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LOWER BOUNDS FOR THE MODULE OF DETERMINANTS OF BLOCK DIAGONALLY MATRICES

JUMAH ZARNAN, MILKO PETKOV

Джума Зарнан, Милко Петков. ОЦЕНКИ СНИЗУ ДЛЯ ДЕТЕРМИНАНТ КЛЕТОЧНЫХ МАТРИЦ С ДОМИНИРУЮЩЕЙ КВАЗИДИАГОНАЛЬЮ

Получены новые оценки снизу для детерминант клеточных матриц с доминирующей квазидиагональю, которые обобщают соответствующие результаты для скалярных матриц.

Jumah Zarnan, Milko Petkov. LOWER BOUNDS FOR THE MODULE OF DETERMINANTS OF BLOCK DIAGONALLY MATRICES

In the present paper a lower bounds for the module of block matrices with strongly dominant diagonal and one-sided dominant diagonal have been obtained.

1. DEFINITIONS

Let

$$A = (A_{ij}) \tag{1}$$

be a $p \times p$ block ($n \times n$ scalar) matrix, where A_{ij} are $\alpha_i \times \alpha_j$ scalar matrices and $\|\cdot\|$ is a matrix norm defined in advance.

Definition 1. Matrix (1) will be called a matrix with a dominant diagonal if at least one of the following two conditions is satisfied:

$$\left\|A_{ii}^{-1}\right\|^{-1} \geqq \sum_{j \neq i} \|A_{ij}\| \quad (i = 1, 2, \dots, p) \tag{2}$$

(3) $$\|A_{jj}^{-1}\|^{-1} \geqq \sum_{i \neq j} \|A_{ij}\| \quad (i = 1, 2, \ldots, p).$$

If (2) holds, then matrix A will be a matrix with a dominant diagonal by rows, and if condition (3) is satisfied, then A will be called a matrix with a dominant diagonal by columns.

Next we are going to study only matrices with a dominant diagonal by rows.

Definition 2. If for every i we have strong inequality in (2), then A will be called a matrix with strong dominant diagonal.

Definition 3. A block matrix P of type $p \times p$ ($n \times n$ scalar), which has a block structure analogous to that of A, will be called a permutational matrix if the n-th columns of P are columns of the identity $n \times n$ matrix chosen in an arbitrary order.

Definition 4. Each matrix of type (1) will be called a reducible matrix if there exists a permutational matrix P such that

$$P^T AP = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{pmatrix},$$

where with P^T the transpose of P is denoted, and $\hat{A}_{11}$ and $\hat{A}_{22}$ are square matrices.

When a matrix of type (1) is not reducible, then it is called a irreducible matrix.

Definition 5. Matrix $A = (A_{ij})$ of type (1) will be called a matrix with irreducible dominant diagonal if (2) holds for at least one inequality and if A is irreducible.

Definition 6. Each matrix A of type (1) will be called a matrix with a left (right) dominant diagonal if it has a dominant diagonal and if

$$\|A_{ii}^{-1}\|^{-1} > \sum_{j<i} \|A_{ij}\| \qquad \left(\|A_{ii}^{-1}\|^{-1} > \sum_{j>i} \|A_{ij}\|\right).$$

2. LOWER BOUNDS FOR $|\det A|$

The following criterion for the nonsingularity of a matrix A of type (1) is known:

Theorem 1 [1, 4]. *If a matrix of type (1) is with a strong dominant diagonal or with a reducible dominant diagonal, then A is nonsingular, i. e.* $\det A \neq 0$.

The following theorems are also known:

Theorem 2 [1, 4]. *Let $A = (a_{ij})$ be an $n \times n$ scalar matrix with a strong dominant diagonal. Then the following inequality holds:*

(4) $$|\det A| \geqq \prod_{i=1}^{n} \Delta_i,$$

where

$$\Delta_i = |a_{ii}| - \sum_{j \neq i} |a_{ij}| \quad (i = 1, 2, \ldots, n).$$

Theorem 3 [3]. *Let $A = (a_{ij})$ be an $n \times n$ scalar matrix with a left dominant diagonal. Then the following inequality holds:*

$$|\det A| \geqq \prod_{i=1}^{n} \delta_i, \tag{5}$$

where

$$\delta_i = |a_{ii}| - \sum_{j<i} |a_{ij}|.$$

Next we are going to obtain some generalizations for the inequalities (4) and (5).

For that purpose let us assume that a block variant of the Gauss method (without choosing the pivot block and with reserving the leading diagonal block) may be applied to the $n \times n$ linear system with a nonsingular block matrix of type (1), and let the matrix obtained after the k-th step be $A_k = (A_{ij}(k))$. Also let

$$\Delta_i^{(k)} = \left\|A_{ii}^{-1}(k)\right\|^{-1} - \sum_{j\neq i} \|A_{ij}(k)\|,$$

$$\delta_i^{(k)} = \left\|A_{ii}^{-1}(k)\right\|^{-1} - \sum_{j<i} \|A_{ij}(k)\|$$

be defined for every $i = 1, 2, \ldots, p$ and for every $k = 0, 1, \ldots, p-1$.

Thus the following lemma is valid.

Lemma 1. *If A is a matrix with a strong dominant diagonal, then*

$$\Delta_i^{(k)} \leqq \Delta_i^{(k+1)} \quad (k = 0, 1, \ldots, p-2;\ i = 1, 2, \ldots, p),$$

where the used matrix norm is such that its value for an arbitrarily chosen identity matrix is equal to 1.

Proof. It is sufficient to prove that $\Delta_i^{(1)} \geqq \Delta_i^{(0)}$. Let us point out also that the matrices $A_{ii}(1)$ are invertible and

$$\|A_{ii}^{-1}\|\ \|A_{i1}\|\ \|A_{11}^{-1}\|\ \|A_{1i}\| < 1 \quad \text{for } i \neq 1.$$

Next we have ($i > 1$)

$$\Delta_i^{(1)} = \left\|A_{ii}^{-1}(1)\right\|^{-1} - \sum_{j\neq 1,i} \|A_{ij}(1)\| =$$

$$= \left\|\left(A_{ii} - A_{i1}A_{11}^{-1}A_{1i}\right)^{-1}\right\|^{-1} - \sum_{j\neq 1,i} \left\|A_{ij} - A_{i1}A_{11}^{-1}A_{1j}\right\| =$$

$$= \left\|A_{ii}^{-1}\left(I - A_{ii}^{-1}A_{i1}A_{11}^{-1}A_{1i}\right)^{-1}\right\|^{-1} - \sum_{j\neq 1,i} \left\|A_{ij} - A_{i1}A_{11}^{-1}A_{1j}\right\| \geqq$$

$$\geqq \left\|A_{ii}^{-1}\right\|^{-1} \left\|\left(I - A_{ii}^{-1}A_{i1}A_{11}^{-1}A_{1i}\right)^{-1}\right\|^{-1} - \sum_{j\neq 1,i} \|A_{ij}\| - \sum_{j\neq 1,i} \left\|A_{i1}A_{11}^{-1}A_{1j}\right\| \geqq$$

$$\geq \|A_{ii}^{-1}\|^{-1}(1-\|A_{ii}^{-1}\|\,\|A_{i1}\|\,\|A_{11}^{-1}\|\,\|A_{1i}\|)-$$
$$-\sum_{j\neq 1,i}\|A_{ij}\|-\sum_{j\neq 1,i}\|A_{i1}\|\,\|A_{11}^{-1}\|\,\|A_{1j}\|=$$
$$=\|A_{ii}^{-1}\|^{-1}-\|A_{i1}\|\,\|A_{11}^{-1}\|\,\|A_{1i}\|-\sum_{j\neq 1,i}\|A_{ij}\|-\sum_{j\neq 1,i}\|A_{i1}\|\,\|A_{11}^{-1}\|\,\|A_{1j}\|=$$
$$=\|A_{ii}^{-1}\|^{-1}-\sum_{j\neq 1}\|A_{i1}\|\,\|A_{11}^{-1}\|\,\|A_{1j}\|-\sum_{j\neq 1,i}\|A_{ij}\|\geq$$
$$\geq \|A_{ii}^{-1}\|^{-1}-\|A_{i1}\|-\sum_{j\neq 1,i}\|A_{ij}\|=$$
$$=\|A_{ii}^{-1}\|^{-1}-\sum_{j\neq i}\|A_{ij}\|=\Delta_i^{(0)}.$$

Thus the proof is completed.

After a number of $p-1$ steps of forward Gaussian elimination we obtain the right quasi-triangular matrix

$$A_{p-1}=\begin{pmatrix} A_{11}(p-1) & \dots\dots\dots\dots & & A_{1p}(p-1) \\ 0 & A_{22}(p-1) & \dots & A_{2p}(p-1) \\ \dots & \dots & \dots & \dots \\ 0 & \dots\dots\dots\dots & & A_{pp}(p-1) \end{pmatrix}$$

for which the following conditions hold:

$$\Delta_i \leqq \Delta_i^{(p-1)} \quad (i=1,2,\dots,p).$$

If we accomplish the backward of the Gaussian method with retaining the diagonal blocks $A_{ii}(p-1)$ of A_{p-1}, we obtain the quasi-diagonal matrix

$$D=\operatorname{diag}(D_1,D_2,\dots,D_p),$$

where $D_i=A_{ii}(p-1)$ and $\Delta_i \leqq \Delta_i^{(p-1)} \leqq \Delta_i(D)=\|D_i^{-1}\|^{-1}$

From Lemma 1 and the assertions made afterwards we have

Theorem 4. *If the matrix $A=(A_{ij})$ from (1) has a strong dominant diagonal, then*

$$|\det A| \geq \prod_{i=1}^{p}\left(\|A_{ii}^{-1}\|^{-1}-\sum_{j\neq i}\|A_{ij}\|\right)^{\alpha_i},$$

where the order of the diagonal matrix A_{ii} is denoted by α_i and the used matrix norm takes the value of 1 for an identity matrix of an arbitrary order.

Proof. If D_i has eigen values λ_{is}, $(s=1,2,\dots,\alpha_i)$, then

$$|\det A|=\prod_{i=1}^{p}|\det D_i|=\prod_{i=1}^{p}\prod_{s=1}^{\alpha_i}|\lambda_{is}|\geq$$
$$\geq \prod_{i=1}^{p}\prod_{s=1}^{\alpha_i}\|D_i^{-1}\|^{-1}=\prod_{i=1}^{p}\left(\|D_i^{-1}\|^{-1}\right)^{\alpha_i}\geq \prod_{i=1}^{p}\left(\|A_{ii}^{-1}\|^{-1}-\sum_{j\neq i}\|A_{ij}\|\right)^{\alpha_i}.$$

Thus the theorem is proved.

Lemma 2. *If $A = (A_{ij})$ from* (1) *is with a left dominant diagonal and*

$$\delta_i^{(k)} = \|A_{ii}^{-1}(k)\|^{-1} - \sum_{j<i} \|A_{ij}(k)\|,$$

then

$$\delta_i^{(k)} \leqq \delta_i^{(k+1)} \quad (i = 1, 2, \ldots, p;\ k = 0, 1, \ldots, p-2).$$

The proof is similar to that of Lemma 1.

Theorem 5. *If the matrix A from* (1) *is with a left dominant diagonal, then*

$$|\det A| \geqq \prod_{i=1}^{p} \left(\|A_{ii}^{-1}\|^{-1} - \sum_{j<i} \|A_{ij}\| \right)^{\alpha_i}$$

with the same assumptions made for the matrix norm.

The proof is similar to that of Theorem 4.

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ОЦЕНКИ СНИЗУ ДЛЯ ЧИСЛА ВЕРШИН НЕКОТОРЫХ ГРАФОВ РАМСЕЯ

НЕДЯЛКО НЕНОВ

Недялко Ненов. ОЦЕНКИ СНИЗУ ДЛЯ ЧИСЛА ВЕРШИН НЕКОТОРЫХ ГРАФОВ РАМСЕЯ

Для данного графа G символ $V(G)$ обозначает множество его вершин. Граф G называется $(p_1,\dots,p_s)$-графом Рамсея, если в любой s-раскраске его ребер существует монохроматическая p_i-клика i-ого цвета для некоторого i, $1 \leqq i \leqq s$. Символ $R(p_1,\dots,p_s)$ обозначает минимальное натуральное число n, для которого полный граф с n вершинами является $(p_1,\dots,p_s)$-графом Рамсея, а символ $N(R(p_1,\dots,p_s))$ — минимальное натуральное число n, для которого существует $(p_1,\dots,p_s)$-граф Рамсея с $|V(G)| = n$ и несодержащий $R(p_1,\dots,p_s)$-клик. В настоящей работе доказываются нижные оценки для чисел $N(R(p_1,\dots,p_s))$.

Nedjalko Nenov. A LOWER BOUND FOR THE NUMBER OF VERTICES OF SOME RAMSEY GRAPHS

A subset $v_1,\dots,v_p$ of vertices of graph is called a p-clique if any two of them are adjacent. The graph G is called a $(p_1,\dots,p_s)$-Ramsey graph, for some set of integers $p_1,\dots,p_s$ if for every s-colouring of the edges of G there exists an i, $1 \leqq i \leqq s$, such that G contains a monochromatic p_i-clique of the i-colour. The symbol $R(p_1,\dots,p_s)$ denotes the minimal natural number n such that the complete graph with n vertices is $(p_1,\dots,p_s)$-Ramsey graph and the symbol $N(R(p_1,\dots,p_s))$ — the minimal natural number n such that there exists a $(p_1,\dots,p_s)$-Ramsey graph with n vertices and without $R(p_1,\dots,p_s)$-cliques. In this paper it is proved a lower bound for the numbers $N(R(p_1,\dots,p_s))$.

1. ОПРЕДЕЛЕНИЕ ОСНОВНЫХ ПОНЯТИЙ

Будем рассматривать конечные, неориентированные графы, без кратных ребер и петель, т. е. под графом всюду будем понимать упорядоченную пару $G = (V(G), E(G))$, где $V(G)$ — конечное множество, а $E(G)$ — некоторая совокупность 2-элементных подмножеств множества $V(G)$. Элементы $V(G)$ называются вершинами графа G, а элементы $E(G)$ — ребрами графа G. Если $v_1, v_2 \in V(G)$ и $[v_1, v_2] \in E(G)$ будем говорить, что вершины v_1 и v_2 смежные вершины графа G. Дополнение графа G обозначается $\overline{G}$ и определяется следующим образом: $V(G) = V(\overline{G})$ и две вершины смежны в $\overline{G}$ тогда и только тогда, когда они несмежны в G. Множество вершин v_1, v_2, ..., v_p графа G называется p-кликой графа G, если любые две из них смежны. Наибольшее натуральное число p, для которого граф G имеет p-клику, называется кликовым числом графа G и обозначается $\mathrm{cl}(G)$. Множество вершин графа называется независимым множеством вершин, если любые две из них несмежны. Наибольшее натуральное число p, для которого граф G имеет p-вершинное независимое множество вершин, называется числом независимости графа G и обозначается $\alpha(G)$. Ясно, что $\alpha(G) = \mathrm{cl}(\overline{G})$. Разложение $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ называется r-хроматическим разложением вершин графа G, если V_i, $i = 1, \ldots, r$ — независимые множества вершин графа G и $V_i \cap V_j = \varnothing$, $i \neq j$. Наименьшее натуральное число r, для которого граф G обладает r-хроматическим разложением, называется хроматическим числом графа G и обозначается $\chi(G)$.

Граф G_1 называется подграфом графа G, если $V(G_1) \subseteq V(G)$ и $E(G_1) \subseteq E(G)$. Пусть $M \subseteq V(G)$. Тогда через $\langle M \rangle$ будем обозначать подграф графа G, порожденный множеством вершин M, т. е. $V(\langle M \rangle) = M$, и две вершины множества M смежны в $\langle M \rangle$ тогда и только тогда, когда они смежны в G. Если $v \in V(G)$, то через $G - v$ будем обозначать подграф, получающийся из G после удаления вершины v, т. е. $G - v = \langle V(G) - v \rangle$. Если $M \subseteq V(G)$, то через $\mathrm{Ad}(M)$ будем обозначать множество всех вершин графа G, которые смежны всем вершинам множества M. Если $|V(G)| = n$ и любые две вершины графа G смежны, тогда этот граф называется полным графом с n вершинами и обозначается K_n. Упорядоченное множество вершин $v_1, v_2, \ldots, v_n$ данного графа G называется простым циклом длины n графа G, если $[v_1, v_2], [v_2, v_3], \ldots, [v_{n-1}, v_n], [v_n, v_1] \in E(G)$ и обозначается C_n.

Пусть G_1 и G_2 — два графа без общих вершин. Через $G_1 + G_2$ обозначается граф G, для которого $V(G) = V(G_1) \cup V(G_2)$ и $E(G) = E(G_1) \cup E(G_2) \cup E'$, где $E' = \{[v_1, v_2] \mid v_1 \in V(G_1),\ v_2 \in V(G_2)\}$.

2. ОПРЕДЕЛЕНИЕ ГРАФОВ РАМСЕЯ И ФОРМУЛИРОВКА ОСНОВНОГО РЕЗУЛЬТАТА

Определение. Любое разложение $E(G) = E_1 \cup \ldots \cup E_s$, $E_i \cap E_j = \varnothing$, $i \neq j$, называется s-раскраской ребер графа G.

Определение. Будем говорить, что p-клика P графа G является монохроматической p-кликой i-ого цвета s-раскраски $E(G) = E_1 \cup \ldots \cup E_s$ его ребер, если $E(P) \subseteq E_i$.

Определение: Пусть $p_1, \ldots, p_s$ — натуральные числа, $p_i \geqq 2$. Граф G называется $(p_1, \ldots, p_s)$-графом Рамсея, если в любой s-раскраске его ребер существует монохроматическая p_i-клика i-ого цвета для некоторого i, $1 \leqq i \leqq s$.

Определение. Наименьшее натуральное число n, для которого полный граф с n вершинами K_n является $(p_1, \ldots, p_s)$-графом Рамсея, называется числом Рамсея и обозначается $R(p_1, \ldots, p_s)$.

Существование чисел $R(p_1, \ldots, p_s)$ впервые было доказано Рамсеем в [4]. Очевидно $R(p_1, \ldots, p_s)$ — симметрическая функция, $R(p) = p$ и $R(p_1, \ldots, p_s, 2) = R(p_1, \ldots, p_s)$. Поэтому будем рассматривать только такие числа Рамсея $R(p_1, \ldots, p_s)$, для которых $s \geqq 2$ и $p_i \geqq 3$. Известны только следующие нетривиальные числа Рамсея: $R(3,3) = 6$, $R(3,4) = R(4,3) = 9$, $R(3,5) = R(5,3) = 14$, $R(3,6) = R(6,3) = 18$, $R(3,7) = R(7,3) = 23$, $R(3,9) = R(9,3) = 36$, $R(4,4) = 18$, $R(3,3,3) = 18$. По поводу чисел Рамсея см. [7]. Очевидно, что если $\mathrm{cl}(G) \geqq R(p_1, \ldots, p_s)$, тогда граф G является $(p_1, \ldots, p_s)$-графом Рамсея.

Определение. Пусть $p_1, \ldots, p_s$ — натуральные числа. Наименьшее натуральное число n, для которого существует $(p_1, \ldots, p_s)$-граф Рамсея G с n вершинами и $\mathrm{cl}(G) < R(p_1, \ldots, p_s)$, обозначается $N(R(p_1, \ldots, p_s))$.

Числа $N(R(p_1, \ldots, p_s))$ существуют, см. [5]. В [6] доказано, что $N(R(3,3)) = 8$, $N(R(3,5)) = 17$ и $N(R(4,4)) = 20$. Тоже в [6] доказано неравенство

$$N(R(p_1, \ldots, p_s)) \geqq R(p_1, \ldots, p_s) + 2.$$

Это неравенство точное, так что без дополнительных условий усилить его нельзя. В [3] доказана следующая:

Теорема А. *Пусть полный граф с $R(p_1, \ldots, p_s)$ вершинами обладает s-раскраской ребер, в которой для некоторого i, $1 \leqq i \leqq s$, существует единственная монохроматическая p_i-клика i-ого цвета и в которой нет монохроматической p_j-клики j-ого цвета при $i \neq j$. Тогда*

$$N(R(p_1, \ldots, p_s)) \geqq R(p_1, \ldots, p_s) + 3.$$

Если еще потребовать $p_i \geqq 4$, тогда

$$N(R(p_1, \ldots, p_s)) \geqq R(p_1, \ldots, p_s) + 4.$$

В настоящей работе теорема А усиливается следующим образом:

Основная теорема. *Пусть $p_1, \ldots, p_s$ — натуральные числа, $s \geqq 2$, $p_i \geqq 3$. Предположим, что полный граф с $R(p_1, \ldots, p_s)$ вершинами обладает*

s-раскраской ребер со следующими свойствами: для некоторого i, $1 \leqq i \leqq s$, существует единственная монохроматическая p_i-клика i-ого цвета и не существует монохроматическая p_j-клика j-ого цвета при $i \neq j$. Тогда:

а) *если $p_i = 3$, либо граф $K_{R-4} + \overline{C}_7$ является $(p_1, \ldots, p_s)$-графом Рамсея, либо $N(R(p_1, \ldots, p_s)) \geqq R(p_1, \ldots, p_s) + 4$;*

б) *если $p_i \geqq 4$, либо граф $K_{R-5} + \overline{C}_9$ является $(p_1, \ldots, p_s)$-графом Рамсея, либо $N(R(p_1, \ldots, p_s)) \geqq R(p_1, \ldots, p_s) + 5$ $(R = R(p_1, \ldots, p_s))$.*

В [1] доказано, что $N(R(3,4)) = 14$, так, что последнее неравенство точное.

Определение. Будем говорить, что k-хроматическое разложение

$$V(G) = V_1 \cup \ldots \cup V_k$$

множества вершин графа G является t-плотным хроматическим разложением, если объединение любых t из подмножеств V_i, $1 \leqq i \leqq k$, порождает подграф с кликовым числом, равным t.

Очевидно, что:

Предложение 1. *Если некоторое хроматическое разложение множества вершин данного графа не является t-плотным, тогда оно не является и $(t+1)$-плотным.*

Для дальнейшего будет нужно и:

Предложение 2. *Если $\chi(G) = r$, тогда граф G обладает $(r+1)$-хроматическим разложением, которое не является t-плотным для любого t.*

Доказательство. Пусть $V(G) = V_1 \cup \ldots \cup V_r$ является r-хроматическим разложением вершин графа G. Добавляя к этому разложению пустое множество, получаем $(r+1)$-хроматическое разложение, которое очевидно не является t-плотным для любого t.

Для доказательства основной теоремы будет нужна следующая:

Теорема В. *Пусть любое $(r+1)$-хроматическое разложение вершин графа G является 4-плотным и $\mathrm{cl}(G) \leqq r$. Тогда либо $G = K_{r-4} + \overline{C}_9$, либо $|V(G)| \geqq r + 6$.*

3. ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ В И НЕСКОЛЬКИХ НЕОБХОДИМЫХ ДЛЯ ЭТОГО ЛЕММ

Лемма 1 ([2]). *Пусть G — граф, $|V(G)| = r + 4$, $r \geqq 3$, $\mathrm{cl}(G) \leqq r$ и $\chi(G) \geqq r + 1$. Тогда либо $G = K_{r-3} + G_1$, где $\mathrm{cl}(G_1) = 3$ и $\chi(G_1) = 4$, либо существует вершина $v \in V(G)$ такая, что $G - v = K_{r-2} + C_5$.*

Лемма 2. *Пусть $|V(G)| \leqq r + 4$ и $\mathrm{cl}(G) \leqq r$. Тогда граф G обладает $(r+1)$-хроматическим разложением вершин, которое не является 4-плотным.*

Доказательство. Добавлением изолированных вершин можно добиться $|V(G)| = r + 4$. Если $r \leqq 2$, тогда граф G обладает 3-хроматическим разложением вершин и утверждение леммы очевидно. Если $\chi(G) \leqq r$, тогда лемма 2 вытекает из предложения 2. Предположим теперь, что

$r \geqq 3$ и $\chi(G) \geqq r+1$. Тогда для графа можно применить лемму 1. Представляются две возможности:

Случай 1. $G = K_{r-3} + G_1$, где $\chi(G_1) = 4$ и $\mathrm{cl}(G_1) < 4$.

Пусть $V_1 \cup V_2 \cup V_3 \cup V_4$ является 4-хроматическим разложением графа G_1 и $V(K_{r-3}) = \{z_1, \dots, z_{r-3}\}$. Тогда

$$V(G) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup \{z_1\} \cup \ldots \cup \{z_{r-3}\}$$

является $(r+1)$-хроматическим разложением графа G. Из $\mathrm{cl}(G_1) < 4$ вытекает, что это разложение не является 4-плотным.

Случай 2. Существует вершина $v \in V(G)$ такая, что $G - v = K_{r-2} + C_5$.

