/2$, *то*

$$(11) \qquad |e(\lambda, x, y)| \leqq \mathrm{const}\,|x|^{-1}.$$

Замечание 2. В классическом случае, когда $x^2 + y^2 < \mathrm{const}$ имеем более простой результат [3]:

$$(12) \qquad e(\lambda, x, y) = \frac{1}{\pi}\frac{\sin\sqrt{\lambda}(x-y)}{x-y} + O(1), \quad \lambda \to +\infty,$$

равномерно на компактах.

Теорема 2 (случай $1-\delta \leqq x^2 \leqq 1$). *Существуют числа* $\delta > 0$ *и* $\varepsilon > 0$ *такие, что:*

(i) *Если* $|x-y| \leqq \varepsilon$ *и оператор A удовлетворяет предположению* (H_2), *то имеем равномерную асимптотику*

$$(13) \quad E(\lambda, x, y) = a(\lambda, x, y)\lambda^{-1/2} + (b(\lambda, x, y) + c(\lambda, x, y))\,O(\lambda^{-1/6}), \quad \lambda \to +\infty,$$

где

$$(14) \quad a(\lambda, x, y) = \frac{1}{x-y}\left[A_i\left(-B_1\lambda^{2/3}\right)A_i'\left(-B_2\lambda^{2/3}\right) - A_i'\left(-B_1\lambda^{2/3}\right)A_i\left(-B_2\lambda^{2/3}\right)\right],$$

(15) $$b(\lambda,x,y)=(b(\lambda,x)b(\lambda,y))^{1/2},\quad c(\lambda,x,y)=(c_1(\lambda,x,y)c_2(\lambda,x,y))^{1/2},$$

(16) $$b(\lambda,x)=\left(A_i\left(-B(x)\lambda^{2/3}\right)\right)^2+\lambda^{-2/3}\left(A_i'\left(-B(x)\lambda^{2/3}\right)\right)^2,$$

(17) $$c_j(\lambda,x)=\left(A_i\left(-B_j\lambda^{2/3}\right)\right)^2+\lambda^{-2/3}\left(A_i'\left(-B_j\lambda^{2/3}\right)\right)^2,\quad j=1,2.$$

Здесь $B_j=B_j(x,y)$ являются гладкими функциями со свойствами:

(18) $$\begin{cases}B_1(x,y)=4^{-1/3}(1-y^2)+O\left(r(x,y)\right),\\ B_2(x,y)=4^{-1/3}(1-x^2)+O\left(r(x,y)\right),\end{cases}\qquad r(x,y)\to 0,$$

где $r(x,y)=(1-x^2)^2+(x-y)^2$. При этом $B(x)=B_1(x,x)=B_2(x,x)$ и $A_i(s)=\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty}e^{-i(st+t^3/3)}dt$ есть функция Эйри;

(ii) *Если $|x-y|>\varepsilon$ и оператор A удовлетворяет предположению* (H_1), *то выполнена равномерная оценка*

(19) $$E(\lambda,x,y)=b(\lambda,x,y)O(\lambda^{-1/6}),\quad \lambda\to+\infty.$$

Замечание 3. При $y=x$ имеем равенство

(20) $$a(\lambda,x,x)=\left(\frac{1-x^2}{B(x)}\right)^{1/2}f\left(-B(x)\lambda^{2/3}\right)\lambda^{2/3},$$

где $f(s)=-s\left(A_i(s)\right)^2+\left(A_i'(s)\right)^2$, в частности $f(s)=\dfrac{1}{\pi}(-s)^{1/2}+O\left(|s|^{-1}\right)$ при $s\to-\infty$.

Следствие 2. *Пусть оператор A удовлетворяет предположению* (H_1). *Если $(1-\delta)\lambda<x^2<\lambda-\lambda^{1/3+\delta}$, где $\delta>0$ достаточно мало и $y^2<c$, то*

(21) $$|e(\lambda,x,y)|\le \mathrm{const}\,(1-x^2/\lambda)^{-1/4}\lambda^{-1/2}.$$

Теорема 3 (случай $1\le x^2\le 1+\lambda^{-2/3+\sigma}$, $\sigma>0$). *Для любого малого $\sigma>0$ существует $\varepsilon>0$ так, чтобы:*

(i) *Если $|x-y|\le\varepsilon$ и оператор A удовлетворяет предположению* (H_2), *то выполнена равномерная асимптотика*

(22) $$E(\lambda,x,y)=a(\lambda,x,y)\lambda^{-1/2}+O(\lambda^{-1/6}),\quad \lambda\to+\infty,$$

где функция a дается формулой (14);

(ii) *Если $|x-y|>\varepsilon$ и оператор A удовлетворяет предположению* (H_1), *то имеют место равномерные оценки:*

(23) $$E(\lambda,x,y)=\sqrt{b(\lambda,y)}\,O(\lambda^{-1/6}),\qquad \text{если } y^2\le 1,$$

(24) $$E(\lambda,x,y)=O(\lambda^{-1/6}),\qquad \text{если } y^2\ge 1,$$

Следствие 3. *Пусть оператор A удовлетворяет предположению* (H_1). *Если $\lambda-\lambda^{1/3+\delta}\le x^2\le\lambda+\lambda^{1/3+\delta}$, где $\delta>0$ мало и $y^2<c$, то*

(25) $$|e(\lambda,x,y)|\le \mathrm{const}\,\lambda^{-1/3}.$$

Теорема 4 (случай $x^2 \geq \lambda + \lambda^{1/3+\sigma}$, $\sigma > 0$). *Если оператор A удовлетворяет предположению* (H_1), *то имеет место равномерная оценка*

$$e(\lambda, x, y) = O\left(|x|^{-\infty}\right). \tag{26}$$

2. Доказательство теоремы 1. Будем использовать формулу

$$\int \rho(\lambda - \mu)\, de(\mu, x, y) = \frac{1}{2\pi} \int e^{i\lambda t} \hat{\rho}(t) U(t, x, y)\, dt, \tag{27}$$

где ρ — положительная, четная, быстро убывающая функция, чье преобразование Фурье $\hat{\rho}$ имеет компактный носитель, $\hat{\rho}(t) = 1$ в некоторой окрестности нуля и $U(t, x, y)$ является ядром оператора $U(t) = \exp(-itA)$. Согласно [2] можно построить аппроксимацию этого оператора в виде

$$Q(t)\hat{u}(x) = \frac{1}{2\pi} \int e^{i\varphi(t,\xi,x)} q(t, \xi, x) \hat{u}(\xi)\, d\xi, \quad u \in C_0^\infty(\mathbb{R}), \tag{28}$$

где фаза φ дается формулой (9). В случае предположения (H_1) гладкая функция $\partial_t^k q(t, \xi, x)$, $k \geq 0$ удовлетворяет оценкам (H_1) при $|t| \leq \pi/8$ и $q(0, \xi, x) = 1$. В случае предположения (H_2) имеем асимптотическое равенство $q(t, \xi, x) \sim \sum\limits_{k \geq 0} q_k(t, \xi, x)$, где функция $(\xi, x) \to q_k(t, \xi, x)$ является положительно однородной степени $-2k$, гладкой при $(\xi, x) \neq 0$ и $q_k(0, \xi, x) = 0$, $k \geq 1$,

$$\partial_t q_0 + 2\xi \partial_x q_0 + (ib_0(x, \xi) - \operatorname{tg} 2t)\, q_0 = 0, \quad q_0(0, \xi, x) = 1. \tag{29}$$

Далее, из (27) и (28) следует

$$\int \rho(\lambda - \mu)\, de(\mu, x, y) \sim \frac{1}{4\pi^2} \int e^{i(\lambda t + \varphi(t,\xi,x) - \xi y)} q(t, \xi, x) \hat{\rho}(t)\, dt\, d\xi, \tag{30}$$

где эквивалентность „$A(\lambda, x, y) \sim B(\lambda, x, y)$" означает, что $A(\lambda, x, y) - B(\lambda, x, y) = O\left((|\lambda| + x^2 + y^2)^{-\infty}\right)$. Очевидно

$$\int \rho(\lambda - \mu)\, de(\mu, \sqrt{\lambda} x, \sqrt{\lambda} y) \sim \frac{\sqrt{\lambda}}{4\pi^2} \int e^{i\lambda\psi(t,\xi,x,y)} q(t, \sqrt{\lambda}\xi, \sqrt{\lambda} x) \hat{\rho}(t)\, dt\, d\xi,$$

где функция ψ дается формулой (8). Так как имеет место оценка $|\partial_t \psi(t, \xi, x, y)| \geq c(\xi^2 + x^2)$ при больших ξ^2, если $|t| \leq T$ и T — достаточно мало, то можно интегрировать по частям. Предполагая, что $\operatorname{supp} \hat{\rho} \subset (-T, T)$, получаем

$$\int \rho(\lambda - \mu)\, de(\mu, \sqrt{\lambda} x, \sqrt{\lambda} y) \sim \sqrt{\lambda} \int e^{i\lambda\psi(t,\xi,x,y)} r_0(t, \xi, x, \lambda)\, dt\, d\xi, \tag{31}$$

где $r_0(t, \xi, x, \lambda) = (2\pi)^{-2} q(t, \sqrt{\lambda}\xi, \sqrt{\lambda} x) \hat{\rho}(t) \kappa(\xi)$ и $\kappa \in C_0^\infty(\mathbb{R})$ есть срезающая функция. Далее,

$$\int \rho(\lambda - \mu)\, de(\mu, x, y) \sim \int e^{i(x-y)\xi} g(\xi, x, \lambda)\, d\xi, \tag{32}$$

где функция $g(\xi,x,\lambda) = (2\pi)^{-2}\int e^{i(\lambda t+\varphi(t,\xi,x)-\xi x)}\hat{\rho}(t)q(t,\xi,x)\,dt$ допускает оценку вида $|g(\xi,x,\lambda)| \leqq C_N(|\lambda|+\xi^2)^{-N}$ для любого $N > 0$, если $\lambda < 0$ и ξ^2 — большое. Поэтому, интегрируя по частям в правом интеграле в формуле (32), а затем интегрируя по λ, находим при $y^2 \leqq x^2$ формулу

$$(33)\qquad E_\rho(\lambda,x,y) \sim \frac{1}{x-y}\int e^{i\lambda\psi(t,\xi,x,y)}r(t,\xi,x,\lambda)\,dt\,d\xi,$$

где

$$r(t,\xi,x,\lambda) = \frac{\kappa(\xi)}{4\pi^2}\frac{\hat{\rho}(t)}{t}\left[\sqrt{\lambda}i(\partial_\xi\varphi - x)q(t,\sqrt{\lambda}\xi,\sqrt{\lambda}x) + (\partial_\xi q)(t,\sqrt{\lambda}\xi,\sqrt{\lambda}x)\right]$$

и

$$E_\rho(\lambda,x,y) = \int \rho(\lambda-\mu)\,de(\mu,\sqrt{\lambda}x,\sqrt{\lambda}y).$$

Чтобы найти асимптотику этой функции при $\lambda \to +\infty$, будем применять метод стационарной фазы [6], [7]. Согласно (8), (9) вещественные критические точки (t,ξ) фазы $(t,\xi) \to \psi(t,\xi,x,y)$ существуют только при $x^2 \leqq 1$, $y^2 \leqq 1$ и даются формулами

$$\cos 2t_1 = xy - \omega, \quad \cos 2t = xy + \omega, \quad \xi = \pm d(y),$$

где $\omega = \left[(1-x^2)(1-y^2)\right]^{1/2}$, $d(y) = (1-y^2)^{1/2}$. При этом,

$$(34)\qquad |t_1|^2 \geqq c[\omega + |x-y|], \quad c_1|x-y| \leqq |t| \leqq c_2|x-y|, \quad c_1 > 0,$$

если $|x-y| \leqq \varepsilon$ для некоторого $\varepsilon > 0$. Выбирая ε и T достаточно маленькими (напомним, что $\operatorname{supp}\hat{\rho} \subset (-T,T)$ и $\omega \geqq \delta$), можно предполагать, что $\hat{\rho}(t) = 1$, $\hat{\rho}(t_1) = 0$. Следовательно на носителе подинтегральной функции в (33) лежат только критические точки $(t,d(y))$ и $(-t,-d(y))$. При этом

$$(35)\qquad \det\psi'' = \partial_t^2\psi\partial_\xi^2\psi - (\partial_{\xi t}^2\psi)^2 = -4\omega(\cos 2t)^{-1}$$

в критических точках. В частности, к интегралу

$$(36)\qquad I(\lambda,x,y) = \int e^{i\lambda\psi(t,\xi,x,y)}r(t,\xi,x,\lambda)\,dt\,d\xi$$

применим метод стационарной фазы [6], откуда

$$(37)\ I(\lambda,x,y) = (x-y)(2\pi t)^{-1}(\lambda\omega)^{-1/2}\sin\lambda\psi + R(\lambda,x,y)O(\lambda^{-1/2}), \quad \lambda \to +\infty,$$

где $|x-y| \leqq \varepsilon$ и

$$(38)\qquad \partial_y R(\lambda,x,y) = O(1), \quad \lambda \to +\infty.$$

Теперь (33), (35) – (38) показывают, что при $|x-y| \leqq \varepsilon$

$$(39)\quad E_\rho(\lambda,x,y) = (2\pi t)^{-1}(\lambda\omega)^{-1/2}\sin\lambda\psi(t,\xi,x,y) + O(\lambda^{-1/2}), \quad \lambda \to +\infty.$$

Далее,

$$(40)\quad J(\lambda,x) = \int \rho\left(\frac{\lambda-\mu}{k}\right)de(\mu,\sqrt{\lambda}x,\sqrt{\lambda}x) = |k|^m O(\lambda^{-1/2}), \quad \lambda \to +\infty,$$

для некоторого $m > 1$, где $|k| \geqq 1$. Действительно, согласно (31) имеем

$$(41)\qquad J(\lambda,x) \sim \sqrt{\lambda}\int e^{i\lambda\psi(t,\xi,x,x)}h(t,\xi,\lambda)\,dt\,d\xi,$$

где $h(t,\xi,\lambda) = k(2\pi)^{-2}\hat{\rho}(kt)q(t,\sqrt{\lambda}\xi,\sqrt{\lambda}x)\kappa(\xi)$. Так как критические точки фазы ψ невырождены и выполнены оценки (H_1), то (40) верно в силу метода стационарной фазы [7]. Теперь из (40) получаем оценку

$$|e(\lambda+\mu,\sqrt{\lambda}x,\sqrt{\lambda}x) - E(\lambda,x,x)| \leq c(1+|\mu|)^m\lambda^{-1/2}, \quad \lambda > 1,\ \mu \in \mathbf{R},\ x^2 \leq 1-\delta,$$

откуда [8]

$$(42) \qquad |e(\lambda+\mu,\sqrt{\lambda}x,\sqrt{\lambda}y) - E(\lambda,x,y)| \leq c(1+|\mu|)^m\lambda^{-1/2},$$
$$\lambda > 1,\ \mu \in \mathbf{R},\ y^2 \leq x^2 \leq 1-\delta.$$

Наконец (42) и свойство $\hat{\rho}(0) = 1$ влекут

$$(43) \qquad E_\rho(\lambda,x,y) - E(\lambda,x,y) = O(\lambda^{-1/2}), \quad \lambda \to +\infty,\ y^2 \leq x^2 \leq 1-\delta.$$

Ясно, что асимптотика (7) является следствием (39) и (43).

Остается рассмотреть случай $|x-y| > \varepsilon$. Согласно (34) нет критических точек на носителе подинтегральной функции в (33). Поэтому $E_\rho(\lambda,x,y) = O(\lambda^{-\infty})$, что вместе с (42) доказывает оценку (10).

3. Доказательство теоремы 2. Пусть оператор A удовлетворяет предположению (H_2). Тогда можно использовать однородность функции $q_k(t,\xi,x)$ и написать следующее представление интеграла (36):

$$(44) \qquad I(\lambda,x,y) = \sum_{j=0}^{2} I_j(\lambda,x,y) + R(\lambda,x,y),$$

где

$$(45) \qquad I_j(\lambda,x,y) = \lambda^{1/2-j}\int e^{i\lambda\psi(t,\xi,x,y)}r_j(t,\xi,x)\,dt\,d\xi, \quad j = 0,\ 1,\ 2,$$

$$(46) \qquad \partial_y R(\lambda,x,y) = O(\lambda^{-3/2}), \quad \lambda \to +\infty,$$

$$(47) \qquad r_0(t,\xi,x) = i(2\pi)^{-2}t^{-1}\hat{\rho}(t)(\partial_\xi\varphi - x)q_0(t,\xi,x)\kappa(\xi),$$

$$r_j(t,\xi,x) = (2\pi)^{-2}t^{-1}\hat{\rho}(t)\left[i(\partial_\xi\varphi - x)q_j(t,\xi,x) + \partial_\xi q_{j-1}(t,\xi,x)\right]\kappa(\xi), \quad j = 1,\ 2.$$

Чтобы найти равномерную асимптотику интеграла $I_j(\lambda,x,y)$ при $\lambda \to +\infty$, применим теорию версальных деформаций [4], [9]. Сначала отметим, что функция $(t,\xi) \to \psi(t,\xi,x,y)$ аналитична при $|t| < \pi/4$, поэтому ее можно продолжить голоморфно для комплексных значений переменных (t,ξ). Тогда эта функция имеет четыре критические точки $(\pm t, \pm d(y))$ и $(\pm t_1, \pm d(y))$ (см. (34)), которые вещественнозначны при $x^2 \leq 1$, $y^2 \leq 1$ и чисто мнимые, если $x^2 > 1$ либо $y^2 > 1$. При $x = y$ и $x^2 = 1$ все эти критические точки сливаются с точкой (0, 0), вырождаясь, а фаза ψ принимает вид

$$\psi(t,\xi,x,x) = -t\left[(\xi - xt)^2 + t^2 g(t,\xi,x)\right], \quad x^2 = 1,$$

поэтому существует голоморфная замена переменных

$$(48) \qquad t = \tau p(\tau,\eta), \quad \xi = \xi(\tau,\eta)$$

такая, что

$$\psi(t(\tau,\eta),\xi(\tau,\eta),x,x) = \tau\eta^2 + \tau^3/3, \quad x^2 = 1.$$

Нетрудно убедиться [4], [9], что семейство $\{ct + t\xi^2 + t^3/3\}$ является версальной деформацией функции $t\xi^2 + t^3/3$ в классе всех голоморфных, нечетных функций $g(t,\xi)$, $g(0,\xi) = 0$, определенных в окрестности начала, которые вещественнозначны при вещественных значениях переменных (t,ξ). Этот класс инвариантен относительно нечетных локальных диффеоморфизмов $(\tau,\eta) = v(t,\xi)$, $v(0,\xi) = (0,\eta)$, которые вещественнозначны при вещественных значениях переменных (t,ξ). Следовательно, поскольку функция $\psi(t,\xi,x,x)$ принадлежит этому классу, то существует нечетная голоморфная замена переменных вида (48), для которой

(49) $$\psi(t(\tau,\eta),\xi(\tau,\eta),x,x) = -B_0(x)\tau + \tau\eta^2 + \tau^3/3$$

если $|x^2-1| \leqq \delta$ для некоторого $\delta > 0$. При этом

(50) $$B_0(x) = \left(\frac{3}{2}\psi(t_1(x),d(x),x,x)\right)^{2/3}.$$

Равенство (50) следует из того факта, что критическая точка $(\sqrt{B_0},0)$ является образом критической точки $(t_1(x),d(x))$. В частности,

(51) $$B_0(x) = 1 - x^2 + O\left((1-x^2)^2\right) \qquad \text{при } x^2 \to 1.$$

Далее, линейная замена переменных $\tau \mapsto 4^{-1/3}(\tau+\eta)$, $\eta \mapsto 4^{-1/3}(\tau-\eta)$ приводит функцию (49) к виду

(52) $$\psi(t(\tau,\eta),\xi(\tau,\eta),x,x) = -B(x)(\tau+\eta) + \tau^3/3 + \eta^3/3, \quad |x^2-1| \leqq \delta,$$

где

(53) $$B(x) = 4^{-1/3}B_0(x).$$

В частности,

(54) $$\begin{cases} (\sqrt{B},-\sqrt{B}) & \text{является образом точки } (0,d(x)), \\ (\sqrt{B},\sqrt{B}) & \text{есть образ точки } (t_1(x),d(x)). \end{cases}$$

С другой стороны, семейство $\{-B_1\tau - B_2\eta + \tau^3/3 + \eta^3/3\}$ является версальной деформацией для функции $-B(\tau+\eta) + \tau^3/3 + \eta^3/3$ в классе всех нечетных, гладких функций, определенных в окрестности начала, относительно нечетных локальных диффеоморфизмов. Поэтому существует нечетная замена переменных $(t,\xi) \mapsto (\tau,\eta)$ такая, что

(55) $$\psi(t(\tau,\eta),\xi(\tau,\eta),x,y) = -B_1(x,y)\tau - B_2(x,y)\eta + \tau^3/3 + \eta^3/3,$$

если $|x^2-1| \leqq \delta$, $|x-y| \leqq \varepsilon$ для некоторых $\delta > 0$, $\varepsilon > 0$. При этом, $B_1(x,x) = B_2(x,x) = B(x)$. Далее, если δ и ε достаточно малы, то критические точки находятся вблизи начала, поэтому используя принцип стационарной фазы и замену переменных (55), получаем

(56) $$I_j(\lambda,x,y) = \lambda^{1/2-j}\int e^{i\lambda(-B_1\tau - B_2\eta + \tau^3/3 + \eta^3/3)}g_j(\tau,\eta)\,d\tau\,d\eta,$$

где

(57) $$g_j(\tau,\eta) = r_j(t(\tau,\eta),\xi(\tau,\eta),x)J(\tau,\eta)$$

и $J(\tau,\eta)$ является якобианом отображения $(\tau,\eta) \mapsto (t(\tau,\eta),\xi(\tau,\eta))$.

Так как согласно (29) и (H_2) функция g_0 — нечетная, то подготовительная теорема Мальгранжа [7], [9] дает

$$(58)\qquad g_0(\tau,\eta) = a_1(x,y)\tau + a_2(x,y)\eta + (\tau^2 - B_1)f_1 + (\eta^2 - B_2)f_2.$$

Таким образом(56) — (58) и интегрировоние по частям влечет

$$(59)\qquad I_0(\lambda,x,y) = a(\lambda,x,y)\lambda^{-1/2} + (x-y)c(\lambda,x,y)O(\lambda^{-1/6}) + R_0(\lambda,x,y),$$

где

$$(60)\qquad a(\lambda,x,y) = \frac{4\pi^2}{i}\Big[a_1(x,x)A_i'\left(-B_1\lambda^{2/3}\right)A_i\left(-B_2\lambda^{2/3}\right) + a_2(x,x)A_i\left(-B_1\lambda^{2/3}\right)A_i'\left(-B_2\lambda^{2/3}\right)\Big],$$

$$(61)\qquad \partial_y R_0(\lambda,x,y) = c(\lambda,x,y)O(\lambda^{-1/6}), \quad \lambda \to +\infty.$$

При этом здесь были учтены представление $a_j(x,y) = a_j(x,x) + O(x-y)$, неравенство Коши–Буняковского и определения (15), (17). Коэффициенты $a_j(x,x)$ можно подсчитать,используя (58) при $y = x$, откуда $g_0(\sqrt{B},\sqrt{B})' = (a_1(x,x) + a_2(x,x))\sqrt{B}$, $g_0(\sqrt{B},-\sqrt{B}) = (a_1(x,x) - a_2(x,x))\sqrt{B}$, если $x^2 \leq 1$. Поэтому, учитывая еще (54), (57), (47), заключаем, что $a_1(x,x) + a_2(x,x) = 0$ и

$$(62)\qquad a_1(x,x) = -i(2\pi)^{-2}d(x)J(\sqrt{B},-\sqrt{B}).$$

Якобиан $J(\sqrt{B},-\sqrt{B})$ можно найти из равенства

$$(63)\qquad J^2(\tau,\eta)\det\psi'' = 4\tau\eta,$$

которое выполняется в критических точках и следует из (52). Имея ввиду (54) и (35), из (63) вытекает

$$(64)\qquad J(\sqrt{B},-\sqrt{B}) = \sqrt{B}(d(x))^{-1}.$$

Следовательно, (62) и (64) влекут

$$(65)\qquad a_1(x,x) = -i/4\pi^2, \quad a_2(x,x) = i/4\pi^2, \quad \text{если } x^2 \leq 1.$$

Далее, аналогично (59), но отправляясь от представления

$$g_1(\tau,\eta) = b_0 + b_1\tau + b_2\eta + b_3\tau\eta + (\tau^2 - B_1)h_1 + (\eta^2 - B_2)h_2,$$

находим оценку

$$(66)\qquad \partial_y I_1(\lambda,x,y) = c(\lambda,x,y)(\lambda^{-1/6}), \quad \lambda \to +\infty$$

и, тем более, оценку

$$(67)\qquad \partial_y I_2(\lambda,x,y) = c(\lambda,x,y)O(\lambda^{-1/6}), \quad \lambda \to +\infty.$$

Отметим, что $c(\lambda,x,y) \geq c\lambda^{-2/3}$, $c > 0$, как следствие из определений (15), (17) и того факта, что нули функции Эйри и нули ее производной перемежаются. Поэтому (46) переписывается в виде

$$(68)\qquad \partial_y R(\lambda,x,y) = c(\lambda,x,y)O(\lambda^{-1/6}), \quad \lambda \to +\infty.$$

Теперь ясно, что (33), (36), (44), (59) – (61), (65) – (68) влекут асимптотику

(69) $$E_\rho(\lambda,x,y) = a(\lambda,x,y)\lambda^{-1/2} + c(\lambda,x,y)O(\lambda^{-1/6}), \quad \lambda \to +\infty,$$

где коэффициент a дается формулой (14).

Далее, докажем при условии (H_1), что интеграл (40) допускает оценку

(70) $$J(\lambda,x) = |k|^m b(\lambda,x)O(\lambda^{-1/6}), \quad \lambda \to +\infty, \quad \text{если } x^2 \leqq 1,\ |k| \geqq 1$$

для некоторого $m > 1$. Действительно, если $x^2 \leqq 1-\delta$, $\delta > 0$, то имеем оценку (40), а асимптотика функции Эйри [6] и определение (16) показывают, что

(71) $$b(\lambda,x) \geqq c\lambda^{-1/3} \qquad \text{при } 1-x^2 \geqq \delta > 0.$$

Поэтому в этом случае (70) следует из (40) и (71). Пусть теперь $1-\delta \leqq x^2 \leqq 1$. Тогда в интеграле (41) можно сделать замену переменных (52), откуда

(72) $$J(\lambda,x) \sim \sqrt{\lambda}\int e^{i\lambda(-B\tau-B\eta+\tau^3/3+\eta^3/3)}h(\tau,\eta,\lambda)\,d\tau\,d\eta,$$

где

(73) $$h(\tau,\eta,\lambda) = \frac{k}{4\pi^2}\hat\rho(kt)q(t,\sqrt{\lambda}\xi,\sqrt{\lambda}x)\kappa(\xi)J(\tau,\eta).$$

Чтобы проследить зависимость от параметра к, будем использовать теорему Мальгранжа в следующей форме:

(74) $$h(\tau,\eta,\lambda) = a(\eta,\lambda) + b(\eta,\lambda)\tau + (\tau^2-B)f_1(\tau,\eta,\lambda),$$

где

(75) $$a(\eta,\lambda) = \frac{1}{2}\left[h(\sqrt{B},\eta,\lambda) + h(-\sqrt{B},\eta,\lambda)\right],$$

(76) $$b(\eta,\lambda) = \frac{1}{2\sqrt{B}}\left[h(\sqrt{B},\eta,\lambda) - h(-\sqrt{B},\eta,\lambda)\right],$$

(77) $$f_1(\tau,\eta,\lambda) = \frac{1}{2}\int_0^1\int_0^1\Big[\partial_\tau^2 h\big(s(1-\sigma)\sqrt{B}+\tau\sigma,\eta,\lambda\big)(1-\sigma)$$
$$-\partial_\tau^2 h\big(s(1+\sigma)\sqrt{B}+\tau\sigma,\eta,\lambda\big)(1+\sigma)\Big]\,ds\,d\sigma.$$

Аналогично,

(78) $$a(\eta,\lambda) = a_0 + a_2\eta + (\eta^2-B)h_1(\eta,\lambda),$$

(79) $$b(\eta,\lambda) = a_1 + a_3\eta + (\eta^2-B)h_2(\eta,\lambda).$$

Поэтому,

(80) $$h(\tau,\eta,\lambda) = a_0 + a_1\tau + a_2\eta + a_3\tau\eta + (\tau^2-B)f_1 + (\eta^2-B)f_2,$$

где

(81) $$f_2(\tau,\eta,\lambda) = h_1(\eta,\lambda) + \tau h_2(\eta,\lambda).$$

В частности, (73) – (81) дают оценки

(82) $$a_j = O\left(|k|^3\right), \quad 0 \leqq j \leqq 3,$$

(83) $$\partial^\alpha f_j = O\left(|k|^{|\alpha|+4}\right), \quad j = 1,\, 2.$$

Теперь (70) следует из (72), (80), (82), (83), учитывая (H_1) и определение (16). Из (70) вытекает оценка

$$\left|e(\lambda+\mu, \sqrt{\lambda}x, \sqrt{\lambda}x) - E(\lambda, x, x)\right| \leqq c(1+|\mu|)^m b(\lambda, x)\lambda^{-1/6}, \quad \mu \in \mathbb{R},\ x^2 \leqq 1,$$

откуда

$$\left|e(\lambda+\mu, \sqrt{\lambda}x, \sqrt{\lambda}y) - E(\lambda, x, y)\right| \leqq c(1+|\mu|)^m b(\lambda, x, y)\lambda^{-1/6}, \quad \mu \in \mathbb{R},\ y^2 \leqq x^2 \leqq 1,$$

поэтому, учитывая и равенство $\hat{\rho}(0) = 1$, получаем

(84) $$E_\rho(\lambda, x, y) - E(\lambda, x, y) = b(\lambda, x, y)O(\lambda^{-1/6}), y^2 \leqq x^2 \leqq 1.$$

Сравнивая (69) и (84), находим асимптотику (13). Если же $|x-y| > \varepsilon$, то $E_\rho(\lambda, x, y) = O\left(\lambda^{-\infty}\right)$, что вместе с (84) доказывает оценку (19).

Остается проверить асимптотики (18). Из (55) и (8), (9) следует

(85) $$\begin{cases} -B_1(x,y) = (1-x^2)\dfrac{\partial t}{\partial \tau}(0,0,x,y) + (x-y)\dfrac{\partial \xi}{\partial \tau}(0,0,x,y), \\ -B_2(x,y) = (1-x^2)\dfrac{\partial t}{\partial \eta}(0,0,x,y) + (x-y)\dfrac{\partial \xi}{\partial \eta}(0,0,x,y). \end{cases}$$

В частности, $-B(x) = (1-x^2)\dfrac{\partial t}{\partial \tau}(0,0,x,x) = (1-x^2)\dfrac{\partial t}{\partial \eta}(0,0,x,x)$, что вместе с (53), (51) дает

(86) $$\frac{\partial t}{\partial \eta}(0,0,x,x) = \frac{\partial t}{\partial \tau}(0,0,x,x) = -4^{-1/3} + O(1-x^2) \qquad \text{при } x^2 \to 1.$$

С другой стороны, из (55) и (8), (9) следует также

(87) $$\begin{cases} \dfrac{\partial \xi}{\partial \tau}(0,0,x,x) = 2x\dfrac{\partial t}{\partial \tau}(0,0,x,x) + O(1-x^2), \\ \dfrac{\partial \xi}{\partial \eta}(0,0,x,x) = O(1-x^2) \end{cases} \qquad \text{при } x^2 \to 1.$$

Поэтому (86), (87) и формула Тейлора дают

(88) $$\begin{cases} \dfrac{\partial t}{\partial \tau}(0,0,x,y) = -4^{-1/3} + O(1-x^2+|x-y|), \\ \dfrac{\partial \xi}{\partial \tau}(0,0,x,y) = -4^{-1/3}2x + O(1-x^2+|x-y|), \\ \dfrac{\partial t}{\partial \eta}(0,0,x,y) = -4^{-1/3} + O(1-x^2+|x-y|), \\ \dfrac{\partial \xi}{\partial \eta}(0,0,x,y) = O(1-x^2+|x-y|). \end{cases}$$

Теперь ясно, что (85) и (88) влекут асимптотики (18). Теорема 2 доказана.