Очевидно $\chi(C_5) = 3$. Пусть $V_1 \cup V_2 \cup V_3$ является 3-хроматическим разложением графа C_5 и $V(K_{r-2}) = \{z_1, \dots, z_{r-2}\}$. Тогда

$$(1) \qquad V_1 \cup V_2 \cup V_3 \cup \{z_1\} \cup \ldots \cup \{z_{r-2}\}$$

является $(r+1)$-хроматическим разложением графа $G - v$. Так как $\mathrm{cl}(C_5) = 2$, то это разложение не является 3-плотным и согласно предположению 2 не является и 4-плотным. Если вершина v несмежна некоторой вершине z_i, $1 \leqq i \leqq r-2$, тогда, групируя вершину v с этой вершиной из (1), получим $(r+1)$-хроматическое разложение вершин графа G, которое не является 3-плотным. Если вершина v смежна всем вершинам z_i, $1 \leqq i \leqq r-2$, тогда попадаем в условия случая 1.

Лемма 3. *Пусть u и v несмежные вершины графа G и $\mathrm{Ad}(u) \subseteq \mathrm{Ad}(v)$. Если $|V(G)| = r+5$ и $\mathrm{cl}(G) \leqq r$, тогда граф G обладает $(r+1)$-хроматическим разложением, которое не является 4-плотным.*

Доказательство. Согласно лемме 2 подграф $G - u$ обладает $(r+1)$-хроматическим разложением вершин $V_1 \cup \ldots \cup V_{r+1}$, которое не является 4-плотным. Добавим вершину u к тому из множеств V_i, $1 \leqq i \leqq r+1$, которому принадлежит вершина v. Из $\mathrm{Ad}(u) \subseteq \mathrm{Ad}(v)$ следует, что таким образом получается $(r+1)$-хроматическое разложение вершин графа G. Ясно, что это разложение тоже не является 4-плотным.

Лемма 4. *Пусть $|V(G)| = r+2$, $\mathrm{cl}(G) \leqq r$, $r \geqq 2$ и $\alpha(G) = 2$. Тогда дополнение $\overline{G}$ графа G имеет два ребра без общей вершины.*

Доказательство. Из $\mathrm{cl}(G) \leqq r$ следует, что в G есть две несмежные вершины v_1 и v_2. Если две из остальных вершин $v_3, \dots, v_{r+2}$ несмежны, тогда лемма доказана. Предположим, что любые две из вершин $v_3, \dots, v_{r+2}$ — смежны. Из $\mathrm{cl}(G) \leqq r$ следует, что v_1 несмежна некоторой вершине v_i, $3 \leqq i \leqq r+2$. Тоже самое относится и к вершине v_2. Из $\alpha(G) = 2$ следует, что v_1 и v_2 несмежны разным вершинам множества $\{v_3, \dots, v_{r+2}\}$. Этим лемма доказана.

Лемма 5. *Пусть $|V(G)| = r+5$, $\mathrm{cl}(G) \leqq r$, $r \geqq 3$ и $\alpha(G) = 2$. Тогда дополнение $\overline{G}$ графа G имеет четыре ребра, любые два из которых не имеют общую вершину.*

Доказательство. Из $\mathrm{cl}(G) \leqq r$ следует, что в G есть вершины $v_1, v_2, \dots, v_6$ такие, что $[v_1, v_2] \notin E(G)$, $[v_3, v_4] \notin E(G)$ и $[v_5, v_6] \notin E(G)$. Остальные вершины графа G обозначим через $v_7, \dots, v_{r+5}$. Положим $T = \{v_7, \dots, v_{r+5}\}$. Если две вершины множества T несмежны, то утверждение доказано.

Предположим, что любые две вершины множества T — смежны, т. е. $\langle T\rangle = K_{r-1}$. Если $v_1 \notin \mathrm{Ad}(T)$ и $v_2 \notin \mathrm{Ad}(T)$, тогда из $\alpha(G)=2$ следует, что v_1 и v_2 несмежны разным вершинам множества T и утверждение доказано. Следовательно можно предположить, что $v_1 \in \mathrm{Ad}(T)$. Из аналогичных соображений можно предположить еще, что $v_3 \in \mathrm{Ad}(T)$ и $v_5 \in \mathrm{Ad}(T)$. Так как $\langle T\rangle = K_{r-1}$ и $\mathrm{cl(G)} \leqq \mathrm{r}$, то $\{v_1, v_3, v_5\}$ — независимое множество вершин графа G, что противоречит условию $\alpha(G)=2$.

Лемма 6. *Пусть* $|V(G)| = r+5$, $\mathrm{cl}(G) \leqq r$ *и* $\alpha(G) \geqq 3$. *Тогда граф* G *обладает* $(r+1)$*-хроматическим разложением вершин, которое не является 4-плотным.*

Доказательство. Если $\chi(G) \leqq r$, лемма 6 вытекает из предложения 2. Если $r \leqq 3$, тогда граф G очевидно обладает $(r+1)$-хроматическим разложением. Так как $\mathrm{cl}(G) \leqq r \leqq 3$, то это разложение не является 4-плотным. Предположим, что $r \geqq 4$ и $\chi(G) > r$. Пусть $V(G) = \{v_1, \ldots, v_{r+5}\}$. Так как $\alpha(G) \geqq 3$, то можно предположить, что $\{v_1, v_2, v_3\}$ — независимое множество. Из $\mathrm{cl}(G) \leqq r$ следует, что $\alpha(\langle v_4, \ldots, v_{r+5}\rangle) \geqq 2$ и, поэтому представляются две возможности:

Случай 1. $\alpha(\langle v_4, \ldots, v_{r+5}\rangle) \geqq 3$. Предположим, что $\{v_4, v_5, v_6\}$ — независимое множество, тогда

$$V(G) = \{v_1, v_2, v_3\} \cup \{v_4, v_5, v_6\} \cup \{v_7\} \cup \ldots \cup \{v_{r+5}\} \tag{2}$$

является $(r+1)$-хроматическим разложением вершин графа G. Положим $T = \{v_7, \ldots, v_{r+5}\}$. Если в T есть несмежные вершины, группируя две такие вершины, из (2) получим r-хроматическое разложение, что противоречит допущению $\chi(G) \geqq r+1$. Предположим, что любые две вершины множества T — смежны, т. е. $\langle T\rangle = K_{r-1}$. Из $\chi(G) \geqq r+1$ вытекает, что одна из вершин v_1, v_2, v_3 смежна всем вершинам множества T (иначе группируя v_1, v_2, v_3 с несмежными им вершинами множества T, из (2) получим r-хроматическое разложение). Аналогичным образом заключаем, что и одна из вершин v_4, v_5, v_6 смежна всем вершинам множества T. Итак, можно предположить, что $v_1 \in \mathrm{Ad}(T)$ и $v_4 \in \mathrm{Ad}(T)$. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-1}$ следует, что $[v_1, v_4] \notin E(G)$. Если вершины v_5 и v_6 несмежны вершине v_1, тогда $\mathrm{Ad}(v_1) \subseteq \mathrm{Ad}(v_4)$ и лемма 6 вытекает из леммы 3. Предположим, что хотя бы одна из вершин v_5, v_6 смежна v_1 и пусть например $[v_1, v_5] \in E(G)$. Из аналогичных соображений можно предположить еще, что $[v_4, v_2] \in E(G)$. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-1}$ следует, что $v_2 \notin \mathrm{Ad}(T)$ и $v_5 \notin \mathrm{Ad}(T)$. Поэтому для вершин v_2 и v_5 представляются две возможности:

Подслучай 1.a. В T нет вершины, несмежной одновременно вершинам v_2 и v_5 (и следовательно они несмежны разным вершинам множества T). Без ограничения общности можно предположить, что $[v_2, v_7] \notin E(G)$ и $[v_5, v_8] \notin E(G)$. Из условия рассматриваемого подслучая вытекает, что $[v_5, v_7] \in E(G)$, $[v_2, v_8] \in E(G)$. Если $[v_6, v_7] \in E(G)$, тогда $\mathrm{Ad}(v_2) \subseteq \mathrm{Ad}(v_7)$ и лемма 6 вытекает из леммы 3. Если $[v_3, v_8] \in E(G)$, тогда $\mathrm{Ad}(v_5) \subseteq \mathrm{Ad}(v_8)$ и опять лемма 6 вытекает из леммы 3. Если $[v_3, v_8] \notin E(G)$ и $[v_6, v_7] \notin E(G)$, тогда $(\mathrm{Ad}(v_7) \cap \mathrm{Ad}(v_8)) \setminus T = \{v_1, v_4\}$ и так как $[v_1, v_4] \notin E(G)$, то первые четыре подмножества $(r+1)$-хроматического разложения (2) порождают подграф без 4-клик и следовательно оно не является 4-плотным.

Подслучай 1.6. В T есть вершина, которая несмежна одновременно v_2 и v_5. Пусть например $[v_2, v_7] \notin E(G)$ и $[v_5, v_7] \notin E(G)$. Для пар $[v_3, v_7]$ и $[v_6, v_7]$ представляются следующие возможности:

I. $[v_3, v_7] \notin E(G)$ и $[v_6, v_7] \notin E(G)$. В этой ситуации $\mathrm{Ad}(v_7)\setminus T = \{v_1, v_4\}$. Так как $[v_1, v_4] \notin E(G)$, то первые три подмножества $(r+1)$-хроматического разложения (2) порождают подграф без 3-клик и следовательно это разложение не является 3-плотным. Согласно предложению 1 разложение (2) не *является и 4-плотным.*

II. Одна из вершин v_3, v_6 смежна v_7, а другая — нет. Пусть например $[v_6, v_7] \in E(G)$ и $[v_3, v_7] \notin E(G)$. Если v_6 несмежна некоторой вершине из множества $\{v_8, \ldots, v_{r+5}\}$, например $[v_6, v_8] \notin E(G)$, тогда $(\mathrm{Ad}(v_7) \cap \mathrm{Ad}(v_8)) \setminus T = \{v_3, v_4\}$. Как уже было отмечено выше, из этого вытекает, что первые четыре подмножества $(r+1)$-хроматического разложения (2) порождают подграф без 4-клик и следовательно оно не является 4-плотным. Если $v_6 \in \mathrm{Ad}(T)$, тогда из $\mathrm{cl}(G) \leqq r$ следует, что $\{v_1, v_4, v_6\}$ — независимое множество. Так как $\mathrm{Ad}(v_7) \setminus T = \{v_1, v_4, v_6\}$, то из этого вытекает, что первые три подмножества разложения (2) пораждают подграф без 3-клик и следовательно оно не является 3-плотным. Из предложения 1 вытекает, что это разложение не является 4-плотным.

III. $[v_3, v_7] \in E(G)$ и $[v_6, v_7] \in E(G)$. Если v_3 и v_6 несмежны разным вершинам множества $\{v_8, \ldots, v_{r+5}\}$ и например — $[v_3, v_8]$, $[v_6, v_9] \notin E(G)$, тогда из разложения (2) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_7\} \cup \{v_8, v_3\} \cup \{v_9, v_6\} \cup \ldots,$$

в котором первые три подмножества порождают подграф без 3-клик. Согласно предложению 1 новополученное $(r+1)$-хроматическое разложение не является 4-плотным. Если v_3 и v_6 несмежны одной и той же вершине множества $\{v_8, \ldots, v_{r+5}\}$ и например $[v_3, v_8]$, $[v_6, v_8] \notin E(G)$, тогда $(\mathrm{Ad}(v_7) \cap \mathrm{Ad}(v_8)) \setminus T = \{v_1, v_4\}$. Так как $[v_1, v_4] \notin E(G)$, то первые четыре подмножества разложения (2) порождают подграф без 4-клик. Остается рассмотреть ситуацию когда хотя бы одна из вершин v_3, v_6 смежна всем вершинам множества T. Без ограничения общности можно предположить, что $v_3 \in \mathrm{Ad}(T)$. Если еще $v_6 \in \mathrm{Ad}(T)$, тогда из $\mathrm{cl}(G) \leqq r$ вытекает, что $\{v_1, v_3, v_4, v_6\}$ — независимое множество вершин графа G. Из этого вытекает, что первые три подмножества разложения (2) порождают подграф без 3-клик и согласно предложению 1 это разложение не является 4-плотным. Если вершина $v_6 \notin E(G)$ и например $[v_6, v_8] \notin E(G)$, тогда $(\mathrm{Ad}(v_7) \cap \mathrm{Ad}(v_8)) \setminus T = \{v_1, v_3, v_4\}$. Так как $\{v_1, v_3, v_4\}$ — независимое множество вершин графа G (иначе $\mathrm{cl}(G) > r$), то из последнего равенства вытекает, что первые четыре подмножества разложения (2) порождают подграф без 4-клик.

Случай 2. $\alpha(\langle v_4, \ldots, v_{r+5}\rangle) = 2$. Применяя лемму 4 к подграфу $\langle v_4, \ldots, v_{r+5}\rangle$, заключаем, что можно предположить $[v_4, v_5] \notin E(G)$ и

$[v_6, v_7] \notin E(G)$. Тогда

$$(3) \qquad V(G) = \{v_1, v_2, v_3\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \ldots \cup \{v_{r+5}\}$$

является $(r+1)$-хроматическим разложением вершин графа G. Положим $T = \{v_8, \ldots, v_{r+5}\}$. Так как $r \geqq 4$, то $T \neq \varnothing$. Рассуждая так же как и для разложения (2), заключаем, что из $\chi(G) > r$ вытекает $\langle T \rangle = K_{r-2}$ и что можно предположить $v_1, v_4, v_6 \in \mathrm{Ad}\,(T)$. Из $\mathrm{cl}(G) \leqq r$ следует, что $\{v_1, v_4, v_6\}$ не является 3-кликой графа G. Для вершин v_5 и v_7 представляются следующие возможности:

Подслучай 2.а. В T есть вершина, которая несмежна вершинам v_5 и v_7. Предположим для определенности, что $[v_5, v_8] \notin E(G)$ и $[v_7, v_8] \notin E(G)$. В этой ситуации для вершин v_2 и v_3 есть следующие возможности:

I. $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \notin E(G)$. Ясно, что $\mathrm{Ad}\,(v_8) \backslash T = \{v_1, v_4, v_6\}$. Так как $\{v_1, v_4, v_6\}$ не является 3-кликой, то объединение первых четырех подмножеств разложения (3) порождает подграф без 4-клик и следовательно это разложение не является 4-плотным.

II. Одна из вершин v_2, v_3 смежна v_8, а другая — нет. Пусть например $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \in E(G)$. Если вершина v_3 несмежна некоторой из вершин $v_9, \ldots, v_{r+5}$ и например $[v_3, v_9] \notin E(G)$, тогда из (3) получаем новое $(r+1)$-хроматическое разложение вершин графа G

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \{v_3, v_9\} \cup \ldots,$$

в котором объединение первых четырех подмножеств порождает подграф без 4-клик и следовательно это разложение не является 4-плотным. Если $v_3 \in \mathrm{Ad}\,(T)$, тогда из $\mathrm{cl}(G) \leqq r$ следует, что $\langle v_1, v_3, v_4, v_6 \rangle$ не содержит 3-клик. Так как $\mathrm{Ad}\,(v_8) \backslash T = \{v_1, v_3, v_4, v_6\}$, то объединение первых четырех подмножеств разложения (3) порождает подграф без 4-клик.

III. Вершина v_8 смежна вершинам v_2 и v_3. Если $v_2 \in \mathrm{Ad}\,(T)$ и $v_3 \in \mathrm{Ad}\,(T)$, тогда из $\langle T \rangle = K_{r-2}$ и $\mathrm{cl}(G) \leqq r$ следует, что $\{v_1, v_2, v_3, v_4, v_6\} = \mathrm{Ad}\,(v_8) \setminus T$ не содержит 3-клик и следовательно объединение первых четырех подмножеств разложения (3) порождает подграф без 4-клик. Если $v_2 \in \mathrm{Ad}\,(T)$ и $v_3 \notin E(G)$ и например $[v_3, v_9] \notin E(G)$, тогда из разложения (3) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \{v_3, v_9\} \cup \ldots,$$

в котором объединение первых четырех подмножеств порождает подграф без 4-клик. Если $v_2 \notin \mathrm{Ad}\,(T)$ и $v_3 \notin \mathrm{Ad}\,(T)$, тогда группируя v_2 и v_3 с несмежными им вершинами множества T, из разложения (3) получаем новое $(r+1)$-хроматическое разложение, в котором объединение первых четырех подмножеств порождает подграф без 4-клик.

Подслучай 2.б. Вершины v_5 и v_7 несмежны разным вершинам множества T. Пусть например $[v_5, v_8] \notin E(G)$ и $[v_7, v_9] \notin E(G)$. Можно предположить еще, что $[v_5, v_9] \in E(G)$ и $[v_7, v_8] \in E(G)$ так как иначе попадаем в условиях подслучая 2.а. Если $[v_4, v_6] \notin E(G)$, тогда $\langle v_4, v_5, v_6, v_7, v_8, v_9 \rangle$ не содержит 4-клик и следовательно $(r+1)$-хроматическое разложение (3) вершин графа G не является 4-плотным. Предположим, что $[v_4, v_6] \in E(G)$.

Но тогда либо $[v_1, v_6] \notin E(G)$, либо $[v_1, v_4] \notin E(G)$, так как $\{v_1, v_4, v_6\}$ не является 3-кликой. Без ограничения общности можно предположить, что $[v_1, v_4] \notin E(G)$. Для вершин v_2 и v_3 представляются следующие возможности:

I. $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \notin E(G)$. В этой ситуации объединение вершины v_8 с первыми двумя подмножествами разложения (3) порождает подграф без 3-клик и следовательно это разложение не является 3-плотным. Согласно предложению 1 разложение (3) не является и 4-плотным.

II. Одна из вершин v_2, v_3 смежна вершине v_8, а другая — нет. Пусть например $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \in E(G)$. Если вершина v_3 несмежна некоторой вершине множества $\{v_9, \ldots, v_{r+5}\}$, тогда группируя v_3 с такой вершиной, из разложения (3) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \ldots$$

Так как $\langle v_1, v_2, v_4, v_5, v_8 \rangle$ не содержит 3-клик, то последнее $(r+1)$-хроматическое разложение не является 3-плотным. Согласно предложению 1 это разложение не является и 4-плотным. Если $v_3 \in \mathrm{Ad}\,(T)$ из $\mathrm{cl}(G) \leqq r$ и $\langle T \rangle = K_{r-2}$ следует, что $\{v_3, v_4, v_6\}$ не является 3-кликой. Так как $[v_4, v_6] \in E(G)$, то либо $[v_3, v_4] \notin E(G)$, либо $[v_3, v_6] \notin E(G)$. Если $[v_3, v_4] \notin E(G)$, тогда вершина v_8 вместе с первыми двумя подмножествами разложения (3) порождают подграф без 3-клик. Согласно предложению 1 $(r+1)$-хроматическое разложение (3) не является 4-плотным. Если $[v_3, v_6] \notin E(G)$, тогда из разложения (3) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_8\} \cup \{v_3, v_6\} \cup \{v_7, v_9\} \cup \ldots,$$

в котором первые три подмножества порождают подграф без 3-клик. Согласно предложению 1 последнее $(r+1)$-хроматическое разложение не является 4-плотным.

III. $[v_2, v_8] \in E(G)$ и $[v_3, v_8] \in E(G)$. В этой ситуации очевидно, что $\mathrm{Ad}\,(v_5) \subseteq \mathrm{Ad}\,(v_8)$ и лемма 6 вытекает из леммы 3.

Подслучай 2.в. Одна из вершин v_5, v_7 принадлежит $\mathrm{Ad}\,(T)$, а другая — нет. Пусть например $v_7 \in \mathrm{Ad}\,(T)$ и $[v_5, v_8] \notin E(G)$. Из $\mathrm{cl}(G) \leqq r$ следует, что $\langle v_1, v_4, v_6, v_7 \rangle$ не содержит 3-клик. Если $[v_2, v_8] \in E(G)$ и $[v_3, v_8] \in E(G)$, тогда $\mathrm{Ad}\,(v_5) \subseteq \mathrm{Ad}\,(v_8)$ и лемма 6 вытекает из леммы 3. Если $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \notin E(G)$, тогда $\mathrm{Ad}\,(v_8) \setminus T = \{v_1, v_4, v_6, v_7\}$. Так как $\{v_1, v_4, v_6, v_7\}$ не содержит 4-клик, то объединение первых четырех подмножеств разложения (3) порождает подграф без 4-клик. Если $[v_2, v_8] \notin E(G)$ и $[v_3, v_8] \in E(G)$, тогда для v_3 представляются две возможности:

I. Вершина v_3 несмежна некоторой из вершин $v_9, \ldots, v_{r+5}$. Пусть например $[v_3, v_9] \notin E(G)$. Тогда из (3) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1, v_2\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \{v_3, v_9\} \cup \ldots,$$

в котором объединение первых четырех подмножеств порождает подграф без 4-клик.

II. $v_3 \in \mathrm{Ad}(T)$. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-2}$ следует, что $\langle v_1, v_3, v_4, v_6, v_7\rangle$ не содержит 3-клик. Так как $\mathrm{Ad}(v_8) \setminus T = \{v_1, v_3, v_4, v_6, v_7\}$, то первые четыре подмножества разложения (3) порождают подграф без 4-клик.

Подслучай 2.г. $v_5 \in \mathrm{Ad}(T)$ и $v_7 \in \mathrm{Ad}(T)$. Если $v_2 \notin \mathrm{Ad}(T)$ и $[v_2, v_i] \notin E(G)$, $8 \leqq i \leqq r+5$, тогда $\mathrm{Ad}(v_2) \subseteq \mathrm{Ad}(v_i)$ и лемма 6 вытекает из леммы 3. Если $v_3 \notin \mathrm{Ad}(T)$ рассуждаем аналогично. Если $v_2 \in \mathrm{Ad}(T)$ и $v_3 \in \mathrm{Ad}(T)$, тогда из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-2}$ следует, что объединение первых трех подмножеств $(r+1)$-хроматического разложения (3) порождает подграф без 3-клик. Согласно предложению 1 разложение (3) не является 4-плотным.

Лемма 6 доказана.

Доказательство теоремы B. Допустим, что $|V(G)| \leqq r+5$. Докажем, что $G = K_{r-4} + \overline{C}_9$. Согласно лемме 2 $|V(G)| = r+5$, а согласно предложению 2

$$\chi(G) \geqq r+1. \tag{4}$$

Согласно лемме 6 $\alpha(G) = 2$. Из $\alpha(G) = 2$ и $R(3,3) = 6$ вытекает $\mathrm{cl}(G) \geqq 3$. Так как $\mathrm{cl}(G) \leqq r$, то $r \geqq 3$. Сделанные рассуждения показывают, что граф G удовлетворяет условиям леммы 5 и следовательно существуют вершины $v_1, \ldots, v_8 \in V(G)$ такие, что $[v_1, v_2] \notin E(G)$, $[v_3, v_4] \notin E(G)$, $[v_5, v_6] \notin E(G)$, $[v_7, v_8] \notin E(G)$. Остальные вершины графа G обозначим $T = \{v_9, \ldots, v_{r+5}\}$. Тогда

$$V(G) = \{v_1, v_2\} \cup \{v_3, v_4\} \cup \{v_5, v_6\} \cup \{v_7, v_8\} \cup \{v_9\} \cup \ldots \cup \{v_{r+5}\} \tag{5}$$

является $(r+1)$-хроматическим разложением вершин графа G. Заметим, что $T \neq \varnothing$, т. е. $r \geqq 4$ (иначе $r = 3$ и так как $\mathrm{cl}(G) \leqq r$, граф G не содержит 4-клик и следовательно разложение (5) не является 4-плотным). Любые две вершины множества T смежны, т. е. $\langle T\rangle = K_{r-3}$ (иначе группируя две несмежные вершины множества T, из (5) получим r-хроматическое разложение графа G, что противоречит неравенству (4)). Одна из вершин v_1, v_2 смежна всем вершинам множества T (иначе группируя v_1 и v_2 с несмежными им вершинами множества T, из (5) получим r-хроматическое разложение, что сново противоречит неравенству (4)). Без ограничения общности можно предположить, что $v_1 \in \mathrm{Ad}(T)$. Из аналогичных соображений можно предположить еще, что $v_3, v_5, v_7 \in \mathrm{Ad}(T)$. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-3}$ следует, что $\{v_1, v_3, v_5, v_7\}$ не является 4-кликой. Прежде всего докажем, что две из вершин v_2, v_4, v_6, v_8 принадлежат $\mathrm{Ad}(T)$, а другие две — нет. Допустим, что это не так. Тогда для вершин v_2, v_4, v_6, v_8 представляются следующие возможности:

Случай 1. $v_2, v_4, v_6, v_8 \notin \mathrm{Ad}(T)$. Так как $\{v_1, v_3, v_5, v_7\}$ не является 4-кликой графа G, то можно предположить, что $[v_1, v_3] \notin E(G)$. Если в T есть вершина, которая несмежна вершинам v_2 и v_4 и например $[v_2, v_9]$, $[v_4, v_9] \notin E(G)$, тогда $\langle v_1, v_2, v_3, v_4, v_9\rangle$ не содержит 3-клик и следовательно разложение (5) не является 3-плотным. Это противоречит предложению 1. Если v_2 и v_4 несмежны разным вершинам множества T и например $[v_2, v_9] \notin E(G)$, $[v_4, v_{10}] \notin E(G)$, тогда $\langle v_1, v_2, v_3, v_4, v_9, v_{10}\rangle$ не содержит

4-клик. Получили, что разложение (5) не является 4-плотным, что является противоречием.