4. Доказательство следствия 2. При условиях следствия 2 выполнена оценка (19). Так как $(1-x^2)\lambda^{2/3} \geq \lambda^\delta$, то используя асимптотику функции Эйри и учитывая (16), получаем

$$(89) \qquad b(\lambda, x) = (1-x^2)^{-1/2}O(\lambda^{-1/3}), \qquad \text{если } (1-x^2)\lambda^{2/3} > \lambda^\delta,\ \delta > 0,$$

$$(90) \qquad b(\lambda, y) = O(\lambda^{-1/3}), \qquad \text{если } y^2 < 1-\delta.$$

Теперь оценка (21) следует из (19), (15), (16) и (89), (90).

5. Доказательство теоремы 3. Как и при доказательстве теоремы 2 мы имеем (59)–(61). Но в случае $x^2 > 1$ вещественные критические точки фазы ψ исчезают, поэтому коэффициенты $a_j(x,x)$ нельзя подсчитать как раньше. Однако можно заметить, что (65) и формула Тейлора дают асимптотики

$$(91) \qquad \begin{cases} a_1(x,x) = -i/4\pi^2 + O(x^2-1), \\ a_2(x,x) = i/4\pi^2 + O(x^2-1) \end{cases} \qquad \text{при } x^2 \to 1.$$

Так как $x^2 - 1 \leq \lambda^{-2/3+\delta}$ и $\delta > 0$ достаточно мало, то из (59)–(61), (91), (66)–(68), (33), (36), (44), учитывая оценку $c(\lambda,x,y) \leq \text{const}$, получаем

$$(92) \qquad E_\rho(\lambda,x,y) = a(\lambda,x.y)\lambda^{-1/2} + O(\lambda^{-1/6}), \quad \lambda \to +\infty.$$

Далее, применяя в формуле (31) замену переменных (52), получаем оценку

$$\int \rho(\lambda-\mu)\,de(\mu, \sqrt{\lambda}x, \sqrt{\lambda}x) = O(\lambda^{-1/6}), \quad \lambda \to +\infty,\ 1 \leq x^2 \leq 1+\delta.$$

только при условии (H_1), откуда

$$(93) \quad |e(\lambda+\sigma, \sqrt{\lambda}x, \sqrt{\lambda}x) - E(\lambda,x,x)| \leq c.\lambda^{-1/6}, \text{ если } |\sigma| \leq 1,\ 1 \leq x^2 \leq 1+\delta.$$

С другой стороны, обычные оценки собственных чисел и собственных функций оператора A [2] показывают, что

$$(94) \qquad |e(\lambda,x,y)| \leq c(1+|\lambda|)^3, \quad \lambda \in \mathbb{R}.$$

Поэтому

$$(95) \qquad E_\rho(\lambda,x,y) - E(\lambda,x,y) = \int_0^{\lambda/2} [\Delta(\mu)E + \Delta(-\mu)E]\rho(\mu)\,d\mu + O(\lambda^{-\infty}),$$

где $\Delta(\mu)E = e(\lambda+\mu, \sqrt{\lambda}x, \sqrt{\lambda}y) - E(\lambda,x,y)$. Чтобы оценить $\Delta(\pm\mu)E$ при $0 < \mu < \lambda/2$ будем использовать (93) при $x^2 \geq 1$, $y^2 \geq 1$ и (70) при $y^2 \leq 1$. В результате получаем

$$(96) \qquad |\Delta(\pm\mu)E| \leq c(1+|\mu|)^m\lambda^{-1/6}, \quad 0 < \mu < \lambda/2.$$

Теперь (92) и (95), (96) доказывают асимптотику (22).

Наконец, если $|x-y| > \varepsilon$, то $E_\rho(\lambda,x,y) = O(\lambda^{-\infty})$. В случае $y^2 \geqq 1$ оценка (24) следует отсюда и (95), (96). Если же $y^2 < 1$, то (70) и (93) влекут оценку

$$(97) \qquad |\Delta(\pm\mu)E| \leqq c(1+|\mu|)^m \sqrt{b(\lambda,y)}\lambda^{-1/6}, \quad \text{если } 0<\mu<\lambda/2.$$

Поэтому (23) следует из (95) и (97). Теорема 3 доказана.

6. Доказательство следствия 3. При условиях этого следствия имеют место оценки (19) или (23) соответственно. Так как $b(\lambda,x) \leqq \text{const}$ и выполнено (90), то $E(\lambda,x,y) = O(\lambda^{-1/3})$, откуда (25) следует.

7. Доказательство теоремы 4. Учитывая (94), достаточно оценить функцию $e(\lambda,x,x)$. Для этого будем использовать представление

$$(98) \qquad E_\rho(\lambda,x,x) \sim \lambda^{1/2}\int e^{i\lambda\psi(t,\xi,x,x)} g(t,\sqrt{\lambda}\xi,\sqrt{\lambda}x)\kappa(\xi)\,dt\,d\xi,$$

где
$$g(t,\xi,x) = \frac{1}{4\pi^2}\frac{\hat\rho(t)}{t}\left[\xi(x-\partial_\xi\varphi)q(t,\xi,x) + i\xi\partial_\xi q(t,\xi,x)\right],$$

которое выводится из (32) аналогично (33) (или прямо из (33)). Если $|x^2-1| \leqq \delta$, то можно сделать замену переменных (52) и получить оценку

$$(99) \qquad E_\rho(\lambda,x,x) = b(\lambda,x)O(\lambda^{1/6}), \quad \lambda\to+\infty.$$

Так как $(x^2-1)\lambda^{2/3} \geqq \lambda^\sigma$, $\sigma>0$, то асимптотика функции Эйри и (16), (18) показывают, что $b(\lambda,x) = O(\lambda^{-\infty})$, следовательно (99) переписывается в виде

$$(100) \qquad \int \rho(\lambda-\mu)e(\mu,x,x)\,d\mu = O(|x|^{-\infty}), \quad \lambda+\lambda^{1/3+\sigma} \leqq x^2 \leqq (1+\delta)\lambda.$$

Аналогично,

$$(101) \qquad \int \rho(\lambda-\mu)\,de(\mu,x,x) = O(|x|^{-\infty}), \quad \lambda+\lambda^{1/3+\sigma} \leqq x^2 \leqq (1+\delta)\lambda.$$

Пусть теперь $x^2 \geqq 1+\delta$. Тогда фазовая функция ψ не имеет вещественных критических точек и

$$(102) \qquad |\partial_t\psi| + |\partial_\xi\psi| \geqq c.x^2, \quad c>0.$$

Действительно, если $1+\delta \leqq x^2 \leqq b$, то (102) очевидно выполнено. Если же $x^2 \geqq b$ и b — достаточно большое число, то (102) следует из неравенства $|\partial_t\psi| \geqq c(x^2+\xi^2-1)$, где $|t|<T$ и T достаточно мало. Оценка (102) позволяет интегрировать по частям в интеграле (98), что в итоге приводить снова к оценкам (100) и (101). Таким образом они доказаны при $x^2 > \lambda+\lambda^{1/3+\sigma}$, $\sigma>0$. Наконец из (100) и (101) следует оценка $e(\lambda,x,x) = O(|x|^{-\infty})$ стандартным образом. Теорема 4 доказана.

8. Доказательство теоремы о равносходимости. Следуя схеме доказательства теоремы о равносходимости из [1, стр. 264], достаточно установить оценку вида

$$R_n(f,y) = O(1)\left(\int_{|x|>1} |x|^{-1}|f(x)|\,dx + \int_{x^2<c} |f(x)|\,dx\right) + o(1) \tag{103}$$

при $n \to +\infty$, $y^2 < c/2$, где $c > 1$ и

$$R_n(f,y) = s_n(f,y) - \frac{1}{\pi}\int_{y-\delta}^{y+\delta} f(x)\frac{\sin\sqrt{n}(x-y)}{x-y}\,dx.$$

Именно, соотношение (5) верно для каждой собственной функции, следовательно для плотного в $L^2(\mathbf{R})$ множества функций $\{f\}$. С другой стороны, нетрудно сообразить, что функцию класса

$$\left\{f \in L^1_{\text{loc}}(\mathbf{R}) : \int_{|x|>1} |x|^{-1}|f(x)|\,dx + \int_{x^2<c} |f(x)|\,dx < \infty\right\}$$

можно аппроксимировать линейными комбинациями собственных функций.

Далее, (4) и (12) показывают, что

$$R_n(f,y) = O(1)\left(\int_{x^2<c} |f(x)|\,dx + \int_{x^2>c} f(x)e(n,x,y)\,dx\right), \quad y^2 < c/2. \tag{104}$$

Поэтому остается оценить интегралы

$$K_j(\lambda,y) = \int a_j(\lambda,x)f(x)e(\lambda,x,y)\,dx, \quad 1 \le j \le 4, \tag{105}$$

где $a_1(\lambda,x)$ — характеристическая функция множества $\{x : c < x^2 < (1-\delta)\lambda\}$, $\delta > 0$; $a_2(\lambda,x) = a(\lambda,x)$, $a_3(\lambda,x) = b(\lambda,x)$ и $a_4(\lambda,x)$ — характеристическая функция множества $\{x : x^2 > \lambda + \lambda^{1/3+\delta}\}$.

a) Оценка интеграла K_1 делается с помощью следствия 1:

$$K_1(\lambda,y) = O(1)\int_{|x|>1} |x|^{-1}|f(x)|\,dx, \quad y^2 < c/2. \tag{106}$$

b) Оценка интеграла K_2 следует из (21) и (S_2):

$$K_2(\lambda,y) = o(1), \quad \lambda \to +\infty,\ y^2 < c. \tag{107}$$

c) К интегралу K_3 применяем следствие 3 и (S_3):

$$K_3(\lambda,y) = o(1), \quad \lambda \to +\infty,\ y^2 < c. \tag{108}$$

d) Интеграл K_4 оценивается с помощью теоремы 4:

$$K_4(\lambda, y) = O(1) \int\limits_{|x|>1} |x|^{-1} |f(x)| \, dx. \tag{109}$$

Теперь ясно, что оценка (103) следует из (104)–(109). Теорема доказана.

ЛИТЕРАТУРА

1. Szegö, G. Orthogonal polynomials. AMS, 1959.
2. Helffer, B. Theorie spectrale pour des operateurs globalement elliptiques. — Asterisque, 112, 1984.
3. Hörmander, L. The spectral function of an elliptic operator. — *Acta Math.*, 121, 1968, 173–218.
4. Poenaru, V. Singularites en presence de symetrie. Lect. notes in mathem. 510, Springer – Verlag, 1976.
5. Караджов, Г. Е. О равносходимости рядов Эрмита. — *Сердика*, 16, 1990, 217–229.
6. Федорюк, М. В. Метод перевала. М., 1977.
7. Хермандер, Л. Анализ линейных дифференциальных операторов с частными производными. Т. 1, М., 1986.
8. Шубин, М. А. Псевдодифференциальные операторы и спектральная теория. М., 1978.
9. Арнольд, В. И., А. Н. Варченко, С. М. Гусейн-Заде. Особенности дифференцируемых отображений. М., 1982.

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ROUNDING ANALYSIS OF PARALLEL ALGORITHM FOR THE SOLUTION OF A TRIDIAGONAL LINEAR SYSTEM OF EQUATIONS

Plamen Piskulijski

Пламен Пискулийски. АНАЛИЗ ОШИБКИ ОКРУГЛЕНИЯ В ПАРАЛЛЕЛЬНОМ АЛГОРИТМЕ РЕШЕНИЯ ТРЕХДИАГОНАЛЬНЫХ СИСТЕМ ЛИНЕЙНЫХ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ

В этой статье мы представляем параллельный алгоритм решения систем линейных алгебраических уравнений с трехдиагональными матрицами. Тоже проводим детальный анализ ошибки округления. Наконец, приведены результаты некоторых численных экспериментов и сделано сравнение между ошибками параллельных и последовательных алгоритмов.

Plamen Piskulijski. ROUNDING ANALYSIS OF PARALLEL ALGORITHM FOR THE SOLUTION OF A TRIDIAGONAL LINEAR SYSTEM OF EQUATIONS

In this paper we present a parallel algorithm for the solution of a tridiagonal linear system of equations.We also give a detailed rounding error analysis. Finaly we present some numerical experiments and give a comparison between the errors of parallel and consecutive algorithms.

1. INTRODUCTION

In this paper we examin the solution of tridiagonal systems of linear equations and estimate the rounding error. It is well known that such systems can be solved using a conventional serial computer in a time proportional to N where N is the number of equations.

Stone [13] first discussed the solution of a tridiagonal system on a parallel computer, relating the LDU decomposition using the first and second order linear recurrences. He developed recursive doubling algorithms to compute the necessary terms in $O(\log N)$ steps with $O(N)$ processors. The methods of odd-even elimination [14], [9] are another class of parallel algorithms with some quite different characteristics. There are many other algorithms for solving tridiagonal systems [7], [10], [5], [6], [3], which solve them in $O(\log N)$ steps with $O(N)$ processors.

Several authors have noted that cyclic reduction is just Gaussian elimination applied to PAP^T for a particular permutation matrix P (see for example [9]). Thus the algorithm is numerically stable for matrices for which Gaussian elimination is stable without pivoting, for example, symmetric positive definite or diagonally dominant matrices. The situation is not as attractive for Stone's algorithm. Using a stability analysis technique for recurrence relations introduced in [8], [11] the authors have shown that the algorithm is in general unstable, suffering from exponential error growth.

Sameh and Kuck [12] present two algorithms for tridiagonal systems using $O(N)$ processors. One of the algorithms requires $\log N$ steps but can suffer from exponential growth of errors; the more stable version requires $O[(\log\log N)(\log N)]$ steps.

There are consistent algorithms different from cyclic reduction. Swarztrauber [15], [16] introduced an algorithm for tridiagonal systems based on an efficient implementation of Cramer's rule. The algorithm requires $O(\log N)$ steps on $O(N)$ processors but only $O(N)$ total operations are performed. Unlike cyclic reduction the algorithm is well defined for general nonsingular systems.

Our goal in this paper is an error analysis of the parallel algorithm given in [3] using the approach from [2]. In section 2 we present the algorithm for solving tridiagonal systems of linear equations, which solve the system of order N in $O(n \log_n N)$ steps, where $N = n^s$. In section 3 we analize the rounding error in numerical solution of the system, using forward analysis. In section 4 we have performed some numerical experiments on an IBM–PC in machine precision $\approx 10^{-7}$ and we have given a comparison between the errors of parallel and consecutive algorithms.

2. PARALLEL ALGORITHM

The parallel algorithm from [3] will be applied for solving the tridiagonal system of equations

$$(2.1) \quad A_i u_{i-1} + B_i u_i + C_i u_{i+1} = f_i, \quad u_0 = \alpha, \quad u_N = \beta, \quad i = 1, 2, \dots, N-1$$

under the assumption that the following conditions hold

$$(2.2) \qquad |A_i| + |C_i| \leq |B_i|, \quad i = 1, 2, \dots, N-1.$$

Conditions (2.2) ensure the existence of the algorithm of the system (2.1) and the stability of the problem [4]. For simplicity the problem (2.1), (2.2) can be written in the form:

$$(2.3) \qquad \Lambda u_i = f_i, \quad i = 1, 2, \dots, N-1, \quad u_0 = \alpha, \quad u_N = \beta.$$

We represent the solution of the system (2.3) in the form

$$(2.4) \qquad u = \alpha u^{1,0} + \beta u^{0,1} + u^{0,0}$$

where the vectors $u^{1,0}$, $u^{0,1}$, $u^{0,0}$ are solutions of the systems :

$$\begin{aligned}
&\Lambda u_i^{1,0} = 0, \quad \Lambda u_i^{0,1} = 0, \quad \Lambda u_i^{0,0} = f_i, \qquad i = 1, 2, \dots, N-1,\\
&u_0^{1,0} = 1, \qquad u_0^{0,1} = 0, \qquad u_0^{0,0} = 0,\\
&u_N^{1,0} = 0, \qquad u_N^{0,1} = 1, \qquad u_N^{0,0} = 0
\end{aligned}$$

respectively.

Choosing the following knots of parallelism of the problem (2.3) $0 = i_0 < i_1 < \ldots < i_k = N$, we set

$$\begin{aligned} &\alpha_m = u_{i_m}, \quad m = 0, 1, \ldots, k; \\ &u^m = \alpha_{m-1} u^{m,1,0} + \alpha_m u^{m,0,1} + u^{m,0,0}, \quad m = 1, \ldots, k \end{aligned} \tag{2.5}$$

where u^m, $u^{m,1,0}$, $u^{m,0,1}$, $u^{m,0,0}$ are solutions of linear systems

$$\Lambda u_i^m = f_i, \quad i = i_{m-1} + 1, \ldots, i_m - 1, \quad u_{i_{m-1}}^m = \alpha_{m-1}, \quad u_{i_m}^m = \alpha_m, \tag{2.6}$$

$$\Lambda u_i^{m,1,0} = 0, \quad i = i_{m-1} + 1, \ldots, i_m - 1, \quad u_{i_{m-1}}^{m,1,0} = 1, \quad u_{i_m}^{m,1,0} = 0, \tag{2.7}$$

$$\Lambda u_i^{m,0,1} = 0, \quad i = i_{m-1} + 1, \ldots, i_m - 1, \quad u_{i_{m-1}}^{m,0,1} = 0, \quad u_{i_m}^{m,0,1} = 1, \tag{2.8}$$

$$\Lambda u_i^{m,0,0} = f_i, \quad i = i_{m-1} + 1, \ldots, i_m - 1, \quad u_{i_{m-1}}^{m,0,0} = 0, \quad u_{i_m}^{m,0,0} = 0. \tag{2.9}$$

It follows from (2.5)–(2.9) that the solution u of (2.3) satisfies $u_i = u_i^m$ for $i = i_{m-1}, i_m$, i.e. the solution of the system (2.3) consists of the steps:

a) to solve the systems (2.7), (2.8), (2.9) in each interval (i_{m-1}, i_m);

b) to obtain α_i, $i = 0, 1, \ldots, k$;

c) to derive the solution by formulas (2.5).

The unknown quantities α_i are defined in the following way: from (2.1) and (2.5) we obtain

$$\begin{aligned} u^m &= \alpha_{m-1} u^{m,1,0} + \alpha_m u^{m,0,1} + u^{m,0,0}, \\ u^{m+1} &= \alpha_m u^{m+1,1,0} + \alpha_{m+1} u^{m+1,0,1} + u^{m+1,0,0}, \end{aligned} \tag{2.10}$$

$$A_{i_m} u_{i_m-1} + B_{i_m} u_{i_m} + C_{i_m} u_{i_m+1} = A_{i_m} u_{i_m-1}^m + B_{i_m} \alpha_m + C_{i_m} u_{i_m+1}^{m+1} = f_{i_m} \tag{2.11}$$

and substituting (2.10) in (2.11) we get

$$a_m \alpha_{m-1} + b_m \alpha_m + c_m \alpha_{m+1} = \varphi_m, \quad m = 1, \ldots, k-1, \quad \alpha_0 = \alpha, \quad \alpha_k = \beta \tag{2.12}$$

where

$$\begin{aligned} a_m &= A_{i_m} u_{i_m-1}^{m,1,0}, \quad b_m = B_{i_m} + A_{i_m} u_{i_m-1}^{m,0,1} + C_{i_m} u_{i_m+1}^{m+1,1,0}, \\ c_m &= C_{i_m} u_{i_m+1}^{m+1,0,1}, \quad \varphi_m = f_{i_m} - A_{i_m} u_{i_m-1}^{m,0,0} - C_{i_m} u_{i_m+1}^{m+1,0,0}. \end{aligned} \tag{2.13}$$

So the obtained parallel algorithm for solving the system (2.3) is:

a) to solve in parallel the systems (2.7) – (2.9) in each interval (i_m, i_{m+1});

b) to obtain a_m, b_m, c_m, φ_m for $m = 1, \ldots, k-1$ in parallel in each interval by formulas (2.13);

c) to solved the system (2.12) by some method;

d) to derive in parallel the solution of the problem by formulas (2.5).

Let us suppose for convenience, that $N = n^s$, where n and s are integers, such that $n, s \geqq 2$. Choosing $k = N/n = n^{s-1}$ equidistant knots of parallelism i_m, i.e. $i_m = mn$, $m = 0, 1, \ldots, k$ and solving (9) by the described method, we come to the problem (2.12) for the determination of α_m, $m = 0, 1, \ldots, k$. The problem (2.12) is in the same form as (2.3) but it is of size n-times smaller than (2.3). We again apply the same parallel scheme for the system (2.12), but this time with

$k/n = n^{s-2}$ equidistant points j_l, $l = 0, \ldots, n^{s-2}$ and get the tridiagonal system of size n^2 times smaller than (2.3). Finaly we obtain the system of the kind (2.12) which is of size n and we solve it by the same method. After that we deduce the solution of the system (2.3) from formulas (2.5).

3. ERROR ANALYSIS

Let us have the system

$$Au = f \tag{3.1}$$

and instead of this system let us solve the system

$$\tilde{A}\tilde{u} = \tilde{f} \tag{3.2}$$

where $\tilde{A} = A + \delta A$, $\tilde{f} = f + \delta f$, $\overline{u} = u + \delta u$. Then the relative data error satisfies

$$\frac{\|\delta u\|}{\|u\|} \leq \nu(\tilde{A})\frac{\|\delta A\|}{\|\tilde{A}\|} + \nu(\tilde{A})\frac{\|\delta A\|\,\|\delta f\|}{\|\tilde{A}\|\,\|f\|} + \nu(\tilde{A})\frac{\|\delta f\|}{\|f\|} \tag{3.3}$$

where $\nu(\tilde{A}) = \|\tilde{A}\| \cdot \|\tilde{A}^{-1}\|$ is the condition number of $\tilde{A}$. Consequently, the full error Δu satisfies $\|\Delta u\| \leq \|u - \tilde{u}\| + \|\tilde{u} - \overline{u}\|$ where the first member is the error from inexactness of coeficients (data error) and the second member is rounding error.

First we will examine the tridiagonal systems (2.7) and (2.8). The system (2.7) has the form

$$A_j u_{j-1} + B_j u_j + C_j u_{j+1} = 0,\ j = 2, \ldots, n-1, \quad u_1 = 1, \quad u_n = 0. \tag{3.4}$$

We will apply for it the following formulas:

$$\begin{aligned} &\alpha_j = -\frac{A_j}{B_j - C_j\alpha_{j+1}}, \quad \beta_j = \frac{f_j - C_j\beta_{j+1}}{B_j - C_j\alpha_{j+1}}, \quad j = n, n-1, \ldots, 1, \\ &u_j = \alpha_j u_{j-1} + \beta_j, \quad j = 1, \ldots, n, \end{aligned} \tag{3.5}$$

in order to evaluate a solution of the system. Since $|\alpha_n| = 0$ and $|\alpha_{n-1}| = |C_{n-1}/B_{n-1}| \leq 1$, we obtain by induction that

$$|\alpha_j| \leq |C_j|/(|B_j| - |A_j|) \leq 1, \quad j = n-2, \ldots, 1. \tag{3.6}$$

Since $\beta_n = \beta_{n-1} = \cdots = \beta_2 = 0$, $\beta_1 = 1$ and $u_1 = \beta_1 = 1$, it is easily derived that

$$|u_j| \leq 1,\ j = 1, \ldots, n. \tag{3.7}$$

For the system (2.8), which is in the form

$$A_j u_{j-1} + B_j u_j + C_j u_{j+1} = 0,\ j = 2, \ldots, n-1, \quad u_1 = 0, \quad u_n = 1, \tag{3.8}$$

we apply the following formulas

$$\begin{aligned} &\alpha_j = -\frac{C_j}{B_j + A_j\alpha_{j-1}}, \quad \beta_j = \frac{f_j - A_j\beta_{j-1}}{B_j + A_j\alpha_{j-1}}, \qquad j = 1, \ldots, n, \\ &u_j = \alpha_j u_{j+1} + \beta_j, \quad j = n, \ldots, 1, \end{aligned} \tag{3.9}$$

in order to evaluate a solution of the system. By analogy with (3.6) we deduce that $|\alpha_j| \leq 1$, $j = 1, \ldots, n$, $\beta_1 = \cdots = \beta_{n-1} = 0$, $\beta_n = 1$, and since $|u_n| = 1$ it is easily seen that

$$|u_j| \leq 1, \quad j = n, \ldots, 1, \tag{3.10}$$

We obtained that the solutions of the systems (2.7) and (2.8) do not exceed unity in absolute value.

Using the results of [2] we get that the rounding error of the system (2.7) satisfies

$$\begin{aligned}\left|\delta u_{n-i}^{m,1,0}\right| &\leqq \varepsilon\{i + C\kappa[n(n+1) - (n-i)(n-i+1)]/2 \\ (3.11)\quad &+ C_1\kappa[(n+1)(n+2)(2n+3) - (n-i+1)(n-i+2)(2n-2i+3)]/12\}, \\ \left|\delta u_n^{m,1,0}\right| &\leqq C_1\kappa\frac{n(n+1)}{2}\varepsilon, \quad i = 1, \dots, n-1\end{aligned}$$

where ε is an unit roundoff, C does not depend on ε and n, and

$$(3.12)\qquad \varepsilon\kappa n(n-1)/2 \leqq 1 - 1/C$$

where

$$\kappa = \max_{1 \leqq j \leqq s}\{\kappa_j\}, \quad \kappa_j = \max_{2 \leqq k \leqq n-1}\left\{\prod_{i=2}^{k}|\gamma_i|\right\}, \quad \gamma_j = \alpha_j A_j/C_j,$$

and

$$C_1 = 1 + \max_{2 \leqq j \leqq n-1}\{|A_j\overline{\beta}_{j-1}/C_j|\}.$$

It turns out that $|\delta u_s^{m,1,0}| = O(\varepsilon n^3)$, if the conditions (2.2) are valid.

Now we examine the system (2.8) which is of order n. It is of the same type (3.8) and condition (2.2) holds for it. Using [2] and since $\beta_1 = \beta_2 = \cdots = \beta_{n-1} = 0$, $\beta_n = 1$ for the rounding error of the system (2.8) it follows

$$\begin{aligned}\left|\delta u_{n-i}^{m,0,1}\right| &\leqq \varepsilon\left\{i + C\frac{\kappa}{2}[n(n+1) - (n-i)(n-i+1)]\right\}, \; i = 1, \dots, n-1, \\ (3.13)\quad \left|\delta u_n^{m,0,1}\right| &\leqq \varepsilon,\end{aligned}$$

i.e. $|\delta u_s^{m,0,1}| = O(\varepsilon n^2)$.

Now it remains to evaluate the error for the system (2.9)

$$(3.14)\qquad A_j u_{j-1} + B_j u_j + C_j u_{j+1} = f_j, \; j = 2, \dots, n-1, \quad u_1 = 0, \quad u_n = 0.$$

Conditions (2.2) are valid for it, too. Similarly, for this system (3.14) we find the following estimations

$$\begin{aligned}\left|\delta u_{n-i}^{m,0,0}\right| &\leqq \varepsilon\{i + C\kappa[n(n+1) - (n-i)(n-i+1)]\overline{u}/2 \\ (3.15)\quad &+ C_1\kappa[(n+1)(n+2)(2n+3) - (n-i+1)(n-i+2)(2n-2i+3)]/12\}. \\ \left|\delta u_n^{m,0,0}\right| &\leqq C_1\kappa\frac{n(n+1)}{2}\varepsilon, \quad i = 1, \dots, n-1,\end{aligned}$$

under the assumption that $\overline{u}_j^{m,0,0}$ are bounded, i.e. $\overline{u} = \max_j\left\{|\overline{u}_j^{m,0,0}|\right\}$. We obtained that $|\delta u_j^{m,0,0}| = O(\varepsilon n^3)$.

On the next stage of the algorithm the system (2.12) must be solved with coeficients (2.13). This system is tridiagonal and is of size n times smaller than (2.3). Conditions (2.2) hold for it.

Now we will derive rounding error of the coeficients (2.13). Instead of the exact value $a_m^{(2)}$, where the superscripts denote the stage of the algorithm, let some approximation value $\overline{a}_m^{(2)}$ be taken, such that $\overline{a}_m^{(2)} = A_{i_m} \overline{u}_{i_m-1}^{m,1,0} + \varepsilon$. Then

$$\left|\delta a_m^{(2)}\right| = \left|\overline{a}_m^{(2)} - a_m^{(2)}\right| \leqq |A_{i_m}| \left|\delta u_{i_m-1}^{m,1,0}\right| + \varepsilon,$$

and from (3.11) it follows

(3.16) $$\left|\delta a_m^{(2)}\right| \leqq \varepsilon\,(1+|A_{i_m}|) + |A_{i_m}|\varepsilon C\kappa n + |A_{i_m}|\varepsilon C_1\kappa(n^2+2n+1)/2.$$

By analogy with (3.16), we derive the following estimations for $|\delta c_m^{(2)}|$, $|\delta b_m^{(2)}|$, $|\delta \varphi_m^{(2)}|$:

(3.17) $$\left|\delta c_m^{(2)}\right| \leqq \varepsilon\,(n-2+|C_{i_m}|) + |C_{i_m}|\varepsilon C\frac{\kappa}{2}(n^2+n-6)$$

(3.18) $$\left|\delta b_m^{(2)}\right| \leqq \varepsilon\,(1+|A_{i_m}|+(n-2)\,|C_{i_m}|) + |A_{i_m}|\,(n^2+3n-6)\varepsilon C\frac{\kappa}{2} + |C_{i_m}|\varepsilon C_1\kappa(2n^3+9n^2+13n-78)/12$$

(3.19) $$\left|\delta \varphi_m^{(2)}\right| \leqq \varepsilon\,(1+|A_{i_m}|+(n-2)\,|C_{i_m}|) + |A_{i_m}|\,(n^2+3n-6)\varepsilon C\frac{\kappa}{2}\overline{u} + \left(|A_{i_m}|\,6(n^2+2n+1) + |C_{i_m}|\,(2n^3+9n^2+13n-78)\right)\varepsilon C_1\kappa/12.$$

We again apply the same parallel scheme for the system (2.12), but this time with $k/n = n^{s-2}$ equidistant points j_l, $l = 0, \ldots, n^{s-1}$. Thus, the obtained tridiagonal systems is of the same kind as (2.7) – (2.9), but the coeficients of these systems of order n are the coeficients of the system (2.12). And they already have some error and an error in the new systems appears from the inexactness of coeficients (data error). This error must be estimated.

The full error Δu satisfies $\|\Delta u\| \leqq \|u-\tilde{u}\|+\|\tilde{u}-\overline{u}\|$ and now we have to estimate the full error Δu. The rounding error is evaluated similarly to the rounding error of the systems (2.7) – (2.9). So we evaluate the data error by (3.3) using that the condition number for tridiagonal matrices is $O(n^2)$ [2].