Случай 2. Одна из вершин v_2, v_4, v_6, v_8 принадлежит $\mathrm{Ad}(T)$, а остальные три — нет. Пусть например $v_2, v_4, v_6 \notin \mathrm{Ad}(T)$ и $v_8 \in \mathrm{Ad}(T)$. Если хотя бы одна из пар $[v_1, v_3]$, $[v_1, v_5]$, $[v_3, v_5]$ не является ребром графа G, повторяя буквально рассуждения случая 1, достигаем до противоречия. Поэтому предположим, что $\{v_1, v_3, v_5\}$ является 3-кликой графа G. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-3}$ вытекает $v_7 \notin \mathrm{Ad}(v_1, v_3, v_5)$ и $v_8 \notin \mathrm{Ad}(v_1, v_3, v_5)$. Вершины v_7 и v_8 несмежны разным вершинам 3-клики $\{v_1, v_3, v_5\}$, так как иначе $\alpha(G) \geqq 3$, что противоречит условию $\alpha(G) = 2$. Без ограничения общности можно предположить, что $[v_1, v_7] \notin E(G)$ и $[v_3, v_8] \notin E(G)$. Для вершин v_2 и v_4 представляются две возможности:

I. Некоторая вершина множества T несмежна вершинам v_2 и v_4. Пусть например $[v_2, v_9]$, $[v_4, v_9] \notin E(G)$. Тогда $\langle v_1, v_2, v_3, v_4, v_7, v_8, v_9\rangle$ не содержит 4-клик, т. е. разложение (5) не является 4-плотным, что является противоречием.

II. Вершины v_2 и v_4 несмежны разным вершинам множества T. Пусть например $[v_2, v_9] \notin E(G)$ и $[v_4, v_{10}] \notin E(G)$. Тогда

$$\{v_1, v_7\} \cup \{v_3, v_8\} \cup \{v_2, v_9\} \cup \{v_4, v_{10}\} \cup \{v_5, v_6\} \cup \ldots$$

является r-хроматическим разложением вершин графа G, что противоречит неравенству (4).

Случай 3. Три из вершин v_2, v_4, v_6, v_8 принадлежат $\mathrm{Ad}(T)$, а четвертая — нет. Пусть например $v_4, v_6, v_8 \in \mathrm{Ad}(T)$ и $[v_2, v_9] \notin E(G)$. В этом случае $\mathrm{Ad}(v_2) \subseteq \mathrm{Ad}(v_9)$, что противоречит лемме 3.

Случай 4. $v_2, v_4, v_6, v_8 \in \mathrm{Ad}(T)$. Из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-3}$ вытекает, что $\langle v_1, v_2, \ldots, v_8\rangle$ не содержит 4-клик и следовательно разложение (5) не является 4-плотным.

Итак, доказано, что две из вершин v_2, v_4, v_6, v_8 принадлежат $\mathrm{Ad}(T)$, а другие две — нет. Без ограничения общности можно предположить, что $v_2, v_4 \notin \mathrm{Ad}(T)$ и $v_6, v_8 \in \mathrm{Ad}(T)$. В этой ситуации из $\mathrm{cl}(G) \leqq r$ и $\langle T\rangle = K_{r-3}$ следует, что подграф $\langle v_1, v_3, v_5, v_6, v_7, v_8\rangle$ не содержит 4-клик. Покажем, что из этого вытекает, что v_2 и v_4 несмежны одной и той же вершине множества T. Допустим, что это неверно и пусть например $[v_2, v_9] \notin E(G)$ и $[v_4, v_{10}] \notin E(G)$. Тогда из (5) получаем новое $(r+1)$-хроматическое разложение

$$\{v_1\} \cup \{v_3\} \cup \{v_5, v_6\} \cup \{v_7, v_8\} \cup \{v_2, v_9\} \cup \{v_4, v_{10}\} \cup \ldots,$$

в котором первые четыре подмножества порождают подграф без 4-клик. Итак, можно предположить, что $[v_2, v_9] \notin E(G)$ и $[v_4, v_9] \notin E(G)$. Вершина v_9 вместе с первыми тремя подмножествами разложения (5) порождает подграф, содержащий 4-клику. Из этого вытекает, что либо $\{v_1, v_3, v_5\}$, либо $\{v_1, v_3, v_6\}$ является 3-кликой графа G. Без ограничения общности можно предположить, что $\{v_1, v_3, v_5\}$ является 3-кликой графа G. Из аналогичных рассуждений можно предположить, что $\{v_1, v_3, v_7\}$ тоже является 3-кликой графа G. Из $\mathrm{cl}(G) \leqq r$ вытекает $[v_5, v_7] \notin E(G)$, а из последнего и $\alpha(G) = 2$ следует $[v_5, v_8] \in E(G)$ и $[v_6, v_7] \in E(G)$. Так как подграф $\langle v_1, v_3, v_5, v_6, v_7, v_8 \rangle$ не содержит 4-клик, то $\{v_1, v_3, v_5, v_8\}$ и $\{v_1, v_3, v_6, v_7\}$ не являются 4-кликами. Вместе с $[v_5, v_8]$, $[v_6, v_7] \in E(G)$ это дает, что $v_6, v_8 \notin \mathrm{Ad}(v_1, v_3)$. Представляются две возможности:

I. Вершины v_6 и v_8 несмежны одновременно хотя бы одной из вершин v_1, v_3. Пусть например $[v_1, v_6] \notin E(G)$ и $[v_1, v_8] \notin E(G)$. Тогда $\langle v_1, v_2, v_5, v_6, v_7, v_8, v_9 \rangle$ не содержит 4-клик. Это означает, что разложение (5) не является 4-плотным.

II. Одна из вершин v_6, v_8 несмежна вершине v_1, а другая — вершине v_3. Без ограничения общности можно предположить, что $[v_1, v_6] \notin E(G)$ и $[v_3, v_8] \notin E(G)$. В этой ситуации упорядоченное подмножество $\{v_1, v_2, v_9, v_4, v_3, v_8, v_7, v_5, v_6\}$ вершин графа G является 9-циклом его дополнения $\overline{G}$. Этим доказано, что $G \subseteq K_{r-4} + \overline{C}_9$.

Осталось доказать, что $G = K_{r-4} + \overline{C}_9$. Для этого достаточно доказать, что после удаления произвольного ребра графа $K_{r-4} + \overline{C}_9$ получается подграф, обладающий $(r+1)$-хроматическим разложением вершин, которое не является 4-плотным. Из-за существующих симметрий графа $K_{r-4} + \overline{C}_9$ достаточно рассмотреть только следующие случаи:

Случай 1. От графа $K_{r-4} + \overline{C}_9$ удаляется ребро подграфа K_{r-4}. Пусть $V(K_{r-4}) = \{w_1, \ldots, w_{r-4}\}$ и $\overline{C}_9 = \{u_1, \ldots, u_9\}$ (рис. 1). Без ограничения общности можно предположить, что удаляется ребро $[w_1, w_2]$. Тогда

$$\{u_1, u_2\} \cup \{u_3, u_4\} \cup \{u_5, u_6\} \cup \{u_7, u_8\} \cup \{u_9\} \cup \{w_1, w_2\} \cup \{w_3\} \cup \ldots \cup \{w_{r-4}\}$$

является r-хроматическим разложением полученного подграфа. Согласно

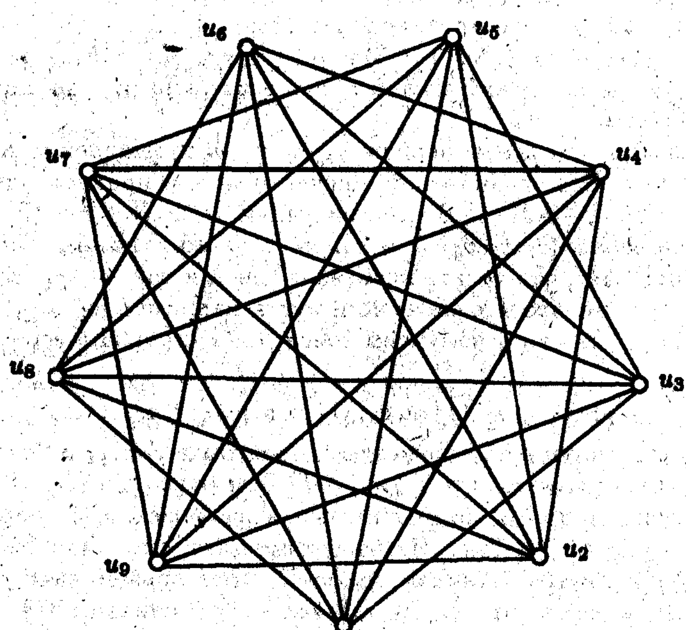

Рис. 1

предложению 2 этот подграф обладает $(r+1)$-хроматическим разложением, которое не является 4-плотным.

Случай 2. Удаляется ребро вида $[w_i, u_j]$, $1 \leqq i \leqq r-4$, $1 \leqq j \leqq 9$. Без ограничения общности можно предположить, что удаляется ребро $[w_1, u_1]$. Тогда

$$\{w_1, u_1\} \cup \{u_2, u_3\} \cup \{u_4, u_5\} \cup \{u_6, u_7\} \cup \{u_8, u_9\} \cup \{w_2\} \cup \ldots \cup \{w_{r-4}\}$$

является r-хроматическим разложением полученного подграфа. Согласно предложению 2 этот подграф обладает $(r+1)$-хроматическим разложением, которое не является 4-плотным.

Случай 3. Удаляется ребро $[u_1, u_3]$. Тогда

$$\{u_1, u_2, u_3\} \cup \{u_4, u_5\} \cup \{u_6, u_7\} \cup \{u_8, u_9\} \cup \{w_1\} \cup \ldots \cup \{w_{r-4}\}$$

является r-хроматическим разложением полученного подграфа. Согласно предложению 2 этот подграф обладает $(r+1)$-хроматическим разложением, которое не является 4-плотным.

Случай 4. От графа $K_{r-4}+\overline{C}_9$ удаляется ребро $[u_1, u_4]$. В этом случае

$$\{u_1, u_4\} \cup \{u_5, u_6\} \cup \{u_7, u_8\} \cup \{u_9\} \cup \{u_2, u_3\} \cup \{w_1\} \cup \ldots \cup \{w_{r-4}\}$$

является $(r+1)$-хроматическим разложением полученного подграфа. Первые четыре подмножества этого разложения порождают $\overline{C}_7$. Очевидно $\overline{C}_7$ не содержит 4-клик, так что это $(r+1)$-хроматическое разложение не является 4-плотным.

Случай 5. От графа $K_{r-4}+\overline{C}_9$ удаляется ребро $[u_1, u_5]$. В этом случае

$$\{u_1, u_2\} \cup \{u_3, u_4\} \cup \{u_5\} \cup \{u_6, u_7\} \cup \{u_8, u_9\} \cup \{w_1\} \cup \ldots \cup \{w_{r-4}\}$$

является $(r+1)$-хроматическим разложением вершин полученного подграфа. Первые три подмножества этого разложения порождают подграф, изоморфный $\overline{C}_5$. Так как $\overline{C}_5$ не содержит 3-клик, то полученное $(r+1)$-хроматическое разложение не является 3-плотным. Согласно предложению 1 оно не является и 4-плотным.

4. ДОКАЗАТЕЛЬСТВО ОСНОВНОЙ ТЕОРЕМЫ

Пусть $p_1, \ldots, p_s$ — натуральные числа, $s \geqq 2$, $p_i \geqq 3$, $1 \leqq i \leqq s$. Для краткости положим $R = R(p_1, \ldots, p_s)$. Будут нужны следующие леммы:

Лемма 7 ([5]). *Если* $\chi(G) < R(p_1, \ldots, p_s)$, *тогда граф* G *не является* $(p_1, \ldots, p_s)$*-графом Рамсея.*

Лемма 8. *Пусть* G *является* $(p_1, \ldots, p_s)$*-графом Рамсея с* $\mathrm{cl}(G) < R$ *и* $|V(G)| \leqq R+4$. *Тогда* $\chi(G) = R$.

Доказательство. Добавлением изолированных вершин можно добиться $|V(G)| = R+4$. Согласно лемме 7 достаточно доказать неравенство $\chi(G) \leqq R$. Пусть $V(G) = \{v_1, \ldots, v_{R+4}\}$. Рассмотрим два случая:

Случай 1. $\alpha(G) \geqq 3$. Пусть $\{v_1, v_2, v_3\}$ независимое множество вершин графа G. Если $\alpha(\langle v_4, \ldots, v_{R+4}\rangle) \geqq 3$ и $\{v_4, v_5, v_6\}$ тоже независимое множество, тогда

$$\{v_1, v_2, v_3\} \cup \{v_4, v_5, v_6\} \cup \{v_7\} \cup \ldots \cup \{v_{R+4}\}$$

является R-хроматическим разложением графа G и следовательно $\chi(G) \leqq R$. Если $\alpha(\langle v_4, \ldots, v_{R+4}\rangle) = 2$, тогда применяя лемму 4 к подграфу $\langle v_4, \ldots, v_{R+4}\rangle$, где $r = R-1$, убеждаемся, что можно предположить $[v_4, v_5]$, $[v_6, v_7] \notin E(G)$. Разложение

$$\{v_1, v_2, v_3\} \cup \{v_4, v_5\} \cup \{v_6, v_7\} \cup \{v_8\} \cup \ldots \cup \{v_{R+4}\}$$

является R-хроматическим разложением вершин графа G и следовательно $\chi(G) \leqq R$.

Случай 2. $\alpha(G) = 2$. Согласно лемме 5 ($r = R-1$) можно предположить, что $[v_1, v_2]$, $[v_3, v_4]$, $[v_5, v_6]$ и $[v_7, v_8]$ не являются ребрами графа G. Но тогда

$$\{v_1, v_2\} \cup \{v_3, v_4\} \cup \{v_5, v_6\} \cup \{v_7, v_8\} \cup \{v_9\} \cup \ldots \cup \{v_{R+4}\}$$

является R-хроматическим разложением вершин и следовательно $\chi(G) \leqq R$.

Лемма 9. *Пусть $p_1, \ldots, p_s$ — натуральные числа, $p_i \geqq 3$, $s \geqq 2$ и полный граф с $R = R(p_1, \ldots, p_s)$ вершинами обладает s-раскраской ребер со следующим свойством: для некоторого i, $1 \leqq i \leqq s$, существует единственная монохроматическая p_i-клика i-ого цвета и не существует монохроматическая p_j-клика j-ого цвета при $i \neq j$. Тогда, если граф G является $(p_1, \ldots, p_s)$-графом Рамсея и $\chi(G) = R$, то любое R-хроматическое разложение вершин графа G является p_i-плотным.*

Доказательство. Рассмотрим s-раскраску ребер K_R, в которой существует единственная монохроматическая p_i-клика i-ого цвета и не существует монохроматическая p_j-клика j-ого цвета при $i \neq j$. Пусть $V(K_R) = \{z_1, \ldots, z_R\}$ и предположим, что $\{z_1, \ldots, z_{p_i}\}$ является единственной p_i-кликой i-ого цвета. Допустим, что существует R-хроматическое разложение

$$V(G) = V_1 \cup \ldots \cup V_R$$

вершин графа G, которое не является p_i-плотным. Без ограничения общности можно предположить, что первые p_i подмножества этого R-хроматического разложения порождают подграф без p_i-клик, т. е. $\mathrm{cl}(\langle V_1 \cup \ldots \cup V_{p_i}\rangle) < p_i$. Рассмотрим отображение $\varphi : V(G) \longrightarrow V(K_R)$, определенное следующим образом: если $v \in V_i$, тогда $\varphi(v) = z_i$. При помощи отображения φ и данной s-раскраски ребер K_R строим s-раскраску ребер графа G следующим образом: ребро $[v_i, v_j] \in E(G)$ имеет такой же цвет, что и ребро $[\varphi(v_i), \varphi(v_j)]$ графа K_R. Покажем, что полученная s-раскраска ребер графа G не содержит монохроматическая p_k-клика k-ого цвета для любого k, $1 \leqq k \leqq s$. Допустим противное, т. е. для некоторого k существует монохроматическая p_k-клика k-ого цвета Q. Тогда $\varphi(Q)$ является монохроматической p_k-кликой k-ого цвета данной s-раскраски ребер K_R. Из свойств этой раскраски вытекает, что $k = i$ и $\varphi(Q) = \{z_1, \ldots, z_{p_i}\}$. Это означает, что $Q \subseteq V_1 \cup \ldots \cup V_{p_i}$, что является противоречием, так как мы предположили, что подграф $\langle V_1 \cup \ldots \cup V_{p_i}\rangle$ не содержит p_i-клик.

Доказательство основной теоремы. Предположим, что выполнено требование $p_i = 3$ подусловия а). Допустим, что существует $(p_1, \ldots, p_s)$-граф Рамсея G с $\mathrm{cl}(G) < R$ и $|V(G)| \leqq R+3$. Добавлением изолированных вершин можно добиться $|V(G)| = R+3$. Из леммы 8 вытекает $\chi(G) = R$. Согласно лемме 9 любое R-хроматическое разложение вершин графа G является 3-плотным. С другой стороны, согласно лемме 1 ($r = R-1$) либо существует вершина $v \in V(G)$, такая что $G-v = K_{R-3}+C_5$, либо $G = K_{R-4} + G_1$. Пусть $G - v = K_{R-3} + C_5$. В этом случае G обладает R-хроматическим разложением, которое не яляется 3-плотным (см. доказательство леммы 2). Пусть теперь $G = K_{R-4} + G_1$. В [2] доказано, что либо G_1 является подграфом графа $\overline{K}_2 + C_5$, либо $G_1 = \overline{C}_7$. Если $G \subseteq \overline{K}_2 + C_5$, тогда $G \subseteq K_{R-4} + \overline{K}_2 + C_5$. Граф $K_{R-4} + \overline{K}_2 + C_5$ очевидно обладает R-хроматическим разложением вершин, которое не является 3-плотным. Но тогда и G обладает таким R-хроматическим разложением. Окончательно получаем $G = K_{R-4} + \overline{C}_7$.

Предположим, что выполнено требование $p_i \geqq 4$ подусловия б). Допустим, что существует $(p_1, \ldots, p_s)$-граф Рамсея G с $\mathrm{cl}(G) < R$ и $|V(G)| \leqq R+4$. Добавлением изолированных вершин можно добиться $|V(G)| = R+4$. Согласно лемме 8 $\chi(G) = R$. Согласно лемме 9 любое R-хроматическое разложение вершин графа G является p_i-плотным и так как $p_i \geqq 4$, то оно является 4-плотным. Согласно теореме B ($r = R-1$) граф $G = K_{R-5} + \overline{C}_9$.

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ON A GENERALIZATION OF THE JACOBI OPERATOR IN THE RIEMANNIAN GEOMETRY*

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Грозьо Станилов, Веселин Видев. ОБОБЩЕНИЕ ОПЕРАТОРА ЯКОБИ В РИМАНОВОЙ ГЕОМЕТРИИ

Пусть (M,g) римановое многообразие размерности n с тензором кривизны R. Если X, Y произвольная пара касательных векторов в точке $p \in M$, мы вводим в рассмотрении оператора кривизны $\lambda_{X,Y}(u) = \frac{1}{2}(R(u,X,Y) + R(u,Y,X))$. Это линейный симметрический оператор. Эго собственные значения зависят от базисе X, Y касательного подпространство $E^2(p;X,Y)$ натянутое на векторов X, Y. Мы доказываем: 1) эйнштейновые многообразия одинственные для которых следа этого оператора не зависит от базисе подпространства $E^2(p;X,Y)$; 2) вещественные пространственные формы размерности четыре одинственные для которых спектр этого оператора не зависит от $E^2(p;X,Y)$. В обоих случаев p произвольная точка многообразии.

Grozio Stanilov, Veselin Videv. ON A GENERALIZATION OF THE JACOBI OPERATOR IN THE RIEMANNIAN GEOMETRY

Let (M,g) be a Riemannian manifold of dimension n with curvature tensor R. If X, Y is an orthonormal pair of tangent vectors at a point p of M, we define the curvature operator $\lambda_{X,Y}(u) = \frac{1}{2}(R(u,X,Y) + R(u,Y,X))$. It is a symmetric operator but its eigen values depend on the base X, Y of the tangent subspace $E^2(p;X,Y)$ spanned by X, Y. We prove: 1) the Einstein manifolds are the unique for which the trace of this operator does not depend on the base of

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$E^2(p;X,Y)$; 2) the real space forms of dimension 4 are the unique for which the spectrum of this operator does not depend on the base of $E^2(p;X,Y)$. Of course in both assertions p is an arbitrary point of M.

1. INTRODUCTION

Let (M,g) be an n-dimensional Riemannian manifold, p — a point of M, M_p — the tangent space in p. We denote by R the curvature tensor of the manifold. The well-known Jacobi operator in the Riemannian geometry is defined in the following way: if X is an unit tangent vector of M_p, then we consider the linear mapping

$$\lambda_X : M_p \to M_p$$

defined by

$$\lambda_X(u) = R(u,X,X).$$

From the properties of R it follows that λ_X is a linear symmetrical operator. Namely, it is the Jacobi operator in respect to X. The geometry of this operator is investigated very actively in the last years, see, for example, [1–3].

In this paper we state at first the problem about a generalization of this operator.

Let X, Y is an arbitrary pair of unit tangent vectors. Then we define the mapping

$$\lambda_{X,Y} : M_p \to M_p$$

by

$$\lambda_{X,Y}(u) = \frac{1}{2}[R(u,X,Y) + R(u,Y,X)].$$

Evidently, it is a generalization of the Jacobi operator because of

$$\lambda_{X,X} = \lambda_X.$$

Further we propose that X, Y is an arbitrary orthonormal pair of tangent vectors. If x, y is another pair of tangent vectors in the 2-dimensional tangent subspace $E^2(p;X,Y)$ of M_p spanned by X, Y, we have the relations

$$\begin{aligned} x &= \cos\varphi.X - \varepsilon.\sin\varphi.Y, \\ y &= \sin\varphi.X + \varepsilon.\cos\varphi.Y, \quad \varepsilon = \pm 1. \end{aligned} \tag{1}$$

By these formulas all orthogonal transformations in $E^2(p;X,Y)$ are represented. Now we establish the relation

$$\lambda_{x,y}(u) = \cos 2\varphi.\lambda_{X,Y}(u) + \varepsilon.\frac{\sin 2\varphi}{2}[R(u,X,X) - R(u,Y,Y)].$$

Hence the operator $\lambda_{X,Y}$ is not invariant under the orthogonal transformations in $E^2(p;X,Y)$.

It is evident that $\lambda_{X,Y} = \lambda_{Y,X}$ and we can take $\varepsilon = 1$.

Let now $e_1, e_2, \ldots, e_n$ is an arbitrary orthonormal base of the tangent space M_p. Then we have ($i = 1, 2, \ldots, n$)

$$\lambda_{x,y}(e_i) = \cos 2\varphi . \lambda_{X,Y}(e_i) + \frac{\sin 2\varphi}{2}[R(e_i, X, X) - R(e_i, Y, Y)].$$

Hence we obtain

$$R(e_i, x, y, e_i) = \cos 2\varphi . R(e_i, X, Y, e_i) + \frac{\sin 2\varphi}{2}[R(e_i, X, X, e_i) - R(e_i, Y, Y, e_i)].$$

From this equalities we get

$$S(x, y) = \cos 2\varphi . S(X, Y) + \frac{\sin 2\varphi}{2}[S(X, X) - S(Y, Y)], \tag{2}$$

where S is the classical Ricci tensor of the manifold M.

2. A CHARACTERIZATION OF THE EINSTEIN MANIFOLDS BY THE GENERALIZED JACOBI OPERATOR

In this part we prove the following

Theorem 1. *Let (M, g) be an n-dimensional Riemannian manifold and $n \geqq 2$. Then the following assertions are equivalent:*

1) *(M, g) is an Einstein manifold;*

2) *The trace of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base X, Y of the tangent subspace $E^2(p; X, Y)$ (and it is equal to zero).*

Proof. Let (M, g) be an n-dimensional Einstein manifold. Then at every point $p \in M$ and for any tangent vectors x, y we have

$$S(x, y) = c'.g(x, y), \quad c' = \text{const},$$

whence it follows that

$$S(X, Y) = 0 \tag{3}$$

for any arbitrary pair of tangent vectors X, Y. Also from the definition of the operator $\lambda_{X,Y}$ we have

$$c = R(u, X, Y, u),$$

where u is an eigen vector of $\lambda_{X,Y}$ and c is the corresponding eigen value, i. e.

$$\lambda_{X,Y}(u) = \frac{1}{2}[R(u, X, Y) + R(u, Y, X)] = c.u.$$

Hence, if $u_1, u_2, \ldots, u_n$ are the eigen vectors of $\lambda_{X,Y}$ with corresponding eigen values $c_1, c_2, \ldots, c_n$, then

$$\operatorname{tr} \lambda_{X,Y} = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} R(u_i, X, Y, u_i) = S(X, Y).$$

According to (3) we obtain

$$\operatorname{tr} \lambda_{X,Y} = 0.$$

Conversely, let at any point $p \in M$ and for any orthonormal pair of tangent vectors X, Y of M_p the trace of the operator $\lambda_{X,Y}$ not depend on the base of the plane $E^2(p; X, Y)$. If the tangent vectors x, y are related to X, Y by (1), then

$$S(x, y) = S(X, Y),$$

whence from (2) we have

$$S(X, Y) = \cos 2\varphi . S(X, Y) + \frac{\sin 2\varphi}{2}[S(X, X) - S(Y, Y)],$$

or

$$S(X, Y) = \operatorname{cotg}\varphi[S(X, X) - S(Y, Y)].$$

From the last equality by $\varphi = -\frac{\pi}{4}$ and $\varphi = \frac{\pi}{4}$ we obtain correspondingly

$$S(X, Y) = -S(X, X) + S(Y, Y), \quad S(X, Y) = S(X, X) - S(Y, Y),$$

whence it follows that (3) is satisfied for every orthonormal pair of tangent vectors X, Y of M_p and at any point $p \in M$. Now we can apply (3) for the orthonormal pair

$$\frac{u}{|u|}, \quad \frac{v.g(u, u) - u.g(u, v)}{|u|.\sqrt{g(u, u).g(v, v) - g^2(u, v)}},$$

where u, v are arbitrary tangent vectors of M_p. We have

$$S\left(\frac{u}{|u|}, \frac{v.g(u, u) - u.g(u, v)}{|u|.\sqrt{g(u, u).g(v, v) - g^2(u, v)}}\right) = 0,$$

whence it follows that

$$\frac{S(u, v)}{g(u, v)} = \frac{S(u, u)}{g(u, u)}.$$

From the last equality, if we change u by v and v by u, we obtain

$$\frac{S(v, u)}{g(v, u)} = \frac{S(v, v)}{g(v, v)}.$$

Then the last two equalities lead to

$$\frac{S(u, u)}{g(u, u)} = \frac{S(v, v)}{g(v, v)},$$

which is satisfied for arbitrary tangent vectors u, v of M_p and at every point $p \in M$. It means that (M, g) is an Einstein manifold.

From the theorem we get the following

Corollary. *Let (M, g) be a 3-dimensional Riemannian manifold. Then (M, g) is a real space form iff the trace of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base X, Y of the plane $E^2(p; X, Y)$.*

Proof. Let (m, g) be a 3-dimensional real space form, i. e. Riemannian manifold of constant sectional curvature μ. Then

$$R(x, y, z) = \mu.[g(y, z).x - g(x, z).y].$$

Hence

$$\lambda_{X,Y}(u) = -\frac{1}{2}\mu[g(u,X).Y + g(u,Y).X] \tag{4}$$

for any orthonormal pair X, Y of tangent vectors of M_p and at any point $p \in M$. Then for the spectrum $\Omega_{X,Y}$ of $\lambda_{X,Y}$ we have

$$\Omega_{X,Y} = \left\{-\frac{\mu}{2}, \frac{\mu}{2}, 0\right\},$$

whence it follows that $\operatorname{tr}\lambda_{X,Y} = 0$.

Conversely, if $\operatorname{tr}\lambda_{X,Y}$ does not depend on the plane $E^2_\bullet(p; X, Y)$, then from Theorem 1 follows that (M,g) is an Einstein manifold and, since $\dim M = 3$, (M,g) is a real space form [6].

3. A CHARACTERIZATION OF THE REAL SPACE FORMS BY THE SPECTRUM OF THE GENERALIZED JACOBI OPERATOR

Now we investigate the Riemannian manifolds with the following properties: at every point $p \in M$ and for any orthonormal pair X, Y of tangent vectors of M_p the spectrum of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p; X, Y)$.

At first we remark that from this property it follows that the trace of $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p; X, Y)$ and according to Theorem 1 we get that (M,g) is an Einstein manifold. Also from this property follows that the spectrum of the operator $\lambda_{X,Y}$ has the form

$$\Omega_{X,Y} = \{c_1, -c_1, c_2, -c_2, \ldots, c_{2k}, -c_{2k}, 0, \ldots, 0\}, \quad 2k \leq n. \tag{5}$$

Indeed, let u be an arbitrary eigen vector of $\lambda_{X,Y}$ with a corresponding eigen value c. It follows that u is also an eigen vector of the operator $-\lambda_{X,Y}$ with a corresponding eigen value $-c$. But from the definition of $\lambda_{X,Y}$ we have

$$\lambda_{-X,Y} = -\lambda_{X,Y},$$

whence we obtain that $-c$ is also an eigen value of $\lambda_{-X,Y}$. Since the spectra of $\lambda_{X,Y}$ and $\lambda_{-X,Y}$ coincide, then $-c$ is also an eigen value of the operator $\lambda_{X,Y}$. That means that c and $-c$ are eigen values of $\lambda_{X,Y}$, and $\Omega_{X,Y}$ has the form (5).

Let now (M,g) be an n-dimensional Riemannian manifold of a constant sectional curvature μ. Then from (4) we directly obtain that

$$\Omega_{X,Y} = \left\{-\frac{\mu}{2}, \frac{\mu}{2}, 0, \ldots, 0\right\}, \tag{6}$$

which is of the form (5). Our conjecture is that the converse is also true. In this paper we can prove it only in the four-dimensional case.

From now on let (M,g) be a 4-dimensional manifold with the property given at the beginning of this section: p is an arbitrary point of M, X, Y is an arbitrary orthonormal pair of tangent vectors of M_p. Let x, y be another orthonormal base of the plane $E^2(p;X,Y)$. Further, if e_1, e_2, e_3, e_4 is an arbitrary orthonormal base of the tangent space M_p, then every eigen vector v of the operator $\lambda_{x,y}$ can be represented in the form

$$v = v^1.e_1 + v^2.e_2 + v^3.e_3 + v^4.e_4.$$

From the definition of $\lambda_{x,y}$ we get the relation

$$R(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4, \cos\varphi.X - \sin\varphi.Y)$$
$$+ R(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4, \sin\varphi.X + \cos\varphi.Y) = c(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4),$$

whence after scalar multiplying by e_1, e_2, e_3, e_4 we obtain

(7) $$a_{i1}v^1 + a_{i2}v^2 + a_{i3}v^3 + a_{i4}v^4 = 0,$$

where

$$a_{ii} = \sin 2\varphi.[R(e_i,X,X,e_i) - R(e_i,Y,Y,e_i)] + 2.\cos 2\varphi.R(e_i,X,Y,e_i) - 2c,$$
$$a_{kj} = \sin 2\varphi.[R(e_k,X,X,e_j) - R(e_k,Y,Y,e_j)]$$
$$+ \cos 2\varphi.[R(e_k,X,Y,e_j) + R(e_k,Y,X,e_j)], \quad k \neq j, \quad i,j,k = 1,2,3,4.$$

Since v is a non-zero vector, then

(8) $$\det(a_{ij}) = 0, \quad i,j = 1,2,3,4.$$

From here, by $X = e_1$, $Y = e_2$, $\varphi = 0$, we obtain the following equation:

$$\begin{vmatrix} -2c & -K_{12} & R_{3121} & R_{4121} \\ -K_{12} & -2c & R_{2123} & R_{2124} \\ R_{3121} & R_{2123} & 2R_{3123} - 2c & R_{4123} + R_{4213} \\ R_{4121} & R_{2124} & R_{4123} + R_{4213} & 2R_{4124} - 2c \end{vmatrix} = 0,$$

or

$$16c^4 - 4I_1c^2 - 2I_2c + I_3 = 0,$$

where

$$I_1 = 4R^2_{4124} + (R_{4123} + R_{3124})^2 + R^2_{2123} + R^2_{2124} + K^2_{12} + R^2_{3121} + R^2_{4121},$$
$$I_2 = 2R_{2123}(R_{4123} + R_{3124})R_{2124} - 2R^2_{2124}R_{3123} - 2R_{2123}R_{4124} - 2K_{12}R_{4121}R_{2124}$$
$$- 2K_{12}R_{3121}R_{2123} - R_{3121}(R_{4123} + R_{3124})R_{4121} + 2R_{3121}R_{4124} - 2R_{4121}R_{3123},$$
$$I_3 = 2K^2_{12}R_{4124} + K_{12}R_{4121}R_{2123}(R_{4123} + R_{3124}) + K_{12}R_{3121}(R_{4123} + R_{3124})R_{2124}$$
$$- 4K_{12}R_{4121}R_{2124}R_{3123} + K^2_{12}(R_{4123} + R_{3124})^2 - 2K_{12}R_{3121}R_{2123}R_{4124}$$
$$+ R^2_{3121}R^2_{2124} - 2R_{4121}R_{2123}R_{2124}R_{3121} + K_{12}R_{4121}R_{2123}(R_{4123} + R_{3124})$$
$$+ R^2_{4121}R^2_{2123}.$$

Here

$$R_{ijkt} = R(e_i,e_j,e_k,e_t), \quad i,j,k,t = 1,2,3,4,$$

are the components of the curvature tensor R in respect to the base e_1, e_2, e_3, e_4. Also from (7) and (8) by $X = e_1$, $Y = e_2$ and $\varphi = \frac{\pi}{4}$ we have

$$\begin{vmatrix} -K_{12} - 2c & 0 & -R_{3221} & -R_{4221} \\ 0 & K_{12} - 2c & R_{3112} & R_{4112} \\ -R_{3221} & R_{3112} & K_{13} - K_{23} - 2c & 2R_{4113} \\ -R_{4221} & R_{4112} & 2R_{4113} & K_{14} - K_{24} - 2c \end{vmatrix} = 0,$$

or

$$16c^4 - 4J_1c^2 - 2J_2c + J_3 = 0,$$

where

$$J_1 = K_{12}^2 + (K_{13} - K_{23})^2 + R_{4112}^2 + 4R_{4113}^2 + R_{3112}^2 + R_{3221}^2 + R_{4221}^2,$$
$$J_2 = K_{12}(R_{3221}^2 + R_{4221}^2 - R_{3112}^2 - R_{4112}^2) + (K_{13} - K_{23})(R_{4112}^2 + R_{4221}^2 - R_{3112}^2 - R_{3221}^2) - 4R_{3114}(R_{4112}R_{3112} + R_{4221}R_{3221}),$$
$$J_3 = -K_{12}\left[4R_{3112}R_{4112}R_{4113} - R_{4112}^2(K_{13} - K_{23}) - 4R_{4113}K_{12} - R_{3112}^2(K_{14} - K_{24})\right] + K_{12}^2(K_{13} - K_{23})^2 + 4K_{12}R_{3221}R_{4221}R_{4113} - 2R_{3221}R_{4112}R_{4221}R_{3112} + R_{4112}^2R_{3221}^2 + K_{12}(K_{14} - K_{24})^2R_{3221}^2 - R_{4221}^2K_{12}(K_{13} - K_{23}) + R_{3112}^2R_{4221}^2.$$

Since the spectrum of $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p;X,Y)$, we have

$$I_1 = J_1, \quad I_2 = J_2 = 0, \quad I_3 = J_3.$$

Further we use the first of these equalities. Then

$$(9) \qquad 4R_{4124}^2 + (R_{4123} + R_{3124})^2 + R_{2123}^2 + R_{2124}^2 + R_{3121}^2 + R_{4121}^2 = (K_{13} - K_{23})^2 + R_{4112}^2 + 4R_{4113}^2 + R_{3112}^2 + R_{3221}^2 + R_{4221}^2.$$

Since (M,g) is an Einstein manifold, then:

a) the sectional curvature of every 2-dimensional subspace of the tangent space M_p is equal to the sectional curvature of its orthogonal complements [4];

b) $R_{issj} + R_{ittj} = 0$ (cf. [3]), $i \neq j$, $i \neq s$, $i \neq t$, $j \neq s$, $j \neq t$, $s \neq t$; $i,j,s,t = 1,2,3,4$.

Now from (8) we obtain

$$(10) \qquad (K_{13} - K_{23})^2 + 4R_{4113}^2 = 4R_{1442}^2 + (R_{4123} + R_{4213})^2,$$

which is hold for any orthonormal base e_4, e_3, e_2, e_1 of M_p, i. e.

$$(K_{42} - K_{32})^2 + 4R_{1442}^2 = 4R_{4113}^2 + (R_{1432} + R_{1342})^2,$$

or

$$(11) \qquad (K_{13} - K_{23})^2 + 4R_{1442}^2 = 4R_{4113}^2 + (R_{4123} + R_{4213})^2.$$

Now from (10) we get

$$R_{1442}^2 = R_{4113}^2,$$

and whence from (9) it follows that

$$(12) \qquad R_{4123} + R_{4213} = \varepsilon(K_{13} - K_{23}), \quad \varepsilon = \pm 1.$$

From (12) and the first Bianchi identity we obtain

$$(13) \qquad 2R_{4123} + R_{4312} = \varepsilon(K_{13} - K_{23}).$$

If we change e_3 by e_4, we get

$$2R_{3124} + R_{3412} = \varepsilon(K_{14} - K_{24}).$$

Hence

$$(14) \qquad 2R_{4213} - R_{4312} = \varepsilon(K_{23} - K_{13}).$$

Now from (13) and (14) it follows that

$$R_{4123} + R_{4213} = 0,$$

whence

$$K_{13} - K_{23} = 0.$$

This equality can be written in the form

$$K(X \wedge Z) = K(Y \wedge Z).$$

It holds for every orthonormal tripple of tangent vectors X, Y, Z in M_p and at any point $p \in M$. We conclude that (M, g) is a real space form [6].

Thus we have proved the following

Theorem 2. *Let (M, g) be a 4-dimensional Riemannian manifold. Then the following assertions are equivalent:*

1) *(M, g) is a real space form;*

2) *The spectrum of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base X, Y of the plane $E^2(p; X, Y)$ for every plane E^2 and at any point $p \in M$.*

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A GENERALIZATION OF THE LEVI–CIVITAE CONNECTION ON RIEMANNIAN MANIFOLDS*

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Ирина Петрова. ОДНО ОБОБЩЕНИЕ СВЯЗНОСТИ ЛЕВИ–ЧИВИТА НА РИМАНОВЫХ МНОГООБРАЗИЯХ

Пусть M римановое многообразие с метричным тензором g. Линейная связность $\widetilde{\nabla}$ на M, обладающая свойствами

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = 0, \qquad \underset{X,Y,Z}{\sigma} \left(\widetilde{\nabla}_X g\right)(Y,Z) = 0,$$

называется Обобщенной связностью Леви–Чивита для g. В статье даны примеры таких связностей. Одна из них это (0)-связность Картана и Схоутена на Ли-группе для каждой левоинвариантной метрики. Описана связь этих связностей с метричными связностями. Доказано, что на каждое римановое многообразие существует Обобщенная Леви–Чивита связность, которая не является Леви–Чивита связностью.

Irina Petrova. A GENERALIZATION OF THE LEVI–CIVITAE CONNECTION ON RIEMANNIAN MANIFOLDS

Let M be a Riemannian manifold with a metric tensor g. A linear connection $\widetilde{\nabla}$ on M with the properties

$$\widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y] = 0, \qquad \underset{X,Y,Z}{\sigma} \left(\widetilde{\nabla}_X g\right)(Y,Z) = 0$$

is called a Generalised Levi–Civitae connection for g. In the paper are given examples of such connections. One of them is the (0)-connection of Cartan and Schouten on a Lie group for any left invariant metric. A relation is described between these connections and the metric connections. It is proved that there is a Generalized Levi–Civitae connection that is not a Levi–Civitae connection on Riemannian manifolds.

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1. GENERALIZED LEVI–CIVITAE CONNECTIONS. EXAMPLES

Let M be a Riemannian manifold with a metric tensor g. That means M is considered with the tensor field g of type $(0,2)$, which has the properties:

1) g is symmetric, i. e. $g(X,Y) = g(Y,X)$ for $X, Y \in \mathfrak{X}M$;

2) for a point p of M and for $X \in \mathfrak{X}M$, such that $X_p \neq o$, $g(X,X)(p) > 0$ holds.

The metric tensor g is also called a (Riemannian) metric on M.

If M is a Riemannian manifold with a metric tensor g, there is an unique linear connection ∇:

1) ∇ is symmetric, i. e. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$ for $X, Y \in \mathfrak{X}M$;

2) $\nabla g = 0$.

∇ is called a Levi–Civitae connection for g (see [1]).

N o t a t i o n s. Let M be a Riemannian manifold with a metric tensor g, ∇ — the Levi–Civitae connection for g. If $\tilde{\nabla}$ is a linear connection on M, the symbols $\sigma\tilde{\nabla}g$ and $\sigma g\tilde{\nabla}$ are used for a denoting

$$\underset{X,Y,Z}{\sigma} \left(\tilde{\nabla}_X g\right)(Y,Z) \quad \text{and} \quad \underset{X,Y,Z}{\sigma} g\left(\tilde{\nabla}_X Y, Z\right).$$

If t is a tensor field of type $(1,2)$, we denote by the symbols gt and σgt the tensors

$$g(t(X,Y),Z) \quad \text{and} \quad \underset{X,Y,Z}{\sigma} g(t(X,Y),Z)$$

correspondingly.

Definition. Let M be a Riemannian manifold with a metric tensor g. A linear connection $\tilde{\nabla}$ with the properties:

1) $\tilde{\nabla}$ is symmetric;

2) $\sigma\tilde{\nabla}g = 0$,

is called a Generalized Levi–Civitae connection for g.

It is well-known that all linear connections on M are given by $\nabla + T$, where T is a tensor field of type $(1,2)$.

Proposition. *The connection $\tilde{\nabla} = \nabla + \tilde{T}$ is a generalized Levi–Civitae connection iff $\tilde{T}$ is symmetric, $\sigma g\tilde{T} = 0$. We have $\tilde{\nabla} \equiv \nabla$ iff $\tilde{T} = 0$.*

It can be proved by trivial calculations.

Remark. Let s be a symmetric tensor of type $(1,2)$. It is easy to check that the conditions:

1) $\sigma gs = 0$;

2) $g(s(X,X),X) = 0$, $X \in \mathfrak{X}M$,

are equivalent.

We propose two examples of Generalized Levi–Civitae connections.

Example 1. Let M be a Riemannian manifold with a metric tensor g, for which exists $\xi \in \mathfrak{X}M$, not identically equal to zero, globally defined on M. Then

$$\tilde{\nabla} = \nabla + \tilde{T}, \quad \text{where } \tilde{T}(X,Y) = 2g(X,Y)\xi - g(X,\xi)Y - g(Y,\xi)X$$

is a Generalized Levi–Civitae connection, which is not Levi–Civitae connection.

It is so, because $\widetilde{T}$ is a symmetric tensor, $\widetilde{T} \not\equiv 0$, $g(\widetilde{T}(X,X),X) = 0$. By the above remark and the proposition it follows that $\widetilde{\nabla}$ is a Generalized Levi–Civitae connection, $\widetilde{\nabla} \not\equiv \nabla$, and $\widetilde{\nabla}$ has the following properties:

1) if $X, Y \in \mathfrak{X}M$ are such that X, Y, ξ form an orthogonal triple, then $\widetilde{\nabla}_X Y = \nabla_X Y$;

2) on the FM-submodule of $\mathfrak{X}M$, which is orthogonal to ξ, we have $\widetilde{\nabla} g = \nabla g = 0$.

Example 2. Let G be a Lie group and $\mathcal{G}$ be the algebra of the left-invariant vector fields on G. From the theory of the Lie groups is known that G is parallelizable and it has bases of left-invariant vector fields. Hence, we always can construct a Riemannian metric tensor on G. Let we recall that a metric tensor g is left-invariant iff it is invariant for the left translations. It is well-known that a metric g on G is left-invariant iff for an arbitrary chosen base $X_1, \ldots, X_n$, $X_i \in \mathcal{G}$,

$$g(X_i, X_j) = \text{const}.$$

Let $X_1, \ldots, X_n$ be a base on G, $X_i \in \mathcal{G}$. We consider the linear connection

$$\overline{\nabla}_X Y = X(\varphi_i) X_i, \quad Y = \varphi_i X_i.$$

This connection is independent on the choice of the base of left-invariant fields and it is called a left-invariant connection. It is proved in [2] that $\overline{\nabla}$ is complete, i. e. its geodesics can be continued, being defined for all real values of their parameters.

Let $\overline{T}$ be the tensor of torsion for $\overline{\nabla}$. We consider the symmetrization of $\overline{\nabla}$:

$$\widetilde{\nabla} = \overline{\nabla} - \frac{1}{2}\overline{T}.$$

It is also complete, because the geodesics for the both connections coincide. $\widetilde{\nabla}$ is introduced by Cartan and Schouten, and it is called (0)-connection (see [1] or [6]).

Proposition. *Let G be a Lie group:*

1. *For each left-invariant metric g the equation $\sigma\widetilde{\nabla} g = 0$ holds (i. e. $\widetilde{\nabla}$ is a Generalized Levi–Civitae connection for any left-invariant metric).*

2. *$\widetilde{\nabla}$ is the unique linear symmetric connection on G with the property* 1.

3. *For a left-invariant metric g the connection $\widetilde{\nabla}$ is the Levi–Civitae connection iff g is right-invariant.*

Proof. 1. Let g be a left-invariant metric. Since $\sigma\widetilde{\nabla} g$ is a tensor field, to check $\sigma\widetilde{\nabla} g = 0$ it is enough to prove it for the elements of a base on G. G admits a base of left-invariant vector fields, so it is enough to prove $\sigma\widetilde{\nabla} g = 0$ for left-invariant vector fields.

Let $X, Y, Z \in \mathcal{G}$. We have

$$\overline{\nabla}_X Y = X(\text{const}).Y = 0,$$

$$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = -[X,Y],$$

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\overline{T}(X,Y) = \frac{1}{2}[X,Y],$$

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = X \circ g(Y,Z) - g\left(\widetilde{\nabla}_X Y, Z\right) - g\left(Y, \widetilde{\nabla}_X Z\right).$$

If $X, Y \in \mathcal{G}$, then $g(X,Y) = \text{const}$. Hence

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = \frac{1}{2}(g([Z,X],Y) - g([X,Y],Z)).$$

Now it can be checked easy that

$$\underset{X,Y,Z}{\sigma}\left(\widetilde{\nabla}_X g\right)(Y,Z) = 0,$$

which proves 1.

2. Let $\hat{\nabla}$ be a linear symmetric connection on G such that for each left-invariant metric g

$$\sigma\hat{\nabla}g = 0.$$

Let $X_1, \ldots, X_n$, $X_i \in \mathcal{G}$, be a base on G. Then

$$\hat{\nabla}_{X_i}^{X_j} = \Gamma_{ij}^k X_k, \quad \Gamma_{ij}^k \in FM.$$

We shall prove that Γ_{ij}^k are defined uniquely and hence $\widetilde{\nabla} \equiv \hat{\nabla}$. Let we consider only the left-invariant metrics, for which the base $X_1, \ldots, X_n$ is orthogonal, i. e.

$$g(X_i, X_j) = 0, \quad i \neq j.$$

Let Ω is the set, formed by them. From the identity $\sigma\hat{\nabla}g = 0$ we obtain after some transformations:

$$\underset{i,j,k}{\sigma}\Gamma_{ij}^k g(X_k, X_k) = \underset{i,j,k}{\sigma} g\left(\hat{\nabla}_{X_i}^{X_j}, X_k\right),$$

for each $g \in \Omega$ and each i, j, k from 1 to n. We set

$$F_{g(i,j,k)} = \underset{i,j,k}{\sigma} g\left(\hat{\nabla}_{X_i}^{X_j}, X_k\right), \quad g_{kk} = g(X_k, X_k).$$

Then

$$\Gamma_{ij}^k g_{kk} + \Gamma_{jk}^i g_{ii} + \Gamma_{ki}^j g_{jj} = F_{g(i,j,k)}.$$

2.1. Let $i = j = k$. We have $3\Gamma_{ii}^i g_{ii} = F_{g(i,i,i)}$. That means Γ_{ii}^i are uniquely defined.

2.2. Let $i \neq j \neq k \neq i$. We consider three left-invariant Ω-metrics g^1, g^2, g^3, for which

$$\begin{pmatrix} g_{kk}^1 & g_{ii}^1 & g_{jj}^1 \\ g_{kk}^2 & g_{ii}^2 & g_{jj}^2 \\ g_{kk}^3 & g_{ii}^3 & g_{jj}^3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then we have

$$1.\Gamma_{ij}^k + 1.\Gamma_{jk}^i + 1.\Gamma_{ki}^j = F_{g^1(i,j,k)},$$
$$1.\Gamma_{ij}^k + 2.\Gamma_{jk}^i + 1.\Gamma_{ki}^j = F_{g^2(i,j,k)},$$
$$1.\Gamma_{ij}^k + 1.\Gamma_{jk}^i + 2.\Gamma_{ki}^j = F_{g^3(i,j,k)}.$$

These relations can be considered like a system of linear equations about Γ_{ij}^k, Γ_{jk}^i, Γ_{ki}^j with determinant different from zero. Hence Γ_{ij}^k, Γ_{jk}^i, Γ_{ki}^j are uniquely defined.