Throughout this section the superscripts in brackets denote the stage of the algorithm and $\|\varphi\|$, $\|\tilde{A}\|$ are a maximum of the norms of φ and the matrix of the system (2.9) at different steps respectively. First, let us examine the system of the kind (2.7). For it $\|\delta\varphi\| = 0$ and then

$$\frac{\|\overline{\delta}\alpha^{m,1,0}\|^{(2)}}{\|\alpha^{m,1,0}\|^{(2)}} \leqq \nu(\tilde{A})\frac{\|\delta A\|^{(2)}}{\|\tilde{A}\|} \leqq \frac{\nu(\tilde{A})}{\|\tilde{A}\|}A_n,$$

where

$$A_n = |C_{i_m}|\,\varepsilon\frac{\kappa}{2}\left[C_1(2n^3+9n^2+13n-78)/6 + 2C(n^2+n-6)\right] + |C_{i_m}|\,\varepsilon(n-2) + |A_{i_m}|\,\varepsilon\frac{\kappa}{2}\left[C_1(n^2+2n+1)+4Cn\right] + \varepsilon\left[2+2\,|A_{i_m}|+|C_{i_m}|\right],$$

and A is the matrix of corresponding system. Since the rounding error of this system is evaluated by (3.11), then

(3.20) $$\|\Delta\alpha^{m,1,0}\|^{(2)} \leqq \|\overline{\delta}\alpha^{m,1,0}\|^{(2)} + \|\delta\alpha^{m,1,0}\|^{(2)} \leqq \nu(\tilde{A})A_n/\|\tilde{A}\| + \overline{A}_n,$$

$$\overline{A}_n = \varepsilon[n-1+C\frac{\kappa}{2}(n^2+n-2)+C_1\kappa(2n^3+9n^2+13n-24)/12].$$

For the system (2.8) also $\|\delta\varphi\|=0$ and the corresponding error is

$$\|\Delta\alpha^{m,0,1}\|^{(2)} \leqq \|\overline{\delta}\alpha^{m,0,1}\|^{(2)}+\|\delta\alpha^{m,0,1}\|^{(2)} \leqq \nu(\tilde{A})A_n/\|\tilde{A}\|+\tilde{A}_n, \tag{3.21}$$

$$\tilde{A}_n = \varepsilon[n-1+C\frac{\kappa}{2}(n^2+n-2)].$$

It remains to examine the system of kind (2.9). For it $\|\delta\varphi\|\neq 0$ and then from (3.3) it is derived

$$\frac{\|\overline{\delta}\alpha^{m,0,0}\|^{(2)}}{\|\alpha^{m,0,0}\|^{(2)}} \leqq \nu(\tilde{A})\frac{\|\delta A\|^{(2)}}{\|\tilde{A}\|}+\nu(\tilde{A})\frac{\|\delta A\|^{(2)}\|\delta\varphi\|^{(2)}}{\|\tilde{A}\|\,\|\varphi\|}+\nu(\tilde{A})\frac{\|\delta\varphi\|^{(2)}}{\|\varphi\|}$$
$$\leqq \nu(\tilde{A})\frac{A_n}{\|\tilde{A}\|}+\nu(\tilde{A})\frac{A_nB_n}{\|\tilde{A}\|\,\|\varphi\|}+\nu(\tilde{A})\frac{B_n}{\|\varphi\|},$$

$$B_n = \varepsilon\,(1+|A_{i_m}|+(n-2)\,|C_{i_m}|)+\left(|A_{i_m}|\,2n+|C_{i_m}|\,(n^2+n-6)\right)\varepsilon C\,\overline{u}\kappa/2$$
$$+\left(|A_{i_m}|\,6(n^2+2n+1)+|C_{i_m}|\,(2n^3+9n^2+13n-78)\right)\varepsilon C_1\kappa/12.$$

Then for the system (2.9) we find

$$\|\Delta\alpha^{m,0,0}\|^{(2)} \leqq \left(\nu(\tilde{A})\frac{A_n}{\|\tilde{A}\|}+\nu(\tilde{A})\frac{A_nB_n}{\|\tilde{A}\|\,\|\varphi\|}+\nu(\tilde{A})\frac{B_n}{\|\varphi\|}\right)\|\alpha^{m,0,0}\|^{(2)}+\overline{B}_n, \tag{3.22}$$

$$\overline{B}_n = \varepsilon[n-1+C\frac{\kappa}{2}(n^2+n-2)\overline{u}+C_1\kappa(2n^3+9n^2+13n-24)/12].$$

Now we derive the rounding error of coeficients (2.13) at the third stage. Let instead of $a_m^{(3)}$ an approximation value $\overline{a}_m^{(3)}$ be taken, such that

$$\overline{a}_m^{(3)} = \overline{a}_{i_m}^{(2)}\left(\overline{\alpha}_{i_m-1}^{m,1,0}\right)^{(2)}+\varepsilon.$$

Then

$$\left|\delta a_m^{(3)}\right| = \left|\overline{a}_m^{(3)}-a_m^{(3)}\right| \leqq \left|a_m^{(2)}\right|\,\|\delta\alpha^{m,1,0}\|^{(2)}+\left|\delta a_m^{(2)}\right|\,\left|\overline{\alpha}_{i_m-1}^{m,1,0}\right|^{(2)}+\varepsilon,$$

and since (3.7) holds, then

$$\left|\delta a_m^{(3)}\right| \leqq \left|a_m^{(2)}\right|\,\|\Delta\alpha^{m,1,0}\|^{(2)}+\left|\delta a_m^{(2)}\right|+\varepsilon.$$

Then from (3.20) it follows that

$$\left|\delta a_m^{(3)}\right| \leqq \left|a_m^{(2)}\right|\left(\nu(\tilde{A})A_n/\|\tilde{A}\|+\overline{A}_n\right)+\left|\delta a_m^{(2)}\right|+\varepsilon. \tag{3.23}$$

By analogy with (3.23), we derive the following estimations for $\left|\delta c_m^{(3)}\right|$, $\left|\delta b_m^{(3)}\right|$, $\left|\delta\varphi_m^{(3)}\right|$

$$\left|\delta c_m^{(3)}\right| \leqq \left|c_m^{(2)}\right|\left(\nu(\tilde{A})A_n/\|\tilde{A}\|+\tilde{A}_n\right)+\left|\delta c_m^{(2)}\right|+\varepsilon, \tag{3.24}$$

(3.25) $$\left|\delta b_m^{(3)}\right| \leq \left(\left|c_m^{(2)}\right| + \left|a_m^{(2)}\right|\right) \nu(\tilde{A}) \frac{A_n}{\|\tilde{A}\|} + \left|a_m^{(2)}\right| \tilde{A}_n + \left|c_m^{(2)}\right| \overline{A}_n + \left|\delta a_m^{(2)}\right| + \left|\delta c_m^{(2)}\right| + \left|\delta b_m^{(2)}\right| + \varepsilon,$$

(3.26) $$\left|\delta \varphi_m^{(3)}\right| \leq \left(\left|c_m^{(2)}\right| + \left|a_m^{(2)}\right|\right) \nu(\tilde{A}) M_1 + \left(\left|\delta a_m^{(2)}\right| + \left|\delta c_m^{(2)}\right|\right) \overline{\alpha} + \left|\delta \varphi_m^{(2)}\right| + \varepsilon,$$

(3.27) $$M_1 = A_n/\|\tilde{A}\| + A_n B_n/\|\tilde{A}\|\,\|\varphi\| + B_n/\|\varphi\| + \overline{B}_n.$$

On the next stage from the formulas (3.20) – (3.22) it follows

$$\|\Delta\alpha^{m,1,0}\|^{(3)} \leq \frac{\nu(\tilde{A})}{\|\tilde{A}\|} \left[\frac{\nu(\tilde{A})}{\|\tilde{A}\|} A_n \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) + \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right)(\overline{A}_n + \tilde{A}_n) + 2\left|\delta a_m^{(2)}\right| + 2\left|\delta c_m^{(2)}\right| + \left|\delta b_m^{(2)}\right| + 3\varepsilon \right] + \overline{A}_n,$$

$$\|\Delta\alpha^{m,0,1}\|^{(3)} \leq \frac{\nu(\tilde{A})}{\|\tilde{A}\|} \left[\frac{\nu(\tilde{A})}{\|\tilde{A}\|} A_n \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) + \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right)(\overline{A}_n + \tilde{A}_n) + 2\left|\delta a_m^{(2)}\right| + 2\left|\delta c_m^{(2)}\right| + \left|\delta b_m^{(2)}\right| + 3\varepsilon \right] + \tilde{A}_n,$$

$$\|\Delta\alpha^{m,0,0}\|^{(3)} \leq \nu(\tilde{A})^2 \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) M_2 + 2\frac{\nu(\tilde{A})^3}{\|\tilde{A}\|^2\|\varphi\|} \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) A_n M_1 + \nu(\tilde{A}) \left\{ \frac{1}{\|\varphi\|} \left[\left(\left|\delta a_m^{(2)}\right| + \left|\delta c_m^{(2)}\right|\right) \overline{\alpha} + \left|\delta\varphi_m^{(2)}\right| + \varepsilon \right] + \frac{1}{\|A\|} \left[\left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) \left(\overline{A}_n + \tilde{A}_n\right) + 2\left|\delta a_m^{(2)}\right| + 2\left|\delta c_m^{(2)}\right| + \left|\delta b_m^{(2)}\right| + 3\varepsilon \right] \right\} + O(\varepsilon n^3),$$

where $M_2 = M_1/\|\varphi\| + 2A_n/\|\tilde{A}\|^2$.

Finally, at the $(s-1)$-th stage the system of order n and of type (2.12) is obtained and it can be written in the form

(3.28) $$a_m^{(s)}\alpha_{m-1} + b_m^{(s)}\alpha_m + c_m^{(s)}\alpha_{m+1} = \varphi_m^{(s)}, \quad m = 1, \ldots, n-1, \qquad \alpha_0 = \alpha, \quad \alpha_n = \beta,$$

where the error for the coeficients is:

$$
\begin{aligned}
\left|\delta a_m^{(s)}\right| &\leqq \left|a_m^{(s-1)}\right| \|\Delta\alpha^{m,1,0}\|^{(s-1)} + \left|\delta a_m^{(s-1)}\right| + \varepsilon,\\
\left|\delta c_m^{(s)}\right| &\leqq \left|c_m^{(s-1)}\right| \|\Delta\alpha^{m,0,1}\|^{(s-1)} + \left|\delta c_m^{(s-1)}\right| + \varepsilon,\\
\left|\delta b_m^{(s)}\right| &\leqq \left|c_m^{(s-1)}\right| \|\Delta\alpha^{m,1,0}\|^{(s-1)} + \left|a_m^{(s-1)}\right| \|\Delta\alpha^{m,0,1}\|^{(s-1)}\\
&\quad + \left|\delta a_m^{(s-1)}\right| + \left|\delta c_m^{(s-1)}\right| + \left|\delta b_m^{(s-1)}\right| + \varepsilon,\\
\left|\delta\varphi_m^{(s)}\right| &\leqq \left(\left|c_m^{(s-1)}\right| + \left|a_m^{(s-1)}\right|\right) \|\Delta\alpha^{m,0,0}\|^{(s-1)}\\
&\quad + \left(\left|\delta a_m^{(s-1)}\right| + \left|\delta c_m^{(s-1)}\right|\right)\overline{\alpha} + \left|\delta\varphi_m^{(s-1)}\right| + \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
\|\Delta\alpha^{m,1,0}\|^{(s-1)} \leqq \frac{\nu(\tilde{A})}{\|\tilde{A}\|}\Bigg(&2^{s-3}\frac{\nu(\tilde{A})^{s-3}}{\|\tilde{A}\|^{s-3}}A_n C_{s-2}\\
&+ 2^{s-5}3\frac{\nu(\tilde{A})^{s-4}}{\|\tilde{A}\|^{s-4}}A_n C_{s-3} + O(\varepsilon n^{2s-7})\Bigg) + \overline{A}_n,
\end{aligned}
$$

$$
\begin{aligned}
\|\Delta\alpha^{m,0,1}\|^{(s-1)} \leqq \frac{\nu(\tilde{A})}{\|\tilde{A}\|}\Bigg(&2^{s-3}\frac{\nu(\tilde{A})^{s-3}}{\|\tilde{A}\|^{s-3}}A_n \dot{C}_{s-2}\\
&+ 2^{s-5}3\frac{\nu(\tilde{A})^{s-4}}{\|\tilde{A}\|^{s-4}}A_n C_{s-3} + O(\varepsilon n^{2s-7})\Bigg) + \tilde{A}_n,
\end{aligned}
$$

$$
\|\Delta\alpha^{m,0,0}\|^{(s-1)} \leqq \nu(\tilde{A})^{s-2}M_{s-2}C_{s-2} + 2^{s-3}\frac{\nu(\tilde{A})^{2s-5}}{\|\tilde{A}\|^{s-2}\|\varphi\|}A_n M_{s-3}C_{s-2}^2 + O(\varepsilon n^{2s-3}),
$$

$$
C_{s-i} = \left(\left|a_m^{(s-i)}\right| + \left|c_m^{(s-i)}\right|\right)\dots\left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right), \quad i = 1,\dots,s-2,
$$

$$
M_{s-i} = M_{s-i-1}/\|\varphi\| + A_n 2^{s-i+1}/\|\tilde{A}\|^{s-i}, \quad i = 1,\dots,s-2,
$$

and M_1 is defined by (3.28).

Now we have to evaluate the solution of the system (3.28) in order to find the rounding error of the system (2.3). The total error can be estimated by $\|\Delta\alpha\|^{(s)} \leqq \|\alpha - \tilde{\alpha}\| + \|\tilde{\alpha} - \overline{\alpha}\|$ and consequently

$$
\|\Delta\alpha\|^{(s)} \leqq \nu(\tilde{A})^{s-1}M_{s-1}C_{s-1} + 2^{s-2}\frac{\nu(\tilde{A})^{2s-3}}{\|\tilde{A}\|^{s-1}\|\varphi\|}A_n M_{s-2}C_{s-1}^2 + O(\varepsilon n^{2s-1}).
$$

To estimate the product C_{s-i} let us set

$$
M = \max_{1 \leqq k \leqq N}(|A_k|, |C_k|). \tag{3.29}
$$

Then from (3.7), (3.10) and (3.29) we get

$$\begin{aligned} C_{s-i} &= \left(\left|a_m^{(s-i)}\right| + \left|c_m^{(s-i)}\right|\right) \dots \left(\left|a_m^{(2)}\right| + \left|c_m^{(2)}\right|\right) \\ &\leq \left(M \left|u_{i_m-1}^{m,1,0}\right|^{(1)} \dots \left|\alpha_{i_m-1}^{m,1,0}\right|^{(s-i)} + M \left|u_{i_m+1}^{m+1,0,1}\right|^{(1)}\right. \\ &\quad \left. \dots \left|\alpha_{i_m+1}^{m+1,0,1}\right|^{(s-i)}\right) \dots \left(M \left|u_{i_m-1}^{m,1,0}\right|^{(1)} + M \left|u_{i_m+1}^{m+1,0,1}\right|^{(1)}\right) \\ &\leq M^{s-2} 2^{s-2}, \qquad i = 1, \dots, s-2. \end{aligned}$$

Developing M_{s-2} and M_{s-1} we obtain that the error of the system (2.3) can be estimated by

$$\begin{aligned} \|\Delta\alpha\|^{(s)} &\leq \nu(\tilde{A})^{s-1} M^{s-2} C_1 n^3 2^{s-2} K_1 \frac{\kappa}{6} \\ &\quad + \nu(\tilde{A})^{s-1} M^{s-1} 2^{s-2} \varepsilon n^2 \frac{\kappa}{12} (6CK_2 + 15C_1K_3) \\ &\quad + 2^s \nu(\tilde{A})^{2s-3} \left(M^{s-1} 2^{s-2} \varepsilon C_1 n^3 \frac{\kappa}{12}\right)^2 K_4 / \|\tilde{A}\|^{s-1} \|\varphi\|^2 \\ &\quad + O(\varepsilon n^{2s-1}), \end{aligned}$$

$$\begin{aligned} K_1 &= (\|\tilde{A}\|M + \|\varphi\|M + \|\varphi\|\,\|\tilde{A}\|)/\|\varphi\|^{s-1}\|\tilde{A}\| + P_{s-1}M/2\|\varphi\|, \\ K_2 &= (2\|\varphi\| + \|\varphi\|\,\|\tilde{A}\|\overline{u}/M)/\|\varphi\|^{s-1}\|\tilde{A}\| + P_{s-1}/\|\varphi\|, \\ K_3 &= (\|\varphi\| + \|\tilde{A}\|^2 12\overline{u}/C_1\kappa)/\|\varphi\|^{s-1}\|\tilde{A}\| + P_{s-1}/2\|\varphi\|, \\ K_4 &= (\|\tilde{A}\| + \|\varphi\| + \|\varphi\|\,\|\tilde{A}\|/M)/\|\varphi\|^{s-3}\|\tilde{A}\| + P_{s-2}/2, \\ P_{s-i} &= 4(\|\tilde{A}\|^{s-i-1} - 2^{s-i-1})/\|\tilde{A}\|^{s-i}(\|\tilde{A}\|^2 - 2), \quad i = 1, 2. \end{aligned}$$

And finally we have to estimate the rounding error from (2.5) at each stage. By analogy with (3.23)–(3.26) it follows for the rounding error of the solution of the system (2.3):

$$\begin{aligned} \text{(3.30)} \quad \|\delta u\| &\leq 2^{s-1}\|\delta\alpha\|^{(s)} + \|\overline{\alpha}\| \sum_{i=1}^{s-2} 2^{s-i-1} \left(\|\delta\alpha^{m,1,0}\|^{(s-i)} + \|\delta\alpha^{m,0,1}\|^{(s-i)}\right) \\ &\quad + \sum_{i=1}^{s-2} 2^{s-i-1} \|\delta\alpha^{m,0,0}\|^{(s-i)}, \end{aligned}$$

where $\|\overline{\alpha}\|$ is a maximum of the norms of solutions α at the systems of kind (2.12) at the different steps. And consequently

$$\begin{aligned} \|\Delta u\| &\leq \nu(\tilde{A})^{s-1} M^{s-2} C_1 n^3 2^{2s-3} K_1 \frac{\kappa}{6} \\ &\quad + \nu(\tilde{A})^{s-1} M^{s-1} 2^{2s-3} \varepsilon n^2 \frac{\kappa}{12} (6CK_2 + 15C_1K_3) \\ &\quad + 2^{2s-1} \nu(\tilde{A})^{2s-3} \left(M^{s-1} 2^{s-2} \varepsilon C_1 n^3 \frac{\kappa}{12}\right)^2 K_4 + O(\varepsilon n^{2s-1}) / \|\tilde{A}\|^{s-1} \|\varphi\|^2. \end{aligned}$$

N	$\varepsilon \approx 10^{-7}$, $\max \lvert x_i - \overline{x}_i \rvert$	
	parallel alg.	conseq. alg.
$3^6 + 1 = 730$	79,64	1,539
$9^3 + 1 = 730$	12,77344	
$27^2 + 1 = 730$	5,09375	
$6^4 + 1 = 1297$	188,6	7,09375
$36^2 + 1 = 1297$	51,5	
$2^{12} + 1 = 4097$	0	5226,626
$4^6 + 1 = 4097$	0	
$8^4 + 1 = 4097$	0	
$16^3 + 1 = 4097$	2508,875	
$64^2 + 1 = 4097$	1712	
$3^8 + 1 = 6562$	574586	45210,5
$9^4 + 1 = 6562$	84836,5	
$81^2 + 1 = 6562$	34623,5	
$10^4 + 1 = 10001$	309374	1365653
$100^2 + 1 = 10001$	167126	
$5^6 + 1 = 15626$	9342526	$1,134279 . 10^7$
$25^3 + 1 = 15626$	117528	
$125^2 + 1 = 15626$	926624	
$12^4 + 1 = 20737$	3919304	$2,955723 . 10^7$
$144^2 + 1 = 20737$	463156	

Table 1

We have from [2] that

$$|\delta u_1| \leq \varepsilon\{N - 1 + C\kappa(N^2 + N - 2)\overline{u}/2 + C_1\kappa(2N^3 + 9N^2 + 13N - 24)/12\}. \tag{3.31}$$

Since $\nu(A) = O(n^2)$ for tridiagonal matrices [2], it is easily seen that if

$$2^{2s-3}M^{s-2}K_1 \approx n^{s-1} \quad \text{and} \quad 2^{2s-3}M^{s-1}(K_2 + K_4/18 + 15K_3/9) \approx n^s,$$

then the errors of parallel and consecutive algorithms will be approximately equal.

Note that if instead of (2.2) we have

$$|A_i| + |C_i| < |B_i|, \quad i = 1, \ldots, N - 1, \tag{3.32}$$

then the rounding error for solving the tridiagonal systems is $|\delta u_i| \leq O(\varepsilon N)$, $i = 0, \ldots, N$. Since the condition number for such tridiagonal matrices is constant [1], the rounding error of the parallel algorithm given in [3] satisfies $\|\delta u\| \leq O(\varepsilon n)$.

4. NUMERICAL EXPERIMENTS

In this section a comparison of error bounds is given between the parallel and consequtive algorithms. For illustration of the above estimations with respect to the rounding error we have performed some numerical experiments for solving the

N	$\varepsilon \approx 10^{-7}$, $\max \lvert x_i - \overline{x}_i \rvert$	
	parallel alg.	conseq. alg.
$3^6 + 1 = 730$	$1{,}43 . 10^{-3}$	
$9^3 + 1 = 730$	$2{,}30 . 10^{-4}$	$2{,}60 . 10^{-5}$
$27^2 + 1 = 730$	$9{,}19 . 10^{-5}$	
$6^4 + 1 = 1297$	$1{,}08 . 10^{-3}$	$4{,}82 . 10^{-5}$
$36^2 + 1 = 1297$	$2{,}94 . 10^{-4}$	
$2^{12} + 1 = 4097$	0	
$4^6 + 1 = 4097$	0	
$8^4 + 1 = 4097$	0	$3{,}96 . 10^{-3}$
$16^3 + 1 = 4097$	$1{,}43 . 10^{-3}$	
$64^2 + 1 = 4097$	$9{,}79 . 10^{-4}$	
$3^8 + 1 = 6562$	$1{,}28 . 10^{-1}$	
$9^4 + 1 = 6562$	$1{,}89 . 10^{-2}$	$1{,}08 . 10^{-2}$
$81^2 + 1 = 6562$	$7{,}72 . 10^{-3}$	
$10^4 + 1 = 10001$	$2{,}95 . 10^{-2}$	$1{,}45 . 10^{-1}$
$100^2 + 1 = 10001$	$1{,}58 . 10^{-2}$	
$5^6 + 1 = 15626$	$3{,}69 . 10^{-1}$	
$25^3 + 1 = 15626$	$4{,}62 . 10^{-3}$	$4{,}24 . 10^{-1}$
$125^2 + 1 = 15626$	$3{,}63 . 10^{-2}$	
$12^4 + 1 = 20737$	$8{,}73 . 10^{-2}$	$6{,}19 . 10^{-1}$
$144^2 + 1 = 20737$	$1{,}03 . 10^{-2}$	

Table 2

following tridiagonal systems of equations

$$(4.1) \qquad \begin{aligned} &u_{i-1} - 2u_i + u_{i+1} = -1, \quad i = 2, \ldots, N-1, \\ &u_1 = 0, \quad u_N = N - 1, \end{aligned}$$

$$(4.2) \qquad \begin{aligned} &-u_{i-1} + 2u_i - u_{i+1} = 0, \quad i = 2, \ldots, N-1, \\ &2u_1 - u_2 = 1, \quad -u_{N-1} + 2u_N = 1 \end{aligned}$$

for different choices of N. Let $\overline{u}_i$ denote the computed solution of the systems. The exact solutions of the systems (4.1) and (4.2) are $u_{N+1-k} = k(n-k)/2$, $k = 1, \ldots, N$ and $u_i = 1$, $i = 1, \ldots, N$, respectively.

In most of the tested cases, the absolute error of the parallel algorithm is better than this of the consequtive one. It turns out to be advisable to use parallel algorithm when N is large. Also if n_1 and n_2 are such that $N = n_1^a = n_2^b$ and $n_1 < n_2$, then the error of parallel algorithm is better for n_2.

Finally, in view of the above, it seems to be advisable to use a double lenght accumulator (see [17]). This will have a favorable practical effect on stabilizing the algorithm and reducing the error bounds.

Here are given two tables illustrating the results of this paper. The table 1 and table 2 concern the systems (4.1) and (4.2), respectively.

REFERENCES

1. Завьялов, Ю. С., В. И. Квасов, В. Л. Мирошниченко. Методы сплайн-функции. М., 1980.
2. Ильин, В. П., Ю. И. Кузнецов. Трехдиагональные матрицы и их приложения. М., Наука, 1985, 52–59.
3. Коновалов, А. Н., Н. Н. Яненко. Некоторые вопросы теории модульного анализа параллельного программирования для задач математической физики и механики сплошной среды. Современные проблемы мат. физики и вычислительной математики, Наука, 1982.
4. Самарский, А. А. Теория разностных схем. М., Наука, 1977.
5. Buneman, O. A compact non-iterative Poisson solver. Inst. for Plasma Res., Stanford Univ., Rep. SU-IPR-294, Stanford, 1969.
6. Buzbee, B. L., G. H. Golub, C. W. Nielson. On direct methods for solving Poisson's equations. *SIAM J. Numer. Anal.*, vol. 7, № 4, 627–656.
7. Csanky, L. On the parallel complexity of some computational problems. Diss. Comput. Sci. Div., Univ. of California, Berkeley, 1974.
8. Dubois, P., G. Rodrigue. An analysis of the recursive doubling algorithm. *In*: Kuck et al., 1977, 299–305.
9. Lambiotte, J., R. Voigt. The solution of tridiagonal systems on the CDS STAR-100 computer. *ACM Trans. Math. Software*, 1, 1975, 308–329.
10. Orcutt, S. E. Computer organisation and algorithms for very high-speed computations. Diss., Stanford Univ., 1974.
11. Sameh, A., D. Kuck. A parallel QR algorithm for symmetric tridiagonal matrices. *IEEE Tras. Comput.*, 1977, C-26, 147–153.
12. Sameh, A., D. Kuck. On stable parallel linear systems solvers. *J. ACM*, 25, 1978, 81–91.
13. Stone, H. S. An efficient parallel algorithm for the solution of a tridiagonal system of equations. *J. ACM*, 20, 1973, 27–38.
14. Stone, H. S. Parallel tridiagonal solvers. *ACM Tras. Math. Software*, vol. 1, № 4, 1975, 289–307.
15. Swarztrauber, P. A parallel algorithm for solving general tridiagonal equations. *Math. Comp.*, 33, 1979, 189–199.
16. Swarztrauber, P. The solution of tridiagonal systems on the CRAY-1. *In*: Jesshope and Hockney, vol. 2, 1979, 343–358.
17. Wilkinson, J. H. Rounding errors in algebraic proceses. Prentice-Hall, 1963.

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ESTIMATION OF THE ERROR OF RUNGE–KUTTA'S METHOD IN MULTIVARIATE CASE

Hussain Ali Al–Juboury

Хусеин Али Ал–Джубури. ОЦЕНКА ПОГРЕШНОСТЕЙ МНОГОМЕРНОГО МЕТОДА РУНГЕ–КУТТА

В работе получена оценка погрешностей многомерного метода Рунге–Кутта первого и второго порядка через усредненными модулями. Оценка получается без дополнительных ограничениях о гладкости решения. Пользуясь свойствами модулей можно вывести разные порядки сходимости.

Hussain Ali Al–Juboury. ESTIMATION OF THE ERROR OF RUNGE–KUTTA'S METHOD IN MULTIVARIATE CASE

The error estimates are obtained for Runge–Kutta's methods of first and second order in the multidimensional case by means of averaged moduli of smoothness without any additional assumptions on the solutions of the equations. The different orders of convergence can be derived from these estimates using the properties of the moduli of smoothness.

1. INTRODUCTION

In this paper we shall obtain error estimates of the numerical solution for the m-dimensional Cauchy problem of the Runge–Kutta's methods with local error of second and third degree using the averaged modulus of smoothness (which is denoted by $\tau(f;\delta)_{L_p}$, see [4]). All notations and definitions, which are used here are involved in [4].

From the properties of the averaged moduli of smoothness, which are mentioned at the end of this paragraph, together with theorems (1) and (2) below, we can obtain many consequences such as that the classical orders of the error $O(h)$ and $O(h^2)$ are obtained under weaker assumptions on the solutions.

Here we list the main properties of averaged moduli of smoothness (see [1]–[3]):

1) $\tau_k(f;\delta')_{L_p} \leq \tau_k(f;\delta'')_{L_p}$ for $\delta' \leq \delta''$;

2) $\tau_k(f+g;\delta)_{L_p} \leq \tau_k(f;\delta)_{L_p} + \tau_k(g;\delta)_{L_p}$;

3) $\tau_k(f;\delta)_{L_p} \leq 2\tau_{k-1}(f;\frac{k}{k-1}\delta)_{L_p}$;

4) $\tau_k(f;\delta)_{L_p} \leq \delta\tau_{k-1}(f';\frac{k}{k-1}\delta)_{L_p}$;

5) $\tau_k(f;n\delta)_{L_p} \leq (2n)^{k+1}\tau_k(f;\delta)_{L_p}$;

6) $\tau_k(f;\lambda\delta)_{L_p} \leq (2(\lambda+1))^{k+1}\tau_k(f;\delta)_{L_p}, \quad \lambda > 0$;

7) $\tau(f;\delta)_{L_p} \leq \delta\|f'\|_{L_p}$;

8) $\tau(f;\delta)_{L_p} \leq \delta V_a^b f$ (where $V_a^b f$ is the variation of the function f between a and b).

2. RUNGE–KUTTA'S METHODS

We shall mention briefly the result of the one dimensional case. Consider the following ordinary differential equation with the initial value:

$$y' = f(x,y), \quad x \in [0, A],\ A > 0,$$

$$y(0) = y_0,$$

and assume that the right hand side of the equation satisfies a Lipschitz condition with respect to the variable y, i.e.:

$$|f(x,y) - f(x,z)| \leq K\,|y - z|,$$

where K is an absolute constant and $x_i = ih$, $h = A/n$, $i = 0, 1, 2, \ldots, n$. If we apply Euler's method then the following estimate holds:

$$\tilde{y}_{i+1} = \tilde{y}_i + hf(x_i, \tilde{y}_i), \quad \tilde{y}_0 = y_0,$$

$$\max_{0\leq i\leq n} |y_i - \tilde{y}_i| \leq 2e^{AK}\tau(y';h)_{L_p}$$

(see [4]).

Suppose that we have a system of m ordinary differential equations with initial conditions as follows:

$$(y^1)' = f^1(x, y^1, \ldots, y^m), \quad y^1(0) = y_0^1,$$

$$(y^2)' = f^2(x, y^1, \ldots, y^m), \quad y^2(0) = y_0^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \ldots\ldots$$

$$(y^m)' = f^m(x, y^1, \ldots, y^m), \quad y^m(0) = y_0^m.$$

We shall need the following generalized Lipschitz condition:

$$|f^m(x, y^1, y^2, \ldots, y^m) - f^m(x, \tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^m)|$$

$$\leq K\left\{|y^1 - \tilde{y}^1| + |y^2 - \tilde{y}^2| + \cdots + |y^m - \tilde{y}^m|\right\}.$$

By Euler's method:

$$\begin{aligned}
\widetilde{y}^1_{i+1} &= \widetilde{y}^1_i + hf^1(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \ldots, \widetilde{y}^m_i),\\
\widetilde{y}^2_{i+1} &= \widetilde{y}^2_i + hf^2(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \ldots, \widetilde{y}^m_i),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\widetilde{y}^m_{i+1} &= \widetilde{y}^m_i + hf^m(x_i, \widetilde{y}^1_i, \widetilde{y}^2_i, \ldots, \widetilde{y}^m_i),
\end{aligned}$$

estimating the error in the i-th step by means of the error in the $(i-1)$-th step, it follows:

$$\begin{aligned}
&\left|y^1_{i+1} - \widetilde{y}^1_{i+1}\right|\\
&= \left|y^1_{i+1} - \widetilde{y}^1_i - hf^1(x_i, \widetilde{y}^1_i, \ldots, \widetilde{y}^m_i) + hf^1(x_i, y^1_i, \ldots, y^m_i) - hf^1(x_i, y^1_i, \ldots, y^m_i)\right|\\
&\leq \left|y^1_{i+1} - \widetilde{y}^1_i + y^1_i - y^1_i - hf^1(x_i, \widetilde{y}^1_i, \ldots, \widetilde{y}^m_i) + hf^1(x_i, y^1_i, \ldots, y^m_i) - hf^1(x_i, y^1_i, \ldots, y^m_i)\right|\\
&\leq \left|y^1_{i+1} - y^1_i - h(y^1_i)'\right| + \left|hf^1(x_i, \widetilde{y}^1_i, \ldots, \widetilde{y}^m_i) - hf^1(x_i, \widetilde{y}^1_i, \ldots, \widetilde{y}^m_i)\right| + \left|y^1_i - \widetilde{y}^1_i\right|,
\end{aligned}$$

and hence

$$\begin{aligned}
&\left|y^1_{i+1} - \widetilde{y}^1_{i+1}\right|\\
&\leq h\omega((y^1)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \cdots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^1_i - \widetilde{y}^1_i\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|y^2_{i+1} - \widetilde{y}^2_{i+1}\right|\\
&\leq h\omega((y^2)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \cdots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^2_i - \widetilde{y}^2_i\right|,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\left|y^m_{i+1} - \widetilde{y}^m_{i+1}\right|\\
&\leq h\omega((y^m)', x_{i+1/2}; h) + Kh\left\{\left|y^1_i - \widetilde{y}^1_i\right| + \cdots + \left|y^m_i - \widetilde{y}^m_i\right|\right\} + \left|y^m_i - \widetilde{y}^m_i\right|.
\end{aligned}$$

Let $\psi_i = \left|y^1_i - \widetilde{y}^1_i\right| + \cdots + \left|y^m_i - \widetilde{y}^m_i\right|$, then

$$\psi_{i+1} \leq h\left[\omega((y^1)', x_{i+1/2}; h) + \cdots + \omega((y^m)', x_{i+1/2}; h)\right] + (1 + mKh)\psi_i,$$

and

$$\psi_i \leq h\left[\omega((y^1)', x_{i-1/2}; h) + \cdots + \omega((y^m)', x_{i-1/2}; h)\right] + (1 + mKh)\psi_{i-1},$$

therefore

$$\begin{aligned}
\psi_{i+1} &\leq h(1 + mKh)\left[\omega((y^1)', x_{i-1/2}; h) + \cdots + \omega((y^m)', x_{i-1/2}; h)\right]\\
&\quad + \left[\omega((y^1)', x_{i+1/2}; h) + \cdots + \omega((y^m)', x_{i+1/2}; h)\right] + (1 + mKh)^2\psi_{i-1}.
\end{aligned}$$

If we repeat this inequality recursively on i we get

$$\psi_{i+1} \leq (1 + mKh)^i \sum_{j=0}^{i} h\left[\omega((y^1)', x_{j+1/2}; h) + \cdots + \omega((y^m)', x_{j+1/2}; h)\right]$$

$$\leq \left(1 + \frac{mAK}{n}\right)^n \sum_{j=0}^{i}\left[\int_{x_j}^{x_{j+1}} \omega((y^1)', x_{j+1/2}; h)\,dx + \cdots + \int_{x_j}^{x_{j+1}} \omega((y^m)', x_{j+1/2}; h)\,dx\right]$$

$$\leq e^{mAK}\left[\int_0^A \omega((y^1)', x; 2h)\, dx + \cdots + \int_0^A \omega((y^m)', x; 2h)\, dx\right]$$

$$\leq 2e^{mAK}\left[\tau((y^1)'; h) + \cdots + \tau((y^m)'; h)\right].$$

From the last estimations it follows that

$$\psi_{i+1} \leq 2e^{mAK} \sum_{r=1}^{m} \tau((y^r)'; h)_{L_p}.$$

So we have proved the following theorem.

Theorem 1. *The following estimation is true*

$$\max\{|y_i^r - \widetilde{y}_i^r| : 1 \leq r \leq m,\ 0 \leq i \leq n\} \leq 2e^{mAK} \sum_{r=1}^{m} [\tau((y^r)'; h)_{L_p}].$$

To estimate the error for those Runge–Kutta's methods which have local error $O(h^3)$ in multivariate case we restrict ourselves to two dependent variables as follows

$$\text{(1)} \qquad \begin{aligned} y' &= f(x, y, z), \quad y(0) = y_0, \\ z' &= g(x, y, z), \quad z(0) = z_0. \end{aligned}$$

Using the formulae

$$\text{(2)} \qquad \begin{aligned} \widetilde{y}_{i+1} &= \widetilde{y}_i + phf(x_i, \widetilde{y}_i, \widetilde{z}_i) + qhf(x_i + \alpha h, \widetilde{y}_i + \beta hf(x_i, \widetilde{y}_i, \widetilde{z}_i), \widetilde{z}_i + \beta hg(x_i, \widetilde{y}_i, \widetilde{z}_i)), \\ \widetilde{z}_{i+1} &= \widetilde{z}_i + phg(x_i, \widetilde{y}_i, \widetilde{z}_i) + qhg(x_i + \alpha h, \widetilde{y}_i + \beta hf(x_i, \widetilde{y}_i, \widetilde{z}_i), \widetilde{z}_i + \beta hg(x_i, \widetilde{y}_i, \widetilde{z}_i)) \end{aligned}$$

where the constants p, q, α, β satisfy the system

$$\text{(3)} \qquad p + q = 1, \quad q\alpha = \frac{1}{2}, \quad q\beta = \frac{1}{2},$$

it follows that this system has one-parameter solution of the form

$$p = 1 - s, \quad q = s, \quad \alpha = \beta = \frac{1}{2s}.$$

For the sake of simplicity we shall put $s = \frac{1}{2}$ (the general case can be considered in a similar way), i.e.

$$p = q = \frac{1}{2}, \quad \alpha = \beta = 1,$$

then from (1) and (2) we obtain

$$|y_{i+1} - \widetilde{y}_{i+1}|$$

$$= \left|y_{i+1} - \widetilde{y}_i - phf(x_i, \widetilde{y}_i, \widetilde{z}_i) - qhf(x_i + \alpha h, \widetilde{y}_i + \beta hf(x_i, \widetilde{y}_i, \widetilde{z}_i), \widetilde{z}_i + \beta hg(x_i, \widetilde{y}_i, \widetilde{z}_i))\right|$$

$$= \left|y_{i+1} - \widetilde{y}_i - \frac{h}{2} f(x_i, \widetilde{y}_i, \widetilde{z}_i) - \frac{h}{2} f(x_i + h, \widetilde{y}_i + hf(x_i, \widetilde{y}_i, \widetilde{z}_i), \widetilde{z}_i + hg(x_i, \widetilde{y}_i, \widetilde{z}_i))\right|$$

$$\leq \Big|y_{i+1} - \widetilde{y}_i - \frac{h}{2} f(x_i, \widetilde{y}_i, \widetilde{z}_i) + \frac{h}{2} f(x_i, y_i, z_i) - \frac{h}{2}(x_i, y_i, z_i)$$

$$- \frac{h}{2} f(x_i + h, \widetilde{y}_i + hf(x_i, \widetilde{y}_i, \widetilde{z}_i), \widetilde{z}_i + hg(x_i, \widetilde{y}_i, \widetilde{z}_i))$$

$$+\frac{h}{2}f\big(x_i+h, y_i+hf(x_i,y_i,z_i), z_i+hg(x_i,y_i,z_i)\big)$$

$$-\frac{h}{2}f\big(x_i+h, y_i+hf(x_i,y_i,z_i), z_i+hg(x_i,y_i,z_i)\big)\Big|.$$

By applying Lipschitz condition we obtain

$$
\begin{aligned}
|y_{i+1}-\widetilde{y}_{i+1}| \le{} & \frac{Kh}{2}\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\} \\
& + \frac{Kh}{2}\Big\{\big|\widetilde{y}_i+hf(x_i,\widetilde{y}_i,\widetilde{z}_i)-y_i-hf(x_i,y_i,z_i)\big| \\
& \qquad + \big|\widetilde{z}_i+hg(x_i,\widetilde{y}_i,\widetilde{z}_i)-z_i-hg(x_i,y_i,z_i)\big|\Big\} \\
& + \Big|y_{i+1}-\widetilde{y}_i+y_i-y_i-\frac{h}{2}f(x_i,y_i,z_i) \\
& \qquad -\frac{h}{2}f\big(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\Big| \\
\le{} & \frac{Kh}{2}\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\}+\frac{Kh}{2}\Big\{\big|\widetilde{y}_i+hf(x_i,\widetilde{y}_i,\widetilde{z}_i)-y_i-hf(x_i,y_i,z_i)\big| \\
& \qquad + \big|\widetilde{z}_i+hg(x_i,\widetilde{y}_i,\widetilde{z}_i)-z_i-hg(x_i,y_i,z_i)\big|\Big\} \\
& \qquad + \Big|y_{i+1}-\widetilde{y}_i+y_i-y_i-\frac{h}{2}y_i'-\frac{h}{2}f\big(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\Big| \\
\le{} & \frac{Kh}{2}\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\}+\frac{Kh}{2}\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\} \\
& \qquad + \frac{K^2h^2}{2}\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\}+|y_i-\widetilde{y}_i| \\
& \qquad + \Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f\big(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\Big| \\
\le{} & c_1\{|y_i-\widetilde{y}_i|+|z_i-\widetilde{z}_i|\}+|y_i-\widetilde{y}_i|+c_2
\end{aligned}
$$

$$\text{(4)}\quad \le (1+c_1)\,|y_i-\widetilde{y}_i|+c_1\,|z_i-\widetilde{z}_i|+c_2,$$

where $c_1=\dfrac{2Kh+K^2h^2}{2}$, and

$$c_2=\Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f\big(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\Big|.$$

Let us estimate c_2:

$$
\begin{aligned}
c_2 = {} & \Big|y_{i+1}-y_i-\frac{h}{2}y_i'-\frac{h}{2}f\big(x_i+h,y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big) \\
\text{(5)}\quad & -hy'_{i+1/2}+hy'_{i+1/2}-\frac{h}{2}f(x_i+h,y_{i+1},z_{i+1})+\frac{h}{2}f(x_i+h,y_{i+1},z_{i+1})\Big| \\
\le{} & \big|y_{i+1}-y_i-hy'_{i+1/2}\big|+h\Big|y'_{i+1/2}-\frac{1}{2}y_i'-\frac{1}{2}y'_{i+1}\Big|
\end{aligned}
$$

$$+\frac{h}{2}\left|f(x_{i+1},y_{i+1},z_{i+1})-f\big(x_{i+1},y_i+hf(x_i,y_i,z_i),z_i+hg(x_i,y_i,z_i)\big)\right|,$$

where $x_{i+1}=x_i+h, y_{i+1}=y(x_{i+1}), y'_{i+1/2}=y'(x_{i+1/2})$. Now estimating the first term in the right hand side of inequality (5) we get

$$\left|y_{i+1}-y_i-hy'_{i+1/2}\right|=h\left|\frac{y_{i+1}-y_i}{h}-y'_{i+1/2}\right|$$

$$=\left|\int\limits_{x_i}^{x_{i+1}}\left[y'(t)-y'_{i+1/2}\right]dt\right|=h\left|\int\limits_{-1/2}^{1/2}\left[y'(x_{i+1/2}+th)-y'_{i+1/2}\right]dt\right|$$

$$=h\left|\int\limits_{0}^{1/2}\left[y'(x_{i+1/2}+th)-2y'_{i+1/2}+y'(x_{i+1/2}-th)\right]dt\right|$$

$$\leq h\int\limits_{0}^{1/2}\omega_2\left(y',x_{i+1/2};\frac{h}{2}\right)dt=\frac{h}{2}\omega_2\left(y',x_{i+1/2};\frac{h}{2}\right). \tag{6}$$

In order to estimate the following term $\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|$, let p be the algebraic polynomial of first degree, which interpolates the function y' at the points x_i and x_{i+1}. We have (see [4], lemma 2.3, p. 30)

$$\|y'-p\|_{C[x_i,\,x_{i+1}]}\leq\omega_2(y',x_{i+1/2};h/2), \tag{7}$$

where $p(x_i)=p_i,\ p(x_{i+1})=p_{i+1}$.

From (7) we get

$$\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|$$

$$\leq\left|y'_{i+1/2}-p_{i+1/2}-\frac{1}{2}(y'_i-p_i)-\frac{1}{2}(y'_{i+1}-p_{i+1/2})\right|+\left|p_{i+1/2}-\frac{1}{2}(p_i+p_{i+1/2})\right|$$

$$\leq\left|y'_{i+1/2}-p_{i+1/2}\right|+\frac{1}{2}\left|y'_i-p_i\right|+\frac{1}{2}\left|y'_{i+1}-p_{i+1}\right|\leq 2\omega_2\left(y',x_{i+1/2};h/2\right).$$

Since $p_{i+1/2}-\frac{1}{2}p_i-\frac{1}{2}p_{i+1}=0$, p being of first degree, we obtain

$$\left|y'_{i+1/2}-\frac{1}{2}(y'_i+y'_{i+1})\right|\leq\frac{1}{2}\omega_2(y',x_{i+1/2};h/2)\leq\frac{1}{2}\omega_2(y',x_{i+1/2};h).$$

Similarly we get

$$\left|z'_{i+1/2}-\frac{1}{2}(z'_i+z'_{i+1})\right|\leq\frac{1}{2}\omega_2(z',x_{i+1/2};h/2)\leq\frac{1}{2}\omega_2(z',x_{i+1/2};h/2).$$

Now

$$|f(x_{i+1}, y_{i+1}, z_{i+1}) - f(x_{i+1}, y_i + hf(x_i, y_i, z_i), z_i + hg(x_i, y_i, z_i))|$$

$$\leqq K\{|y_{i+1} - y_i - hy_i'| + |z_{i+1} - z_i - hz_i'|\}$$

$$\leqq \dots \leqq Kh\left(\left|\frac{y_{i+1} - y_i}{h} - y_i'\right| + \left|\frac{z_{i+1} - z_i}{h} - z_i'\right|\right)$$

$$\leqq Kh\left\{\left|\frac{1}{h}\int_{x_i}^{x_{i+1}} [y'(t) - y_i']\, dt\right| + \left|\frac{1}{h}\int_{x_i}^{x_{i+1}} [z'(t) - z_i']\, dt\right|\right\}$$

$$\leqq Kh\{\omega(y', x_{i+1/2}; h) + \omega(z', x_{i+1/2}; h)\}. \tag{8}$$

From (4), (6), (5) and (8) we obtain

$$|y_{i+1} - \tilde{y}_{i+1}| + |z_{i+1} - \tilde{z}_{i+1}| \leqq \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right] |y_i - \tilde{y}_i| + \frac{h}{2}\omega_2\left(y', x_{i+1/2}; \frac{h}{2}\right)$$

$$+ \frac{Kh^2}{2}\omega(y', x_{i+1/2}; h) + 2h\omega_2(y', x_{i+1/2}; h) + \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right] |z_i - \tilde{z}_i|$$

$$+ \frac{h}{2}\omega_2\left(z', x_{i+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega\,(z', x_{i+1/2}; h) + 2h\omega_2\,(z', x_{i+1/2}; h).$$

Applying the above inequality recursively on i, we obtain

$$|y_{i+1} - \tilde{y}_{i+1}| + |z_{i+1} - \tilde{z}_{i+1}|$$

$$\leqq \sum_{k=0}^{i} \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]^{i-k} \left[\frac{h}{2}\omega_2\left(y', x_{k+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega(y', x_{k+1/2}; h)\right. \tag{9}$$

$$\left. + 2h\omega_2(y', x_{k+1/2}; h)\right] + \sum_{k=0}^{i} \left[1 + Kh\left(1 + \frac{Kh}{2}\right)\right]^{i+k} \left[\frac{h}{2}\omega_2\left(z', x_{k+1/2}; \frac{h}{2}\right)\right.$$

$$\left. + \frac{Kh^2}{2}\omega(z', x_{k+1/2}; h) + 2h\omega_2(z', x_{k+1/2}; h)\right].$$

Set $1 + \dfrac{kA}{2} = c_3$, then from (9) we get

$$\max\{|y_i - \tilde{y}_i| + |z_i - \tilde{z}_i| \ : \ 0 \leqq i \leqq n\}$$

$$\leqq \left(1 + \frac{c_3AK}{n}\right)^n \sum_{k=1}^{n-1} \left[\frac{h}{2}\omega_2\left(y', x_{k+1/2}; \frac{h}{2}\right) + \frac{Kh^2}{2}\omega\,(y', x_{k+1/2}; h)\right.$$

$$\left. + 2h\omega_2(y', x_{k+1/2}; h)\right] + \left(1 + \frac{c_3AK}{n}\right)^n \sum_{k=1}^{n-1} \left[\frac{h}{2}\omega_2\left(z', x_{k+1/2}; \frac{h}{2}\right)\right.$$

$$\left. + \frac{Kh^2}{2}\omega(z', x_{k+1/2}; h) + 2h\omega_2(z', x_{k+1/2}; h)\right]$$

$$\le e^{c_3AK}\sum_{k=0}^{n-1}\left[\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(y',x;h)\,dx+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(y',x;h)\,dx+2\int_{x_k}^{x_{k+1}}\omega_2(y',x;h)\,dx\right]$$

$$+\,e^{c_3AK}\sum_{k=0}^{n-1}\left[\frac{1}{2}\int_{x_k}^{x_{k+1}}\omega_2(z',x;h)\,dx+\frac{Kh}{2}\int_{x_k}^{x_{k+1}}\omega(z',x;h)\,dx+2\int_{x_k}^{x_{k+1}}\omega_2(z',x;h)\,dx\right]$$

$$=e^{c_3AK}\int_0^A\left[\frac{1}{2}\omega_2(y',x;h)+\frac{Kh}{2}\omega(y',x;h)+2\omega_2(y',x;h)\right]dx$$

$$+\,e^{c_3AK}\int_0^A\left[\frac{1}{2}\omega_2(z',x;h)+\frac{Kh}{2}\omega(z',x;h)+2\omega_2(z',x;h)\right]dx$$

$$=e^{c_3AK}A\left[\frac{1}{2}\tau_2(y';h)_{L_1}+\frac{Kh}{2}\tau(y';h)_{L_1}+2\tau_2(y';h)_{L_1}\right]$$

$$+\,e^{c_3AK}A\left[\frac{1}{2}\tau_2(z';h)_{L_1}+\frac{Kh}{2}\tau(z';h)_{L_1}+2\tau_2(z';h)_{L_1}\right]$$

$$=e^{c_3AK}A\left[\frac{1}{2}\left(\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}\right)+\frac{Kh}{2}\left(\tau(y';h)_{L_1}+\tau(z';h)_{L_1}\right)\right.$$

$$\left.+\,2\left(\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}\right)\right].$$

Therefore we have proved the following theorem.

Theorem 2. *For the solution of the problem (1) the estimation*

$$\max\{|y_i-\tilde{y}_i|+|z_i-\tilde{z}_i|\,:\,0\le i\le n\}$$

$$\le c\left[\tau_2(y';h)_{L_1}+\tau_2(z';h)_{L_1}+h\left[\tau(y';h)+\tau(z';h)\right]\right]$$

holds, where c is a constant depending on A and K only.

REFERENCES

1. **A n d r e e v, A. S.** Some estimations of the approximate solution of differential equations. Approximation and function Spaces, Z. Ciesielski (ed.), PWN — Warszawa — North Holland pub. company, Amsterdam — New York — Oxford, 1981, 16–24.
2. **A n d r e e v, A. S.** Convergence rate for spline collocation to Fredholm integral equation of the second kind. *Pliska*, Studia math. Bulg., 5, 1983, 84–92.
3. **A n d r e e v, A. S., D. G i c h e v.** Error estimates for the numerical solution of the Cauchy problem. *Serdica*, Bulg. math. publ., 10, 1984, 206–215.
4. **S e n d o v, B., V. A. P o p o v.** The averaged moduli of smoothness. A. Wiley–Interscience publ., 1988, 117–123.

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AN EXTERNAL CHARACTERIZATION OF THE PRIME COMPUTABILITY *

Ivan N. Soskov

Иван Н. Сосков. ВНЕШНЯЯ ХАРАКТЕРИЗАЦИЯ ПРОСТОЙ ВЫЧИСЛИМОСТИ

В настоящей работе продолжены исследования автора по максимальной вычислимости в классах алгебраических систем.

Основной результат работы состоит в том, что в любом достаточно богатом классе алгебраических систем простая вычислимость Московакиса является самой сильной среди вычислимостей, которые являются инвариантными, последовательными и обладающими подструктурным свойством.

Ivan N. Soskov. AN EXTERNAL CHARACTERIZATION OF THE PRIME COMPUTABILITY

In the paper we continue the study of the maximal concepts of computability on classes of first order structures.

The main result is that on each rich enough class of denumerable structures the Prime computability of Moschovakis is the strongest among all computabilities which are sequential, invariant and have the substructure property.

In [1] we begin the study of the so called maximal concepts of computability. With these investigations we aim at obtaining a classification of the concepts of "effective" computability on first order structures. The main idea, on which this classification is based, is to consider the behavior of a computability not only on a single structure but on a class of structures. Then one can formulate some properties and in some cases to prove that there exists a strongest computability among the computabilities which have this properties. Such computabilities are called maximal. The results in [1] are connected with the characterization of some maximal non-deterministic computabilities. It is proved there that on each rich

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enough class $\mathcal{A}$ of denumerable structures Search computability of Moschovakis [2] is the strongest among the effective and invariant on $\mathcal{A}$ computabilities and the Computability by means of recursively enumerable definitional schemes (REDS-computability) [3], [4] is the strongest among the effective and invariant on $\mathcal{A}$ computabilities which have the substructure property on $\mathcal{A}$.

The exact definitions of the notions of effectiveness and invariance of a computability and of the substructure property are given in section 1.3 of the present paper.

Here we begin the study of the maximal sequential computabilities. The main result in the paper is that on each rich enough class $\mathcal{A}$ of denumerable structures the Prime computability of Moschovakis [2] is the strongest computability among the sequential and invariant on $\mathcal{A}$ computabilities which have the substructure property on $\mathcal{A}$.

This result follows from an appropriate external characterization of the Prime computability, obtained in section 2.2.

As in [1], all results in the paper are formulated and proved only for (classes of) denumerable partial structures. The problem of generalizing the present results for classes of arbitrary structures is still open.

1. PRELIMINARIES

1.1. Notation and basic definitions.

Let $\mathcal{L} = \{f_1, \ldots, f_n; T_1, \ldots, T_k\}$ be a first order language, here $f_1, \ldots, f_n$ are functional symbols and $T_1, \ldots, T_k$ are predicate symbols. Let each f_i be a_i-ary and let each T_j be b_j-ary.

In what follows we shall consider only partial structures of the language $\mathcal{L}$ with denumerable domains, i.e. structures $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$, where A — the domain of $\mathfrak{A}$ — is a denumerable set of objects, each θ_i is an a_i-ary partial function on A and each Σ_j is a b_j-ary partial predicate on A. The structure $\mathfrak{A}$ will be called total if all initial functions $\theta_1, \ldots, \theta_n$ and all initial predicates $\Sigma_1, \ldots, \Sigma_k$ are totally defined on A.

By $|\mathfrak{A}|$ we shall denote the domain of the structure $\mathfrak{A}$.

Throughout the paper by a structure we shall mean a denumerable partial structure of the language $\mathcal{L}$ and by a total structure we shall mean a total denumerable structure of the language $\mathcal{L}$.

The partial predicates on the domain A of a structure will be identified with the partial mappings which obtain values in $\{0, 1\}$, taking 0 for true and 1 for false.

Let $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ and $\mathfrak{B} = (B; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k)$ be structures.

The surjective mapping κ of A onto B is called a *strong homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff the following conditions are fulfilled:

(i) $\kappa(\theta_i(s_1, \ldots, s_{a_i})) \cong \varphi_i(\kappa(s_1), \ldots, \kappa(s_{a_i}))$ for all $s_1, \ldots, s_{a_i}$ of A.

(ii) $\Sigma_j(s_1, \ldots, s_{b_j}) \cong \sigma_j(\kappa(s_1), \ldots, \kappa(s_{b_j}))$ for all $s_1, \ldots, s_{b_j}$ of A.

Obviously if κ is an injective strong homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ then κ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

The structure $\mathfrak{A}$ is called a *substructure* of $\mathfrak{B}$ and $\mathfrak{B}$ is called an *extension* of $\mathfrak{A}$ if the following conditions are satisfied:

(i) $A \subseteq B$;

(ii) $\theta_i(s_1,\ldots,s_{a_i})) \cong \varphi_i(s_1,\ldots,s_{a_i})$ for all $s_1, \ldots, s_{a_i}$ of A.

(iii) $\Sigma_j(s_1,\ldots,s_{b_j})) \cong \sigma_j(s_1,\ldots,s_{b_j})$ for all $s_1, \ldots, s_{b_j}$ of A.

By $\mathfrak{A} \subseteq \mathfrak{B}$ we shall denote that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$.

The structure $\mathfrak{B}$ is said to be a *total extension* of $\mathfrak{A}$, in symbols $\mathfrak{A} \subseteq_t \mathfrak{B}$, if $\mathfrak{A} \subseteq \mathfrak{B}$ and the following two conditions hold:

(i) if $1 \leq i \leq n$, then φ_i is totally defined on $B^{a_i} \setminus A^{a_i}$,

(ii) if $1 \leq j \leq n$, then σ_j is totally defined on $B^{b_j} \setminus A^{b_j}$.

Notice that if $\mathfrak{A}$ is a total structure and $\mathfrak{A} \subseteq_t \mathfrak{B}$, then $\mathfrak{B}$ is also total.

1.2. The μ-recursive operators and the μ-recursive functions.

Let $\mathbb{N}$ be the set of all natural numbers. By $\mathfrak{F}_n$, $n \geq 1$, we shall denote the set of all partial functions of n arguments on $\mathbb{N}$.

Let $n_1,\ldots,n_k$ and m be positive natural numbers. A *μ-recursive operator* of type $(n_1,\ldots,n_k \Rightarrow m)$ is called each total mapping Γ of $\mathfrak{F}_{n_1} \times \mathfrak{F}_{n_2} \times \cdots \times \mathfrak{F}_{n_k}$ into $\mathfrak{F}_m$ such that whenever $\theta_1,\ldots,\theta_k$ are elements of $\mathfrak{F}_{n_1}, \mathfrak{F}_{n_2}, \ldots, \mathfrak{F}_{n_k}$, respectively, then $\Gamma(\theta_1,\ldots,\theta_k)$ is defined uniformly with respect to $\theta_1,\ldots,\theta_k$ trough an explicit expression build up from $\theta_1,\ldots,\theta_k$ and the initial primitive recursive functions by means of the operations substitution, primitive recursion and minimization.

The μ-recursive operators are studied by Skordev in [5], [6], by Sasso in [7] and recently by Cooper [8]. In particular in [6] it is shown that the μ-recursive operators coincide with the Turing computable ones, where an operator Γ is Turing computable if there exists a Turing machine which computes the value of $\Gamma(\theta_1,\ldots,\theta_k)(x_1,\ldots,x_m)$ using oracles for $\theta_1,\ldots,\theta_k$ in a sequential way.

The close connections between the μ-recursive operators and some sequential concepts of computability on first order structures are established in [4].

Here we shall use a definition of the μ-recursive operators which is a reformulation of that one given in [4].

Let us fix the positive natural numbers $n_1,\ldots,n_k$.

Let $R^1,\ldots,R^k$ be new predicate symbols and let each R^i be $n_i + 1$-ary.

The *number theoretic predicates* (n.t. predicates) of type $(n_1,\ldots,n_k)$ are defined by means of the following inductive clauses:

(i) The empty expression Λ is a n.t. predicate;

(ii) Each expression of the form $R^i(z_1,\ldots,z_{n_i},y)$, where $1 \leq i \leq k$ and $z_1,\ldots,z_{n_i},y$ are arbitrary natural numbers is a n.t. predicate;

(iii) If E^1 and E^2 are n.t. predicates then so is and $(E^1 \& E^2)$.

We shall assume that $(E^1 \& \Lambda) = E^1$, and $(\Lambda \& E^2) = E^2$.

Let $\theta_1,\ldots,\theta_k$ be elements of $\mathfrak{F}_{n_1}, \mathfrak{F}_{n_2}, \ldots, \mathfrak{F}_{n_k}$, respectively. Then the k-tuple $\theta^* = (\theta_1,\ldots,\theta_k)$ will be called a *functional system* of type $(n_1,\ldots,n_k)$.

Let E be a n.t. predicate of type $(n_1,\ldots,n_k)$, and $\theta^* = (\theta_1,\ldots,\theta_k)$ be a functional system of the same type. Then the value E_{θ^*} of E over θ^* is defined by means of the following inductive clauses:

(i) If $E = \Lambda$, then $E_{\theta^*} \cong 0$;

(ii) If $E = R^i(z_1,\ldots,z_{n_i},y)$, $1 \leq i \leq k$, then

$$E_{\theta^*} \cong \begin{cases} 0, & \text{if } \theta_i(z_1,\ldots,z_{n_i}) \cong y, \\ 1, & \text{if } \theta_i(z_1,\ldots,z_{n_i}) \text{ is defined and } \theta_i(z_1,\ldots,z_{n_i}) \neq y, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

(iii) If $E = (E^1 \& E^2)$, then

$$E_{\theta^*} \cong \begin{cases} 1, & \text{if } E^1_{\theta^*} \cong 1, \\ E^2_{\theta^*}, & \text{if } E^1_{\theta^*} \cong 0, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Notice that it may happen that $(E^1 \& E^2)_{\theta^*}$ is defined though $E^2_{\theta^*}$ is not defined. On the other hand, $(E^1 \& E^2)_{\theta^*} \cong 0$ iff $E^1_{\theta^*} \cong 0$ and $E^2_{\theta^*} \cong 0$.

Obviously the operation "&" is associative though not commutative. In what follows we shall write the n.t. predicates in the form $E^1 \& E^2 \& \dots \& E^l$, $l \geq 0$, omitting the brackets.

Let E^1 and E^2 be n.t. predicates of type $(n_1, \dots, n_k)$. Then E^1 and E^2 are said to be contrary if

$$E^1 = E^0 \& R^i(z_1, \dots, z_{n_i}, y) \& Q^1 \quad \text{and} \quad E^2 = E^0 \& R^i(z_1, \dots, z_{n_i}, x) \& Q^2,$$

where $x \neq y$ and E^0, Q^1, Q^2 are n.t. predicates.

Clearly if E^1 and E^2 are contrary, then for each functional system θ^* of type $(n_1, \dots, n_k)$, if $E^1_{\theta^*} \cong 0$, then $E^2_{\theta^*}$ is defined and $E^2_{\theta^*} \cong 1$.

If E is a n.t. predicate of type $(n_1, \dots, n_k)$ and $y \in \mathbb{N}$, then $E \supset y$ will be called a *number theoretic expression* of type $(n_1, \dots, n_k)$.

Assume that an effective coding of the n.t. expressions is fixed. We shall use $(E^v \supset y^v)$ to denote the n.t. expression with code v.

A *treelike number theoretic scheme* of type $(n_1, \dots, n_k)$ will be called each r.e. set $\{E^v \supset y^v\}_{v \in V}$ of n.t. expressions of type $(n_1, \dots, n_k)$ such that if v_1 and v_2 are distinct elements of V, then E^{v_1} and E^{v_2} are contrary.

Given a treelike n.t. scheme $S = \{E^v \supset y^v\}_{v \in V}$ and a functional system θ^* of the same type as S, define the value S_{θ^*} of S over θ^* by the equivalence $S_{\theta^*} \cong y \iff \exists v (v \in V \;\&\; E^v_{\theta^*} \cong 0 \;\&\; y = y^v)$.

Notice that if $S = \{E^v \supset y^v\}_{v \in V}$ is a treelike n.t. scheme, θ^* is a functional system of the same type as S and $S_{\theta^*} \cong y$, then for all $v \in V$, $E^v_{\theta^*}$ is defined.