2.3. Let $i = j \neq k$. We have

$$\Gamma^k_{ii} g_{kk} + (\Gamma^i_{ik} + \Gamma^i_{ki}) g_{ii} = F_{g(i,i,k)}, \quad g \in \Omega.$$

Analogously to 2.2 we can prove that Γ^k_{ii}, $\Gamma^i_{ik} + \Gamma^i_{ki}$ are uniquely defined. But $\hat{\nabla}$ is symmetrical. Hence, for each $g \in \Omega$:

$$g\left(\hat{\nabla}^{X_k}_{X_i} - \hat{\nabla}^{X_i}_{X_k}, X_i\right) = g([X_i, X_k], X_i),$$

$$(\Gamma^i_{ik} - \Gamma^i_{ki}) g_{ii} = g([X_i, X_k], X_i).$$

That means $\Gamma^i_{ik} - \Gamma^i_{ki}$ are uniquely defined too. Hence Γ^i_{ik}, Γ^i_{ki} are uniquely defined; that proves 2.

3. Let g be a left-invariant metric, ∇ — the Levi–Civitae connection for g. Since G has bases of left-invariant vector fields, the condition $\nabla \equiv \tilde{\nabla}$ is equivalent to

$$g(\nabla_X Y, Z) = g\left(\tilde{\nabla}_X Y, Z\right), \quad X, Y, Z \in \mathcal{G}.$$

But from the proof of 1 we know that for each $X, Y, Z \in \mathcal{G}$

$$2g\left(\tilde{\nabla}_X Y, Z\right) = g([X, Y], Z),$$

$$2g(\nabla_X Y, Z) = g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X).$$

Hence $g(\nabla_X Y, Z) = g\left(\tilde{\nabla}_X Y, Z\right)$ is equivalent to

$$-g([X, Z], Y) - g([Y, Z], X) = 0, \quad g([Z, X], Y) + g(X, [Z, Y]) = 0,$$

i. e. for each $Z \in \mathcal{G}$ the operator

$$\mathrm{ad}_Z : \mathcal{G} \to \mathcal{G}$$
$$X \to [Z, X]$$

is antisymmetric.

So we have got that if g is left-invariant then $\nabla \equiv \tilde{\nabla}$ iff ad_Z is antisymmetric for each $Z \in \mathcal{G}$. But in [3] is proved if g is left-invariant then ad_Z is antisymmetric for each $Z \in \mathcal{G}$ iff g is right-invariant. That proves 3.

From the proposition we receive the following

Corollary. *Let G be a Lie group that admits a bi-invariant metric:*

1. *All bi-invariant metrics induce the same Levi–Civitae connection.* (*Let we call this common connection bi-invariant connection.*)

2. *The symmetrization of the left-invariant connection, the symmetrization of the right-invariant connection and the bi-invariant connection coincide.*

2. A RELATION BETWEEN THE METRIC CONNECTIONS AND THE GENERALIZED LEVI–CIVITAE CONNECTIONS. AN EXISTENCE THEOREM

Let M be a Riemannian manifold with metric tensor g. We recall the following

Definition. A linear connection D on M is metric, if $Dg = 0$.

N o t a t i o n s. M_1 is the set of the metric connections on M, M_2 — the set of the Generalized Levi–Civitae connections on M, (D,τ) — a metric connection D with tensor of torsion τ, $M_1' = \{(D,\tau) \mid \sigma g\tau = 0\}$.

Proposition 1. *Let $D \in M_1$, τ is the torsion for D. Then $\tilde{\nabla} = D - \frac{1}{2}\tau \in M_2$, $\tilde{\nabla}$ and D have the same geodesics.*

The proposition can be proved by trivial calculations.

Remark 1. Let G be a Lie group, $\overline{\nabla}$ is the left-invariant connection on G. It is easy to see that $\overline{\nabla}$ is a metric connection for any left-invariant metric g, i. e. Proposition 1 is a generalization of Example 2 from Section 1.

Remark 2. Let $D \in M_1$, τ is the torsion of D. Let T be the tensor, $D = \nabla + T$, aT is the antisymmetric part of T, $\tilde{T}$ — the symmetric part of T. Then

$$aT = \frac{1}{2}\tau, \quad \tilde{\nabla} = D - \frac{1}{2}\tau = \nabla + \tilde{T}.$$

Proposition 2. *Let $D \in M_1$, τ is the torsion of D, $\tilde{\nabla} = D - \frac{1}{2}\tau$. Then $\nabla \equiv \tilde{\nabla}$ iff $g\tau$ is antisymmetric.*

It is easy to check the assertion of the proposition.

Remark 3. Let T be a tensor of type $(1,2)$, $D = \nabla + T$. Then D is metric iff $g(T(X,Y),Z) + g(T(X,Z),Y) = 0$.

Proposition 3. *The correspondence*

$$M_1' \to M_2,$$
$$(D,\tau) \to \tilde{\nabla} = D - \frac{1}{2}\tau$$

is bijective. By it ∇ (considered as a generalized Levi–Civitae connection) is generated by itself (considered as a metric connection).

Proof. Let $\tilde{\nabla} = \nabla + \tilde{T}$ be a Generalized Levi–Civitae connection. We search for the metric connections D with torsion τ, such that

$$(s) \qquad \left| \begin{array}{l} D - \frac{1}{2}\tau = \tilde{\nabla}, \\ \sigma g\tau = 0. \end{array} \right.$$

(We show below there is an unique connection D with the property (s).) It is enough to find all tensors τ of type $(1,2)$:

$$(s') \qquad \left| \begin{array}{l} \tau \text{ is antisymmetric}, \\ g(T(X,Y),Z) + g(T(X,Z),Y) = 0, \text{ where } T = \frac{1}{2}\tau + \tilde{T}, \\ \sigma g\tau = 0. \end{array} \right.$$

Then $D = \nabla + T$, $T = \frac{1}{2}\tau + \tilde{T}$, will give all the metric connections with the property (s).

It is easy to see the conditions (s') are equivalent to

$$(s'') \qquad \left| \begin{array}{l} \tau(X,Y) = -\tau(Y,X), \\ g(\tau(X,Y),Z) + g(\tau(X,Z),Y) = 2g(\widetilde{T}(Y,Z),X), \\ \sigma g \tau = 0, \end{array} \right.$$

and (s'') are equivalent to

$$(s''') \qquad g(\tau(X,Y),Z) = \frac{2}{3}(g(\widetilde{T}(Y,Z),X) - g(\widetilde{T}(Z,X),Y).$$

Hence there is an unique tensor τ, for which holds (s).

That proves the proposition.

Remark 4. If τ is an antisymmetric tensor of type $(1,2)$, there is a (unique) metric connection D with tensor of torsion τ (see [5]).

From this remark and Proposition 3 we receive

Corollary. *Let M be a Riemannian manifold with metric tensor g. Then the following conditions are equivalent:*

1. *On M there is a symmetric tensor $\widetilde{T} \neq 0$ of type $(1,2)$, for which holds $\sigma g \widetilde{T} = 0$.*

2. *On M there is an antisymmetric tensor $\tau \neq 0$ of type $(1,2)$, for which holds $\sigma g \tau = 0$.*

Theorem. *Let M be a Riemannian manifold with metric tensor g. Then:*

1. *On M there is a Generalized Levi–Civitae connection that is not a Levi–Civitae one.*

2. *On M there is a metric connection that is not a Levi–Civitae one.*

If M is a Hausdorff space, for each $p \in M$ can be found a Generalized Levi–Civitae connection and a metric connection, which coincide with the Levi–Civitae connection locally around p.

Proof. In [1, 4] is proved the following

Lemma. *Let M be a manifold, p is an arbitrary chosen point of M, U is a co-ordinate neighbourhood of p. There is a function $f \in FM$, such that $f(p) = 1$, $f(M \setminus U) = 0$.*

From the proof of the lemma in [4] is clear that we can choose $f \neq$ const even on M. If f is such a function, then $\omega(X) = X \circ f$ is a differential 1-form, not identically equal to zero. We set

$$\tau(X,Y) = \omega(X)Y - \omega(Y)X.$$

Obviously, τ is an antisymmetric tensor of type $(1,2)$ that is not identically equal to zero. It is easy to check $\sigma g \tau = 0$. From the last corollary we receive the theorem.

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ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

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ON THE THREE-SPACES PROBLEM AND EXTENSION OF MLUR NORMS ON BANACH SPACES

GEORGE ALEXANDROV

Георги Александров. О ЗАДАЧЕ ТРЕХ ПРОСТРАНСТВ И ПРОДОЛЖЕНИЯХ СИММЕТРИЧНО ЛОКАЛЬНО РАВНОМЕРНО ВЫПУКЛЫХ НОРМ В ПРОСТРАНСТВАХ БАНАХА

Показано, что если X банахово пространство, Y его подпространство, которое имеет эквивалентной симметрично локально равномерно выпуклой (СЛРВ) нормой $\|\cdot\|$ и факторпространство X/Y сепарабельно, тогда норму $\|\cdot\|$ можно продолжить до СЛРВ нормой на всем пространстве X.

George Alexandrov. ON THE THREE-SPACES PROBLEM AND EXTENSION OF MLUR NORMS ON BANACH SPACES

We show that if X is a Banach space, Y is a subspace of X which admits an equivalent midpoint locally uniformly rotund (MLUR) norm $\|\cdot\|$, and if X/Y is separable, then the norm $\|\cdot\|$ has an extension which is a MLUR norm on X.

1. INTRODUCTION

The three-space problem for a property A of Banach space X consists in the question: If two of three spaces X, Y, X/Y (Y is a subspace of X) possess the property A, then does the third space also have the same property A? Also, the following question is close to the three-space problem: If the norm $\|\cdot\|$ on the subspace Y of a Banach space X possesses the property A, then can the norm $\|\cdot\|$ be extended to such a norm $\|\cdot\|_0$ on X (i. e. the restriction of $\|\cdot\|_0$ on Y is equal to $\|\cdot\|$) with the same property A?

These problems are treated in [A1, A2, GTWZ, JZ1, JZ2] for locally uniformly rotund and rotund renorming of Banach spaces. Here we discuss the same problems for the MLUR property.

2. DEFINITIONS AND REMARKS

A norm $\|\cdot\|$ of a Banach space X is called *midpoint locally uniformly rotund* (MLUR) if

$$\lim_n \left(\|x+x_n\|^2 + \|x-x_n\|^2 - 2\|x\|^2\right) = 0, \quad x, x_n \in X,$$

implies $\lim_n \|x_n\| = 0$.

A norm $\|\cdot\|$ of a Banach space X is called *locally uniformly rotund* (LUR) if

$$\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x+x_n\|^2\right) = 0, \quad x, x_n \in X,$$

implies $\lim_n \|x - x_n\| = 0$.

Obviously LUR $\Rightarrow$ MLUR.

If Y is a subspace of the Banach space X, then $\hat{x}$ means the element of X/Y given by x.

Lemma. *Let X be a* MLUR *Banach space. Then for each $x \in X$ and $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, x) > 0$ such that whenever $y \in X$, $\|x - y\| < \delta$ and $z \in X$, $\|y+z\|^2 + \|y-z\|^2 - 2\|y\|^2 < \delta$, we have $\|z\| < \varepsilon$.*

3. MAIN RESULTS

Theorem 1 ([A3]). *Let X be a Banach space and let Y be a subspace of X such that Y and X/Y admit, respectively, an equivalent* MLUR *and* LUR *norm. Then X admits an equivalent* MLUR *norm.*

Theorem 2. *Let X be a Banach space and let Y be a subspace of X which admits an equivalent* MLUR *norm $\|\cdot\|$, and let X/Y be separable. Then the norm $\|\cdot\|$ can be extended to an equivalent* MLUR *norm on X.*

Proof. We construct the extension of the norm $\|\cdot\|$ on X following the method of [JZ2].

First, we extend the given MLUR norm $\|\cdot\|$ on Y to an equivalent norm $\|\cdot\|$ on X. (For a simple construction of such a norm see e. g. [JZ2].)

Since X/Y is separable, then, as known ([K]), the space X/Y admits an equivalent MLUR norm $\|\cdot\|_0$.

Let $B : X/Y \to X$ be the Bartle–Graves continuous selection map (i. e. $B\hat{x} \in \hat{x}$) [BP].

Let $\{\hat{a}_n\}_{n=1}^{\infty}$, $\hat{a}_n \neq 0$, be a dense subset of X/Y. We assume that $a_n = B\hat{a}_n$.

For each $n \in \mathbb{N}$ ($\mathbb{N}$ — positive integers) choose $f_n \in X^*$ such that $f_n(a_n) = 1$, $\|f_n\| = \|\hat{a}_n\|^{-1}$, $f_n = 0$ on Y and denote by $P_n(x) = f_n(x)a_n$, $Q_n = I - P_n$ (I is the identity map on X) and $T_n = Q_n/(1 + \|P_n\|)$.

For every $x \in X$ we put

$$\|x\|_1^2 = (1-b)\|x\|^2 + \|\hat{x}\|_0^2 + \sum_{n=1}^{\infty} \|T_n(x)\|^2/2^n,$$

where

$$b = \sum_{n=1}^{\infty} 1/2^n(1+\|P_n\|)^2, \quad 0 < b < 1.$$

Then $\|\cdot\|_1$ is an equivalent norm on X whose restriction on Y coincides with the MLUR norm $\|\cdot\|$.

We now are going to show that $\|\cdot\|_1$ is a MLUR norm.

For this purpose we assume there are ε, $0 < \varepsilon < 1$, $x \in X$, and sequence $\{y_m\}$, such that

(1) $$\|x + y_m\|_1^2 + \|x - y_m\|_1^2 - 2\|x\|_1^2 \to 0$$

but

(2) $$\|y_m\| > \varepsilon,$$

and shall find a contradiction.

From (1) and a convexity argument we get

(3) $$\|x + y_m\|^2 + \|x - y_m\|^2 - 2\|x\|^2 \to 0,$$

(4) $$\|\hat{x} + \hat{y}_m\|_0^2 + \|\hat{x} - \hat{y}_m\|_0^2 - 2\|\hat{x}\|_0^2 \to 0$$

and

(5) $$\|T_n(x + y_m)\|^2 + \|T_n(x - y_m)\|^2 - 2\|T_n(x)\|^2 \xrightarrow[m]{} 0$$

for each $n \in \mathbf{N}$.

The norm $\|\cdot\|_0$ is MLUR on X/Y and therefore from (4) we have

(6) $$\|\hat{y}_m\|_0 \to 0.$$

C a s e i) Let $x \in Y$. According to (6) for every m there is $y'_m \in Y$ such that

(7) $$\|y_m - y'_m\| \to 0.$$

From (3) and (7) we receive that

$$\|x + y'_m\|^2 + \|x - y'_m\|^2 - 2\|x\|^2 \to 0,$$

and since the norm $\|\cdot\|$ is MLUR on Y, then

(8) $$\|y'_m\| \to 0.$$

Therefore from (7) and (8) $\|y_m\| \to 0$, which contradicts (2).

C a s e ii) Let $x \notin Y$, $\hat{x} \neq 0$. Put $x = x_0 + y_0$, $x_0 = B\hat{x}$, $y_0 \in Y$. Choose $\hat{a}_n \in \{\hat{a}_n\}$ such that

(9) $$\hat{a}_n \to \hat{x},$$

and since B is a continuous map, then

(10) $$a_n \to x_0.$$

For each $n \in \mathbb{N}$, let $z_n \in \hat{a}_n$ and

(11) $$z_n \to x.$$

Put $z_n = a_n + v_n$, $v_n \in Y$, and from (10) and (11) we have

(12) $$v_n \to y_0.$$

Since $\|P_n\| = \|a_n\|/\|\hat{a}_n\|$ and $Q_n(x_0) = (x_0 - a_n) + f_n(a_n - x_0)a_n$, then

(13) $$\|P_n\| \to \mathrm{d}$$

and

(14) $$\|Q_n(x_0)\| \to 0,$$

where $d = \|x_0\|/\|\hat{x}\|$.

The assumption that $\|\cdot\|$ is a MLUR norm on Y and the Lemma imply that for our $y_0 \in Y$ and $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon/6$, such that if $y \in Y$, $\|y_0 - y\| < \delta$ and $z \in Y$, $\|y + z\|^2 + \|y - z\|^2 - 2\|y\|^2 < \delta$, then

(15) $$\|z\| < \varepsilon/6.$$

Choose δ_1 such that

$$0 < \delta_1 < \delta/[1 + 14(d+2)^2(3K+1)],$$

where $K = \max(\sup\|y_m\|, \|x\|, \|y_0\|)$.

According to (6), (9) and (11)–(14) there is an $n_0 \in \mathbb{N}$ such that for each $n, m \geqq n_0$ we have

(16) $$\|Q_n\| < d + 2,$$

(17) $$\|Q_n(x_0)\| < \delta_1,$$

(18) $$\|x - z_n\| < \delta_1,$$

(19) $$\|y_0 - v_n\| < \delta_1$$

and

(20) $$\|(\hat{x} + \hat{y}_m) - \hat{a}_n\|_0 < \delta_1/2.$$

We fix $n \geqq n_0$ until the end of the proof.

From (5)

$$\|Q_n(x + y_m)\|^2 + \|Q_n(x - y_m)\|^2 - 2\|Q_n(x)\|^2 \xrightarrow[m]{} 0.$$

Therefore, there is an $m \geqq n$ such that

(21) $$D_m = \|Q_n(x + y_m)\|^2 + \|Q_n(x - y_m)\|^2 - 2\|Q_n(x)\|^2 < \delta_1.$$

Choose $t_n \in \hat{a}_n$ (use (20)) such that

(22) $$\|(x + y_m) - t_n\| < \delta_1.$$

Put $t_n = a_n + y_0 + u_n$, $u_n \in Y$. Obviously,

(23) $$Q_n(z_n) = v_n, \quad Q_n(t_n) = y_0 + u_n \quad \text{and} \quad Q_n(x - x_0) = y_0.$$

Furthermore, we have (use (23), (16), (17), (18), (21) and (22))

$$\|v_n + u_n\|^2 + \|v_n - u_n\|^2 - 2\|v_n\|^2 =$$

$$= \|Q_n(z_n) + Q_n(t_n) - y_0\|^2 + \|Q_n(z_n) - Q_n(t_n) + y_0\|^2 - 2\|Q_n(z_n)\|^2 \leqq$$

$$\leqq D_m + 2\Big(\|Q_n\|(\|x - z_n\| + \|(x + y_m) - t_n\|) + \|Q_n(x_0)\|\Big) \times \tag{24}$$

$$\times \Big(\|Q_n\|(\|z_n\| + \|t_n\| + \|x\| + \|y_m\|) + \|y_0\|\Big) +$$

$$+ 2\|Q_n\|^2\|x - z_n\|(\|x\| + \|z_n\|) <$$

$$< \delta_1 \left[1 + 14(d+2)^2(3K+1)\right] < \delta.$$

Therefore, by (19), (24) and (15) we get $\|u_n\| < \varepsilon/6$.

Then

$$\|z_n - t_n\| \leqq \|y_0 - v_n\| + \|u_n\| < \delta_1 + \varepsilon/6 < \varepsilon/3. \tag{25}$$

Thus, by (18), (22) and (25)

$$\|y_m\| \leqq \|(x + y_m) - t_n\| + \|t_n - z_n\| + \|z_n - x\| < 2\delta_1 + \varepsilon/3 < 2\varepsilon/3 < \varepsilon,$$

which contradicts (2).

The theorem is proved.

Remark. Let $E(X)$ be the metric space of all equivalent norms on the Banach space X, endowed with the metric of an uniform convergence on unit ball. If there exists at least one equivalent MLUR norm p on the space X, then the set of all equivalent MLUR norms $M(X)$ is dense in $E(X)$. Really, the set

$$R(X) = \left\{r = \sqrt{q^2 + \varepsilon^2 p^2} \ : \ q \in E(X),\ \varepsilon > 0\right\}$$

is subset of $M(X)$ and dense in $E(X)$. In this case, if Y is a subspace of X, obviously "almost all" equivalent MLUR norms on Y can be extended to such norms on X. Indeed, the set of all restrictions of norms from $R(X)$ on Y is dense in $E(X)$.

We finish the paper with the following

Questions. Let X be a Banach space and let Y be a subspace of X.

1) If both Y and X/Y admit equivalent MLUR norms, does X admit an equivalent MLUR norm too?

2) What are the conditions which the space Y has to satisfy, so that the equivalent MLUR norm on Y could be extended to an equivalent MLUR norm on X?

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ON REITERATION BETWEEN FAMILIES OF BANACH SPACES

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Людмила Николова, Ларс Ерик Персон. О РЕИТЕРАЦИИ В СЕМЕЙСТВЕ БАНАХОВЫХ ПРОСТРАНСТВ

Статья начинается с кратким обзором теории вещественной и комплексной интерполяции между конечным числом и даже в семействе банаховых пространств. Представлены некоторые теоремы реитерации и сделано применение этих теорем, например дано интерполяционное доказательство версии неравенства Гельдера для семейства X^p-пространств и рассмотрена техника помагающая характеризировать интерполяционные пространства в некоторых конкретных семействах банаховых пространств. В конце имеются примеры, нерешенные проблемы и рассматривается отношение представленных результатов к другим связанным с этой тематикой результатов.

Ljudmila Nikolova, Lars Erik Persson. ON REITERATION BETWEEN FAMILIES OF BANACH SPACES

In this paper we briefly discuss some developments concerning real and complex interpolation between finite many or even between general families of Banach spaces. Moreover, we present and apply some reiteration theorems. In particular, we give an interpolation proof of a version of Hölder's inequality for families of X^p-spaces and a technique to characterize the interpolation spaces between some concrete families of Banach spaces. Some examples, open questions, and the relations to other connected results are pointed out.

0. INTRODUCTION

In the theory and applications of interpolation spaces we usually consider a Banach couple (A_0, A_1), i. e. A_0 and A_1 are Banach spaces, which are embedded in a Hausdorff topological vector space U. There are several studied constructions for obtaining interpolation spaces with respect to the couple (A_0, A_1) and the most well-studied and applied such methods are the complex method $[A_0, A_1]_\theta$ $(0 \leqq \theta \leqq 1)$ and the real method $(A_0, A_1)_{\theta,q}$ $(0 < \theta < 1,\ 0 < q \leqq \infty)$. See e. g. the books [1–3, 16, 25, 33] and also the Bibliography of Maligranda [18] (including approximately 2500 references).

Parts of the theory concerning interpolation between two Banach spaces can be generalized to cover also cases where we interpolate between finite many Banach spaces and even between general families of Banach spaces. Here we mention the following developments:

1) A theory for *complex interpolation* between families of Banach spaces was developed by Coifman–Cwikel–Rochberg–Sagher–Weiss (see [6–8]) and, independently, by Krein–Nikolova (see [14, 15]). Another complex interpolation method between n-tuples of Banach spaces was suggested by Lions [17] and studied in detail by Favini [10]. This method of Favini–Lions was extended by Cwikel–Janson [9] to cover also complex interpolation between very general families of spaces.

2) A theory for *real interpolation* between n-tuples of Banach spaces was introduced and studied by Sparr [31]. A similar theory for real interpolation between 2^k-tuples of Banach spaces was studied by Fernandez [11]. In this connection we also mention early works by Yoshikawa, Kerzman and Foias–Lions (cf. the discussion in [31, p. 248]). Moreover, Cobos–Peetre [5] have developed a theory for real interpolation between finite many Banach spaces, which, in particular, covers both Sparr's and Fernandez' constructions for the cases $n = 3$ and $n = 4$, respectively. The construction of Sparr was extended by Cwikel–Janson [9] to cover also real interpolation between a fairly general family $A = \{A_t\}_{t\in\Gamma}$, where A_t are Banach spaces and Γ is a general probability measure space. Some very new constructions have recently been introduced by Carro [4] and the present authors [21] (see also [20] and [22]).

In this paper we discuss and apply some reiteration results for families of Banach spaces. In Section 1 we present some well-known reiteration theorems. In Section 2 we expose an interpolation proof of a version of Hölder's inequality for families of X^p-spaces (cf. [23]). In Section 3 we point out a technique to characterize the interpolation spaces between families of Banach Spaces in several cases of practical interest. Some concrete examples are given and discussed. Finally, Section 4 is reserved for some further examples, concluding remarks and open questions.

1. SOME REITERATION RESULTS

In the sequel we let D and $P(z,t)$, $z \in D$, $0 \leq t < 2\pi$, denote the open unit disc and the Poisson kernel, respectively. Moreover, we let $\alpha = \alpha(e^{it})$, $p = p(e^{it})$ and $q = q(e^{it})$ denote measurable functions on $[0, 2\pi)$ such that $0 < \alpha(e^{it}) \leq 1$ and

$p(e^{it}),\ q(e^{it}) \geqq 1$. The functions $\alpha(z)$, $p(z)$ and $q(z)$, $z \in D$, are defined by

$$(1.1)\qquad \alpha(z) = \frac{1}{2\pi}\int_0^{2\pi} \alpha(e^{it})P(z,t)\,dt, \quad \frac{1}{p(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(e^{it})}P(z,t)\,dt,$$

$$\frac{1}{q(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{q(e^{it})}P(z,t)\,dt,$$

respectively. $B = \{B_t\}$, $t \in [0, 2\pi]$, denotes an *interpolation family* in the sense of R. R. Coifman et al. [6] (see also [7, 8]).