The following proposition is a reformulation of Theorem 7.11 of [4].

Proposition 1. *The total mapping* Γ *of* $\mathfrak{F}_{n_1} \times \mathfrak{F}_{n_2} \times \dots \times \mathfrak{F}_{n_k}$ *into* $\mathfrak{F}_m$ *is a* μ*-recursive operator if and only if there exists a recursive function* $\gamma(n, x_1, \dots, x_m)$ *such that the following is true for all* $x_1, \dots, x_m$ *of* $\mathbb{N}$ *:*

(i) *For each* n, $\gamma(n, x_1, \dots, x_m)$ *yields a code of a n.t. expression* $E^{\gamma(n,x_1,\dots,x_m)} \supset y^{\gamma(n,x_1,\dots,x_m)}$ *of type* $(n_1, \dots, n_k)$.

(ii) *The set* $S^{x_1,\dots,x_m} = \{E^{\gamma(n,x_1,\dots,x_m)} \supset y^{\gamma(n,x_1,\dots,x_m)}\}_{n \in \mathbb{N}}$ *is a treelike scheme.*

(iii) *If* $\theta^* = (\theta_1, \dots, \theta_k)$ *is a functional system of type* $(n_1, \dots, n_k)$, *then* $\Gamma(\theta_1, \dots, \theta_k)(x_1, \dots, x_m) \cong y \iff S^{x_1,\dots,x_m}_{\theta^*} \cong y$.

For our purposes the following evident consequence of Proposition 1 is sufficient.

Proposition 2. *Let* Γ *be a* μ*-recursive operator of type* $(n_1, \dots, n_k \Rightarrow m)$. *Suppose that* $x_1, \dots, x_m$ *are fixed elements of* $\mathbb{N}$. *Then there exists a treelike n.t. scheme* S *of type* $(n_1, \dots, n_k)$ *such that whenever* $\theta^* = (\theta_1, \dots, \theta_k)$ *is a functional system of type* $(n_1, \dots, n_k)$, *then* $\Gamma(\theta_1, \dots, \theta_k)(x_1, \dots, x_m) \cong y \iff S_{\theta^*} \cong y$.

Let $\mathfrak{B} = (\mathbb{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k)$ be a structure over the natural numbers. We shall use $\mathfrak{B}$ to denote and the functional system $(\varphi_1, \dots, \varphi_n, \sigma_1, \dots, \sigma_k)$ of type $(a_1, \dots, a_n, b_1, \dots, b_k)$.

A partial a-ary function φ on $\mathbb{N}$ is said to be *μ-recursive* in $\mathfrak{B}$ if there exists a μ-recursive operator Γ of type $(a_1, \ldots, a_n, b_1, \ldots, b_k \Rightarrow a)$ such that $\Gamma(\mathfrak{B}) = \varphi$.

Clearly the μ-recursive in $\mathfrak{B}$ functions are exactly those which can be obtained from the initial primitive recursive functions and from the basic functions and predicates of $\mathfrak{B}$ by means of substitution, primitive recursion and minimization.

We are finishing this section with some words concerning the relationships between the relative μ-recursiveness and the relative partial recursiveness.

Let $\mathfrak{B}$ be a structure and let $|\mathfrak{B}| = \mathbb{N}$. A partial function φ on $\mathbb{N}$ is said to be partial recursive in $\mathfrak{B}$ if $\varphi = \Delta(\mathfrak{B})$, where Δ is a partial recursive operator. For the definitions of the recursive, of the partial recursive operators and of the enumeration operators the reader may consult [9].

Since each μ-recursive operator is partial recursive and even recursive, each μ-recursive in $\mathfrak{B}$ function is partial recursive in $\mathfrak{B}$.

If the structure $\mathfrak{B}$ is total then the μ-recursive in $\mathfrak{B}$ functions coincide with the partial recursive in $\mathfrak{B}$ functions. But there are examples of partial structures in which the μ-recursive functions are a proper subclass of the partial recursive functions. Such examples are given in [10] and [5].

Intuitively, the μ-recursive in $\mathfrak{B}$ functions are those which are computable by means of sequential procedures using the basic functions and predicates of $\mathfrak{B}$ while the partial recursive in $\mathfrak{B}$ functions are those which are computable by means of arbitrary non-deterministic (parallel) procedures using the basic functions and predicates of $\mathfrak{B}$.

Speaking about non-deterministic procedures it seems natural to admit not only single-valued but also partial multiple-valued (p.m.v.) functions to be computable by means of such procedures.

A p.m.v. function φ on $\mathbb{N}$ is said to be partial recursive in the structure $\mathfrak{B}$ iff there exists an enumeration operator Γ such that $\Gamma(\mathfrak{B})$ is the graph of φ.

1.3. Computability on a class of structures.

Let $\mathcal{A}$ be a class of structures. A *computability* on $\mathcal{A}$ is called every mapping C of $\mathcal{A}$ such that if $\mathfrak{A} \in \mathcal{A}$, then $C(\mathfrak{A})$ is a set of p.m.v. functions on $\mathfrak{A}$.

The computability C is said to be *effective* on $\mathcal{A}$ if for each element $\mathfrak{B}$ of $\mathcal{A}$, if $|\mathfrak{B}| = \mathbb{N}$, then all elements of $C(\mathfrak{B})$ are partial recursive in $\mathfrak{B}$ p.m.v. functions.

A computability C is said to be *invariant* on $\mathcal{A}$ iff whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, κ is a strong homomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ and $\theta \in C(\mathfrak{A})$, there exists a $\varphi \in C(\mathfrak{B})$ of the same arity as θ such that $\kappa(\varphi(s_1, \ldots, s_a)) \cong \theta(\kappa(s_1), \ldots, \kappa(s_a))$ for all elements $s_1, \ldots, s_a$ of $|\mathfrak{B}|$.

A computability C on $\mathcal{A}$ has the *substructure property* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, $\mathfrak{A} \subseteq \mathfrak{B}$ and $\theta \in C(\mathfrak{A})$, there exists a φ in $C(\mathfrak{B})$ of the same arity as θ and such that $\theta(s_1, \ldots, s_a) \cong \varphi(s_1, \ldots, s_a)$ for all elements $s_1, \ldots, s_a$ of $|\mathfrak{A}|$.

To explain the substructure property we have to think that there exists a computational process which computes the value of θ over the arguments $s_1, \ldots, s_a$. Now, the substructure property follows from the assumption that in the course of the computation no additional information but the arguments is needed.

Let C be a computability on $\mathcal{A}$. Then C is said to be *sequential* on $\mathcal{A}$ iff whenever $\mathfrak{B} \in \mathcal{A}$ and $|\mathfrak{B}| = \mathbb{N}$, then all elements of $C(\mathfrak{B})$ are μ-recursive in $\mathfrak{B}$.

Clearly if C is a sequential computability on a class $\mathcal{A}$ of structures, then C is also effective on $\mathcal{A}$.

Let C_1 and C_2 be two computabilities on A. The computability C_1 is said to be *weaker* than C_2 on A, in symbols $C_1 \subseteq_A C_2$, iff for all $\mathfrak{A}$ in A, $C_1(\mathfrak{A}) \subseteq C_2(\mathfrak{A})$.

As we shall see in the next section, if A is a rich enough class of structures, then each sequential and invariant on A computability which has the substructure property on A is weaker than Prime computability on A.

2. PRIME COMPUTABILITY

2.1. Definition and some properties of the Prime computability.

Let $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ be a structure.

Let $\{X_1, X_2, \ldots\}$ be a denumerable set of variables. We shall use the capital letters X, Y, Z to denote variables.

If τ is a term in the language $\mathcal{L}$, then we shall write $\tau(X_1, X_2, \ldots, X_a)$ to denote that all of the variables in τ are among X_1, X_2, $\ldots$, X_a.

If $\tau(X_1, X_2, \ldots, X_a)$ is a term, s_1, $\ldots$, s_a are arbitrary elements of A, then with $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ we shall denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements s_1, $\ldots$, s_a.

Let T_0 be a new unary predicate symbol and Σ_0 be the total predicate $\lambda s.0$ on A.

Termal predicates in the language $\mathcal{L}$ are defined by the inductive clauses:

If $T \in \{T_0, \ldots, T_k\}$, T is b-ary and τ^1, $\ldots$, τ^b are terms, then any of $T(\tau^1, \ldots, \tau^b)$ and $\neg T(\tau^1, \ldots, \tau^b)$ is a termal predicate;

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(X_1, X_2, \ldots, X_a)$ be a termal predicate whose variables are among X_1, $X_2, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of A. The value $\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Π over s_1, $\ldots$, s_a in $\mathfrak{A}$ is defined by the inductive clauses:

If $\Pi = T_j(\tau^1, \ldots, \tau^{b_j})$, $0 \leq j \leq k$, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \Sigma_j(\tau^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), \ldots, \tau^{b_j}_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a));$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \begin{cases} 0, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 1, \\ 1, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0, \\ \text{undefined}, & \text{otherwise}; \end{cases}$$

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong \begin{cases} 1, & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 1, \\ \Pi^2_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a), & \text{if } \Pi^1_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0, \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

If Π is a termal predicate and τ is a term, then $Q = (\Pi \supset \tau)$ is called a *conditional term*.

Let $Q(X_1, \ldots, X_a)$ be a conditional term with variables among X_1, $\ldots$, X_a and let s_1, $\ldots$, s_a be elements of A. Then the value $Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of Q over s_1, $\ldots$, s_a in $\mathfrak{A}$ is defined by

$$Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong t$$
$$\iff \Pi_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong 0 \;\&\; \tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong t.$$

Assume that an effective coding of the expressions of the language $\mathcal{L}$ is fixed. By $Q^v = (\Pi^v \supset \tau^v)$ we shall denote the conditional term with code v.

An a-ary partial function θ on A is called *prime computable* on $\mathfrak{A}$ iff for some recursive sequence $\{\Pi^{\rho(n)} \supset \tau^{\rho(n)}\}_{n\in\mathbb{N}}$ of conditional expressions with variables among $Z_1,\ldots,Z_r$, X_1, ..., X_a and for some fixed elements t_1, ..., t_r of A the following equivalence is true for all s_1, ..., s_a and t in A:

$$\theta(s_1,\ldots,s_a) \cong t$$

$$\iff \exists n(n \in \mathbb{N} \ \& \ \Pi_{\mathfrak{A}}^{\rho(n)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong 0$$

$$\& \ \tau_{\mathfrak{A}}^{\rho(n)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong t$$

$$\& \ \forall m(m < n \Rightarrow \Pi_{\mathfrak{A}}^{\rho(m)}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong 1)).$$

Notice that each prime computable on $\mathfrak{A}$ function is single-valued.

The original definition of the prime computable functions in [2] looks out somewhat different. There the prime computable functions are defined as a subclass of the p.m.v. functions on the set A^*, where A^* is an appropriate extension of A. However for partial functions on A both definitions are equivalent, see [11] or [12].

Given a structure $\mathfrak{A}$, denote by PC($\mathfrak{A}$) the class of all prime computable on $\mathfrak{A}$ functions.

Proposition 3. *The computability* PC *is sequential, invariant and has the substructure property on each class* A *of structures.*

Let $Q = (\Pi \supset \tau)$ be a conditional term with variables among X_1, ..., X_a. Then Q is said to be *definite* over the elements s_1, ..., s_a, in symbols $!!Q_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$, iff $\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong 1$ or $(\Pi(X_1/s_1,\ldots,X_a/s_a) \cong 0$ and $\tau_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$ is defined).

An a-ary single-valued partial function θ is said to be *definable* on $\mathfrak{A}$ iff for some r.e. set $\{Q^v\}_{v\in V}$ of conditional terms with variables among Z_1, ..., Z_r; X_1, ..., X_a and some fixed elements t_1, ..., t_r of A the following conditions hold for all elements s_1, ..., s_a and t of A:

(i) If $\theta(s_1,\ldots,s_a)$ is defined then for each v in V,

$$!!Q^v_{\mathfrak{A}}(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a).$$

(ii) $\theta(s_1,\ldots,s_a) \cong t$

$$\iff \exists v(v \in V \& Q^v(Z_1/t_1,\ldots,Z_r/t_r,X_1/s_1,\ldots,X_a/s_a) \cong t).$$

Proposition 4. *Each definable on* $\mathfrak{A}$ *single-valued function is prime computable on* $\mathfrak{A}$.

As we shall see later, the definable on $\mathfrak{A}$ single-valued functions coincide with the prime computable ones.

2.2. External characterisation of the Prime computability.

Let $\mathfrak{A} = (A; \theta_1,\ldots,\theta_n; \Sigma_1,\ldots,\Sigma_k)$ be a structure.

An ordered pair $(\alpha, \mathfrak{B})$ is called *enumeration* of $\mathfrak{A}$ iff α is a partial surjective mapping of $\mathbb{N}$ onto A, $\mathfrak{B} = (\mathbb{N}; \varphi_1,\ldots,\varphi_n; \sigma_1,\ldots,\sigma_k)$ is a structure and the following conditions are fulfilled:

(i) The domain of α (dom(α)) is closed with respect to the partial functions $\varphi_1,\ldots,\varphi_n$;

(ii) $\alpha(\varphi_i(x_1,\ldots,x_{a_i})) \cong \theta_i(\alpha(x_1),\ldots,\alpha(x_{a_i}))$ for all $x_1,\ldots,x_{a_i}$ of dom(α), $1 \leq i \leq n$;

(iii) $\sigma_j(x_1,\ldots,x_{b_j}) \cong \Sigma_j(\alpha(x_1),\ldots,\alpha(x_{b_j}))$ for all $x_1,\ldots,x_{b_j}$ of $\mathrm{dom}(\alpha)$, $1 \leq j \leq k$;

(iv) Each φ_i is totally defined on $\mathbf{N}^{a_i} \setminus (\mathrm{dom}(\alpha))^{a_i}$ and each σ_j is totally defined on $\mathbf{N}^{b_j} \setminus (\mathrm{dom}(\alpha))^{b_j}$.

Let $\langle\alpha, \mathfrak{B}\rangle$ be an enumeration of $\mathfrak{A}$. Denote by $\mathfrak{B}^*$ the structure $(\mathrm{dom}(\alpha); \varphi_1^*,\ldots,\varphi_n^*; \sigma_1^*,\ldots,\sigma_k^*)$, where each φ_i^* is the restriction of φ_i on $\mathrm{dom}(\alpha)$ and each σ_j^* is the restriction of σ_j on $\mathrm{dom}(\alpha)$. It follows from the definition that α is a strong homomorphism from $\mathfrak{B}^*$ to $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq_t \mathfrak{B}$.

A partial a-ary function θ on A is called *μ-admissible* in the enumeration $\langle\alpha, \mathfrak{B}\rangle$ iff for some μ-recursive in $\mathfrak{B}$ partial function φ of a arguments on $\mathbf{N}$ and for all $x_1,\ldots,x_a$ in $\mathrm{dom}(\alpha)$, the following is true:

(i) If $\varphi(x_1,\ldots,x_a) \cong y$, then $y \in \mathrm{dom}(\alpha)$;

(ii) $\theta(\alpha(x_1),\ldots,\alpha(x_a)) \cong \alpha(\varphi(x_1,\ldots,x_a))$.

Here we shall prove the following theorem which gives an external characterization of the prime computability.

Theorem 1. *A partial function θ on A is prime computable on $\mathfrak{A}$ iff it is μ-admissible in all enumerations of $\mathfrak{A}$.*

Similar external characterizations of Search computability and of Computability by means of effectively definable schemes are obtained in [13], [14], [15] and [1].

Let $\langle.,.\rangle$ be an effective coding of the ordered pairs of natural numbers and let $\lambda x.(x)_0$ and $\lambda x.(x)_1$ be recursive functions such that $(\langle x_0, x_1\rangle)_0 = x_0$ and $(\langle x_0, x_1\rangle)_1 = x_1$. If $n \geq 2$, then by $\langle x_0,\ldots,x_n\rangle$ we shall denote $\langle x_0, \langle x_1,\ldots, \langle x_{n-1}, x_n\rangle\ldots\rangle\rangle$.

We shall assume that the coding $\langle.,.\rangle$ is chosen so that $\langle x,y\rangle > x$ and $\langle x,y\rangle > y$. For example, let $\langle x,y\rangle = 2^x3^y$.

Let $\langle\alpha, \mathfrak{B}\rangle$ be an enumeration of $\mathfrak{A}$. Suppose that $\mathfrak{B} = (\mathbf{N}; \varphi_1,\ldots,\varphi_n; \sigma_1,\ldots,\sigma_k)$. The enumeration $\langle\alpha, \mathfrak{B}\rangle$ is called *special* if whenever $1 \leq i \leq n$ and $\varphi_i(x_1,\ldots,x_{a_i}) \cong y$, then $y = \langle i,j,x_1,\ldots,x_{a_i}\rangle$ for some natural j.

A $n+k+2$-tuple $(H_1, \alpha_1, \varphi_1^1,\ldots,\varphi_n^1, \sigma_1^1,\ldots,\sigma_k^1)$ is called *finite part* (of a special enumeration) iff the following conditions are satisfied:

(i) H_1 is a finite subset of N;

(ii) α_1 is a partial mapping with a finite domain of $\mathbf{N}$ into A, $\mathrm{dom}(\alpha_1) \cap H_1 = \varnothing$;

(iii) Each φ_i^1 is an a_i-ary partial function on $H_1 \cup \mathrm{dom}(\alpha_1)$ and:

(a) $\mathrm{dom}(\alpha_1)$ is closed with respect to $\varphi_1^1,\ldots,\varphi_n^1$;

(b) if $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$, then $y = \langle i,j,x_1,\ldots,x_{a_i}\rangle$ for some natural j;

(c) if $x_1,\ldots,x_{a_i}$ are elements of $\mathrm{dom}(\alpha_1)$ and $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$, then $\theta_i(\alpha_1(x_1),\ldots,\alpha_1(x_{a_i})) \cong \alpha_1(y)$;

(iv) Each σ_j^1 is a partial predicate on $(H_1 \cup \mathrm{dom}(\alpha_1))^{b_j} \setminus (\mathrm{dom}(\alpha_1))^{b_j}$.

Let $\Delta_1 = (H_1, \alpha_1, \varphi_1^1,\ldots,\varphi_n^1, \sigma_1^1,\ldots,\sigma_k^1)$ and $\Delta_2 = (H_2, \alpha_2, \varphi_1^2,\ldots,\varphi_n^2, \sigma_1^2,\ldots,\sigma_k^2)$ be finite parts. Then $\Delta_1 \subseteq \Delta_2$ iff the following conditions are true:

(i) $H_1 \subseteq H_2$ and $\alpha_1 \subseteq \alpha_2$;

(ii) $\varphi_i^1 \subseteq \varphi_i^2$, $i = 1,\ldots,n$, and $\sigma_j^1 \subseteq \sigma_j^2$, $j = 1,\ldots,k$;

(iii) If $\varphi_i^2(x_1,\ldots,x_{a_i}) \cong y$ and $y \in \mathrm{dom}(\alpha_1)$, then $\varphi_i^1(x_1,\ldots,x_{a_i}) \cong y$.

If $\Delta = (H_1, \alpha_1, \varphi_1^1, \ldots, \varphi_n^1, \sigma_1^1, \ldots, \sigma_k^1)$ is a finite part and $\langle \alpha, \mathfrak{B} \rangle = (\mathbb{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k))$ is a special enumeration, then $\Delta \subseteq \langle \alpha, \mathfrak{B} \rangle$ iff the following is true:

(i) $H_1 \cap \mathrm{dom}(\alpha) = \varnothing$ and $\alpha_1 \subseteq \alpha$;

(ii) $\varphi_i^1 \subseteq \varphi_i$, $i = 1, \ldots, n$, and $\sigma_j^1 \subseteq \sigma_j$, $j = 1, \ldots, k$;

(iii) If $\varphi_i(x_1, \ldots, x_{a_i}) \cong y$ and $y \in \mathrm{dom}(\alpha_1)$, then $\varphi_i^1(x_1, \ldots, x_{a_i}) \cong y$.

The proofs of the following two propositions can be found in [1].

Proposition 5. *Let Δ_1, Δ_2 and Δ_3 be finite parts and let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration. Then $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \Delta_3$ implies $\Delta_1 \subseteq \Delta_3$, and $\Delta_1 \subseteq \Delta_2$ and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$ implies $\Delta_1 \subseteq \langle \alpha, \mathfrak{B} \rangle$.*

Proposition 6. *Let $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_q \subseteq \cdots$ be a sequence of finite parts. Let $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots \sigma_k^q)$ and suppose that the following is true:*

(i) *for each $s \in A$ there exists a q, such that $s \in \mathrm{range}(\alpha_q)$;*

(ii) *if $1 \leqq i \leqq n$, $x_1, \ldots, x_{a_i}$ are elements of $\mathrm{dom}(\alpha_q)$ and $\theta_i(\alpha_q(x_1), \ldots, \alpha_q(x_{a_i}))$ is defined, then for some $p \geqq q$, $\varphi_i^p(x_1, \ldots, x_{a_i})$ is defined;*

(iii) *if $(x_1, \ldots, x_{a_i})$ belongs to $(H_q \cup \mathrm{dom}(\alpha_q))^{a_i} \setminus (\mathrm{dom}(\alpha_q))^{a_i}$, then for some $p \geqq q$, $\varphi_i^p(x_1, \ldots, x_{a_i})$ is defined;*

(iv) *for each natural number x there exists a q such that $x \in H_q$ or $x \in \mathrm{dom}(\alpha_q)$.*

Then there exists a special enumeration $\langle \alpha, \mathfrak{B} \rangle$ such that for all natural q, $\Delta_q \subseteq \langle \alpha, \mathfrak{B} \rangle$.

Let val be an effective one to one mapping of the set of all natural numbers onto the set of all variables.

Let $\mathbb{N}_0$ be the set of those natural numbers which are not of the form $\langle i, j, x_1, \ldots, x_{a_i} \rangle$, where $1 \leqq i \leqq n$.

Suppose that $\Delta_1 = (H_1, \alpha_1, \varphi_1^1, \varphi_2^1, \ldots, \varphi_n^1, \sigma_1^1, \sigma_2^1, \ldots, \sigma_k^1)$ is a finite part. Let $\mathrm{dom}(\alpha_1) = \{w_1, \ldots, w_r\}$, $\alpha_1(w_i) \cong t_i$ and $\mathrm{val}(w_i) = W_i$, $i = 1, \ldots, r$. Let $x_1, \ldots, x_a$ be distinct elements of $\mathbb{N}_0 \setminus (H_1 \cup \mathrm{dom}(\alpha_1))$ and let $\mathrm{val}(x_i) = X_i$, $i = 1, \ldots, a$.

Proposition 7. *There exists an effective way to define for each n.t. expression $E \supset y$ of type $(a_1, \ldots, a_n, b_1, \ldots, b_k)$ a conditional term Q with variables among $W_1, \ldots, W_r, X_1, \ldots, X_a$ such that for all elements $s_1, \ldots, s_a$ and t of A, the following conditions are satisfied:*

(1) *If $Q_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong t$, then there exists a finite part $\Delta_2 \supseteq \Delta_1$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$, $\alpha(y) \cong t$ and $E_{\mathfrak{B}} \cong 0$;*

(2) *If $Q_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \not\cong t$, then at least one of the following is true:*

(2.1) *There exists a finite part $\Delta_2 \supseteq \Delta_1$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $x_1, \ldots, x_a$ belong to $\mathrm{dom}(\alpha)$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \mathrm{dom}(\alpha)$;*

(2.2) *For each special enumeration $\langle \alpha, \mathfrak{B} \rangle$, if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_1$, $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$ and $E_{\mathfrak{B}} \cong 0$, then $\alpha(y) \not\cong t$;*

(3) *If $Q_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is not definite, then there exists a finite part $\Delta_2 \supseteq \Delta_1$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, and $E_{\mathfrak{B}}$ is not defined.*

A detailed proof of this proposition is given in the Appendix.

Now we are ready for the proof of Theorem 1.

Proof of Theorem 1. Let θ be an a-ary partial function on A. If θ is prime computable on $\mathfrak{A}$, then clearly θ is μ-admissible in each enumeration of $\mathfrak{A}$.

Suppose now that θ is not prime computable on $\mathfrak{A}$. We can assume that θ is single-valued. Obviously if θ is not single-valued, then θ is not μ-admissible in any enumeration $\langle\alpha, \mathfrak{B}\rangle$ of $\mathfrak{A}$. By Proposition 4, θ is not definable on $\mathfrak{A}$.

Let us fix an enumeration of the μ-recursive operators of type $(a_1, \ldots, a_n, b_1, \ldots, b_k \Rightarrow a)$. By Γ_n we shall denote the μ-recursive operator with number n.

We shall construct a special enumeration $\langle\alpha^0, \mathfrak{B}^0\rangle$ such that θ is not admissible in it. The definition of $\langle\alpha^0, \mathfrak{B}^0\rangle$ will be performed by steps. In each step q we shall define a finite part $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \ldots, \varphi_n^q; \sigma_1^q, \ldots, \sigma_k^q)$ so that $\Delta_q \subseteq \Delta_{q+1}$. After that we shall define $\langle\alpha^0, \mathfrak{B}^0\rangle$ as a special enumeration such that $\Delta_q \subseteq \langle\alpha^0, \mathfrak{B}^0\rangle$ for all q.

We shall consider four kinds of steps. With the first three kinds we shall ensure that the conditions (i) — (iv) of the hypothesis of Proposition 6 are true. With the steps $q = \langle 4m+3, n\rangle$ we shall ensure that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_{q+1}$ is a special enumeration and $\zeta = \Gamma_n(\mathfrak{B})$, then for some $x_1, \ldots, x_a$ of $\mathrm{dom}(\alpha)$, at least one of the following two conditions is not true:

(4) $\zeta(x_1, \ldots, x_a) \cong y \Rightarrow y \in \mathrm{dom}(\alpha)$,

(5) $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \alpha(\zeta(x_1, \ldots, x_a))$.

Let $s_0, s_1, \ldots$ be an arbitrary enumeration of A.

Let $\Delta_0 = (\varnothing, \alpha_0, \varphi_1^0, \ldots, \varphi_n^0, \sigma_1^0, \ldots, \sigma_k^0)$, where $\alpha_0, \varphi_1^0, \ldots, \varphi_n^0, \sigma_1^0, \ldots, \sigma_k^0$ are totally undefined.

Suppose that $\Delta_q = (H_q, \alpha_q, \varphi_1^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots, \sigma_k^q)$ is defined. We shall consider the following cases:

1. $(q)_0 = 4m$, for some $m \in \mathbb{N}$. Let x be the first natural number, which does not belong to $\mathrm{dom}(\alpha_q) \cup H_q$ and let s be the first element of the sequence $s_0, s_1, \ldots$ which does not belong to $\mathrm{range}(\alpha_q)$. If such s does not exist then let s be an arbitrary element of A. Define $\alpha_{q+1}(x) \cong s$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$ for $z \neq x$ and let $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi_1^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots, \sigma_k^q)$.

2. $q = \langle 4m+1, \langle i, x_1, \ldots, x_{a_i}\rangle\rangle$, where $m \in \mathbb{N}$, $1 \leq i \leq n$, $x_1, \ldots, x_{a_i}$ are elements of $\mathrm{dom}(\alpha_q)$, $\varphi_i^q(x_1, \ldots, x_{a_i})$ is not defined and $\theta_i(\alpha_q(x_1), \ldots, \alpha_q(x_{a_i}))$ is defined. Let $y = \langle i, j, x_1, \ldots, x_{a_i}\rangle$ be an element of $\mathbb{N} \setminus (\mathrm{dom}(\alpha_q) \cup H_q)$. Let $\alpha_{q+1}(y) \cong \theta_i(\alpha_q(x_1), \ldots, \alpha(x_{a_i}))$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$ for $z \neq x$. Let $\varphi_i^{q+1}(x_1, \ldots, x_{a_i}) \cong y$ and $\varphi_i^{q+1}(z_1, \ldots, z_{a_i}) \cong \varphi_i^q(z_1, \ldots, z_{a_i})$ for $(z_1, \ldots, z_{a_i}) \neq (x_1, \ldots, x_{a_i})$. Define $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi_1^q, \ldots, \varphi_{i-1}^q, \varphi_i^{q+1}, \varphi_{i+1}^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots, \sigma_k^q)$.

3. $q = \langle 4m+2, \langle i, x_1, \ldots, x_{a_i}\rangle\rangle$, where $m \in \mathbb{N}$, $(x_1, \ldots, x_{a_i})$ belongs to $(H_q \cup \mathrm{dom}(\alpha_q))^{a_i} \setminus (\mathrm{dom}(\alpha_q))^{a_i}$ and $\varphi_i^q(x_1, \ldots, x_{a_i})$ is not defined. Let $y = \langle i, j, x_1, \ldots, x_{a_i}\rangle$ be an element of $\mathbb{N} \setminus (\mathrm{dom}(\alpha_q) \cup H_q)$ and let $H_{q+1} = H_q \cup \{y\}$. Let $\varphi_i^{q+1}(x_1, \ldots, x_{a_i}) \cong y$ and $\varphi_i^{q+1}(z_1, \ldots, z_{a_i}) \cong \varphi_i^q(z_1, \ldots, z_{a_i})$ for $(z_1, \ldots, z_{a_i}) \neq (x_1, \ldots, x_{a_i})$. Let $\Delta_{q+1} = (H_{q+1}, \alpha_q, \varphi_1^q, \ldots, \varphi_{i-1}^q, \varphi_i^{q+1}, \varphi_{i+1}^q, \ldots, \varphi_n^q, \sigma_1^q, \ldots, \sigma_k^q)$.

4. $q = \langle 4m+3, n\rangle$, for some $m, n \in \mathbb{N}$. Let $\mathrm{dom}(\alpha_q) = \{w_1, \ldots, w_r\}$, and let us fix some distinct elements $x_1, \ldots, x_a$ of $\mathbb{N}_0 \setminus (H_q \cup \mathrm{dom}(\alpha_q))$. Let $\mathrm{val}(w_i) = W_i$, $i = 1, \ldots, r$, and $\mathrm{val}(x_i) = X_i$, $i = 1, \ldots, a$.

By Proposition 2, there exists a treelike n.t. scheme $S = \{E^v \supset y^v\}_{v \in V}$ such that for each structure $\mathfrak{B}$ over the natural numbers

(6) $$\Gamma_n(\mathfrak{B})(x_1, \ldots, x_a) \cong y \Longleftrightarrow \exists v(v \in V \,\&\, E^v_{\mathfrak{B}} \cong 0 \,\&\, y = y^v).$$

By Proposition 7, there exists an effective way to define for each $v \in V$ a conditional term $Q^v(W_1, \ldots, W_r, X_1, \ldots, X_a)$ satisfying the conditions (1), (2) and (3) with respect to $E^v \supset y^v$, $x_1, \ldots, x_a$ and Δ_q. So we obtain the r.e. set $\{Q^v\}_{v \in V}$ of conditional terms.

Let $\alpha(w_i) \cong t_i$, $i = 1, \ldots, r$. Define the a-ary p.m.v. function ζ on A by the equivalence: $t \in \zeta(s_1, \ldots, s_a) \iff \exists v(v \in V \;\&\; Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong t)$.

We shall consider the following cases:

I. For some $s_1, \ldots, s_a$, t of A, $t \in \zeta(s_1, \ldots, s_a)$ but $\theta(s_1, \ldots, s_a) \not\cong t$. Then for some $v \in V$, $Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong t$. By (1), there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$, $\alpha(y^v) \cong t$ and $E_{\mathfrak{B}} \cong 0$.

Let $\Delta_{q+1} = \Delta$. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$ and let $\zeta = \Gamma_n(\mathfrak{B})$. By (6), $\alpha(\zeta(x_1, \ldots, x_a)) \cong \alpha(y^v) \cong t$. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a) \not\cong t$. So in this case (5) is not true.