First we consider the complex interpolation spaces $[B]_z$ and $\{B\}_z$ (see [6–8]). The following reiteration theorem ([8, Theorem 5.1]) may be regarded as a genuine generalization of the "complex" reiteration theorem (see e. g. [2, Theorem 4.6.1]):

Theorem A. *Let (A_0, A_1) be a compatible Banach couple and let $B_t = [A_0, A_1]_{\alpha(e^{it})}$. Then $B = \{B_t\}$, $t \in [0, 2\pi]$, is an interpolation family and, for any $z \in D$, $[B]_z \equiv [A_0, A_1]_{\alpha(z)}$ (with equality of norms).*

Remark. If there exist $t_0, t_1 \in [0, 2\pi]$ such that $\alpha(e^{it_0}) = \sup \alpha(e^{it})$ and $\alpha(e^{it_1}) = \inf \alpha(e^{it})$, then it is well-known that $[B]_z \equiv \{B\}_z$.

Next we consider the following "mixed" reiteration result of Hernandez [12, Main Theorem] (cf. [2, Theorem 4.7.2]):

Theorem B. *Assume that (A_0, A_1) is a compatible Banach couple and let $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$. Then $B = \{B_t\}$, $t \in [0, 2\pi]$, is an interpolation family. Moreover, if*

$$(1.2)\qquad \int_0^{2\pi} \frac{1}{q(e^{it})}\,dt > 0, \quad \int_0^{2\pi} \log \alpha(e^{it})\,dt > -\infty, \quad \textit{and} \quad \int_0^{2\pi} \log\left(1 - \alpha(e^{it})\right)dt > -\infty,$$

then, for any $z \in D$, $[B]_z = (A_0, A_1)_{\alpha(z), q(z)}$ (with equivalence of norms).

Now we consider the real interpolation spaces $\{B\}^S_{z,p,q}$ (the K-method) and $(B)^S_{z,p,q}$ (the J-method) recently introduced by Carro [4] and state her reiteration result (cf. [2, Theorems 3.5.1 and 3.5.4]).

Theorem C. *Let $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$, where (A_0, A_1) is a compatible Banach couple. If $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$, $1 \leqq q < \infty$ and $r \geqq 1$, then, for any $z \in D$,*

$$\{B\}^S_{z,q,1} = (A_0, A_1)_{\alpha(z), q} = (B)^S_{z,q,r}.$$

Remark. The method of proof in [4] ensures the first equality only with $q = 1$, but it is true also for any $q \geq 1$ whenever $\{B\}^S_{z,p,q}$ is continuously imbedded in $A_0 + A_1$.

Remark. In [4] there is also stated a complex variant of Theorem C, where $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$ is replaced by $B_t = [A_0, A_1]_{\alpha(e^{it})}$.

2. REITERATION AND HÖLDER'S INEQUALITY FOR FAMILIES OF X^p SPACES

Let X denotes a Banach lattice on a σ-finite complete measure space Ω. The space X^p, $p \geqq 1$, consists of all $x = x(t)$, $t \in \Omega$, such that $|x|^p \in X$ and equipped with the norm $\|x\|_{X^p} = (\| |x(t)|^p \|_X)^{1/p}$. First we recall the following result (see [23]):

Proposition 2.1. *Assume that the Banach lattice X has the dominated convergence property. Then*

$$\left[X^{p(e^{it})}\right]_z \equiv X^{p(z)}.$$

For the case $X = L_1(\mu)$ see [8, Corollary 5.2]. Proposition 2.1 is proved in [23, p. 96] and here we present an alternative proof based upon Theorem A.

Proof. It is well-known that $[X, L_\infty]_\theta \equiv X^{1/(1-\theta)}$. Therefore we have $X^{p(e^{it})} \equiv [X, L_\infty]_{\alpha(t)}$, where $\alpha(t) = 1 - 1/p(e^{it})$, and, thus, according to Theorem A,

$$\left[X^{p(e^{it})}\right]_z \equiv \left[[X, L_\infty]_{\alpha(t)}\right]_z \equiv [X, L_\infty]_{\alpha(z)} \equiv [X, L_\infty]_{1-1/p(z)} \equiv X^{p(z)}.$$

The proof is complete.

Next we recall the following generalization of Hölder's inequality (see [29]): If $x_j \in X^{p_j}$, $1 \leqq p_j < \infty$, $j = 0, 1, \ldots, N$, then

$$\prod_0^N x_j \in X^r, \quad \text{where} \quad \frac{1}{r} = \sum_0^N x_j \quad \text{and} \quad \|\prod_0^N x_j\|_{X^{p_i}} \leqq \prod_0^N \|x_j\|_{X^{p_i}}. \tag{2.1}$$

By using the interpolation result in Proposition 2.1 we can prove the following extention of this inequality to the case with families of X^pspaces:

Theorem 2.2. *Let X be a Banach lattice having the dominated convergence property. If $x_t \in X^{p(e^{it})}$ and $\log|x_t(s)| \in L[0, 2\pi)$ for $s \in \Omega$, $x_t(s)$ is measurable on $[0, 2\pi] \times \Omega$ and $\log\|x_t\|_{X^{p(e^{it})}} \in L[0, 2\pi)$, then*

$$x = x_z(s) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|x_t(s)| P(z,t)\, dt\right) \in X^{p(z)} \quad \textit{and}$$

$$\|x\|_{X^{p(z)}} \leqq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\|x_t(s)\|_{X^{p(e^{it})}} P(z,t)\, dt\right). \tag{2.2}$$

Remark. We note that (2.2) coincides with (2.1) if $z = 0$ and, for $0 \leqq a_0 < \ldots < a_N = 1$, $E_j = [a_{j-1}, a_j]$, $p(e^{it}) = p_j/m(E_j)$, and $x_t(s) = |y_t(s)|^{1/m(E_j)}$, $t \in 2\pi E_j$, $j = 0, 1, \ldots, N$.

Remark. A somewhat different version of Theorem 2.2 was proved in [23] (cf. also [24]) but the proof we present below is simpler and completely different.

Proof (by interpolation). We adopt the notations and definitions of [23]. In particular $[B]_z$ and $[B]^z$ denote the interpolation spaces of Coifman–Cwikel–Rochberg–Sagher–Weiss and the generalized Calderon construction for the family $B = \{B_t\}$,

$t \in [0, 2\pi]$, of Banach spaces, respectively. In view of Proposition 2.1 we have $\left[X^{p(e^{it})}\right]^{-}_{z} \equiv X^{p(z)}$. Moreover, according to Hernandez [13, Theorem 6.1] it yields that $\left[X^{p(e^{it})}\right]_{z} \equiv \left[X^{p(e^{it})}\right]^{z}$. Now we consider

$$y_z(s) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|y_t(s)| P(z,t)\,dt\right), \quad \text{where} \quad \|y_t(s)\|_{X^{p(e^{it})}} \leqq 1.$$

Then

$$\|y_z(s)\|_{X^{p(z)}} = \|y_z(s)\|_{\left[X^{p(e^{it})}\right]_z} = \|y_z(s)\|_{\left[X^{p(e^{it})}\right]^z} \leqq 1. \tag{2.3}$$

Finally, we obtain (2.2) by inserting

$$y_t(s) = x_t(s)/\|x_t(s)\|_{X^{p(e^{it})}} \quad \text{and} \quad x_z(s) = \exp\left(\frac{1}{2\pi}\log|x_t(s)|P(z,t)\,dt\right)$$

into (2.3).

3. REITERATION AND CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN FAMILIES OF BANACH SPACES

Let $B = \{B_t\}$ be an interpolation family of Banach spaces. Inspired by the proof of Proposition 2.1 we find that it is possible to characterize $[B]_z$ for several concrete interpolation families of Banach spaces by using the following technique:

(a) Write B_t, if possible, as $B_t = [A_0, A_1]_{\alpha(e^{it})}$ or as $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$, where (A_0, A_1) denotes a compatible Banach couple.

(b) Calculate $\alpha(z)$, or $\alpha(z)$ and $q(z)$ (see (1.1)).

(c) Calculate $[A_0, A_1]_{\alpha(z)}$ or $(A_0, A_1)_{\alpha(z), q(z)}$.

(d) Apply Theorem A or Theorem B, respectively.

By using steps (a)–(c) together with Theorem C we can describe the real spaces $\{B\}^S_{z,p,1}$ and $(B)^S_{z,p,q}$ in a similar way.

Nowadays we have very good knowledge concerning the possibilities to carry out the crucial steps (a) and (c) in several cases of practical importance (see e. g. [1–3, 16, 25, 33] and the Bibliography of Maligranda [18]). Here we only give some examples of applications of this technique (cf. also [4, 8]).

3.1. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF LORENTZ $L_{p,q}$-SPACES

Proposition 3.1. *Let* $1 < p(e^{it}) < \infty$ *and* $1 \leqq q(e^{it}) \leqq \infty$. *Then* $B = \left\{L_{p(e^{it}), q(e^{it})}\right\}$, $t \in [0, 2\pi]$, *is an interpolation family of Banach spaces.*

(a) *If the condition* (1.2) *is satisfied with* $\alpha(e^{it}) = 1 - 1/p(e^{it})$, *then* $[B]_z = L_{p(z), q(z)}$.

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = L_{p(z),q}$.

Proof. We use the well-known equality

$$L_{p(e^{it}),q(e^{it})} = (L_1, L_\infty)_{\alpha(e^{it}),q(e^{it})}, \quad \text{where } \alpha(e^{it}) = 1 - 1/p(e^{it}).$$

We also note that $\alpha(z) = 1 - 1/p(z)$ (see (1.1)). Therefore, according to Theorem B,

$$[B]_z = [L_1, L_\infty]_{\alpha(z),q(z)} = [L_1, L_\infty]_{1-1/p(z),q(z)} = L_{p(z),q(z)}.$$

Moreover, by using Theorem C, we find that

$$\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = (L_1, L_\infty)_{\alpha(z),q} = L_{p(z),q},$$

and the proof is complete.

Remark. The first part of Proposition 3.1 is due to Hernandez [12, p. 81]. For the case $p(e^{it}) = q(e^{it})$ we can use Theorem A and a complex variant of Theorem C to find that Proposition 3.1 (b) holds also if we permit that $p(e^{it}) = 1$ or $p(e^{it}) = \infty$. In particular, for any $p(e^{it})$, $q(e^{it}) \geqq 1$ and $r \geqq 1$, it holds that

$$\left\{\{L_{p(e^{it})}\}\right\}^S_{z,p(z),1} = \left(\{L_{p(e^{it})}\}\right)^S_{z,p(z),r} = L_{p(z)}.$$

3.2. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF OPERATOR IDEAL SPACES

Let A_0 and A_1 denote Banach spaces and let $L(A_0, A_1)$ denotes the space of bounded linear operators between A_0 and A_1. We can give more precize information about the operator T by considering the approximation numbers $a_n(T)$, $n = 1, 2, \dots$, where

$$a_n(T) = \inf\{\|T - T_n\| \mid \operatorname{rank} T_n < n,\ T_n \in L(A_0, A_1)\}.$$

We say that $T \in \sigma_{p,q}$, $1 \leq p, q \leq \infty$, if

$$\|T\|_{\sigma_{p,q}} = \left(\sum_1^\infty \left(a_n(T)n^{1/p}\right)^q \frac{1}{n}\right)^{1/q} < \infty,$$

with the usual supremum interpretation of the sum when $q = \infty$. The space $\sigma_{p,p}$ coincides with the usual Schatten–von Neumann class σ_p.

Proposition 3.2. *Let $1 \leq p(e^{it}) \leq \infty$. Then $B = \{\sigma_{p(e^{it})}\}$, $t \in [0, 2\pi)$, is an interpolation family of Banach spaces and $[B]_z \equiv \sigma_{p(z)}$.*

Proof. We use the formula $[\sigma_1, \sigma_\infty]_\theta \equiv \sigma_{1/(1-\theta)}$, $0 \leqq \theta \leqq 1$, (see [30]) and find that

$$\sigma_{p(e^{it})} \equiv [\sigma_1, \sigma_\infty]_{1-1/p(e^{it})}.$$

If $\alpha(e^{it}) = 1 - 1/p(e^{it})$, then $\alpha(z) = 1 - 1/p(z)$. Hence, by using Theorem A, we obtain that

$$[B]_z \equiv [\sigma_1, \sigma_\infty]_{\alpha(z)} \equiv [\sigma_1, \sigma_\infty]_{1-1/p(z)} \equiv \sigma_{p(z)}.$$

Proposition 3.3. *Let $1 < p(e^{it}) < \infty$ and $1 \leq q(e^{it}) \leq \infty$. Then $B = \{\sigma_{p(e^{it}),q(e^{it})}\}$, $t \in [0, 2\pi)$, is an interpolation family of Banach spaces.*

(a) *If the condition (2.1) is satisfied with $\alpha(e^{it}) = 1 - 1/p(e^{it})$, then $[B]_z = \sigma_{p(z),q(z)}$.*

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = \sigma_{p(z),q}$.

Proof. It is well-known that $(\sigma_1, \sigma_\infty)_{\theta,q} \equiv \sigma_{1/(1-\theta),q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$ (see [32]),

$$\sigma_{p(e^{it}),q(e^{it})} = (\sigma_1, \sigma_\infty)_{\alpha(e^{it}),q(e^{it})}, \quad \text{where } \alpha(e^{it}) = 1 - 1/p(e^{it}).$$

Therefore the proof follows by arguing exactly as in the proof of Proposition 3.1.

3.3. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF BESOV SPACES

In this section we assume that the reader is acquainted with Peetre's abstract definitions of the Sobolev spaces H^s_p and the Besov spaces $B^s_{p,q}$ (see [2, 25]). It is a difficult task to describe the real interpolation spaces between Besov spaces for the general (off-diagonal) case (see [25, p. 110] and [26, p. 228]). However, for some special cases we have suitable such interpolation formulas (see [2, Theorem 6.4.5]), e. g. the following ones: Let $-\infty < s_0 < s_1 < \infty$, $0 < \alpha < 1$, $s = (1-\alpha)s_0 + \alpha s_1$ and $1 \leqq p, q \leqq \infty$, then

$$B^s_{p,p} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha,p}, \quad \alpha = 1 - 1/p, \tag{3.1}$$

$$B^s_{p,q} = \left(H^{s_0}_p, H^{s_1}_p\right)_{\alpha,q}. \tag{3.2}$$

We also introduce another measurable function $s(e^{it})$ on $[0, 2\pi)$ and define

$$s(z) = \frac{1}{2\pi} \int_0^{2\pi} s(e^{it}) P(z,t)\, dt.$$

Proposition 3.4. *Assume that* $1 < p(e^{it}) < \infty$, *and* $-\infty < s_0 < s(e^{it}) < s_1 < \infty$. *Then* $\left\{B^{s(u)}_{p(u),p(u)}\right\}$, $u = e^{it}$, $t \in [0, 2\pi)$, *is an interpolation family of Banach spaces.*

(a) *If the condition* (1.2) *is satisfied with* $\alpha(e^{it}) = 1 - 1/p(e^{it})$ *and* $q(e^{it}) = p(e^{it})$, *then* $[B]_z = B^{s(z)}_{p(z),p(z)}$.

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,p(z),1} = (B)^S_{z,p(z),r} = B^{s(z)}_{p(z),p(z)}$.

Proof. According to (3.1) we have

$$B^{s(u)}_{p(u),p(u)} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(u),p(u)}, \quad u = e^{it},$$

where $\alpha(e^{it}) = 1 - 1/p(e^{it})$ and $s(e^{it}) = (1 - \alpha(e^{it}))s_0 + \alpha(e^{it})s_1$. We note that $\alpha(z) = 1 - 1/p(z)$, $s(z) = (1 - \alpha(z))s_0 + \alpha(z)s_1$ and, thus, by using Theorem B and (3.1) once again, we find that

$$[B]_z = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(z),p(z)} = B^{s(z)}_{p(z),p(z)}.$$

Furthermore, according to Theorem C and (3.1),

$$\{B\}^S_{z,p(z),1} = (B)^S_{z,p(z),q} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(z),p(z)} = B^{s(z)}_{p(z),p(z)},$$

and the proof is complete.

Remark. By using the formula (3.2) instead of (3.1) and arguing exactly as in the proof of Proposition 3.4 we find that $B = \left\{B^{s(u)}_{p,q(u)}\right\}$, $u = e^{it}$, $t \in [0, 2\pi)$, $1 \leqq p \leqq \infty$, is an interpolation family of Banach spaces and that the (Hernandez) formula (see [13])

$$[B]_z = B^{s(z)}_{p,q(z)}$$

and the formula

$$\{B\}^S_{z,q(z),1} = (B)^S_{z,q(z),r} = B^{s(z)}_{p,q(z)}$$

hold under the corresponding restrictions on $s(e^{it})$, $q(e^{it})$ and S, respectively.

4. CONCLUDING REMARKS AND EXAMPLES

First we discuss the special case where $B = \{B_t\}$ is a finite family of Banach spaces, for example that $B_t = A_k$, $2\pi(k-1)/N \leqq t \leqq 2\pi k/N$, $k = 1, 2, \ldots, N$. By applying the statements in Section 3 we obtain concrete descriptions of the interpolation spaces between N Banach spaces. Alternatively, we can use the technique in Section 3 and the well-known reiteration results for finite families of Banach spaces (instead of Theorems A–C). We illustrate this idea by considering the Sparr spaces (see [31])

$$\overline{A}_{\overline{\lambda},q}, \quad \overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N), \ 0 \leqq \lambda_i \leqq 1, \ \sum \lambda_i = 1, \ 1 \leqq q \leqq \infty,$$

and $\overline{A} = (A_1, A_2, \ldots, A_N)$ is a N-tuple of compatible Banach spaces. We use the reiteration theorem of Sparr and the arguments used in Section 3.1 to present just one example of this technique:

Example 1. Let $1 \leqq q, p_i, q_i \leqq \infty$, $0 \leqq \lambda_i \leqq 1$ ($i = 1, 2, \ldots, N$), and $\sum \lambda_i = 1$.

(a) If $\dfrac{1}{p} = \displaystyle\sum_1^N \frac{\lambda_i}{p_i}$ and not all of p_i are equal, then $(\sigma_{p_1,q_1}, \sigma_{p_2,q_2}, \ldots, \sigma_{p_N,q_N})_{\overline{\lambda},q} = \sigma_{p,q}$.

(b) If $1 \leqq p \leqq \infty$ and $\dfrac{1}{q} = \displaystyle\sum_1^N \frac{\lambda_i}{q_i}$, then $(\sigma_{p,q_1}, \sigma_{p,q_2}, \ldots, \sigma_{p,q_N})_{\overline{\lambda},q} = \sigma_{p,q}$.

Question 1. Let $1 \leqq q, p, p_i, q_i \leqq \infty$, $0 \leqq \lambda_i \leqq 1$ ($i = 1, 2, \ldots, N$), and $\sum \lambda_i = 1$. Describe the spaces $(\sigma_{p,q_1}, \sigma_{p,q_2}, \ldots, \sigma_{p,q_N})_{\overline{\lambda},q}$ when $\dfrac{1}{q} \neq \displaystyle\sum_1^N \frac{\lambda_i}{q_i}$.

This question is non-trivial also in the case $N = 2$ (see e. g. [26, 28]).

Next we remark that it is well-known that if we interpolate between two Besov spaces, then for the general (off-diagonal) case we need not obtain a Besov space (see the classical question by Peetre [25, p. 110]). A concrete (but somewhat curious) description of these spaces is given in [26, p. 228]. On the other hand, by making reiteration of these (Beurling type) spaces it may happen that we return to the Besov scale of spaces. We illustrate this fact with the following examples:

Example 2. Let $0 < \theta < 1$ and consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$, $q(e^{it}) \neq 1/(1-\theta)$, $t \in [0, 2\pi)$, but $q(z) = 1/(1-\theta)$. Then each space

$$B_t = \left(B^s_{p,1}, B^s_{p,\infty}\right)_{\theta, q(e^{it})}, \quad p \geqq 1, \quad t \in [0, 2\pi), \tag{4.1}$$

in the interpolation family $B = \{B_t\}$ is *not* a Besov space (see [25]). On the other hand, according to Theorems B and C, it yields respectively that

a) If $1 < p < \infty$ and $\int_0^{2\pi} \frac{1}{q(e^{it})}\, dt > 0$, then $[B]_z = B^s_{p,q(z)}$;

b) If $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$ and $r \geqq 1$, then $\{B\}^S_{z,q(z),1} = (B)^S_{z,q(z),r}$ $= B^s_{p,q(z)}$.

Question 2. Consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$ and $B = \{B_t\}$, where B_t is defined by (4.1). Describe $[B]_z$ for the case when $q(z) \neq 1/(1-\theta)$.

Question 3. Describe the spaces $[B]_z$ when $B = \{B_t\}$, $B_t = B^{s(u)}_{p(u),q(u)}$, $u = e^{it}$, $t \in [0, 2\pi]$, without e. g. the restrictions $p(e^{it}) = q(e^{it})$ or $p(e^{it}) = p$ (cf. Proposition 3.4 or the remark after that proposition, respectively).

The observation above, concerning off-diagonal interpolation between Besov spaces, is a special case of the results obtained in [26]. In our next example we present some similar situations with interpolation in off-diagonal cases.

Example 3. Let $1 \leqq p, q_0, q_1, q(e^{it}) \leqq \infty$, $0 < \theta < 1$, and $\dfrac{1}{q_\theta} = \dfrac{1-\theta}{q_0} + \dfrac{\theta}{q_1}$.

(a) $B_t = (L_{p,q_0}, L_{p,q_1})_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a Lorentz $L_{p,q}$-space.

(b) $B_t = (L_{q_0}(\omega_0), L_{q_1}(\omega_1))_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a weighted L_p-space.

(c) $B_t = (l_{q_0}(\{A_k\}), l_{q_1}(\{B_k\}))_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a (strong) sequence space of the type $l_q(\{C_k\})$ (A_k, B_k, C_k, $k = 1, 2, \ldots$, are Banach spaces).

These statements are all special cases of the descriptions given in [19] and [26]. Moreover, by arguing as in Example 2 and considering the case $q(z) = q_\theta$, we find that the reiteration space $[B]_z$, in fact, is a $L_{p,q}$-space, a L_p-space, and a (strong) sequence space of the type $l_q(\{C_k\})$, respectively.

Question 4. Consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$ and $B = \{B_t\}$, where B_t is defined as in Example 3 (a), (b) or (c). Describe $[B]_z$ if $q(z) \neq 1/(1-\theta)$ in each of these cases.

For several reasons it should be interesting to generalize Theorems A–C in various ways. In particular, by using such generalized forms of Theorems A–C and the technique described in Section 3 we get a possibility to describe interpolation spaces between more general families of Banach spaces than those studied in this paper. For example, by replacing the spaces $(A_0, A_1)_{\theta,q}$ with more general parameter function spaces $(A_0, A_1)_{\lambda,q}$ (see [27]) in Theorems B–C it is possible to obtain some weighted versions of the results obtained in Propositions 3.2–3.4.

Finally we remark that the problems posed in Questions 2–4 above are open also for the case when the spaces $[B]_z$ are replaced by $\{B\}^S_{z,q(z),1}$ or $(B)^S_{z,q(z),r}$.

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ON BOUNDED TRUTH–TABLE AND POSITIVE DEGREES*

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Ангел Дичев. ОБ ОГРАНИЧЕНО ТАБЛИЧНЫХ И ПОЗИТИВНЫХ СТЕПЕНЕЙ

В этом статье доказывается, что существует рекурсивно перечислимая btt-степень, которая содержит бесконечная антицепь из рекурсивно перечислимых p-степеней.

Angel Ditchev. ON BOUNDED TRUTH–TABLE AND POSITIVE DEGREES

In the present paper it is shown that there exists a recursively enumerable btt-degree containing an infinite anti-chain of recursively enumerable p-degrees.

In his dissertation [1] Degtev has studied the relationship between different tabular degrees. It is proved there that if $K^* = \{\mathrm{tt}, \mathrm{l}, \mathrm{p}, \mathrm{d}, \mathrm{c}, \mathrm{btt}, \mathrm{bl}, \mathrm{m}\}$, $\mathrm{r}, \mathrm{R} \in K^*$, $\mathrm{r} \neq \mathrm{R}$ and r is weaker than R, then every complete R-degree contains infinitely many recursively enumerable (r.e.) r-degrees. In connection with this he puts the questions, whether there exist a nonrecursive r.e. bp-degree, containing infinitely many bd-degrees, and a nonrecursive r.e. btt-degree, containing infinitely many r.e. bp-degrees. In [5] the first question is answered positively and in this paper the second question is answered positively too. Here, as in [5], something more is shown, namely, there exists an r.e. btt-degree containing an infinite anti-chain of r.e. p-degrees.

In connection with Degtev's questions, mentioned above, the following question arises: Let r and R be different tabular degrees such that R is not weaker than r.

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Does there exist an r.e. R-degree containing an infinite anti-chain of r-degrees. We expect that the answer of this question is positive.

In this paper we use $\mathbb{N}$ to denote the set of all natural numbers $\{0, 1, 2, \ldots\}$. Let $p_0 < p_1 < p_2 \ldots$ be the sequence of all prime numbers and denote by M_i the set $\{x \mid \exists y(x = p_i.y)\}$, $i \in \mathbb{N}$.