II. Let for all $s_1, \ldots, s_a$, t of A, if $\zeta(s_1, \ldots, s_a) \cong t$, then $\theta(s_1, \ldots, s_a) \cong t$. Hence ζ is single-valued. We have the following subcases:

a) For some $s_1, \ldots, s_a$ of A, $\zeta(s_1, \ldots, s_a)$ is defined but there exists a $v \in V$ such that Q^v is not definite over $t_1, \ldots, t_r$, $s_1, \ldots, s_a$.

Clearly in this case $\theta(s_1, \ldots, s_a)$ is also defined.

Since Q^v satisfies (3) with respect to $x_1, \ldots, x_a$, $E^v \supset y^v$ and Δ_q, there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $\alpha(x_i) \cong s_i$, $i = 1, \ldots, n$ and $E^v_{\mathfrak{B}}$ is not defined. Let $\Delta_{q+1} = \Delta$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\Gamma_n(\mathfrak{B}) = \zeta$. Using (6) and the fact that S is a treelike scheme, we obtain that $\zeta(x_1, \ldots, x_a)$ is not defined and, hence, $\alpha(\zeta(x_1, \ldots, x_a))$ is not defined. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a)$ and, hence, $\theta(\alpha(x_1), \ldots, \alpha(x_a))$ is defined.

So we receive that the condition (5) is not true.

b) If $\zeta(s_1, \ldots, s_a)$ is defined then for all $v \in V$, $!!Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$. It follows from here that ζ is definable on $\mathfrak{A}$. Since θ is not definable on $\mathfrak{A}$, $\zeta \not\cong \theta$. Hence there exist $s_1, \ldots, s_a$ and t in A such that $\theta(s_1, \ldots, s_a) \cong t$ and $\zeta(s_1, \ldots, s_a) \not\cong t$. By the definition of ζ, for all $v \in V$, $Q^v_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \not\cong t$. Therefore for all Q^v, at least one of the conditions (2.1) and (2.2) is true with respect to $x_1, \ldots, x_a$, $E^v \supset y^v$ and Δ_q.

Suppose that there exists a $v \in V$ such that (2.1) is true for Q^v. Then there exists a finite part $\Delta \supseteq \Delta_q$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, then $x_1, \ldots, x_a$ belong to $\mathrm{dom}(\alpha)$, $E^v_{\mathfrak{B}} \cong 0$ and $y^v \notin \mathrm{dom}(\alpha)$. Let $\Delta_{q+1} = \Delta$. Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\zeta = \Gamma_n(\mathfrak{B})$. By (6), $\zeta(x_1, \ldots, x_a) \cong y^v$ but $y^v \notin \mathrm{dom}(\alpha)$. So the condition (4) fails.

Let us suppose that for all Q^v, (2.2) is true.

Let $\alpha_{q+1}(x_1) \cong s_1, \ldots, \alpha_{q+1}(x_a) \cong s_a$ and $\alpha_{q+1}(z) \cong \alpha_q(z)$, for $z \notin \{x_1, \ldots, x_a\}$. Let $\Delta_{q+1} = (H_q, \alpha_{q+1}, \varphi^q_1, \ldots, \varphi^q_n, \sigma^q_1, \ldots, \sigma^q_k)$. Clearly Δ_{q+1} is a finite part and $\Delta_{q+1} \supseteq \Delta_q$.

Let $\langle \alpha, \mathfrak{B} \rangle$ be a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$. Let $\Gamma_n(\mathfrak{B}) = \zeta$. Since $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_{q+1}$, $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$ and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_q$. Then, by (2.2), if $v \in V$ and $E^v_{\mathfrak{B}} \cong 0$, then $\alpha(y^v) \not\cong t$. Therefore $\alpha(\zeta(x_1, \ldots, x_a)) \not\cong t$. On the other hand, $\theta(\alpha(x_1), \ldots, \alpha(x_a)) \cong \theta(s_1, \ldots, s_a) \cong t$. From here it follows that the condition (5) fails.

Define $\Delta_{q+1} = \Delta_q$ in the other cases.

Let $\langle\alpha^0, \mathfrak{B}^0\rangle$ be a special enumeration such that $\langle\alpha^0, \mathfrak{B}^0\rangle \supseteq \Delta_q$ for all q. Let us suppose that θ is μ-admissible in $\langle\alpha^0, \mathfrak{B}^0\rangle$. Then for some μ-recursive in $\mathfrak{B}$ function ζ (4) and (5) are true for all $x_1,\ldots,x_a \in \mathrm{dom}(\alpha^0)$. Let Γ_n be the μ-recursive operator of type $(a_1,\ldots,a_n,\ b_1,\ldots,b_k \Rightarrow a)$ such that $\Gamma_n(\mathfrak{B}) = \zeta$.

Since $\langle\alpha^0, \mathfrak{B}^0\rangle \supseteq \Delta_{(3,n)+1}$, at least one of (4) and (5) is not true for some $x_1,\ldots,x_a$ of $\mathrm{dom}(\alpha^0)$. A contradiction.

2.3. The main result

A class $\mathcal{A}$ of structures is *closed under homomorphic counter-images* iff whenever $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}$ is a structure and there exists a strong homomorphism κ from $\mathfrak{B}$ to $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{A}$.

A class $\mathcal{A}$ of structures is *closed under total extensions* iff whenever $\mathfrak{A} \in \mathcal{A}$, $\mathfrak{B}$ is a structure and $\mathfrak{A} \subseteq_t \mathfrak{B}$, then $\mathfrak{B} \in \mathcal{A}$.

Two natural examples of closed under homomorphic counter-images and under total extensions classes of structures are the class $\mathcal{A}_0$ of all structures and the class $\mathcal{A}_t$ of all total structures.

Theorem 2. *Let $\mathcal{A}$ be a class of structures and let $\mathcal{A}$ be closed with respect to homomorphic counter-images and total extensions. Let C be a computability on $\mathcal{A}$ which is sequential, invariant and let C have the substructure property on $\mathcal{A}$. Then* $C \subseteq_{\mathcal{A}} \mathrm{PC}$.

P r o o f. Let $\mathfrak{A}$ be an element of $\mathcal{A}$ and let $\theta \in C(\mathfrak{A})$. By Theorem 1, to prove that $\theta \in \mathrm{PC}(\mathfrak{A})$ it is sufficient to show that θ is μ-admissible in all enumerations of $\mathfrak{A}$. Let $\langle\alpha, \mathfrak{B} = (\mathrm{N}; \varphi_1,\ldots,\varphi_n; \sigma_1,\ldots,\sigma_k)\rangle$ be an enumeration of $\mathfrak{A}$. Denote by φ_i^* the restriction of φ_i on $\mathrm{dom}(\alpha)$, $i = 1,\ldots,n$ and by σ_j^* the restriction of σ_j on $\mathrm{dom}(\alpha)$. Let $\mathfrak{B}^* = (\mathrm{dom}(\alpha); \varphi_1^*,\ldots,\varphi_n^*; \sigma_1^*,\ldots,\sigma_k^*)$. From the definition of the notion of an enumeration of $\mathfrak{A}$ it follows that α is a strong homomorphism from $\mathfrak{B}^*$ to $\mathfrak{A}$ and $\mathfrak{B}^* \subseteq_t \mathfrak{B}$. Then $\mathfrak{B}^*$ and $\mathfrak{B}$ are elements of $\mathcal{A}$. By the invariance of C, there exists a φ^* in $C(\mathfrak{B}^*)$ such that for all $x_1,\ldots x_a$ of $\mathrm{dom}(\alpha)$, $\alpha(\varphi^*(x_1,\ldots,x_a)) \cong \theta(\alpha(x_1),\ldots,\alpha(x_a))$. From the substructure property of C it follows that there exists an a-ary function φ in $C(\mathfrak{B})$ such that for all $x_1,\ldots,x_a$ of $\mathrm{dom}(\alpha)$, $\varphi(x_1,\ldots,x_a) \cong \varphi^*(x_1,\ldots,x_a)$. Finally, since C is sequential, φ is μ-recursive in $\mathfrak{B}$. So, we obtain that θ is μ-admissible in $\langle\alpha, \mathfrak{B}\rangle$.

This result should be compared with Theorem 4 in [1], which states that if $\mathcal{A}$ is a closed under homomorphic counter-images and total extensions class of structures, then each effective and invariant on $\mathcal{A}$ computability C which has the substructure property on $\mathcal{A}$ is weaker than REDS-computability.

So we obtain that in some sense Prime computability is the sequential counterpart of REDS-computability. Namely both computabilities have the same model-theoretic properties but the first is sequential while the second is non-deterministic.

From the above results it follows that on each total structure the prime computable functions coincide with the single-valued REDS-computable functions. An example of a partial structure on which the prime computable functions are a proper subclass of the single-valued REDS-computable functions is given in [16].

Here we shall give a detailed proof of Proposition 7. The enumeration of the propositions in the Appendix will be independent of that one used in the main text.

We shall suppose fixed a structure $\mathfrak{A} = (A; \theta_1, \ldots, \theta_n, \Sigma_1, \ldots, \Sigma_k)$ and a finite part $\Delta = (H_1, \alpha_1, \varphi_1^1, \ldots, \varphi_n^1, \sigma_1^1, \ldots, \sigma_k^1)$.

Let $\mathrm{dom}(\alpha_1) = \{w_1, \ldots, w_r\}$ and let $x_1, \ldots, x_a$ be distinct elements of $\mathbb{N}_0 \setminus (H_1 \cup \mathrm{dom}(\alpha_1))$.

Recall that $\mathfrak{A}$ is a structure of the first order language $\mathcal{L} = (f_1, \ldots, f_n; T_1, \ldots, T_k)$, where each f_i is a_i-ary and each T_j is b_j-ary.

A finite subset L of $\mathbb{N}^3$ will be called a *termal tree* if the following conditions are true:

(i) If $(i, z, y) \in L$, then $1 \leq i \leq n$, $z = \langle z_1, \ldots, z_{a_i} \rangle$ for some elements $z_1, \ldots, z_{a_i}$ of $\mathbb{N}$ and $y = \langle i, j, z \rangle$ for some $j \in \mathbb{N}$.

(ii) If $(i, z, y_1) \in L$ and $(i, z, y_2) \in L$, then $y_1 = y_2$.

(iii) If $(i, z, y) \in L$, then $y \notin \mathrm{dom}(\alpha_1)$.

(iv) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, and $\varphi_i^1(z_1, \ldots, z_{a_i})$ is defined, then $\varphi_i^1(z_1, \ldots, z_{a_i}) \cong y$.

Let us fix a termal tree L.

Immediate consequences of the condition (i) of the definition and of the choice of the coding function $\langle .,. \rangle$ are:

(T1) If (i_1, z_1, y) and (i_2, z_2, y) are elements of L, then $i_1 = i_2$ and $z_1 = z_2$.

(T2) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $z_j < y$, $j = 1, \ldots, a_i$.

If $(i, z, y) \in L$, then z is called a *L-predecessor* of y. Those natural numbers which have not L-predecessors will be called *L-prime*.

Notice that by (iii) of the definition and by the choice of $x_1, \ldots x_a$, all $w_1, \ldots, w_r, x_1, \ldots, x_a$ are L-prime.

For each natural number y, let $|y|_L = 0$, if y is L-prime, and let $|y|_L = |z_1|_L + \cdots + |z_{a_i}|_L + 1$, if $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$.

Let $y \in \mathbb{N}$. Define the finite subset $[y]$ of $\mathbb{N}$ by means of the following inductive clauses:

If $|y|_L = 0$, then $[y] = \{y\}$.

If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $[y] = \{y\} \cup [z_1] \cup \ldots \cup [z_{a_i}]$.

Define the binary relation "$\leq_L$" on $\mathbb{N}$ by the equivalence $z \leq_L y \iff z \in [y]$.

Now we shall list some properties of the relation "$\leq_L$" which follow easily from the definition.

(R0) If $|y|_L = 0$ and $z \leq_L y$, then $z = y$.

(R1) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$, then $z_j \leq_L y$, $j = 1, \ldots, a_i$.

(R2) If $(i, \langle z_1, \ldots, z_{a_i} \rangle, y) \in L$ and $u \leq_L y$, then $u = y$ or $\exists j (1 \leq j \leq a_i \;\&\; u \leq_L z_j)$.

(R3) $y \leq_L y$.

(R4) $z \leq_L y \;\&\; y \leq_L u \Rightarrow z \leq_L u$.

(R5) $z \leq_L y \;\&\; y \leq_L z \Rightarrow y = z$.

Let $y \in \mathbb{N}$. The subset $(L)_y$ of L is defined by the equivalence

$$(i, z, u) \in (L)_y \iff (i, z, u) \in L \;\&\; u \leq_L y.$$

We have the following properties of $(L)_y$.

(P1) $(L)_y = \varnothing \iff |y|_L = 0$.

(P2) If $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$, then $(L)_y = \{(i,\langle z_1,\dots,z_{a_i}\rangle,y)\}\cup(L)_{z_1}\cup\dots\cup(L)_{z_{a_i}}$.

If $u=\langle z_1,\dots,z_{a_i}\rangle$ and $1\leqq j\leqq a_i$, then we shall use $(u)_j$ to denote z_j.

(P3) $z\leq_L y \iff z=y$ or $\exists i\exists j\exists u\exists v(1\leqq i\leqq n \;\&\; 1\leqq j\leqq a_i \;\&\; z=(u)_j \;\&(i,u,v)\in(L)_y)$.

P r o o f. In the right to left direction (P3) follows from (R1), (R3), (R4) and from the definition of $(L)_y$.

The proof in the other direction is by induction on $|y|_L$.

If $|y|_L=0$, then (P3) follows from (R0).

Let $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$. Suppose that $z\leq_L y$ and $z\neq y$, then, by (R2), $z\leq_L z_j$ for some j, $1\leqq j\leqq a_i$. If $z=z_j$, then let $u=\langle z_1,\dots,z_{a_i}\rangle$ and $y=v$. Clearly $z=(u)_j$. Otherwise, by the induction hypothesis, there exist i_0, j_0, u and v such that $1\leqq i_0\leqq n$, $1\leqq j_0\leqq a_{i_0}$, $z=(u)_{j_0}$ and $(i_0,u,v)\in(L)_{z_j}$. But, by (P2), $(L)_{z_j}\subseteq(L)_y$.

Let $\mathcal{T}(L)$ consists of all natural numbers y satisfying the condition: if $z\leq_L y$ and $|z|_L=0$, then $z\in\{w_1,\dots,w_r,\,x_1,\dots,x_a\}$.

($\mathcal{T}$1) If $|y|_L=0$, then $y\in\mathcal{T}(L)\iff y\in\{w_1,\dots,w_r,\,x_1,\dots,x_a\}$.

($\mathcal{T}$2) If $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$, then $y\in\mathcal{T}(L)$ iff $\forall j(1\leqq j\leqq a_i\Rightarrow z_j\in\mathcal{T}(L))$.

Let $\mathrm{val}(w_i)=W_i$, $i=1,\dots,r$, and $\mathrm{val}(x_i)=X_i$, $i=1,\dots,a$.

For each $y\in\mathcal{T}(L)$, the term $\tau(y,L)$ of the language $\mathcal{L}$ is defined by means of the following inductive clauses:

If $|y|_L=0$, then $\tau(y,L)=\mathrm{val}(y)$.

If $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$, then $\tau(y,L)=f_i(\tau(z_1,L),\dots,\tau(z_{a_i},L))$.

Notice that all variables which occur in $\tau(y,L)$ are among $W_1,\dots,W_r$, $X_1,\dots,X_a$.

Proposition 1. *Let F be a termal tree and $L\subseteq F$. Then the following is true for all natural numbers z and y:*

(a) *L is a termal tree.*

(b) $z\leq_L y\Rightarrow z\leq_F y$.

(c) $(L)_y\subseteq(F)_y$.

(d) $\mathcal{T}(L)\subseteq\mathcal{T}(F)$.

P r o o f. (a) is obvious. The proof of (b) is by induction on $|y|_L$. If $|y|_L=0$ and $z\leq_L y$, then $z=y$. Hence $z\leq_F y$. Suppose that $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$. Then $z=y$ or for some j, $z\leq_L z_j$. Clearly $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in F$. From here using the induction hypothesis we obtain that $z\leq_F y$.

Now (c) follows easily from (b).

Let $y\in\mathcal{T}(L)$. We shall prove that $y\in\mathcal{T}(F)$ by induction on $|y|_L$. If $|y|_L=0$ this follows from ($\mathcal{T}$1). Let $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in L$. Then, by ($\mathcal{T}$ 2), $z_j\in\mathcal{T}(L)$, $j=1,\dots,a_i$. By the induction hypothesis, $z_j\in\mathcal{T}(F)$, $j=1,\dots,a_i$. Clearly $(i,\langle z_1,\dots,z_{a_i}\rangle,y)\in F$. Then using once more ($\mathcal{T}$2) we obtain that $y\in\mathcal{T}(F)$.

Proposition 2. *Let L and F be termal trees and $(L)_y=(F)_y$. Then the following assertions are true for all natural z:*

(a) $z\leq_L y\iff z\leq_F y$.

(b) $z\leq_L y\Rightarrow(L)_z=(F)_z$.

(c) $y\in\mathcal{T}(L)\iff y\in\mathcal{T}(F)$.

(d) *If $y\in\mathcal{T}(L)$, then $\tau(y,L)=\tau(y,F)$.*

P r o o f. (a) follows from (P3).

Let $z \leq_L y$. We shall prove $(L)_z = (F)_z$ by induction on $|z|_L$. Let $|z|_L = 0$. Then $(L)_z = \emptyset$. Suppose that $(F)_z \neq \emptyset$. Then z is not F-prime and, hence, for some i and u, $(i, u, z) \in F$. By (a), $(i, u, z) \in (F)_y$ and, hence, $(i, u, z) \in L$. The last contradicts the fact that $|z|_L = 0$. Thus $(F)_z = \emptyset$. Suppose now that $(i, \langle z_1, \ldots, z_{a_i}\rangle, z) \in L$. By the induction hypothesis, $(L)_{z_j} = (F)_{z_j}$, $j = 1, \ldots, a_i$. Clearly, $(i, \langle z_1, \ldots, z_{a_i}\rangle, z) \in (L)_y$ and, hence, $(i, \langle z_1, \ldots, z_{a_i}\rangle, z) \in F$. Then $(L)_z = (F)_z$ follows from (P2).

The proof of (c) is by induction on $|y|_L$. Let $|y|_L = 0$. Then since $\emptyset = (L)_y = (F)_y$, $|y|_F = 0$. We have $y \in \mathcal{T}(L)$ iff $y \in \{w_1, \ldots, w_r, x_1, \ldots, x_a\}$ iff $y \in \mathcal{T}(F)$. Let $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in L$. By (b), $(L)_{z_j} = (F)_{z_j}$, $j = 1, \ldots, a_i$. Clearly $(i, \langle z_1, \ldots, z_{a_i}\rangle, y) \in F$. Then, by the induction hypothesis and by $(\mathcal{T}\ 2)$, $y \in \mathcal{T}(L) \iff \forall j(1 \leqq j \leqq a_i \Rightarrow z_j \in \mathcal{T}(L)) \iff \forall j(1 \leqq j \leqq a_i \Rightarrow z_j \in \mathcal{T}(F)) \iff y \in \mathcal{T}(F)$.

The proof of (d) proceeds in a similar way.

Proposition 3. *Let F be a termal tree, $L \subseteq F$ and $S = L \cup (F)_y$. Then $(S)_y = (F)_y$.*

Proof. Clearly $S \subseteq F$ and, hence, $(S)_y \subseteq (F)_y$. By means of induction on $|z|_F$ we shall show that if $z \leq_F y$, then $(F)_z \subseteq (S)_z$. Let $|z|_F = 0$. Then $(F)_z = \emptyset$. Let $(i, \langle z_1, \ldots, z_{a_i}\rangle, z) \in F$ and $z \leq_F y$. Then $(i, \langle z_1, \ldots, z_{a_i}\rangle, z) \in (F)_y$ and, hence, $(i, \langle z_1, \ldots, z_{a_i}\rangle, z)$ belongs to S. From here $(F)_z \subseteq (S)_z$ follows from the induction hypothesis and (P2).

Let F be a termal tree. Then L is called a *subtree* of F, in symbols $L \prec F$, iff $L \subseteq F$ and

(ST) If $y \in \mathcal{T}(F)$, $(L)_y \neq (F)_y$ and $(i, z, y) \in F$, then $(i, z, y) \notin L$.

Proposition 4. *Let $L \prec F$ and $y \in \mathcal{T}(L)$. Then $(L)_y = (F)_y$.*

Proof. If y is F-prime then $(L)_y = (F)_y = \emptyset$. Suppose that $(i, z, y) \in F$. Assume that $(L)_y \neq (F)_y$. Clearly, $y \in \mathcal{T}(F)$. Then, by (ST), $(i, z, y) \notin L$. On the other hand, $y \notin \{w_1, \ldots, w_r, x_1, \ldots, x_a\}$ and $y \in \mathcal{T}(L)$. Then for some i_1 and z_1, $(i_1, z_1, y) \in L$. Since $L \subseteq F$, $i_1 = i$ and $z_1 = z$. Hence $(i, z, y) \in L$. A contradiction.

Proposition 5. *Let $L \prec F$ and $S = L \cup (F)_y$. Then $S \prec F$.*

Proof. Let $z \in \mathcal{T}(F)$, $(F)_z \neq (S)_z$ and $(i, u, z) \in F$. It should be clear that $(i, u, z) \notin (F)_y$. Since $(L)_z \subseteq (S)_z$, $(L)_z \neq (F)_z$. Then $(i, u, z) \notin L$. Thus $(i, u, z) \notin S$.

Let $R^1, \ldots, R^{n+k}$ be new predicate symbols. We shall suppose that each R^i is c_i+1-ary, where if $1 \leqq i \leqq n$, then $c_i = a_i$ and if $n+1 \leqq i \leqq n+k$, then $c_i = b_{i-n}$. Let E be a n.t. predicate of type $(a_1, \ldots, a_n, b_1, \ldots, b_k)$. Then E can be written in the form $P^1 \mathbin{\&} \ldots \mathbin{\&} P^m$, where $m \geqq 0$ and each P^l is *atomic* i.e. P^l is in the form $R^i(z_1, \ldots, z_{c_i}, y)$ for some i, $1 \leqq i \leqq n+k$, and some $z_1, \ldots, z_{c_i}, y$ of $\mathbb{N}$.

In what follows we shall consider only n.t. predicates and n.t. expressions of type $(a_1, \ldots, a_n, b_1, \ldots, b_k)$ without stating this explicit each time. We shall use $(\alpha, \mathfrak{B})$ to denote special enumerations of $\mathfrak{A}$. Given an enumeration $(\alpha, \mathfrak{B})$, we shall suppose that the structure $\mathfrak{B}$ is in the form $(\mathbb{N}; \varphi_1, \ldots, \varphi_n, \sigma_1, \ldots, \sigma_k)$. By P^l, $l = 1, 2, \ldots$ we shall denote atomic n.t. predicates.

Let $E = P^1 \mathbin{\&} \ldots \mathbin{\&} P^m$ be a n.t. predicate. Then E is said to be *correct* iff the following conditions are true for $l = 1, \ldots, m$:

1. If $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, $1 \leqq i \leqq n$, then

1.1. $y = \langle i, j, \langle z_1, \ldots, z_{c_i}\rangle\rangle$, for some $j \in \mathbb{N}$;

1.2. If $\varphi_i^1(z_1, \ldots, z_{c_i})$ is defined, then $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$;

1.3. If $y \in \mathrm{dom}(\alpha_1)$, then $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$.

2. If $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, $n+1 \leq i \leq n+k$ then

2.1. $y \in \{0,1\}$;

2.2. If $\sigma_{i-n}^1(z_1, \ldots, z_{c_i})$ is defined, then $\sigma_{i-n}^1(z_1, \ldots, z_{c_i}) \cong y$;

3. If $1 \leq l_1 < l_2 \leq m$, $P^{l_1} = R^i(z_1, \ldots, z_{c_i}, y_1)$ and $P^{l_2} = R^i(z_1, \ldots, z_{c_i}, y_2)$, then $y_1 = y_2$.

Clearly there exists an effective way of recognizing for each n.t. predicate E whether E is correct or not.

Proposition 6. *Let E be a n.t. predicate. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ and $E_{\mathfrak{B}} \cong 0$. Then E is correct.*

P r o o f. Let $E = P^1 \& \ldots \& P^m$. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ and $E_{\mathfrak{B}} \cong 0$. Let $1 \leq l \leq m$ and $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, where $1 \leq i \leq n$. Suppose that $y \in \mathrm{dom}(\alpha_1)$. Since $E_{\mathfrak{B}} \cong 0$, $P^l_{\mathfrak{B}} \cong 0$ and, hence, $\varphi_i(z_1, \ldots, z_{c_i}) \cong y$. Then by the definition of the relation "$\subseteq$" between finite parts and enumerations, $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$. So we obtain that the condition 1.3. is true. The other conditions are obvious.

A n.t. predicate $E = P^1 \& \ldots \& P^m$ is called *simple* iff whenever $1 \leq l \leq m$ and $P^l = R^i(z_1, \ldots, z_{c_i}, y)$, then $n+1 \leq i \leq n+k$.

For each n.t. predicate E, $|E|$ is defined by the inductive clauses:

If E is simple, then $|E| = 0$;

If $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where $1 \leq i \leq n$, then $|E| = |E^1| + |E^2| + 1$.

Clearly if $|E| > 0$, then there exist unique n.t. predicates E^1, $R^i(z_1, \ldots, z_{c_i}, y)$ and E^2 such that E^2 is simple, $1 \leq i \leq n$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$.

Let us fix a correct n.t. predicate $E = P^1 \& \ldots \& P^m$. Define the finite subset L_E of $\mathbb{N}^3$ by the equivalence $(i, z, y) \in L_E \iff \exists i \exists z_1 \ldots \exists z_{c_i} \exists l (1 \leq i \leq n \,\&\, 1 \leq l \leq m \,\&\, z = \langle z_1, \ldots, z_{c_i} \rangle \,\&\, P^l = R^i(z_1, \ldots, z_{c_i}, y) \,\&\, y \notin \mathrm{dom}(\alpha_1))$.

It follows easily from the correctness of E that L_E is a termal tree.

A termal tree L is called *E-consistent* iff the following two conditions are true:

a) $L \cup L_E$ is a termal tree;

b) L is a subtree of $L \cup L_E$.

Notice that if $L = \varnothing$, then L is E-consistent. Notice also that if E is simple, then each termal tree is E-consistent.

Suppose now that $|E| > 0$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let L be an E-consistent termal tree. Denote $L \cup L_E$ by F and let $G = L \cup L_{E^1}$ and $S = L \cup (F)_y$.

Notice that $L \prec F$ and $S \prec F$. Notice also that $G \cup S = F$.

Proposition 7. *If $u \leq_F y$, and $u \neq y$, then $(G)_u = (F)_u$.*

P r o o f. Clearly $(G)_u \subseteq (F)_u$ for all u. Let $u \leq_F y$ and $u \neq y$. We shall show $(F)_u \subseteq (G)_u$ by induction on $|u|_F$. This is obvious if $|u|_F = 0$. Suppose that $(k, \langle u_1, \ldots, u_{c_k} \rangle, u) \in F$. Since $u \neq y$, $(k, \langle u_1, \ldots, u_{c_k} \rangle u) \in G$. Clearly $u_j \leq_F y$, $j = 1, \ldots, k$. Assume that for some j, $u_j = y$. Then $y \leq_F u$ and, hence, $y = u$. A contradiction. From here the inclusion $(F)_u \subseteq (G)_u$ follows from the induction hypothesis and (P2).

Proposition 8. *L is E^1-consistent.*

P r o o f. It is sufficient to show that $L \prec G$. Let $u \in T(G)$, $(L)_u \neq (G)_u$ and $(k, x, u) \in G$. Clearly $(L)_u \subset (G)_u \subseteq (F)_u$. Therefore $(L)_u \neq (F)_u$. Then, since $L \prec F$ and $(k, x, u) \in F$, $(k, x, u) \notin L$.

Proposition 9. *S is $E^1 \& E^2$-consistent.*

P r o o f. $S \cup L_{E^1 \& E^2} = F$.

Proposition 10. *If* $u \in \mathcal{T}(S)$ *and* $u \neq y$, *then* $(S)_u = (G)_u$.

Proof. Let $u \in \mathcal{T}(S)$. Since $S \prec F$, $(S)_u = (F)_u$. We shall show that if $u \neq y$, then $(G)_u = (F)_u$. If u is F-prime, then obviously $(G)_u = (F)_u = \emptyset$. Let $(k, x, u) \in F$. Clearly $(k, x, u) \in (F)_u$. Since, $(S)_u = (F)_u$, $(k, x, u) \in S$.

Let $u \neq y$. If $u \leq_F y$, then $(F)_u = (G)_u$ by Proposition 7. Otherwise, $(k, x, u) \notin (F)_y$ and, hence, $(k, x, u) \in L$. We have that $u \in \mathcal{T}(S)$ and, hence, $u \in \mathcal{T}(F)$. Since $L \prec F$, $(L)_u = (F)_u$. But $(L)_u \subseteq (G)_u \subseteq (F)_u$. Thus $(G)_u = (F)_u$.

Corollary 10.1. *If* $u \in \mathcal{T}(S)$ *and* $u \neq y$, *then* $u \in \mathcal{T}(G)$ *and* $\tau(u,S) = \tau(u,G)$.

Corollary 10.2. *If* $y \in \mathcal{T}(S)$ *and* $y \notin \mathrm{dom}(\alpha_1)$, *then* $(S)_{z_j} = (G)_{z_j}$, $j = 1, \ldots, c_i$, *and* $\tau(y,S) = f_i(\tau(z_1, G), \ldots, \tau(z_{c_i}, G))$.

Now we shall describe an algorithm $\mathcal{R}$ which transforms each correct n.t. predicate E and each E-consistent termal tree L into a termal predicate $\mathcal{R}(E, L)$.

Let $E = P^1 \& \ldots \& P^m$ be a correct n.t. predicate and let L be an E-consistent tree. We shall consider the following cases:

1. Let E be simple. Let $l_1 < l_2 < \ldots < l_h$ be all elements of $\{1, \ldots, m\}$ such that if $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then all $z_1, \ldots, z_{c_i}$ are elements of $\mathcal{T}(L)$.

Let $1 \leqq j \leqq h$ and let $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$. Clearly $y \in \{0, 1\}$ and $n + 1 \leqq i \leqq n + k$. Let $\Pi^{l_j} = T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 0$ and $\Pi^{l_j} = \neg T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 1$.

Let $\mathcal{R}(E, L) = \Pi^{l_1} \& \ldots \& \Pi^{l_h}$, if $h \geqq 1$ and let $\mathcal{R}(E, L) = T_0(X_1)$, otherwise.

2. Let $|E| > 0$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where E^2 is simple and $1 \leqq i \leqq n$. Denote by S the termal tree $L \cup (L \cup L_E)_y$. Clearly L is E^1-consistent and S is $E^1 \& E^2$-consistent.

Let $\mathcal{R}(E, L) = \mathcal{R}(E^1, L) \& \mathcal{R}(E^1 \& E^2, S)$, if $y \notin \mathcal{T}(S)$ and let $\mathcal{R}(E, L) = \mathcal{R}(E^1, L) \& T_0(\tau(y, S)) \& \mathcal{R}(E^1 \& E^2, S)$, if $y \in \mathcal{T}(S)$.

Here T_0 is the unary predicate symbol intended to represent the total unary predicate $\Sigma_0 = \lambda s.0$.

A simple induction on $|E|$ shows that whenever E is a correct n.t. predicate and L is an E-consistent termal tree, then $\mathcal{R}(E, L)$ is defined and $\mathcal{R}(E, L)$ is a termal predicate with variables among $W_1, \ldots, W_r, X_1, \ldots, X_a$.

If $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and L is a termal tree, then $\langle \alpha, \mathfrak{B} \rangle$ satisfies L, in symbols $\langle \alpha, \mathfrak{B} \rangle \vdash L$, iff $x_1, \ldots, x_a$ are elements of $\mathrm{dom}(\alpha)$ and whenever $(i, (z_1, \ldots, z_{c_i}), y) \in L$, then $\varphi_i(z_1, \ldots, z_{c_i}) \cong y$.

Let us suppose that $\alpha_1(w_i) \cong t_i$, $i = 1, \ldots, r$.

Proposition 11. *Let* L *be a termal tree,* $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$ *and* $\langle \alpha, \mathfrak{B} \rangle \vdash L$. *Then* $\mathcal{T}(L) \subseteq \mathrm{dom}(\alpha)$ *and if* $u \in \mathcal{T}(L)$, *then*

$$\alpha(u) \cong \tau(u, L)_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)). \tag{11.1}$$

Proof. By means of induction on $|u|_L$ one can show that if $u \in \mathcal{T}(L)$, then $u \in \mathrm{dom}(\alpha)$ and (11.1) is true.

Let us fix a correct n.t. predicate E, an E-consistent termal tree L and let K be a finite subset of $\mathbb{N}$ containing all natural numbers which occur in E, all elements of $\{w_1, \ldots, w_r, x_1, \ldots, x_a\}$ and such that if $(i, (z_1, \ldots, z_{c_i}), y) \in L$, then $z_1, \ldots, z_{c_i}, y \in K$.

Proposition 12. *Suppose that* E *is simple. Let* $\Pi = \mathcal{R}(E, L)$. *Then the following is true:*

a) *If* $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, $\langle \alpha, \mathfrak{B} \rangle \vdash L$ *and* $E_{\mathfrak{B}} \cong 0$, *then* $\Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$;

b) *Let* $H_1 \cap T(L) = \emptyset$. *Let* $s_1, \ldots, s_a \in A$ *and for each* $u \in T(L)$, $\tau(u, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ *be defined. Then there exists a finite part* $\Delta_2 = (H_2, \alpha_2, \varphi_1^2, \ldots \varphi_n^2, \sigma_1^2, \ldots, \sigma_k^2)$ *such that* $\Delta_2 \supseteq \Delta$ *and the following conditions hold*:

(i) $\mathrm{dom}(\alpha_2) = T(L)$ *and* $H_2 = H_1 \cup (K \setminus T(L))$;

(ii) *If* $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, *then* $\langle \alpha, \mathfrak{B} \rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, *and* $E_{\mathfrak{B}} \cong \Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$.

Proof. Let $E = P^1 \,\&\ldots\&\, P^m$ and let $l_1 < \cdots < l_h$ be the elements of $\{1, \ldots, m\}$ such that if $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then all $z_1, \ldots, z_{c_i}$ belong to $T(L)$. If $1 \leq j \leq h$, and $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$, then let $\Pi^{l_j} = T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, if $y = 0$ and let $\Pi^{l_j} = \neg T_{i-n}(\tau(z_1, L), \ldots, \tau(z_{c_i}, L))$, otherwise. Clearly $\Pi = T_0(X_1)$, if $h = 0$ and $\Pi = \Pi^{l_1} \,\&\ldots\&\, \Pi^{l_h}$, if $h > 0$.

Now a) follows easily from the definition of Π and from Proposition 11.

Suppose that $H_1 \cap T(L) = \emptyset$, $s_1, \ldots, s_a$ are elements of A and for each $u \in T(L)$, $\tau(u, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined.

Let $H_2 = H_1 \cup (K \setminus T(L))$. Clearly $H_2 \supseteq H_1$ and $T(L) \cap H_2 = \emptyset$. Define the mapping α_2 of T (L) into A by the equalities

$$\alpha_2(u) \cong \tau(u, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a), \quad u \in T(L).$$

In particular $\alpha_2(x_i) \cong s_i$, $i = 1, \ldots, a$, and $\alpha_2(w_i) \cong t_i$, $i = 1, \ldots, r$. Hence, $\alpha_1 \subseteq \alpha_2$.

Let $1 \leq i \leq n$. Define the partial function φ_i^2 on $H_2 \cup \mathrm{dom}(\alpha_2)$ by the condition $\varphi_i^2(z_1, \ldots, z_{c_i}) \cong y$ iff $(i, \langle z_1, \ldots, z_{c_i} \rangle, y) \in L$ or $\varphi_i^1(z_1, \ldots, z_{c_i}) \cong y$. Since L is a termal tree, the definition of φ_i^2 is correct. Clearly $\varphi_i^2 \supseteq \varphi_i^1$.

Let $1 \leq j \leq k$. The partial predicate σ_j^2 on $(H_2 \cup \mathrm{dom}(\alpha_2))^{b_j} \setminus (\mathrm{dom}(\alpha_2))^{b_j}$ is defined by the equivalence:

$$\sigma_j^2(z_1, \ldots, z_{b_j}) \cong y$$
$$\iff (\exists l(1 \leq l \leq m \,\&\, P^l = R^{n+j}(z_1, \ldots, z_{b_j}, y)) \text{ or } \sigma_j^1(z_1, \ldots, z_{b_j}) \cong y)$$
$$\&\, (z_1, \ldots, z_{b_j}) \in (H_2 \cup \mathrm{dom}(\alpha_2))^{b_j} \setminus (\mathrm{dom}(\alpha_2))^{b_j}.$$

Since E is a correct n.t. predicate, the definition of σ_j^2 is also correct.

Let $\Delta_2 = (H_2, \alpha_2, \varphi_1^2, \ldots, \varphi_n^2, \sigma_1^2, \ldots, \sigma_k^2)$. It follows easily from the definition of Δ_2 that Δ_2 is a finite part and $\Delta_2 \supseteq \Delta$.

Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$. We have to show that $\langle \alpha, \mathfrak{B} \rangle \vdash L$ and

(12.1) $$E_{\mathfrak{B}} \cong \Pi_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a).$$

The fact $\langle \alpha, \mathfrak{B} \rangle \vdash L$ follows immediately from the definition of the partial functions φ_i^2, $i = 1, \ldots, n$, and from $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$.

Let us turn to the proof of (12.1). From the definition of the predicates $\sigma_1^2, \ldots, \sigma_k^2$ and from $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$ it follows that if l is an element of $\{1, \ldots, m\} \setminus \{l_1, \ldots, l_h\}$, then $P^l_{\mathfrak{B}} \cong 0$. So to prove (12.1) it is sufficient to show that if $1 \leq j \leq h$, then $P^{l_j}_{\mathfrak{B}} \cong \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$. Let $1 \leq j \leq h$ and $P^{l_j} = R^i(z_1, \ldots, z_{c_i}, y)$. Then $z_1, \ldots, z_{c_j}$ are elements of $T(L) = \mathrm{dom}(\alpha_2)$ and

$\alpha(z_p) \cong \tau(z_p, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$, $p = 1, \ldots, c_i$. We have the following equivalencies:

$$\begin{aligned} P^{l_j}_{\mathfrak{B}} \cong 0 &\iff \sigma_{i-n}(z_1, \ldots, z_{c_i}) \cong y \\ &\iff \Sigma_{i-n}(\alpha(z_1), \ldots, \alpha(z_{c_i})) \cong y \\ &\iff \Sigma_{i-n}(\tau(z_1, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a), \ldots, \\ &\qquad \tau(z_{c_i}, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)) \cong y \\ &\iff \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 0. \end{aligned}$$

The equivalence

$$P^{l_j}_{\mathfrak{B}} \cong 1 \iff \Pi^{l_j}_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 1$$

can be proved in a similar way.

Proposition 13. *Let* $\Pi = \mathcal{R}(E, L)$, $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, $\langle \alpha, \mathfrak{B} \rangle \vdash L$ *and* $E_{\mathfrak{B}} \cong 0$. *Then*

(13.1) $$\Pi\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0.$$

P r o o f. Induction on $|E|$. If $|E| = 0$, then (13.1) follows from the previous proposition. Suppose that $|E| > 0$ and $E = E^1 \;\&\; R^i(z_1, \ldots, z_{c_i}, y) \;\&\; E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$, $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \;\&\; E^2, S)$. Since $E_{\mathfrak{B}} \cong 0$, $\langle \alpha, \mathfrak{B} \rangle \vdash L \cup L_E$ and, hence, $\langle \alpha, \mathfrak{B} \rangle \vdash S$. Then, by the induction hypothesis, $\Pi^1_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$ and $\Pi^2_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a)) \cong 0$

If $y \notin \mathcal{T}(S)$, then $\Pi = \Pi^1 \& \Pi^2$. Therefore (13.1) is true. Suppose that $y \in \mathcal{T}(S)$. Then $\Pi = \Pi^1 \;\&\; T_0(\tau(y, S)) \;\&\; \Pi^2$. Now we have to show that $\tau(y, S)\mathfrak{A}(W_1/, \ldots, W_r/t_r, X_1/\alpha(x_1), \ldots, X_a/\alpha(x_a))$ is defined. This follows from $\langle \alpha, \mathfrak{B} \rangle \vdash S$ and from Proposition 11.

Proposition 14. *Let* $\Pi = \mathcal{R}(E, L)$ *and* $s_1, \ldots, s_a$ *be elements of* A. *Suppose that* $\Pi\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 0$ *and for each* $u \in \mathcal{T}(L)$, $\tau(u, L)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ *is defined. Then*

(14.1) $$\forall u (u \in \mathcal{T}(L \cup L_E) \Rightarrow \tau(u, L \cup L_E)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \text{ is defined}).$$

P r o o f. Induction on $|E|$. If $|E| = 0$, then $L \cup L_E = L$. The proposition is trivial. Let $|E| > 0$ and $E = E^1 \& R^i(z_1, \ldots, z_{c_i}, y) \& E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$. Clearly $S \cup L_{E^1 \& E^2} = L \cup L_E$. Let $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \;\&\; E^2, S)$. We have that $\Pi^1_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 0$ and $\Pi^2_{\mathfrak{A}}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \cong 0$.

Using the induction hypothesis we obtain that if $u \in \mathcal{T}(L \cup L_{E^1})$, then $\tau(u, L \cup L_{E^1})\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined.

Suppose that $y \notin \mathcal{T}(S)$. Then by Proposition 10, for $u \in \mathcal{T}(S)$, $(L \cup L_{E^1})_u = (S)_u$. Hence, by Proposition 2, if $u \in \mathcal{T}(S)$, then $\tau(u, S) = \tau(u, L \cup L_{E^1})$. Therefore, if $u \in \mathcal{T}(S)$, then $\tau(u, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined. Now applying the induction hypothesis for $E^1 \;\&\; E^2$ and S, we obtain (14.1).

Let us suppose that $y \in \mathcal{T}(S)$. Then $\Pi = \Pi^1 \;\&\; T_0(\tau(y, S)) \;\&\; \Pi^2$ and hence $\tau(y, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined. Using the same arguments as in the previous case, we obtain that if $u \in \mathcal{T}(S)$ and $u \neq y$, then

$\tau(u,S)\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is defined. From here, applying the induction hypothesis for E^1 & E^2 and S, we obtain (14.1).

Proposition 15. *Let $H_1 \cap \mathcal{T}(L \cup L_E) = \varnothing$. Let $s_1,\ldots,s_a \in A$ and for each $u \in \mathcal{T}(L)$, $\tau(u,L)\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ be defined. Let $\Pi = \mathcal{R}(E,L)$ and $\Pi\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a) \cong 0$. Then there exists a finite part $\Delta_2 = (H_2, \alpha_2, \varphi_1^2,\ldots,\varphi_n^2, \sigma_1^2,\ldots,\sigma_k^2)$ such that $\Delta_2 \supseteq \Delta$ and the following conditions hold:*

(i) $\mathrm{dom}(\alpha_2) = \mathcal{T}(L \cup L_E)$ *and* $H_2 = H_1 \cup (K \setminus \mathcal{T}(L \cup L_E))$;

(ii) *If $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B}\rangle \vdash L \cup L_E$, $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$, and $E_{\mathfrak{B}} \cong 0$.*

P r o o f. Induction on $|E|$. If $|E| = 0$, then the proposition follows from Proposition 12.

Let $|E| > 0$ and $E = E^1$ & $R^i(z_1,\ldots,z_{c_i},y)$ & E^2, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$, $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \,\&\, E^2, S)$. Clearly, $\Pi^1_{\mathfrak{a}}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a) \cong \Pi^2_{\mathfrak{a}}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots, X_a/s_a) \cong 0$. Using the previous proposition, we obtain that if $u \in \mathcal{T}(L \cup L_{E^1})$, then

$$\tau(u, L \cup L_{E^1})\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$$

is defined. From here, as in the proof of the previous proposition, it follows that for $u \in \mathcal{T}(S)$, $\tau(u,S)\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is defined. Now applying the induction hypothesis for E^1 & E^2 and S, we obtain that there exists a finite part $\Delta_2 = (H_2, \alpha_2, \varphi_1^2,\ldots,\varphi_n^2, \sigma_1^2,\ldots,\sigma_k^2)$ such that $\Delta_2 \supseteq \Delta$ and the following conditions hold:

a) $\mathrm{dom}(\alpha_2) = \mathcal{T}(L \cup L_E)$ and $H_2 = H_1 \cup (K \setminus \mathcal{T}(L \cup L_E))$;

b) If $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B}\rangle \vdash L \cup L_E$, $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$, and $(E^1 \& E^2)_{\mathfrak{B}} \cong 0$.

It remains to show that if $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $E_{\mathfrak{B}} \cong 0$. Let $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$. If $(i, \langle z_1,\ldots,z_{c_i}\rangle, y) \in L_E$, then $E_{\mathfrak{B}} \cong 0$ follows from $(E^1 \,\&\, E^2)_{\mathfrak{B}} \cong 0$ and $\langle \alpha, \mathfrak{B}\rangle \vdash L \cup L_E$. Otherwise, $y \in \mathrm{dom}(\alpha_1)$. Then, by the correctness of E, $\varphi_i^1(z_1,\ldots,z_{c_i}) \cong y$. Hence, $\varphi_i(z_1,\ldots,z_{c_i}) \cong y$. From here and from $(E^1 \& E^2)_{\mathfrak{B}} \cong 0$ it follows that $E_{\mathfrak{B}} \cong 0$.

Proposition 16. *Let $H_1 \cap \mathcal{T}(L \cup L_E) = \varnothing$. Let $s_1,\ldots,s_a \in A$ and for each $u \in \mathcal{T}(L)$, $\tau(u,L)\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ be defined. Let $\Pi = \mathcal{R}(E,L)$ and $\Pi\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ be not defined. Then there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B}\rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$, and $E_{\mathfrak{B}}$ is not defined.*

P r o o f. Induction on $|E|$. If $|E| = 0$, then the proposition follows from Proposition 12. Let $E = E^1 \& R^i(z_1,\ldots,z_{c_i},y) \& E^2$, where $1 \leq i \leq n$ and E^2 is simple. Let $S = L \cup (L \cup L_E)_y$ and $\Pi^1 = \mathcal{R}(E^1, L)$ and $\Pi^2 = \mathcal{R}(E^1 \& E^2, S)$. We shall consider the following cases:

1. $\Pi^1_{\mathfrak{a}}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is not defined. By the induction hypothesis, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B}\rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1,\ldots,a$, and $E^1_{\mathfrak{B}}$ is not defined. Clearly if $\langle \alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $E_{\mathfrak{B}}$ is also undefined.

2. $\Pi^1_{\mathfrak{a}}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is defined. Hence, $\Pi^1_{\mathfrak{a}}(W_1/t_1,\ldots, W_r/t_r, X_1/s_1,\ldots,X_a/s_a) \cong 0$. From Proposition 14 and from Proposition 10 it follows that if $u \in \mathcal{T}(S)$ and $u \neq y$, then

(16.1) $\quad \tau(u,S)\mathfrak{a}(W_1/t_1,\ldots,W_r/t_r, X_1/s_1,\ldots,X_a/s_a)$ is defined.

2.1. Let $y \notin T(S)$. Then $\Pi = \Pi^1 \& \Pi^2$ and , hence, $\Pi^2(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is not defined. By (16.1) and by the induction hypothesis applied for $E^1 \& E^2$ and S, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B} \rangle \vdash S$, and hence $\langle \alpha, \mathfrak{B} \rangle \vdash L$, $\alpha(x_i) = s_i$, $i = 1, \ldots, a$, and $(E^1 \& E^2)_{\mathfrak{B}}$ is not defined. Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$. Using the same arguments as in the proof of the previous proposition, we obtain that $\varphi_i(z_1, \ldots, z_{c_i}) \cong y$. Then, since $(E^1 \& E^2)_{\mathfrak{B}}$ is not defined, $E_{\mathfrak{B}}$ is also not defined.

2.2. Let $y \in T(S)$. Then $\Pi = \Pi^1 \,\&\, T_0(\tau(y, S)) \,\&\, \Pi^2$. Suppose that $\tau(y, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined. Then using (16.1), we obtain that if $u \in T(S)$, then $\tau(u, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is defined. Now the proof proceeds as in the previous case.

Suppose now that $\tau(y, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$ is not defined. Then $y \notin \mathrm{dom}(\alpha_1)$ and, hence, by Corollary 10.1 and Corollary 10.2, $z_1, \ldots, z_{c_i}$ are elements of $T(L \cup L_{E^1})$ and $\tau(y, S) = f_i(\tau(z_1, L \cup L_{E^1}), \ldots, \tau(z_{c_i}, L \cup L_{E^1}))$. By Proposition 15, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $\langle \alpha, \mathfrak{B} \rangle \vdash L \cup L_{E^1}$, $E^1_{\mathfrak{B}} \cong 0$ and $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$. From here by Proposition 11, we obtain that $T(L \cup L_{E^1}) \subseteq \mathrm{dom}(\alpha)$ and

(16.2) for $u \in T(L \cup L_{E^1})$,
$$\alpha(u) \cong \tau(u, L \cup L_{E^1})\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a).$$

Let $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$. Clearly $\langle \alpha, \mathfrak{B} \rangle \vdash L$. From (16.2) it follows $z_1, \ldots, z_{c_i} \in \mathrm{dom}(\alpha)$ and for $j = 1, \ldots, c_i$, $\alpha(z_j) \cong \tau(z_j, L \cup L_{E^1})\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a)$. We shall prove that $\varphi_i(z_1, \ldots, z_{c_i})$ is not defined. Indeed,

$$\begin{aligned} &\varphi_i(z_1, \ldots, z_{c_i}) \text{ is defined } \iff \theta_i(\alpha(z_1), \ldots, \alpha(z_{c_i})) \text{ is defined} \\ &\iff \theta_i(\tau(z_1, L \cup L_{E^1})\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a), \ldots, \\ &\qquad \tau(z_{c_i}, L \cup L_{E^1})\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)) \text{ is defined} \\ &\iff \tau(y, S)\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r, X_1/s_1, \ldots, X_a/s_a) \text{ is defined.} \end{aligned}$$

From here, since $E^1_{\mathfrak{B}} \cong 0$, it follows that $E_{\mathfrak{B}}$ is not defined.

Proposition 17. *There exists an effective way to define for each n.t. expression $E \supset y$ of type $(a_1, \ldots, a_n, b_1, \ldots, b_k)$ a conditional term Q with variables among $W_1, \ldots, W_r, X_1, \ldots, X_a$ such that for all elements $s_1, \ldots, s_a$ and t of A, the following conditions are satisfied:*

(1) *If $Q\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \cong t$, then there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\Delta_2 \subseteq \langle \alpha, \mathfrak{B} \rangle$, then $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$, $\alpha(y) \cong t$ and $E_{\mathfrak{B}} \cong 0$;*

(2) *If $Q\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a) \not\cong t$, then at least one of the following is true:*

(2.1) *There exists a finite part $\Delta_2 \supseteq \Delta$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $x_1, \ldots, x_a$ belong to $\mathrm{dom}(\alpha)$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \mathrm{dom}(\alpha)$;*

(2.2) *For each special enumeration $\langle \alpha, \mathfrak{B} \rangle$, if $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta$, $\alpha(x_1) \cong s_1, \ldots, \alpha(x_a) \cong s_a$ and $E_{\mathfrak{B}} \cong 0$, then $\alpha(y) \not\cong t$;*

(3) *If $Q\mathfrak{A}(W_1/t_1, \ldots, W_r/t_r,\ X_1/s_1, \ldots, X_a/s_a)$ is not definite, then there exists a finite part $\Delta_2 \supseteq \Delta$, such that if $\langle \alpha, \mathfrak{B} \rangle$ is a special enumeration and $\langle \alpha, \mathfrak{B} \rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1, \ldots, a$, and $E_{\mathfrak{B}}$ is not defined.*

P r o o f. Let E be a n.t. predicate and $y \in \mathbb{N}$. Suppose that E is not correct. Let $Q = (\neg T_0(X_1) \supset X_1)$. Then (1) and (3) are obvious. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Then, by Proposition 6, $E_{\mathfrak{B}} \not\cong 0$. So we receive that (2.2) is true.

Suppose now that E is correct. Let L be the empty termal tree. Let K be the finite set of natural numbers consisting of the elements of $\{w_1, \dots, w_r, x_1, \dots, x_a, y\}$ and of those natural numbers which occur in E. Let $\Pi = \mathcal{R}(E, L)$. Let $Q = (\Pi \supset \tau(y, L_E))$, if $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$. Let $Q = (\neg T_0(X_1) \supset X_1)$, otherwise. We shall show that Q satisfies the conditions (1), (2) and (3).

Suppose that $s_1, \dots, s_a$ and t are elements of A.

Let $Q_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong t$. Then Q is not in the form $(\neg T_0(X_1) \supset X_1)$ and, hence, $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$. Clearly $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong 0$ and $\tau(y, L_E)_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong t$.

Obviously $\mathcal{T}(L) = \{w_1, \dots, w_r,\ x_1, \dots, x_a\}$. Hence, for $u \in \mathcal{T}(L)$, $\tau(u, L)_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a)$ is defined.

By Proposition 15, there exists a finite part $\Delta_2 \supseteq \Delta$ such that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\langle\alpha, \mathfrak{B}\rangle \vdash L$, $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$, and $E_{\mathfrak{B}} \cong 0$. Using Proposition 11, we obtain $\alpha(y) \cong t$. So we receive that (1) is true.

Let us suppose that $Q_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r, X_1/s_1, \dots, X_a/s_a) \not\cong t$.

We shall consider the following cases:

1. $H_1 \cap \mathcal{T}(L_E) \neq \varnothing$. We shall show the validity of (2.2). Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Assume that $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$, and $E_{\mathfrak{B}} \cong 0$. Then $\langle\alpha, \mathfrak{B}\rangle \vdash L_E$. From here, by Proposition 11, it follows that $\mathcal{T}(L_E) \subseteq \mathrm{dom}(\alpha)$. Hence $H_1 \cap \mathrm{dom}(\alpha) \neq \varnothing$. The last contradicts $\Delta \subseteq \langle\alpha, \mathfrak{B}\rangle$. So we receive that (2.2) is true.

2. Let $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $y \notin \mathcal{T}(L_E)$. Then $y \in K \setminus \mathcal{T}(L_E)$. Suppose that $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \not\cong 0$. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$, $\alpha(x_i) \cong s_i$, $i = 1, \dots, n$, and $E_{\mathfrak{B}} \cong 0$. Since $L = \varnothing$, $\langle\alpha, \mathfrak{B}\rangle \vdash L$. By Proposition 13, $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r, X_1/s_1, \dots, X_a/s_a) \cong 0$. A contradiction. So we obtain again that (2.2) is true.

Suppose now that $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong 0$. By Proposition 15, there exists a finite part $\Delta_2 = (H_2,\ \alpha_2,\ \varphi_1^2, \dots, \varphi_n^2, \sigma_1^2, \dots, \sigma_k^2)$ such that $\Delta_2 \supseteq \Delta$, $H_2 = H_1 \cup K \setminus \mathcal{T}(L_E)$ and if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $E_{\mathfrak{B}} \cong 0$ and $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$. So we have that if $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta_2$, then $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$, $E_{\mathfrak{B}} \cong 0$ and $y \notin \mathrm{dom}(\alpha)$. So the validity of (2.1) is proved.

3. Let $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $y \in \mathcal{T}(L_E)$. Let $\langle\alpha, \mathfrak{B}\rangle \supseteq \Delta$. Assume that $\alpha(x_i) \cong s_i$, $i = 1, \dots, a$, $E_{\mathfrak{B}} \cong 0$ and $\alpha(y) \cong t$. By Proposition 13, $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong 0$. Using Proposition 11 and $E_{\mathfrak{B}} \cong 0$, we obtain that $\alpha(y) \cong \tau(y, L_E)_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong t$. Therefore $Q_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong t$. A contradiction. So we have proved (2.2).

Suppose now that $Q_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a)$ is not definite. Then Q is not in the form $(\neg T_0(X_1) \supset X_1)$. Hence $y \in \mathcal{T}(L_E)$ and $H_1 \cap \mathcal{T}(L_E) = \varnothing$ and $Q = (\Pi \supset \tau(y, L_E))$. Assume that $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a) \cong 0$ and $\tau(y, L_E)_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a)$ is not defined. The last contradicts Proposition 14. Hence $\Pi_{\mathfrak{A}}(W_1/t_1, \dots, W_r/t_r,\ X_1/s_1, \dots, X_a/s_a)$ is not defined. From here (3) follows from Proposition 16.

REFERENCES

1. Soskov, I. N. Maximal concepts of computability and maximal programming languages. (Submitted for publication.)
2. Moschovakis, Y. N. Abstract first order computability I. *Trans. Amer. Math. Soc.*, 138, 1969, 427–464.
3. Friedman, H. Algorithmic Procedures, Generalized Turing Algorithms and Elementary Recursion Theory. *In*: Logic Colloquium'69, R. O. Gandy, C. E. M. Yates eds., North-Holland, Amsterdam, 1971, 361–389.
4. Shepherdson, J. C. Computation over Abstract Structures. *In*: Logic Colloquium'73, H. E. Rose, J. C. Shepherdson eds., North-Holland, Amsterdam, 1975, 445–513.
5. Скордев, Д. Г. Изчислими и μ-рекурсивни оператори. *Изв. Мат. инст. БАН*, 7, 1963, 5–43.
6. Skordev, D. G. On Turing computable operators. *Год. Соф. унив., Фак. мат. и мех.*, 67, 1972/73, 103–112.
7. Sasso, L. P. A survey of partial degrees. *J. Symb. Logic*, 40, 1975, 130–140.
8. Cooper, S. B. Enumeration reducibility, non-deterministic computation and relative computability of partial functions. Preprint # 3, Dept. of math., The Univ. of Leeds, Leeds, 1990.
9. Rogers, Jr. H. Theory of Recursive Functions and Effective Computability. McGraw-Hill Book Company, 1967.
10. Myhill, J. Note on degrees of partial functions. *Proc. Amer. Math. Soc.*, 12, 1961, 519–521.
11. Gordon, C. E. Finistically computable functions and relations on abstract structures. *J. Symb. Logic*, 36, 1971, 704.
12. Soskov, I. N. Computability on abstract structures. *Compt. Rend. Acad. Bulg. Sci.*, 36, 1983, 301–303.
13. Moschovakis, Y. N. Abstract computability and invariant definability. *J. Symb. Logic*, 34(4), 1969, 605–633.
14. Soskov, I. N. Definability via enumerations. *J. Symb. Logic*, 54, 1989, 428–440.
15. Soskov, I. N. Computability by means of effectively definable schemes and definability via enumerations. *Archive for Math. Logic*, 29, 1990, 187–200.
16. Soskov, I. N. Prime computability on partial structures. *In*: Mathematical Logic and Its Applications, ed. D. Skordev, Plenum Press, New York and London, 1987, 341–350.

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AN INTELLIGENT OBJECT–ORIENTED DATA MODEL

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Павел Азалов, Фани Златарова. ОДНА ИНТЕЛЛИГЕНТНАЯ ОБЪЕКТНО-ОРИЕНТИРОВАННАЯ МОДЕЛЬ ДАННЫХ

Настоящая работа является коротким введением в одну объектно–ориентированную модель данных. Объекты классифицированы в классы, которые формируют иерархию. Внимание отделено тоже и на описание т. назв. системных классов. Их тип может быть выбран из следующих типов: примитивный, форматный и неформатный. Рассматриваемая модель данных характеризуется включением некоторых знаний в определение объектного класса. Рассматриваются два основных типа знаний: пассивные и активные. Первый тип знаний относится к генерированию некоторых объектов, а второй тип предназначен для автоматического активирования действий, когда удовлетворяются определенные условия, которые заранее специфицированы. Операция Rel_Ship используется для установления некоторых связей между объектами, принадлежащими к разным классам. Правила типа триггеров используются для автоматической регистрации связей, полученных таким способом.

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This paper is a brief introduction to an object-oriented data model. The objects are classified in classes being organised in a class hierarchy. Some attention has been paid to describe the so called system classes. Their type may be chosen among the following ones: primitive, formatted and unformatted. The data model under consideration is characterised by knowledge being incorporated in the object class definition. Two basic types of knowledge are considered: passive and active. The first type of knowledge refers to generating some objects while the second one (the triggers) aims at activating automatically actions when some conditions predetermined in advance are satisfied. The Rel_Ship operation is applied to establish some links among objects from different classes. Triggers are used to registrate automatically the links thus obtained.

1. INTRODUCTION

1.1. Motivation. Nowadays there exists a number of different data models (DM). Each DM has distinct evolutionary applications. However, only few of them, for example, the relational DM, are used quite often in practice and research. The joint processing of text, numbers, graphical images and combinations of them, i.e. the so called unformatted data result in new problems which usually lead to the necessity of extending and sophisticating the existing DM, on one hand, and to elaborating some new ones letting us to process complex informational objects, on the other hand. Such DM have to meet the following basic requirements:

A. The data structures maintained by the DM have to correspond to concrete user's needs. The user should be able to define new data types which should share the same syntax and semantics together with the existing ones. He should be able to define a rich set of operations, permitting the specification of the objects behaviour with respect to the modeled subject area as well [6].

B. To maintain the processing of both traditional formatted data (integers, fixed-length strings, etc.) and unformatted data such as text or graphics [5,8].

These features cannot be discovered in the traditional DM and in particular in the relational DM [3]. The object–oriented DM presented in this paper attempts to meet the above mentioned requirements.

1.2. Object–oriented concepts. Recently the object–oriented programming has become very popular and it has been used to design and elaborate complex application systems in the field of office automation, artificial intelligence, etc. The most important advantage of the object–oriented approach in this case is its letting us to model the entities, typical of the application domain under consideration by using the object as a main concept [9].

The term *object-oriented* can be interpreted in a different way. For example, the *Iris* objects represent entities and concepts from the application domain being modeled [3,4]. They are unique entities in the database with their own identity and existence.

An *OZ+* object is a discrete entity encapsulating both data and an allowable set of operations on these data [10].

In the *GemStone* DM the object is considered as a chunk of private memory with a public interface. *GemStone* merges object-oriented language concepts with those of database systems, and provides an object-oriented database language called *OPAL* which is used for data definition, data manipulation and general computation [6,7].

In general objects represent entities belonging to the modeled subject area [1]. Every object is considered as an instance of an object class, and every object class is a specification of an abstract data type. Objects having the same properties are grouped in one and the same object class. An object class consists of some private memory for a collection of instance variables. The values of these variables determine the state of a particular object. The value of each instance variable is an object too, i.e. this value will have its own private memory for the states it might have with the exception of the simplest objects (i.e. the so called primitive ones, such as a string or an integer). The objects have a well determined behaviour which is expressed through the set of operations applied to them.

The operations or the so called *methods* are elements of the class definition. Each operation is composed of two parts: the interface and the implementation. The first one describes their external characteristics, such as: name, arguments together with their types and result type. The operation implementation indicates how to perform it. Methods as well as instance variables are invisible out of the object. For that reason the objects are treated as incapsulated ones. The operations are performed through the so called *communications*, permitting the communication among objects. Every object can be either an operation argument or a result of such operations. The instances differ from each other by their content and unique identification. The object name identifies the object uniquely at each moment of its existence. Hence, each class is named *collection of objects*. For example, object classes are: *People*, *Books*, *Cars*, etc. Some of their properties might be as follows: *Name*, *Birth_date*, *Editor*, *Car_make*, *Weight*, etc.