If f is a partial function, then by $\mathrm{Dom}(f)$ we shall denote the domain of function f, and by $\mathrm{Ran}(f)$ the range of f. For any partial functions f and g by fg we shall denote the composition of f and g, i.e. $fg = \lambda x.f(g(x))$. If A is a finite set, we shall use $|A|$ to denote the cardinallity of the set A.

The sequence $A_0, A_1, \ldots$ ($\{A_k\}_{k\in\mathbb{N}}$) of sets of natural numbers is said to be recursive (r.e.) iff so is the set $\{(n, x) \mid x \in A_n\}$. The sequence $\varphi_0, \varphi_1, \ldots$ ($\{\varphi_k\}_{k\in\mathbb{N}}$) of unary functions is said to be total recursive (partial recursive) iff so is the binary function $\lambda i \lambda x.\varphi_i(x)$.

It would be useful to remind some definitions from [1, 5, 6].

If β is a Gödel function, then for natural numbers $k, p, p_1, \ldots, p_k, i$ we use the following notations:

$$\langle p_1, \ldots, p_k\rangle = \mu p[\beta(p, 0) = k \,\&\, \beta(p, 1) = p_1 \,\&\, \ldots \,\&\, \beta(p, k) = p_k];$$

$$\mathrm{lh}(p) = \beta(p, 0); \quad (p)_i = \beta(p, i + 1);$$

$$\mathrm{Seq}(p) \iff \forall x\{x < p \Rightarrow [\mathrm{lh}(x) \neq \mathrm{lh}(p) \vee \exists i(i < \mathrm{lh}(p) \,\&\, (x)_i \neq (p)_i)]\};$$

$$\mathrm{Seq}_k(p) \iff \mathrm{Seq}(p) \,\&\, \mathrm{lh}(p) = k.$$

The set A is called *positively reducible* (p-*reducible*) to the set B ($A \leq_{\mathrm{p}} B$) iff there exists a total recursive function f which satisfies the following conditions:

$$\text{(p)} \quad \begin{array}{l} \forall x\{\mathrm{Seq}(f(x)) \,\&\, \forall k[k < \mathrm{lh}(f(x)) \Rightarrow \mathrm{Seq}((f(x))_k)]\}; \\ \forall x\{x \in A \iff \exists k[k < \mathrm{lh}(f(x)) \,\&\, \forall i(i < \mathrm{lh}((f(x))_k) \Rightarrow ((f(x))_k)_i \in B)]\}. \end{array}$$

The set A is said to be *truth table reducible* (tt-*reducible*) to the set B iff there exists a total recursive function f which satisfies the following conditions:

$$\text{(tt)} \quad \begin{array}{l} \forall x\{x \in A \iff \exists k[k < \mathrm{lh}(f(x)) \,\&\, \forall i(i < \mathrm{lh}((f(x))_k) \Rightarrow \\ \qquad \{[(((f(x))_k)_i)_0 \in B \,\&\, \mathrm{Seq}(((f(x))_k)_i)] \vee \\ \qquad [((f(x))_k)_i \notin B \,\&\, \text{not } \mathrm{Seq}(((f(x))_k)_i)]\})]\}. \end{array}$$

If $\mathrm{r} \in \{\mathrm{p}, \mathrm{tt}\}$, then the set A is said to be br-*reducible* to the set B ($A \leq_{\mathrm{br}} B$) iff there exists a natural number m and a total recursive function f which satisfy the conditions (r) and

$$\forall x[|\bigcup_k \bigcup_i ((f(x))_k)_i| \leq m].$$

If r is a reducibility, the set A is said to be r-*equivalent* to the set B ($A \equiv_{\mathrm{r}} B$) iff $A \leq_{\mathrm{r}} B$ and $B \leq_{\mathrm{r}} A$.

For any reducibility r the family $d_{\mathrm{r}}(A) = \{B \mid B \equiv_{\mathrm{r}} A\}$ is called r-*degree* of the set A.

The idea of constructing a btt-degree which contains infinitely many mutually incomparable p-degrees is the same as in [5] and comes from the proof of Skordev's conjecture [cf. 2, 3, 4]. Roughly speaking, we find btt-schemes which are not p-schemes. More precisely, we shall construct an r.e. sequence $\{B_k\}_{k\in\mathbb{N}}$ of sets of natural numbers such that the set B_i has the same btt-degree as the set B_j for any natural numbers i and j. At the same time, if $i \neq j$, then the sets B_i and B_j will be p-incomparable. For this aim we shall prove the following

Theorem 1. *There exist recursive sequences $\{\theta_{1,p}\}_{p\in\mathbb{N}}$, $\{\theta_{2,p}\}_{p\in\mathbb{N}}$ of total recursive functions such that for all natural numbers i and x the equivalences*

$$(*)\qquad \begin{aligned} x \in M_i &\iff \theta_{1,2i}(x) \in M_{i+1} \,\&\, \theta_{2,2i}(x) \notin M_{i+1},\\ x \in M_{i+1} &\iff \theta_{1,2i+1}(x) \in M_i \,\&\, \theta_{2,2i+1}(x) \notin M_i \end{aligned}$$

hold, and such that if i and j are distinct and $(\varphi_1,\dots,\varphi_{l_1}),\dots,(\varphi_{l_{k-1}+1},\dots,\varphi_{l_k})$ is an arbitrary sequence of finite sequences of id *or compositions of $\theta_{1,0},\theta_{2,0},\theta_{1,1}$, $\theta_{2,1},\dots$, then there exists an $x \in \mathbb{N}$ such that the equivalence*

$$(**)\qquad \begin{aligned} x \in M_i \iff &(\varphi_1(x) \in M_j \,\&\, \dots \,\&\, \varphi_{l_1}(x) \in M_j \,\vee \dots \vee\\ &(\varphi_{l_{k-1}+1}(x) \in M_j \,\&\, \dots \,\&\, \varphi_{l_k}(x) \in M_j) \end{aligned}$$

does not hold.

Proof. The construction of the sequences of functions $\{\theta_{1,p}\}_{p\in\mathbb{N}}$, $\{\theta_{2,p}\}_{p\in\mathbb{N}}$ we shall perform by steps. At step s we shall construct finite approximations $\theta^s_{k,p}$ of $\theta_{k,p}$, $k = 1,2$; $p \in \mathbb{N}$, such that $\theta^s_{k,p} \subseteq \theta^{s+1}_{k,p}$ and $\theta^s_{k,p}(x)$ is a primitive recursive function (p.r.f.) of the variables s, p, x, $k = 1,2$. At the end, we shall define $\theta_{k,p} = \bigcup_{s\in\mathbb{N}} \theta^s_{k,p}$, $k = 1,2$; $p \in \mathbb{N}$.

We need some auxiliary definitions and lemmas.

Definition 1. Let $f_{1,0}, f_{2,0}, f_{1,1}, f_{2,1}, \dots$ be an infinite sequence of unary functional symbols. Terms are defined by means of the following inductive clauses:

(a) Every symbol is a term;

(b) If τ_1 and τ_2 are terms, then $(\tau_1\tau_2)$ is a term.

We shall assume that there exist effective codings of all terms, of all finite sequences of terms, and of all finite sequences of finite sequences of terms.

Definition 2. We define the length $\mathrm{l}(\tau)$ of the term τ as follows:

(a) $\mathrm{l}(f_{k,i}) = 1$, $k = 1,2$, $i \in \mathbb{N}$;

(b) If $\tau = (\tau_1\tau_2)$, then $\mathrm{l}(\tau) = \mathrm{l}(\tau_1) + \mathrm{l}(\tau_2)$.

Type and anti-type of a term are defined simultaneously by means of the following inductive definition:

Definition 3.

(a) If $\tau = f_{1,2i}$, then τ has type $i \to i+1$;

(b) If $\tau = f_{1,2i+1}$, then τ has type $i+1 \to i$;

(c) If $\tau = f_{2,2i}$, then τ has anti-type $i \to i+1$;

(d) If $\tau = f_{2,2i+1}$, then τ has anti-type $i+1 \to i$;

(e) If $\tau = (\tau_1\tau_2)$ and τ_2 has type $i \to k$ and τ_1 has type (anti-type) $k \to j$, then τ has type (anti-type) $i \to j$.

We say that τ has type (anti-type) iff τ has type (anti-type) $i \to j$ for some natural i, j; otherwise we say that τ has not type (anti-type).

Definition 4. Let $\mathfrak{A} = \langle \mathsf{N}; \theta_{1,0}, \theta_{2,0}, \theta_{1,1}, \theta_{2,1}, \ldots \rangle$ be a partial structure. The value $\tau_{\mathfrak{A}}$ of the term τ in the structure $\mathfrak{A}$ we define as follows:

(a) If $\tau = f_{k,i}$, then $\tau_{\mathfrak{A}} = \theta_{k,i}$, $k = 1, 2$; $i \in \mathsf{N}$;

(b) If $\tau = (\tau^1\tau^2)$, then $\tau_{\mathfrak{A}} = \tau^1_{\mathfrak{A}}\tau^2_{\mathfrak{A}}$.

Lemma 1. *Let $\mathfrak{A} = \langle \mathsf{N}; \theta_{1,0}, \theta_{2,0}, \theta_{1,1}, \theta_{2,1}, \ldots \rangle$ be a partial structure such that $\theta_{1,i}$, $\theta_{2,i}$ are finite functions and for all natural numbers i and x the following conditions hold:*

(a) $x \in M_i \cap \mathrm{Dom}(\theta_{1,2i}) \Rightarrow \theta_{1,2i}(x) \in M_{i+1}$;

(b) $x \in M_{i+1} \cap \mathrm{Dom}(\theta_{1,2i+1}) \Rightarrow \theta_{1,2i+1}(x) \in M_i$;

(c) $x \in M_i \cap \mathrm{Dom}(\theta_{2,2i}) \Rightarrow \theta_{2,2i}(x) \notin M_{i+1}$;

(d) $x \in M_{i+1} \cap \mathrm{Dom}(\theta_{2,2i+1}) \Rightarrow \theta_{2,2i+1}(x) \notin M_{i+1}$;

(e) $x \in [\mathrm{Dom}(\theta_{1,2i}) \cap \mathrm{Dom}(\theta_{2,2i})] \setminus M_i \Rightarrow \theta_{1,2i}(x) \notin M_{i+1} \vee \theta_{2,2i}(x) \in M_{i+1}$;

(f) $x \in [\mathrm{Dom}(\theta_{1,2i+1}) \cap \mathrm{Dom}(\theta_{2,2i+1})] \setminus M_{i+1} \Rightarrow$
$$\theta_{1,2i+1}(x) \notin M_i \vee \theta_{2,2i+1}(x) \in M_i.$$

Then for every term τ which has type (anti-type) $i_0 \to j_0$ and for every partial structure $\mathfrak{A} = \langle \mathsf{N}; \theta'_{1,0}, \theta'_{2,0}, \theta'_{1,1}, \theta'_{2,1}, \ldots \rangle$ such that

$\theta'_{k,i}$ is an extension of $\theta_{k,i}$,

$\theta'_{k,i}$ satisfies the conditions (a)–(f) (when we replace $\theta'_{k,i}$ instead of $\theta_{k,i}$),

$k = 1, 2$; $i \in \mathsf{N}$ and

$x_0 \in M_{i_0} \cap \mathrm{Dom}(\tau_{\mathfrak{A}'})$

it holds $\tau_{\mathfrak{A}'}(x_0) \in M_{j_0}$ $\quad(\tau_{\mathfrak{A}'}(x_0) \notin M_{j_0})$.

Proof. By induction on the length $l(\tau)$ of the term τ.

If $\theta_{1,0}$, $\theta_{2,0}$, $\theta_{1,1}$, $\theta_{2,1}$, ... are total functions, then the conditions (a)–(f) of Lemma 1 ensure the equivalences $(*)$ to be true for all natural numbers i and x.

Lemma 2. *For every partial structure*

$$\mathfrak{A} = \langle \mathsf{N}; \theta_{1,0}, \theta_{2,0}, \theta_{1,1}, \theta_{2,1}, \ldots \rangle,$$

for all natural numbers $i_1, \ldots, i_l$; $j_1, \ldots, j_l$; $x_1, \ldots, x_l$; $y_1, \ldots, y_l$, for every sequence of terms $\tau^1, \ldots, \tau^l$ such that the following nine properties hold:

1) *$\theta_{1,i}$, $\theta_{2,i}$ are finite functions, $i \in \mathsf{N}$;*

2) *For all natural numbers i and x the six conditions* (a)–(f) *of Lemma 1 hold;*

3) *If τ^p has type $i \to j$, then $i \neq j$ or $j = j_p$;*

4) *If τ^p has anti-type $i \to j$, then $i \neq j$ or $j \neq j_p$;*

5) *If $1 \leqq p, q \leqq l$, $\tau\tau^p = \tau^q$, $x_p = x_q$, and τ^p has type $i \to j$, then $i \neq j_p$ or $j = j_q$;*

6) *If $1 \leqq p, q \leqq l$, $\tau\tau^p = \tau^q$, $x_p = x_q$, and τ^p has an anti-type $i \to j$, then $i \neq j_p$ or $j \neq j_q$;*

7) $x_p \in M_{i_p} \setminus \left[\left(\bigcup_{i\in\mathsf{N}} \mathrm{Dom}(\theta_{1,i})\right) \cup \left(\bigcup_{i\in\mathsf{N}} \mathrm{Dom}(\theta_{2,i})\right)\right]$, $p = 1, \ldots, l$;

8) $y_p \in M_{j_p} \setminus \left[\left(\bigcup_{i\in\mathsf{N}} \mathrm{Dom}(\theta_{1,i})\right) \cup \left(\bigcup_{i\in\mathsf{N}} \mathrm{Dom}(\theta_{2,i})\right)\right]$, $p = 1, \ldots, l$;

9) *If* $x_p = x_q$ *and* $\tau^p = \tau^q$, *then* $y_p = y_q$,
there exists a partial structure $\mathfrak{A}' = \langle \mathsf{N}; \theta'_{1,0}, \theta'_{2,0}, \theta'_{1,1}, \theta'_{2,1}, \dots \rangle$ *such that* $\theta'_{k,i}$ *is a finite extension of* $\theta_{k,i}$, $\theta'_{k,i}$ *satisfies the conditions* (a)–(f), $k = 1, 2$; $i \in \mathsf{N}$, $x_p \in \mathrm{Dom}(\tau^p_{\mathfrak{A}'})$ *and* $\tau^p_{\mathfrak{A}'}(x_p) = y_p$, $p = 1, \dots, l$.

Proof. By induction on $\max(l(\tau^1), \dots, l(\tau^l))$.

First, let $\max(l(\tau^1), \dots, l(\tau^l)) = 1$. Then τ^p has either a type or an anti-type. If for example $\tau^p = f_{1,2i}$, then $i_p \neq i$ or $j_p = i+1$, and if for example $\tau^p = f_{2,2i+1}$, then $i+1 \neq i_p$ and $j_p \neq i$, $p = 1, \dots, l$. We can assume that if $p \neq q$, $1 \leqq p, q \leqq l$, then $\tau^p \neq \tau^q$.

If $f_{k,i} = \tau^p$, then we define $\theta'_{k,i}(x_p) = y_p$ and $\theta'_{k,i}(x) = \theta_{k,i}(x)$ for $x \neq x_p$, and if $f_{k,i} \notin \{\tau^1, \dots, \tau^p\}$, then $\theta'_{k,i} = \theta_{k,i}$. It is easy to check that the structure $\mathfrak{A}' = \langle \mathsf{N}; \theta'_{1,0}, \theta'_{2,0}, \theta'_{1,1}, \theta'_{2,1}, \dots \rangle$ is the needed.

Let us assume that Lemma 2 is true whenever $\max(l(\tau^1), \dots, l(\tau^l)) \leqq n$, and $\tau^1, \dots, \tau^l$ be terms such that $\max(l(\tau^1), \dots, l(\tau^l)) = n+1$. We consider all those p such that $x_p = x_1$ and τ^p has the same last symbol as τ_1. For the sake of simplicity we can assume that $x_1 = x_2 = \dots = x_l$ and $\tau^p = \tau''^p f_{1,2i}$ for some term τ''^p or $\tau^p = f_{1,2i}$, $p = 1, \dots, l$. If for some p, $1 \leqq p \leqq l$, $\tau^p = f_{1,2i}$, then we define $\theta''_{1,2i}(x_p) = y_p$ and $\theta''_{1,2i}(x) = \theta_{1,2i}(x)$ for $x \neq x_p$; $\theta''_{k,j} = \theta_{k,j}$ for $(k,j) \neq (1, 2i)$.

If $\tau^p = \tau''^p f_{1,2i}$ for some term τ''^p, $p = 1, \dots, l$, then let x'' be a natural number satisfying the conditions:

— $x'' \notin \{x_1, \dots, x_l, y_1, \dots, y_l\} \cup \left(\bigcup_{i \in \mathsf{N}} \mathrm{Dom}(\theta_{1,2i})\right) \cup \left(\bigcup_{i \in \mathsf{N}} \mathrm{Dom}(\theta_{2,2i})\right)$;

— if $x_1 \in M_i$, then $x'' \in M_{i+1}$;

— if $x_1 \notin M_i$, then $x'' \notin M_q$ for such a q that τ''^p has the last symbol which has the type or anti-type $q \to r$ for some r.

We define $\theta''_{1,2i}(x_1) = x''$, $x''_1 = \dots = x''_l = x''$ and $\theta''_{1,2i}(x) = \theta_{1,2i}(x)$ for $x \neq x_1$; $\theta''_{k,j} = \theta_{k,j}$ for $(k,j) \neq (1, 2i)$. Then for the structure $\mathfrak{A}'' = \langle \mathsf{N}; \theta''_{1,0}, \theta''_{2,0}, \theta''_{1,1}, \theta''_{2,1}, \dots \rangle$, for the natural numbers $i_1, \dots, i_l$; $j_1, \dots, j_l$; $x_1, \dots, x_l$; $y_1, \dots, y_l$, for the terms $\tau''_1, \dots, \tau''_l$ the conditions of Lemma 2 are satisfied and $\max(l(\tau''^1), \dots, i(\tau''^l)) \leqq n$. Then, according the assumption, there exists such a structure $\mathfrak{A}' = \langle \mathsf{N}; \theta'_{1,0}, \theta'_{2,0}, \theta'_{1,1}, \theta'_{2,1}, \dots \rangle$ which we need, and Lemma 2 is proved.

Let us return to the proof of the Theorem 1.

At step s we shall define a partial structure $\mathfrak{A}^s = \langle \mathsf{N}; \theta^s_{1,0}, \theta^s_{2,0}, \theta^s_{1,1}, \theta^s_{2,1}, \dots \rangle$, and at the end we shall define the structure $\mathfrak{A} = \langle \mathsf{N}; \theta_{1,0}, \theta_{2,0}, \theta_{1,1}, \theta_{2,1}, \dots \rangle$.

By the even steps $s = 2n$ we shall ensure that the function $\theta_{k,i}$ is total, $k = 1, 2$; $i \in \mathsf{N}$. By the odd steps $s = 2n+1$, if $n = \langle n_0, i, j \rangle$, $i \neq j$ and n_0 is a number of the finite sequence $(\tau^1, \dots, \tau^{l_1}), \dots, (\tau^{l_{k-1}+1}, \dots, \tau^{l_k})$ of finite sequences of terms, and φ_p is the value of the term τ^p, $p = 1, \dots, l_k$, we shall find an x such that the equivalence (**) does not hold, i. e. for some x we shall satisfy at least one of the following two conditions:

$$\text{(i) } x \in M_i \;\&\; \Big[(\varphi_1(x) \notin M_j \vee \dots \vee \varphi_{l_1}(x) \notin M_j) \;\&\; \dots \;\&\; (\varphi_{l_{k-1}+1}(x) \notin M_j \vee \dots \vee \varphi_{l_k}(x) \notin M_j)\Big];$$

(ii) $x \notin M_i \,\&\, \Big[(\varphi_1(x) \in M_j \,\&\, \ldots \,\&\, \varphi_{l_1}(x) \in M_j) \vee \ldots \vee$

$$(\varphi_{l_{k-1}+1}(x) \in M_j \,\&\, \ldots \,\&\, \varphi_{l_k}(x) \in M_j)\Big].$$

Let us describe the construction.

Case I. $s = 2n$.

We define $\theta^s_{k,p}(x) = \theta^s_{k,p}(x)$ for $x \neq n$ and:

$$\text{if } n \notin \mathrm{Dom}(\theta_{1,2i}), \text{ then } \theta_{1,2i}(n) = \begin{cases} p_{i+1}, & \text{if } n \in M_i, \\ p_i, & \text{otherwise;} \end{cases}$$

$$\text{if } n \notin \mathrm{Dom}(\theta_{1,2i+1}), \text{ then } \theta_{1,2i+1}(n) = \begin{cases} p_i, & \text{if } n \in M_{i+1}, \\ p_{i+1}, & \text{otherwise;} \end{cases}$$

$$\text{if } n \notin \mathrm{Dom}(\theta_{2,2i}), \text{ then } \theta_{2,2i}(n) = \begin{cases} p_i, & \text{if } n \in M_i, \\ p_{i+1}, & \text{otherwise;} \end{cases}$$

$$\text{if } n \notin \mathrm{Dom}(\theta_{2,2i+1}), \text{ then } \theta_{2,2i+1}(n) = \begin{cases} p_{i+1}, & \text{if } n \in M_{i+1}, \\ p_i, & \text{otherwise.} \end{cases}$$

Case II. $s = 2n + 1$.

If not $\mathrm{Seq}(n)$ or $(n)_1 = (n)_2$, then we do nothing, i. e. $\theta^s_{k,p} = \theta^{s-1}_{k,p}$, $k = 1, 2$, $p \in \mathbb{N}$. If $n = \langle n_0, i, j\rangle$ and $i \neq j$, let n_0 be the number of the finite sequence of finite sequences of terms $(\tau^1, \ldots, \tau^{l_1})$, $\ldots$, $(\tau^{l_{k-1}+1}, \ldots, \tau^{l_k})$. We consider two subcases:

Subcase A. There exists a natural number m, $0 \leq m < k$, such that if $l_m + 1 \leq p, q \leq l_{m+1}$ and $\tau\tau^p = \tau^q$, then for some term τ has a type (anti-type) $k \to l$, then $k \neq j$ or $l = j$ ($k \neq j$ or $l \neq j$). We can assume that $m = 0$ and $l_m = 0$.

Let i_0 be an integer such that $i_0 \neq i$ and $i_0 \neq i'$, and τ^p has a type or anti-type $i' \to j'$ for some j', $1 \leq p \leq l_1$. Let in addition

$$i_1 = \cdots = i_{l_1} = i_0, \;\; j_1 = \cdots = j_{l_1} = j,$$

$$x_1 = \cdots = x_{l_1} = x_0 \in M_{i_0} \setminus \left[\Big(\bigcup_{i \in \mathbb{N}} \mathrm{Dom}(\theta^{s-1}_{1,2i})\Big) \cup \Big(\bigcup_{i \in \mathbb{N}} \mathrm{Dom}(\theta^{s-1}_{2,2i})\Big)\right],$$

$$y_1, \ldots, y_l \in M_j \setminus \left[\Big(\bigcup_{i \in \mathbb{N}} \mathrm{Dom}(\theta^{s-1}_{1,2i})\Big) \cup \Big(\bigcup_{i \in \mathbb{N}} \mathrm{Dom}(\theta^{s-1}_{2,2i})\Big)\right].$$

It is easy to check that the structure $\mathfrak{A}^{s-1} = \langle \mathbb{N}; \theta^{s-1}_{1,0}, \theta^{s-1}_{2,0}, \theta^{s-1}_{1,1}, \theta^{s-1}_{2,1}, \ldots\rangle$, the natural numbers $i_1, \ldots, i_{l_1}$; $j_1, \ldots, j_{l_1}$; $x_1, \ldots, x_{l_1}$; $y_1, \ldots, y_{l_1}$, the terms $\tau^1, \ldots, \tau^{l_1}$ satisfy the conditions of Lemma 2. Therefore, there exists such a structure $\mathfrak{A}^s = \langle \mathbb{N}; \theta^s_{1,0}, \theta^s_{2,0}, \theta^s_{1,1}, \theta^s_{2,1}, \ldots\rangle$ that $\theta^s_{k,p}$ satisfies the conditions (a)–(f) (if we replace $\theta^s_{k,p}$ instead of $\theta_{k,p}$), $k = 1, 2$, $p \in \mathbb{N}$ and $\tau^p_{\mathfrak{A}} s(x_p) = y_p \in M_j$, $p = 1, \ldots, l_1$. It is clear that this structure can be defined effectively.

Subcase B. Assume that subcase A does not hold.

Then we do nothing, i. e. $\theta^s_{k,p} = \theta^{s-1}_{k,p}$, $k = 1, 2$, $p \in \mathbb{N}$.

The construction is completed.

It is easy to check that the following lemmas are correct.