Among the classes defined in an object-oriented system *is_a* relationships can be established. This is achieved by using a concept subclass. For example, if the *Person* class is defined, then it is naturally the *Employee* class to be its subclass.

```
Person
  |
  | is_a
  |
Employee
```

That means, every object of the *Employee* class is an object of the *Person* class, and an object of the *Employee* class can be used wherever an object of the *Person* class can be used too. Thus a class hierarchy might be created. It is a hierarchy of the classes, where a child node is a specialization of the parent node and the parent node is a generalization of the child node [16]. Each subclass might *inherit* various characteristics of the superclass. Thus some of the inherited characteristics may be overridden.

2. OVERVIEW OF THE MODEL

The basic concept of our model named intelligent object oriented DM is the *object*. The object is treated as an instance of an *object class*, which is the structural specification of a class of objects.

2.1. Objects. Every object is an instance of a class of objects. When a user wants to specify and use the system, he pays attention to the nature of the objects under consideration. For that purpose he defines his own classes and methods and then creates instances of these classes. For each of the classes the common structure of all the instances of the class are defined. We will consider the following example:

```
Class Person;
   Attributes:
      Name:string[24];
      Address:string[60];
      Phone:string[12];
   Methods:
      First_name:Name ⟶ string[12];
      Last_name:Name ⟶ string[12];
      City:Address ⟶ string[16];
   Knowledge:
      R2.Ident:Unique(Name,Address);
      R1.Tel_Val:Multivalue(Phone);
      R1.Tel_Exist:Optional(Phone);
   Initialization:
      Phone:="(0073)895370"
End.
```

The *Person* class is characterizad by three parts: **Attributes**, **Methods** and **Knowledge**. Properties, characterizing the objects of the given class are defined in the first part. Such attributes are: *Name*, *Address* and *Phone*. Each of them is specified by its type. It can be either of a primitive type (string, integer, real, boolean) or of a type which has already been defined for another existing object. The objects behaviour typical of all the classes is described in the other parts. The first one includes the operations characterizing the respective objects. For example, in this case they are as follows: *First_name*, *Last_name* and *City*. Thus by using the *City* operation we may obtain the name of the city from the *Address* address. The **Knowledge** part comprises the knowledge needed to describe the objects of that class. They are invariant in respect to the particular instances. The presentation, the utilization and the knowledge types used for the processing of the objects in the classes under consideration are described in Part 3.

The class definition may include the **Initialization** section as well, which indicates the initial values of some of the attributes, if necessary.

2.2. Inheritance and hierarchy. Classes are organized in a hierarchical structure that supports generalization and specialization. A class may be declared as a subclass of another class. The objects of each subclass provide the behaviour of the objects of the superclass plus something extra. In other words, the properties of the superclass objects are one and the same for the objects of its subclasses as well. That is why we say that the properties are *inherited* by the subclass. For example, let us consider the definition of the *Employee* class.

```
Class Employee is_subclass_of Person;
   Attributes:
      Department:string[24];
      Salary:integer
End.
```

In this case the *Employee* class is a subclass of the *Person* class. It inherits all the instance variables and methods from its *Person* superclass and declares additional ones such as *Department* and *Salary*. All algorithms working on the *Person* objects automatically work also on the *Employee* objects too.

3. KNOWLEDGE INCORPORATION IN OBJECTS

The knowledge part of the object class definition comprises a set of rules. It can contain one or more than one rules, or it can be the empty set. Two types of rules are considered: *passive* and *active*. The first one is used in order to present knowledge, needed to generate some rules typical of different classes. Rules of the second type, called *triggers*, are used for the knowledge representation applicable to activating automatically actions when some given conditions take place [5,9,11]. Each rule is named and characterized by two parts: rule type and its specific name which identifies that rule in the set comprising all the rules. Next we will describe briefly each of the rules discussed.

Existential Rules **(E)**

```
E.<name>:Mandatory(<Attr_list>);
E.<name>:Optional(<Attr_list>);
E.<name>:Multivalue(<Attr_list>);
```

Constraints on attribute values independent of the values of other attributes are specified by using the existential rules.

Uniqueness Rules **(U)**

`U.<name>:Unique(<Attr_list>);`

This rule specifies that the value of one or more than one attributes identifies a given object among the other objects of the same class uniquely.

Content Derivation Rules **(C)**

`C.<name>:Attr_name:=Expression;`

This rule is used to specify how a certain attribute is given a value. The attribute value is calculated whenever a query to it appears.

Access Control Rules **(A)**

`A.<name>:Access(<User_list>);`

The user's access rights to the objects of a given class are determined by using the access control rules. Thus the access to all superclasses of this class is defined.

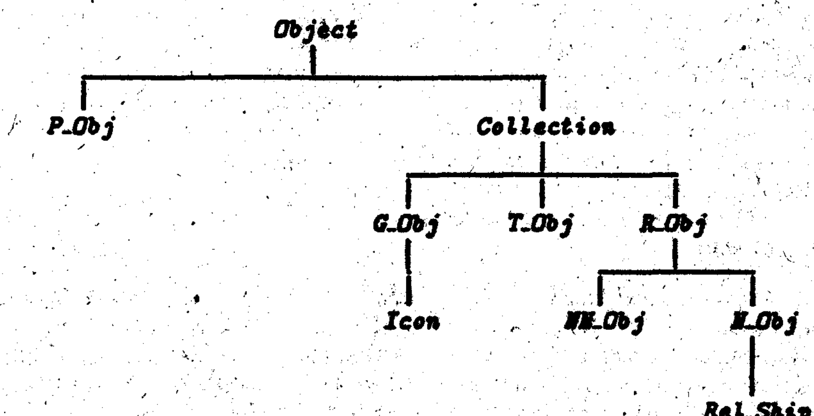
Triggering Rules **(T)**

`T.<name>:if Cond then Action [error <Message>]`

Triggering rules are used to activate automatically the performance of the *Action* action when a given event specified by the *Cond* expression occurs. The error part (which is not obligatory) is used to handle the exceptions.

4. SYSTEM CLASSES

Similarly as in other object-oriented systems we also define a class called the *Object* class which is assumed to be the root of the class hierarchy. The figure given below indicates all the system-defined subclasses of the *Object* class.



The *Object* Class

The *Object* class is a superclass of all the classes and it is used to provide a consistent basic functionality and default behaviour. Many methods in this class are overridden in subclasses. The comparison of two objects and the determination of the class when given an object can illustrate methods of this class being of great significance. The *Object* class has two subclasses: *P_Obj* and *Collection*.

The *P_Obj* Class

This is the class of the primitive objects. The standard defined classes `Integer`, `Real`, `String`, `Boolean` and `Point` are subclasses of it. Here they are used in a trivial way and that is why for that reason we are not going to consider them.

The *Collection* Class

The *Collection* class consists of objects being collections of other objects. The class properties can be described by using two attributes: the maximum number of objects and the maximum object size. The methods typical of this class are as follows:

— determining the number of elements included in a collection;
— sorting the collection elements;
— existence control of a given element belonging to a collection, etc.

The elements of a given collection can be related to different classes.

4.1. Formatted classes. The *R_Obj* class consists of objects which are relations, i.e. sets of elements being of the same type. Each element is an n-tuple. Then two types of relations will be permitted:

— normalized relations (the *N_Rel* class);
— unnormalized relations (the *NN_Rel* class).

The properties of the objects of the *R_Obj* class are described by using the number of attributes in this relation and the maximum number of elements included in a relation. The basic methods typical of this class are as follows: adding rows, deleting rows, updating the existing attribute values in a row, etc.

The *N_Rel* class includes normalized relations. Next we can mention some operations of the relational algebra such as: union, difference, project, restriction and decart product which are among the methods being of great significance. Each user's subclass of the *N_Rel* class defines a particular relational scheme and it may determine some methods typical of the user's applications. Thus we may apply the methods of the classical relational approach to the case under discussion.

The *Rel_Ship* class consists of binary relations having a special function. Links among the objects of the different classes are established by using these relations. But we have to point out that in this case these links are maintained automatically. It is necessary only to define some criteria for the respective classes. This problem is considered in section 5.

The *NN_Rel* class consists of hierarchical relations which sometimes might appear to be of some use.

4.2. Unformatted classes. Many users' problems lead to studying some of the informational objects which cannot be structured and processed by applying the well known and standard methods. Mostly these are the text and graphical informational objects. For instance, a document may consists of: *P_objects* (date, sender); *T_objects* (letter contents); *Icon_objects* (Company Logo).

The *T_Obj* class is the class of the text objects. For example, an abstract of a paper or book; short biographical remarks, etc. They are represented as variable length character strings. The operations typical of such a class are as follows: text comparison; insertion and deletion of a substring in a given text; searche of a substring; exchange of substrings; concatenation of two texts, conversion of a digital text in a number; conversion of a text in a graphical image, etc. The text length is used as a class attribute.

The *G_Obj* class includes the graphical objects such as a graph, a pie chart, a histogram, an icon or an arbitrary picture which can include a text too. All of them are visualized in a rectangular field, whose maximum sizes are presented as attributes in a certain class. The methods typical of this class are, as follows: edition of a graphical image; comparison of graphical images; conversion of graphical images in a text string; graphical image visualization, etc. These operations can be considered as a special subclass of the *G_Obj* class, and the icon processing is typical of them. As an example we can consider the *Student* class:

```
Class Student is_a_subclass_of Person;
   Attributes:
      Photo:Icon[60,80];
      Language:integer;
      Mathematics:integer;
      Autobiography:T_Obj;
   Methods:
      Average:(Language,Mathematics) ⟶ real
End.
```

5. OBJECT ENVIRONMENT

5.1. Object creation and manipulation. Having defined the classes we can begin with generating the instances (objects) from them [2]. The command used in this case is the **new** command. For example, we will consider the instance generation in the *Student* class.

```
new Student (Name,Languages,Mathematics);
      Objects: J_S('John Smith',6.5);
               P_K('Peter Koch',6,6);
```

Using this command two objects *J_S* and *P_K* are created explicitly. Only some of the objects attributes may obtain values. Therefore the names of these attributes have to be indicated. Next the other attributes can be given values by using the **update** command:

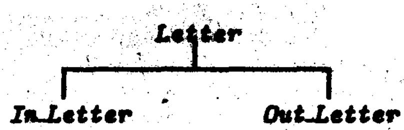
```
update J_S.Phone:='251-167';
```

In this example the *Phone* attribute of the *J_S* object either acquires a new value or it is updated.

The **destroy** command removes an object from a given class; all the objects from the class or the whole class. These functions are illustrated by the following examples:

```
destroy J_S;
destroy Class Student;
```

5.2. A world of linked objects. The class instances may be independent of each other within a certain class and they may be independent of the instances of other classes as well. However there exist some cases, where the objects included in different classes and the objects within a certain class should correspond to each other. For example let us consider the *Letter* class and its subclasses *In_Letter* (input mail)



and *Out_Letter* (output mail). Each of the letters received can result in zero, one or more than one letters to be sent. Then, we may say that between the objects of the classes *In_Letter* and *Out_Letter* there exists an explicit link. Thus the object searche will be facilitated to a great extent by creating and mantaining such links. In our model this link is realized through defining a particular instance which belongs to the *Rel_Ship* class. This link itself is a binary relation. Thus we have:

```
Class Link is_a_subclass_of Rel_Ship;
   Attributes:
      InL:string[12];
      OutL:string[12]
End.
Class Letter;
   Attributes:
      Answer:string[12];
      Sender:Person;
      Receiver:Person;
      Date_Send:Date;
      Body:T_Obj;
   Knowledge:
      E.Ans:Multivalue(Answer)
End.
Class Out_Letter is_a_subclass_of Letter;
   Knowledge:
      T.LO:if Answer<>''
              then update Link.InL:=Answer,
                   update Link.OutL:=?
End.
Class In_Letter is_a_subclass_of Letter;
   Attributes:
      Date_Receive:Date;
   Knowledge:
      T.LI:if Answer<>''
              then update Link.InL:=?,
                   update Link.OutL:=Answer
End.
```

Let us give some explanations. The *Answer* attribute comprises the names (numbers) of the letters received (if any) as its values.

If an output letter is created in reply to some existing letters, then the *LO* trigger will registrates its links with them. The role of the *LI* trigger is similar to that of the *LO* trigger. Thus we accomplish an automatic link between the *In_Letter* class and the *Out_Letter* class.

REFERENCES

1. Banerjee, J., H. Chou, J. Gazza, W. Kim, D. Woelk, N. Ballou, H. Kim. Data Model Issues for Object-Oriented Applications. *ACM TOIS*, vol. 5, № 1, 1989, 3–26.
2. Beech, D. A Foundation for Evolution from Relational to Object Databases. *In*: EDBT'88, Advances in DB Technology, Lecture Notes in Computer Science, vol. 303, 1988, 251–270.
3. Derrett, N., W. Kent, P. Lyngbaek. Some Aspects of Operations in an Object-Oriented Database. *Database Engineering*, vol. 8, № 4, 1985, 66–74.
4. Fishman, D., D. Beech, H. Cate, E. Chow, T. Connors, J. Davis, N. Derret, C. Hoch, W. Kent, P. Lyngbaek, B. Mahbod, M. Neimat, T. Ryan, M. Shan. IRIS: An Object-Oriented Database Management System. *ACM Trans. on Office Inf. Systems*, vol. 5, № 1, January 1987, 48–68.
5. Gibbs, S., D. Tsichritzis. A Data Modeling Approach for Office Information Systems. *ACM TOIS*, vol. 1, № 4, 1983, 299–319.
6. Mayer, D. Object-Oriented Database Development at Servio Logic. *Database Engineering*, vol. 8, № 4, 1985, 58–65.
7. Mayer, D., J. Stein. Development and Implementation of an Object-Oriented DBMS. *In*: Research Directions in Object-Oriented Programming, ed. B. Shriver and P. Wegner, 1987, 355–392.
8. Shek, H., P. Pistor. Data Structures for an Integrated Data Base Management and Information Retrieval System. *In*: Proceedings of the 8th Int. Conf. on VLDB, 1982, 197–207.
9. Tsichritzis, D. Object Species. *Database Engineering*, vol. 8, № 4, 1985, 2–7.
10. Weiser, S., F. Lochovsky. OZ+: An Object-Oriented Database System. *In*: Office and Data Base Systems Research'87, ed. F. Lochovsky, Technical Report CSRI-195, CSRI, University of Toronto, 1987, 174–193.
11. Yang, J. A Proposal for Incorporating Rules in ODA-Documents. *In*: Office Information Systems, ed. B. Pernici and A. Verrijn-Stuart, 1988, 131–147.

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NONEXISTENCE OF ORBITAL MORPHISMS BETWEEN DYNAMICAL SYSTEMS ON SPHERES

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Симеон Т. Стефанов. НЕСУЩЕСТВОВАНИЕ ОРБИТАЛЬНЫХ МОРФИЗМОВ МЕЖДУ ДИНАМИЧЕСКИМИ СИСТЕМАМИ НА СФЕРАХ

Рассмотрены динамические системы на нечетномерных сферах S^{2n+1}, определенные для $t \in \mathbb{R}$, $z \in S^{2n+1}$ формулой

$$tz = \left(e^{i\theta_0 t}z_0, e^{i\theta_1 t}z_1, \ldots, e^{i\theta_n t}z_n\right),$$

где θ_i — действительные числа, $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, $\|z\| = 1$. Такие системы обозначаются через $S^{2n+1}(\theta_0, \ldots, \theta_n)$.

В работе доказано, что в случае рационально независимых θ_i не существуют орбитальных морфизмов

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_{n-1}),$$

т. е. отображения монотонно наматывающие каждую траекторию первой динамической системы на траекторию второй. (Здесь θ'_i — произвольные.)

В случае произвольных θ_i то же самое доказательство проходит с некоторыми усложнениями.

Несуществование орбитальных морфизмов между двумя динамическими системами означает в некотором смысле, что первая система существенно сложнее второй.

Simeon T. Stefanov. NONEXISTENCE OF ORBITAL MORPHISMS BETWEEN DYNAMICAL SYSTEMS ON SPHERES

Some standard dynamical systems in the odd-dimensional spheres S^{2n+1} are considered. The nonexistence of orbital morphisms $f: S^{2n+1} \longrightarrow S^{2n-1}$, i.e. maps winding each trajectory of the first system over some trajectory of the second one, is proved.

We consider, in this note, some standard dynamical systems in the odd-dimensional spheres S^{2n+1} and prove about them a topological fact, which shows that, in some sense, the „chaos" in S^{2n+1} is essentially greater than the „chaos" in S^{2n-1} (in particular, these systems are not semiconjugated). In fact, we shall prove, that there are no orbital morphisms $f: S^{2n+1} \longrightarrow S^{2n-1}$, i.e. smooth maps winding each trajectory of S^{2n+1} over some trajectory of S^{2n-1}. This proposition has the same origin with the „nonexistence" of equivariant maps between G-spaces, namely, the Borsuk-Ulam type theorems (see [1], [2], [4]).

As usually, we shall denote by tx the $\mathbf{R}$-action defined by some dynamical system in X (here $t \in \mathbf{R}$, $x \in X$). The trajectory of an element $x \in X$ is the set

$$\operatorname{traj} x = \{tx | t \in \mathbf{R}\}.$$

Let dynamical systems in the (smooth) manifolds M and N be given.

Definition. A smooth map $f: M \to N$ is called *orbital morphism*, if for any $t \in \mathbf{R}$, $x \in M$

$$f(tx) = \varphi(t, x) f(x),$$

where $\varphi: \mathbf{R} \times M \to \mathbf{R}$ is such that $\frac{\partial \varphi}{\partial t}(t, x) \neq 0$ for any $t \in \mathbf{R}$, $x \in M$.

It is clear that f transforms any trajectory into a trajectory and that the restriction

$$f\restriction_{\operatorname{traj} x}: \operatorname{traj} x \to \operatorname{traj} f(x)$$

is a covering. (This property may be taken for definition in the continuous case.) For compact M, N one obtains also

$$f(\overline{\operatorname{traj} x}) = \overline{f(\operatorname{traj} x)}$$

(where $\overline{A}$ denotes the closure of A). It is easy to see that the composition of two orbital morphisms is also an orbital morphism. This concept is quite more general than „semiconjugacy". The last one is obtained for $\varphi(x, t) = t$.

The nonexistence of orbital morphism between M and N means, in some sense, that the dynamical system in M is essentially more complex than the system in N.

Now we shall define some standard dynamical systems in the odd-dimensional sphere S^{2n+1}. Consider S^{2n+1} as the unit sphere in $\mathbf{C}^{n+1}$

$$S^{2n+1} = \{z = (z_0, \ldots, z_n) \in \mathbf{C}^{n+1} \mid \|z\| = 1\}.$$

Let $\theta_0, \theta_1, \ldots, \theta_n$ be nonzero real numbers. Then set

(1) $$tz = \left(e^{i\theta_0 t} z_0, e^{i\theta_1 t} z_1, \ldots, e^{i\theta_n t} z_n\right).$$

We obtain, in this manner, some dynamical system in S^{2n+1} without stationary points that we shall denote by $S^{2n+1}(\theta_0, \ldots, \theta_n)$. Our aim is to prove that there is no orbital morphism

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_{n-1}).$$

We shall do that in the case of rationally independent $\theta_0, \theta_1, \ldots, \theta_n$ (and arbitrary θ'_i). The same proof works with some complications in the general case of arbitrary θ_i.

The reals $\theta_0, \theta_1, \ldots, \theta_n$ are called *rationally independent*, if the equality

$$\alpha_0\theta_0 + \cdots + \alpha_n\theta_n = 0$$

for some rational $\alpha_i \in \mathbf{Q}$ implies $\alpha_0 = \cdots = \alpha_n = 0$.

Theorem. *Let $\theta_0, \ldots, \theta_n$ be rationally independent real numbers. Then there is no orbital morphism*

$$f: S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_n)$$

where θ'_i are arbitrary.

We shall start with some remarks concerning the system $S^{2n+1}(\theta_0, \ldots, \theta_n)$. The following one is a well-known fact that may be found, for example, in [5].

Lemma 1. *Let $T^{n+1} = (S^1)^{n+1}$ be the $(n+1)$-dimensional torus together with the $\mathbb{R}$-action in it defined by (1) with rationally independent $\theta_0, \theta_1, \ldots, \theta_n$. Then the trajectory of each point z is dense in T^{n+1}:*

$$\overline{\operatorname{traj} z} = T^{n+1}.$$

Consider now the n-dimensional simplex Δ^n as

$$\Delta^n = \{(a_0, \ldots, a_n) \in \mathbb{R}^{n+1} \mid \sum a_i = 1,\ a_i \geqq 0\}$$

and define a map $p : S^{2n+1}(\theta_0, \ldots, \theta_n) \to \Delta^n$ as follows

$$p(z) = (\|z_0\|^2, \ldots, \|z_n\|^2).$$

Evidently, the co-image of some $(a_0, \ldots, a_n) \in \Delta^n$ is the torus $\{z \in S^{2n+1} \mid \|z_i\|^2 = a_i\}$ of dimension k, where k is the number of the nonzero a_i minus 1.

Consider also the factorspace $S^{2n+1}/_{\{\overline{\operatorname{traj} z}\}}$ obtained by the identification of the points of each torus $\overline{\operatorname{traj} z}$ (see lemma 1). It is clear now, that this factor is homeomorphic to Δ^n

$$S^{2n+1}/_{\{\overline{\operatorname{traj} z}\}} \approx \Delta^n.$$

Really, for some $(a_0, \ldots, a_n) \in \Delta^n$ the co-image $p^{-1}(a_0, \ldots, a_n)$ is a torus T^k, but $\overline{\operatorname{traj} z} = T^k$ for any $z \in T^k$ as following by lemma 1, therefore we may identify $\overline{\operatorname{traj} z}$ with $(a_0, \ldots, a_n) \in \Delta^n$. The simplex Δ^n may be considered as the base of a (ramified) bundle with projection $p: S^{2n+1} \to \Delta^n$ and fiberes T^k, $k = 1, \ldots, n+1$.

Before passing to the proof of the theorem, we need two lemmas.

The homologies, we shall make use of hereafter, are Čech ones with integral coefficients. Their properties as well as some other elementary algebraic topology facts may be found, for example, in [3].

A map $f : X \to Y$ is called *trivial in dimension n*, if the induced homomorphism $f_* : H_n(X) \to H_n(Y)$ is trivial: $f_* \equiv 0$.

Lemma 2. *Let X be a compact space and ω be a finite open covering of X, such that the canonical projection $\pi: X \to N_\omega$ into the nerve of ω is nontrivial in dimension n. Suppose $f: X \xrightarrow{\text{on}} Y$ is a map of X onto Y trivial in dimension n. Then for some $y_0 \in Y$ the set $f^{-1}(y_0)$ is not contained in any element of the covering ω.*

P r o o f. Suppose the contrary, i.e. that for any $y \in Y$ the set $f^{-1}(y)$ is contained in some element U_y of ω. Let V_y be an open neighbourhood of y such that $f^{-1}(V_y) \subset U_y$. The covering $\{V_y \mid y \in Y\}$ has a finite subcovering α. Clearly, the simplicial complexes N_α and $N_{f^{-1}(\alpha)}$ are isomorphic (here $f^{-1}(\alpha) = \{f^{-1}(V) \mid V \in \alpha\}$). Since the covering $f^{-1}(\alpha)$ is inscribed in ω, there exists a simplicial map $\lambda: N_{f^{-1}(\alpha)} \to N_\omega$. Consider the diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{f} & Y \\ & \overset{\pi}{\swarrow} & \downarrow \pi_1 & & \downarrow \pi_2 \\ N_\omega & \xleftarrow{\lambda} & N_{f^{-1}(\alpha)} & \overset{\rho}{\approx} & N_\alpha \end{array}$$

where π_1 and π_2 are the canonical projections and ρ is the natural isomorphism. The maps π and $\lambda\pi_1$ are homotopic, since for every $x \in X$ the points $\pi(x)$ and $\lambda\pi_1(x)$ are contained in one and the same simplex. For the same reason, $\rho\pi_1$ and $\pi_2 f$ are also homotopic. Then the diagram

$$\begin{array}{ccccc} & & H_n(X) & \xrightarrow{f_*} & H_n(Y) \\ & \overset{\pi_*}{\swarrow} & \downarrow \pi_{1*} & & \downarrow \pi_{2*} \\ H_n(N_\omega) & \xleftarrow{\lambda_*} & H_n(N_{f^{-1}(\alpha)}) & \overset{\rho_*}{\approx} & H_n(N_\alpha) \end{array}$$

is commutative. Hence $\pi_* = \lambda_* \rho_*^{-1} \pi_{2*} f_* \equiv 0$, which contradicts the condition.

Lemma 3. *Let the n-simplex Δ^n be the union of n open subsets $\Delta^n = \bigcup_{j=1}^{n} V_j$. Then some V_j has a component K intersecting all $n-1$ dimensional faces of Δ^n.*

Proof. Obviously, we may suppose each V_j having a finite number of components (if not, then take their ε-neighbourhoods $O_\varepsilon V_j$ for ε sufficiently small). Consider the covering of Δ^n

$$\gamma = \{K \mid K \text{ is a component of some } V_j\}.$$

It is clear, that $\operatorname{ord}\gamma \leq n$, so the nerve N_γ is a $n-1$ dimensional polyhedron. Consider the canonical projection $\pi_\gamma : \Delta^n \to N_\gamma$, then the restriction $\pi_\gamma \restriction_{\partial\Delta^n} : \partial\Delta^n \to N_\gamma$ is homotopic to a constant and therefore is trivial in dimension $n-1$ ($\partial\Delta^n$ denotes the boundary of Δ^n).

Consider, on the other hand, the covering $\omega = \{U_0, \ldots, U_n\}$ of $\partial\Delta^n$ defined by $U_i = \partial\Delta^n \setminus \Delta_i^{n-1}$ where Δ_i^{n-1} is the (closed) $n-1$ dimensional face of Δ^n opposite to the vertex a_i. It is clear, that $\bigcap U_i = \emptyset$ and the projection $\pi_\omega : \partial\Delta^n \to N_\omega$ is nontrivial in dimension $n-1$. (Really, N_ω may be identified with $\partial\Delta^n$ and π_ω with the identity.) Set $f = \pi_\gamma \restriction_{\partial\Delta^n}$. Then according to lemma 2 there exists $x \in N_\gamma$ such that $f^{-1}(x)$ is not contained in any $\partial\Delta^n \setminus \Delta_i^{n-1}$, which means that $f^{-1}(x)$ intersects all faces Δ_i^{n-1}. But $f^{-1}(x)$ lies in some element of the covering γ (since f is a restriction of the canonical projection π_γ). Hence, by the definition of γ the set $f^{-1}(x)$ lies in the component K of some V_j. Then K intersects all the $n-1$ dimensional faces of Δ^n.

Recalling the notes before lemma 2, let us now pass to the proof of the theorem.

Proof of the theorem. Suppose the contrary, i.e. that there is an orbital morphism

$$f : S^{2n+1}(\theta_0, \ldots, \theta_n) \longrightarrow S^{2n-1}(\theta'_0, \ldots, \theta'_{n-1}).$$

Consider the projection $q : S^{2n-1} \to \Delta^{n-1}$ defined with

$$q(z) = (\|z_0\|^2, \ldots, \|z_{n-1}\|^2).$$

We have

$$\begin{array}{ccc} S^{2n+1} & \xrightarrow{f} & S^{2n-1} \\ \downarrow p & & \downarrow q \\ \Delta^n & & \Delta^{n-1} \end{array}$$

Let $\Delta^{n-1} = \bigcup_{i=0}^{n-1} W_i$ be the union of n open W_i such that W_i does not intersect the $(n-1)$-face opposite to the vertex a_i. Consider the open sets $f^{-1}q^{-1}(W_i)$ which cover S^{2n+1}. We shall prove first that for every i there exists an orbital morphism

$$\varphi_i : f^{-1}q^{-1}(W_i) \longrightarrow S^1.$$

Really, if $z \in f^{-1}q^{-1}(W_i)$, then the i-th coordinate of $f(z)$ is nonzero, and we put $\varphi_i = \pi_i f$ where $\pi_i : q^{-1}(W_i) \to S^1$ is the projection $\pi_i(z) = z_i/\|z_i\|$. Then evidently φ_i is an orbital morphism (as a composition of orbital morphisms)

$$\varphi_i : f^{-1}q^{-1}(W_i) \longrightarrow S^1(\theta_i').$$

Consider furthermore the open sets $V_i = pf^{-1}q^{-1}(W_i)$, which form an open covering of Δ^n. According to lemma 3, some V_i has a component K intersecting all the $(n-1)$-faces of Δ^n. Let $a \in K$ be an arbitrary point of K, then $p^{-1}(a)$ is a torus in $f^{-1}q^{-1}(W_i)$; we shall denote it by T. Our goal is to show, that the homomorphism induced by the embedding

$$j_* : H_1(T) \longrightarrow H_1(f^{-1}q^{-1}(W_i)) \tag{2}$$

is trivial. The group $H_1(T)$ is generated by α_i represented by the circle

$$A_i = \{z \in T \mid \|z_i\|^2 = a_i;\ z_j = z_j^0,\ j \neq i\}$$

where z_j^0 are arbitrary constants with $\|z_j^0\|^2 = a_j$. But the component K intersects the face Δ_i^{n-1}, where $z_i = 0$. Since K is arcwise, we can connect a and Δ_i^{n-1} with a continuous curve lying in K. It enables us to obtain a continuous family of circles A_i^t such that $A_i^0 = A_i$, A_i^1 degenerates to a point and all A_i^t lie in $p^{-1}(K) \subset f^{-1}q^{-1}(W_i)$. Therefore, we found in $f^{-1}q^{-1}(W_i)$ a deformation of A_i into a point, which means that $j_*(\alpha_i) = 0$, hence the homomorphism (2) is trivial.

As it was pointed out before, there exists an orbital morphism $\varphi : f^{-1}q^{-1}(W_i) \to S^1$. Then $\varphi\restriction_T : T \to S^1$ is an orbital morphism, as a composition of j and φ. On the other hand, the homomorphism $(\varphi\restriction_T)_*$ is trivial in dimension 1 (as well as in each other dimension), since $(\varphi\restriction_T)_* = \varphi_* j_* \equiv 0$. But then $\varphi\restriction_T$ induces a trivial homomorphism between the fundamental groups of T and S^1 and therefore $\varphi\restriction_T$ can be lifted to $\tilde{\varphi} : T \to \mathbb{R}$ so that

$$\exp\tilde{\varphi} = \varphi\restriction_T$$

where $\exp(x) = e^{ix}$ is the exponential map (see [3] for details). Since $\varphi\restriction_T$ is an orbital morphism, $\tilde{\varphi}$ is monotone over each trajectory traj z. Fix some $z \in T$. The set $\tilde{\varphi}(T)$ is compact in $\mathbb{R}$, so the following supremum exists:

$$m_0 = \sup \tilde{\varphi}(\text{traj } z).$$

But then the map $\varphi\restriction_{\text{traj } z} = \exp\tilde{\varphi}\restriction_{\text{traj } z}$ is not a covering over the point $\exp(m_0)$, which contradicts the definition of orbital morphism.

REFERENCES

1. Fadell, E. R., P. H. Rabinovitz. Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.*, 45, 1978, 139–174.
2. Munkholm, H. J. Borsuk-Ulam type theorems for proper $\mathbb{Z}_p$-actions on (mod p homology) n-spheres. *Math. Scand.*, 24, 1969, 167–185.
3. Spanier, E. H. Algebraic topology. New York – San Francisco – St. Louis – Toronto – London – Sydney, 1966.
4. Yang, C. T. On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobo and Dyson. I. *Ann. Math.*, 60, 1954, 262–282.
5. Корнфельд, И. П., Я. Г. Синай, С. В. Фомин. Эргодическая теория. М., „Наука", 1980.

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