Lemma 3. *$\theta_{k,p}$ is a total recursive function, $k = 1, 2$, $p \in \mathbb{N}$.*

Lemma 4. *$\theta_{k,p}$ satisfies the equivalences $(*)$, $k = 1, 2$, $p \in \mathbb{N}$.*

Lemma 5. *If $i \neq j$ and $(\varphi_1, \ldots, \varphi_{l_1}), \ldots, (\varphi_{l_{k-1}+1}, \ldots, \varphi_{l_k})$ is an arbitrary sequence of finite sequences of* id *or compositions of the functions $\theta_{1,0}, \theta_{2,0}, \theta_{1,1}, \theta_{2,1}, \ldots$, then there exists an x such that (**) does not hold.*

The proof of Theorem 1 is completed.

Let $\varphi_{k,i}(x) = \langle i, k, x\rangle$, $k = 1, 2$; $i, x \in \mathbb{N}$, and

$$\mathbb{N}_0 = \mathbb{N} \setminus \left[\left(\bigcup_{i \in \mathbb{N}} \mathrm{Ran}(\varphi_{1,i}) \cup \left(\bigcup_{i \in \mathbb{N}} \mathrm{Ran}(\varphi_{2,i}) \right) \right] .$$

Definition 5. Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of disjoint subsets of $\mathbb{N}_0$. We define the sequence $\{[A_k]\}_{k \in \mathbb{N}}$ of disjoint sets of natural numbers by the following rules:

(a) If $p \in A_i$, then $p \in [A_i]$;

(b) If $1 \leqq l \leqq 2$, $i \in \mathbb{N}$, $p \in [A_k]$ and $\theta_{l,i}(k) = n$, then $\varphi_{l,i}(p) \in [A_n]$.

Lemma 6. *If $\{A_k\}_{k \in \mathbb{N}}$ is a recursive (r.e.) sequence of disjoint subsets of $\mathbb{N}_0$, then $\{[A_k]\}_{k \in \mathbb{N}}$ is a recursive (r.e.) sequence of disjoint sets.*

Lemma 7. *For every natural number x, either $x \in \mathbb{N}_0$ or there exists an effective way to find a function φ which is a composition of the functions $\varphi_{1,0}, \varphi_{2,0}, \varphi_{1,1}, \varphi_{2,1}, \ldots$, and a $y \in \mathbb{N}_0$ such that $\varphi(y) = x$.*

Proof. Using induction on $|x|$, where $|x| = 0$ if $x \in \mathbb{N}_0$ and $|\langle k, i, y\rangle| = |y| + 1$, one can easily verify that Lemma 7 is true.

Lemma 8. *Let $\{A_k\}_{k \in \mathbb{N}}$ be a sequence of disjoint subsets of $\mathbb{N}_0$. Then*

(a) *For any function φ, which is a composition of the functions $\varphi_{1,0}, \varphi_{2,0}, \varphi_{1,1}, \varphi_{2,1}, \ldots$, and for any natural number i there exists such a k that $\varphi([A_i]) \subseteq [A_k]$.*

(b) *For any function φ, which is a composition of the functions $\varphi_{1,0}, \varphi_{2,0}, \varphi_{1,1}, \varphi_{2,1}, \ldots$, and for all distinct natural numbers i, j there exists an effective way to verify whether or not $\varphi([A_i]) \subseteq [A_j]$.*

The proof is immediate.

From now on if $\{A_k\}_{k \in \mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{N}_0$, then we denote by B_k the set $\bigcup_{i \in M_k} [A_i]$. It is obvious that if $\{A_k\}_{k \in \mathbb{N}}$ is an r.e. sequence, then $\{B_k\}_{k \in \mathbb{N}}$ is r.e. too.

Lemma 9. *If $\{A_k\}_{k \in \mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{N}_0$, then the following equivalences hold for any natural numbers i and x:*

$$x \in B_i \iff \varphi_{1,2i}(x) \in B_{i+1} \;\&\; \varphi_{2,2i}(x) \notin B_{i+1};$$

$$x \in B_{i+1} \iff \varphi_{1,2i+1}(x) \in B_i \;\&\; \varphi_{2,2i+1}(x) \notin B_i .$$

Proof. This lemma immediately follows from Theorem 1, Lemma 8 and the definitions of A_k, B_k, $k \in \mathbb{N}$.

Corollary. *If $\{A_k\}_{k \in \mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{N}_0$, then the set B_i is* btt*-equivalent to B_j for all natural numbers i and j.*

Lemma 10. *If $\{A_k\}_{k \in \mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{N}_0$, and $i \neq j$, then for every sequence $(\psi_1, \ldots, \psi_{l_1}), \ldots, (\psi_1, \ldots, \psi_{l_1})$ of finite sequences of* id *or* *compositions of the functions $\varphi_{1,0}, \varphi_{2,0}, \varphi_{1,1}, \varphi_{2,1}, \dots$, there exists an effective way to find l, such that for every $x \in A_l$ the equivalence*

$$(***) \qquad x \in B_i \iff (\psi_1(x) \in B_j \;\&\; \dots \;\&\; \psi_{l_1}(x) \in B_j) \vee \dots \vee (\psi_{l_{k-1}+1}(x) \in B_j \;\&\; \dots \;\&\; \psi_{l_k}(x) \in B_j)$$

does not hold.

Proof. This lemma follows again immediately from Theorem 1, Lemma 8 and the definitions of A_k, B_k, $k \in \mathbb{N}$.

Let $\mathbb{N}_0 = \mathbb{N}_1 \cup \mathbb{N}_2$, where $\mathbb{N}_1$ and $\mathbb{N}_2$ are infinite disjoint recursive sets and r' is a monotonically increasing function such that $\mathrm{Ran}(r') = \mathbb{N}_1$ and $r(n) = r'[n.(n+1)/2+n]$.

Additionally, let Φ be a partial recursive function (p.r.f.) which is universal for all unary p.r.f. Let $\Phi_e = \lambda x.\Phi(e,x)$ and $\Phi_{e,s}$ be a finite p.r. approximation of Φ_e, i. e.

$$\Phi_{e,s}(x) = \begin{cases} \Phi_e(x), & \text{if } x \in \mathrm{Dom}(\Phi_e) \;\&\; \Phi_e(x) \text{ is computable in less than } s \text{ steps,} \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

Theorem 2. *There exists an r.e.* btt-*degree which contains an infinite antichain of r.e.* p-*degrees.*

Proof. In order to construct a btt-degree containing infinitely many mutually incomparable p-degrees, we shall construct an r.e. sequence $\{A_k\}_{k\in\mathbb{N}}$ of disjoint subsets of $\mathbb{N}_0$ such that if $i \neq j$, then B_i and B_j are p-incomparable. Then it will follow from Corollary to Lemma 9 that the set B_i has the same btt-degree as the set B_j. Therefore, the proof will be completed.

We construct the sets $\{A_k\}_{k\in\mathbb{N}}$ by steps, building a finite approximation $A_{i,s}$ of A_i, $i \in \mathbb{N}$, on step s. (We shall denote the set $\bigcup_{k\in M_i} [A_{k,s}]$ by $B_{i,s}$.)

At step s, if $(s)_0 = \langle e,i,j\rangle$ and $i \neq j$, our aim is to satisfy the condition that the function Φ_e does not p-reduce B_i to B_j, i. e. to find such an $x \in \mathrm{Dom}(\Phi_e)$ that $\mathrm{Seq}(\Phi_e(x))$, $\forall k[k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \mathrm{Seq}((\Phi_e(x))_k)]$, and at least one of the following two conditions is satisfied:

(i) $x \notin B_i \;\&\; \exists k\{k < \mathrm{lh}(\Phi_e(x)) \;\&\; \forall l[l < \mathrm{lh}((\Phi_e(x))_k) \Rightarrow ((\Phi_e(x))_k)_l \in B_j]\}$;

(ii) $x \in B_i \;\&\; \forall k\{k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \exists l[l < \mathrm{lh}((\Phi_e(x))_k) \;\&\; ((\Phi_e(x))_k)_l \notin B_j]\}$.

For this purpose, on step s, if we find such an x, which satisfies (i), then we would like to put it outside B_i, and if we find an x which satisfies (ii), then we would like to put it in B_i.

If at step s x is placed in some set A_k in order to satisfy either (i) or (ii), then we create an $(s)_0$-*requirement* x. In this case, if x satisfies (ii), then we shall need also some elements $y_1, \dots, y_p$ which do not belong to any set $[A_k]$. So we create a *negative* $(s)_0$-*requirement* $\{y_1, \dots, y_p\}$. To guarantee that, for any e, such that Φ_e is total and satisfies the conditions $\forall x[\mathrm{Seq}(\Phi_e(x))]$ and $\forall k[k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \mathrm{Seq}((\Phi_e(x))_k)]$, and for every i, j, such that $i \neq j$, there exists an x satisfying either (i) or (ii), we shall use a priority argument, so that the smaller $(s)_0$ will have priority.

If x is an $(s)_0$-requirement and $\{y_1, \ldots, y_p\}$ is a negative $(s)_0$-requirement created at step s, and till step t the condition (ii) which is satisfied at step s is not injured, then we shall say that the $(s)_0$-requirement and the negative $(s)_0$-requirement are *active* at step t.

If an $(s)_0$-requirement x satisfies (i), then we call it *active* at every step $t > s$.

If an $(s)_0$-requirement (a negative $(s)_0$-requirement) created at step s is active at every step $t > s$, then we say that it is *constant*.

Now we can describe the construction of the sequence $\{A_k\}_{k \in \mathbb{N}}$.

Step $s = 0$. Let $\mathbb{N}_2 = \{a_0, a_1, \ldots\}$, where $a_0 < a_1 < \ldots$; we take $A_{i,0} = \{a_i\}$. Thus it is ensured that A_i is nonempty.

Step $s > 0$. If not $\mathrm{Seq}((s)_0)$ or $[\mathrm{Seq}((s)_0)$ and $((s)_0)_1 = ((s)_0)_2]$, then we do nothing, i. e. we take $A_{i,s} = A_{i,s-1}$, $i \in \mathbb{N}$, and do not create any requirements.

If $\mathrm{Seq}((s)_0)$ and $(s)_0 = \langle e, i, j \rangle$, where $i \neq j$, we verify whether an active $(s)_0$-requirement exists. If there exists such a requirement, then we do nothing.

If such a requirement does not exist, then we verify whether there exists an $x \in \mathbb{N}_1$ such that

$$x > r((s)_0), \quad x \in \mathrm{Dom}(\Phi_{e,s}), \quad \mathrm{Seq}(\Phi_e(x)),$$

$$\forall k[k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \mathrm{Seq}((\Phi_e(x))_k)], \quad x \notin \bigcup_{i \in \mathbb{N}} A_{i,s-1},$$

and x does not belong to any active negative requirement, created at a step $t < s$ such that $(t)_0 < (s)_0$. If such an x does not exist, then we do nothing.

Otherwise we denote by x_s the least such x and create the $(s)_0$-requirement x_s. Let

$$\Phi_e(x_s) = \langle \langle z_1, \ldots, z_{l_1} \rangle, \ldots, \langle z_{l_{k-1}+1}, \ldots, z_{l_k} \rangle \rangle, \quad \psi_p(y_p) = z_p,$$

where either ψ_p is a composition of the functions $\varphi_{1,0}$, $\varphi_{2,0}$, $\varphi_{1,1}$, $\varphi_{2,1}$, $\ldots$, or $\psi_k = \mathrm{id}$, $1 \leqq p \leqq l_k$, and $y_1, \ldots, y_{l_k} \in \mathbb{N}_0$. We verify, whether there exist natural numbers $z_{i_1}, \ldots, z_{i_k}$ such that $z_{i_p} \notin B_{j,s-1}$, $l_{p-1}+1 \leqq i_p \leqq l_p$ and $x_s \neq y_{i_p}$, $p = 1$, $\ldots$, k. If yes, then

$$A_{p_i,s} = A_{p_i,s-1} \cup \{x_s\}, \quad A_{l,s} = A_{l,s-1} \quad \text{for } l \neq p_i, \ l \in \mathbb{N},$$

and if $\{y_{i_1}, \ldots, y_{i_k}\} \setminus \left(\bigcup_{l \in \mathbb{N}} A_{l,s-1} \cup \{x_s\} \right)$ is nonempty, we create a negative $(s)_0$-requirement

$$\{y_{i_1}, \ldots, y_{i_k}\} \setminus \left(\bigcup_{l \in \mathbb{N}} A_{l,s-1} \cup \{x_s\} \right).$$

Otherwise we consider all those p, $1 \leqq p \leqq k$, such that there does not exist an i_p such that $z_{i_p} \notin B_{j,s-1}$, $l_{p-1} + 1 \leqq i_p \leqq l_p$, and $x_s \neq y_{i_p}$. Let us assume in this case that all these p are $1, \ldots, q$ and $i_{q+1}, \ldots, i_k$ are such that $z_{i_p} \notin B_{j,s-1}$, $l_{p-1} + 1 \leqq i_p \leqq l_p$, $y_{i_p} \neq x_s$, $p = q+1, \ldots, k$. For any p, $1 \leqq p \leqq q$, we consider all those i such that $l_{p-1} + 1 \leqq i \leqq l_p$ and $y_i = x_s$. We assume that for any p, $1 \leqq p \leqq q$, all those i, such that $l_{p-1} + 1 \leqq i \leqq l_p$ and $y_i = x_s$, are $l_{p-1}+1, \ldots, l_p$.

According to Lemma 10 there exists an l such that if $p \in A_l$, then the equivalence $(***)$ does not hold if we replace k by q. We define

$$A_{l,s} = A_{l,s-1} \cup \{x_s\} \text{ and } A_{p,s} = A_{p,s-1} \text{ for } p \neq l, \ p \in \mathbb{N},$$

and we create a negative $(s)_0$-requirement

$$\{y_{i_{q+1}}, \dots, y_{i_k}\} \setminus \left(\bigcup_{l \in \mathbb{N}} A_{l,s-1} \cup \{x_s\} \right)$$

if $\{y_{i_{q+1}}, \dots, y_{i_k}\} \setminus \left(\bigcup_{l \in \mathbb{N}} A_{l,s-1} \cup \{x_s\} \right)$ is nonempty.

Finally, we take $A_i = \bigcup_{s \in \mathbb{N}} A_{i,s}$.

Obviously, this construction is effective, hence the sequence $\{A_k\}_{k \in \mathbb{N}}$ is r.e. Moreover, $\{A_k\}_{k \in \mathbb{N}}$ is a sequence of disjoint subsets of $\mathbb{N}_0$, since one element may be placed in only one A_k.

In order to show that this construction works, we need some lemmas. Let $A = \bigcup_{i \in \mathbb{N}} A_i$.

Lemma 11. *The set* $\mathbb{N}_1 \setminus A$ *is infinite.*

Proof. Let $(\mathbb{N}_1)_n = \{x \mid x \in \mathbb{N}_1 \,\&\, x < r'(n)\}$. We prove that the set $(\mathbb{N}_1)_{r(n)} \cap (\mathbb{N}_1 \setminus A)$ contains at least n elements or, equivalently, $|(\mathbb{N}_1)_{r(n)} \cap A| \leq n.(n+1)/2$.

Indeed, for every $\langle e, i, j\rangle$, $i \neq j$, we have no more than $\langle e, i, j\rangle + 1$ $\langle e, i, j\rangle$-requirements and each of them is greater than $r(\langle e, i, j\rangle)$ and belongs to some $A_k \subseteq A$. Therefore, in $(\mathbb{N}_1)_{r(n)} \cap A$ there are only m-requirements for $m < n$, i. e. in $(\mathbb{N}_1)_{r(n)} \cap A$ there are no more than $1 + 2 + \dots + n = n.(n+1)/2$ elements.

Lemma 11 is proved.

Lemma 12. *The set* $\mathbb{N}_1 \setminus A$ *is immune.*

Proof. Let us assume that there exists a set $C \subseteq \mathbb{N}_1 \setminus A$ which is infinite and r.e., and $x_0 \in \mathbb{N}_2$. Obviously,

$$f(x) = \begin{cases} \langle\langle x_0 \rangle\rangle, & \text{if } x \in C, \\ \text{undefined}, & \text{otherwise} \end{cases}$$

is a p.r.f. Let e be a natural number such that $f = \Phi_e$, and let $x \in \mathrm{Dom}(f)$ such that $x > r(\langle e, 0, 1\rangle)$ and s_0 is the least s which satisfies the equality $\Phi_{e,s}(x) = f(x)$. Then x must be an $\langle e, 0, 1\rangle$-requirement created at some step $s > s_0$ such that $(s)_0 = \langle e, 0, 1\rangle$, i. e. $C \cap A$ is nonempty. This contradicts the assumption. Therefore, $\mathbb{N}_1 \setminus A$ is immune.

Lemma 13. *For any natural number e such that* $\mathbb{N}_1 \subseteq \mathrm{Dom}(\Phi_e)$ *and*

$$\forall x \{x \in \mathbb{N}_1 \Rightarrow \mathrm{Seq}(\Phi_e(x)) \,\&\, \forall k [k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \mathrm{Seq}((\Phi_e(x))_k)]\},$$

and for all distinct i, j, there exists a constant $\langle e, i, j\rangle$-requirement.

Proof. Assume that there is no constant $\langle e, i, j\rangle$-requirement, where $i \neq j$, $\mathbb{N}_1 \subseteq \mathrm{Dom}(\Phi_e)$ and

$$\forall x \{x \in \mathbb{N}_1 \Rightarrow \mathrm{Seq}(\Phi_e(x)) \,\&\, \forall k [k < \mathrm{lh}(\Phi_e(x)) \Rightarrow \mathrm{Seq}((\Phi_e(x))_k)]\}.$$

We find an s_0 such that if $s \geqq s_0$ and $\langle e_1, i_1, j_1\rangle < \langle e, i, j\rangle$, then every $\langle e_1, i_1, j_1\rangle$-requirement is already created. Moreover, let $x \in \mathbb{N}_1 \setminus A$, $x > r(\langle e, i, j\rangle)$ and s be such that $s \geqq s_0$, $\Phi_{e,s}(x) = \Phi_e(x)$ and $(s)_0 = \langle e, i, j\rangle$. Then on step s a constant $\langle e, i, j\rangle$-requirement x is created.

Lemma 13 is proved.

Now we shall prove Theorem 2. Let us assume that $B_i \leq_{\mathrm{p}} B_j$ and $i \neq j$. Therefore, there exists a total recursive function f such that

$$\forall x\{\mathrm{Seq}(f(x))\,\&\,\forall k[k < \mathrm{lh}(f(x)) \Rightarrow \mathrm{Seq}((f(x))_k)]\},$$

and

$$\forall x\{x \in B_i \iff \exists k[k < \mathrm{lh}(f(x))\,\&\,\forall l(l < \mathrm{lh}((f(x))_k) \Rightarrow ((f(x))_k)_l \in B_j)]\}.$$

Let e be an index such that $\Phi_e = f$. It follows from Lemma 13 that there exists a constant $\langle e, i, j\rangle$-requirement x_s created at step s. Then $x_s \in \mathbb{N}_1$,

$$f(x_s) = \langle\langle z_1, \ldots, z_{l_1}\rangle, \ldots, \langle z_{l_{k-1}+1}, \ldots, z_{l_k}\rangle\rangle, \quad \psi_p(y_p) = z_p,$$

where ψ_p is either a composition of the functions $\varphi_{1,0}$, $\varphi_{2,0}$, $\varphi_{1,1}$, $\varphi_{2,1}$, …, or $\psi_p = \mathrm{id}$, $p = 1, \ldots, l_k$, and $y_1, \ldots, y_{l_k} \in \mathbb{N}_0$. We assume that there exist natural numbers $z_{i_1}, \ldots, z_{i_k}$ such that $z_{i_p} \notin B_{j,s-1}$, $l_{p-1} + 1 \leqq i_p \leqq l_p$ and $x_s \neq y_{i_p}$, $p = 1, \ldots, k$. This contradicts the fact that the function f p-reduces B_i to B_j. Therefore, there exists a p, $1 \leqq p \leqq k$, such that there does not exist i_p such that $z_{i_p} \notin B_{j,s-1}$, $l_{p-1} + 1 \leqq i_p \leqq l_p$ and $x_s \neq y_{i_p}$. Let all those p be $1, \ldots, q$ and $z_{i_p} \notin B_{j,s-1}$, $l_{p-1} + 1 \leqq i_p \leqq l_p$, and $x_s \neq y_{i_p}$, $p = q+1, \ldots, k$. For the sake of simplicity we shall assume that $y_1 = \cdots = y_{l_q} = x_s$. Then it is easy to check that x_s does not satisfy the condition (p) if we replace x with x_s, A with B_i, and B with B_j, which contradicts the fact that f p-reduces B_i to B_j.

Theorem 2 has been proved.

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FACTORIZATIONS OF THE GROUPS $PSp_6(q)$ *

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Цанко Генчев, Еленка Генчева. ФАКТОРИЗАЦИИ ГРУПП $PSp_6(q)$

Доказан следующий результат:

Пусть $G = PSp_6(q)$ и $G = AB$, где A, B — собственные неабелевые простые подгруппы G. Тогда имеет место одно из следующих:

(1) $q = 2$ и $A \cong U_3(3)$, $B \cong U_4(2)$;

(2) $q = 4$ и $A \cong J_2$, $B \cong U_4(4)$;

(3) $q = 2^n$ и $A \cong L_2(q^3)$, $B \cong L_4(q)$ или $U_4(q)$;

(4) $q = 2^n > 2$ и $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ или $U_4(q)$.

Tsanko Gentchev, Elenka Gentcheva. FACTORIZATIONS OF THE GROUPS $PSp_6(q)$

The following result is proved.

Let $G = PSp_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G Then one of the following holds:

(1) $q = 2$ and $A \cong U_3(3)$, $B \cong U_4(2)$;

(2) $q = 4$ and $A \cong J_2$, $B \cong U_4(4)$;

(3) $q = 2^n$ and $A \cong L_2(q^3)$, $B \cong L_4(q)$ or $U_4(q)$;

(4) $q = 2^n > 2$ and $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ or $U_4(q)$.

INTRODUCTION

In [5, 6] we determined all the factorizations with two proper simple subgroups of some groups of Lie type of Lie rank 3. In the present work we extend this

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investigation to the simple groups $PSp_6(q)$ of Lie type (C_3) over the finite field $GF(q)$. We prove the following

Theorem. *Let $G = PSp_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 2$ *and* $A \cong U_3(3)$, $B \cong U_4(2)$;

(2) $q = 4$ *and* $A \cong J_2$, $B \cong U_4(4)$;

(3) $q = 2^n$ *and* $A \cong L_2(q^3)$, $B \cong L_4(q)$ *or* $U_4(q)$;

(4) $q = 2^n > 2$ *and* $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ *or* $U_4(q)$.

The factorizations of $PSp_6(q)$ into the product of two maximal subgroups have been determined in [8]. We make use of this result here.

Our notation is standard. Basic information on the known simple groups can be found in [2, 3].

In the proof below we shall freely use the following directly verified properties of the group $G = PSp_6(q)$, $q = 2^n$. Using the symplectic realization

$$G = \{X \in GL_6(q) \mid X^tTX = T\}, \quad \text{where } T = \begin{pmatrix} O & E \\ E & O \end{pmatrix}$$

(E is the indentity matrix), G has four conjugacy classes of involutions denoted here (2_1), (2_2), (2_3), (2_4) with representatives

$$i_1 = \begin{pmatrix} E & P \\ O & E \end{pmatrix}, \quad i_2 = \begin{pmatrix} E & Q \\ O & E \end{pmatrix}, \quad i_3 = \begin{pmatrix} E & E \\ O & E \end{pmatrix}, \quad i_4 = \begin{pmatrix} E & R \\ O & E \end{pmatrix},$$

respectively, where

$$P = \operatorname{diag}(1,0,0), \quad Q = \operatorname{diag}(1,1,0), \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The only involutions of G which are squares (of elements of order 4) are those in the classes (2_2) and (2_4). Further, $|C_G(i_2)| = q^9(q^2-1)$ and $|C_G(i_4)| = q^9(q^2-1)^2$. Lastly, G has no elementary Abelian subgroup of order q^4 all of whose involutions are in the class (2_4).

PROOF OF THE THEOREM

Let $G = PSp_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. The factorizations of $PSp_6(2)$ and $PSp_6(3)$ are determined in [1, 7]; this gives (1) and (3) (with $n = 1$) of the theorem. Thus we can assume that $q \geqq 4$. The list of maximal factorizations of G is given in [8]. This leads, by order considerations, to the following possibilities, where $q = 2^n$:

1) $A \cong L_4(q)$, $B \cong U_3(q)$;

2) $A \cong U_4(q)$, $B \cong L_3(q)$;

3) $A \cong U_4(q)$, $B \cong G_2(\sqrt{q})$, n even > 2;

4) $A \cong L_2(q^3)$, $B \cong L_4(q)$ or $U_4(q)$;

5) $A \cong G_2(q)$, $B \cong PSp_4(q)$, $L_4(q)$ or $U_4(q)$;

6) $A \cong J_2$, $B \cong U_4(4)$, $n = 2$.

Note that $PSp_6(2^n)$ contains subgroups isomorphic to $U_3(2^n)$ or $L_3(2^n)$ if and only if n is even or odd, respectively.

We consider these possibilities case by case.

Cases 1), 2), 3). Here $|A \cap B| = q^2-1$, q^2-1, and $q-1$, respectively. Each of the groups $U_3(q)$, $L_3(q)$, and $G_2(\sqrt{q})$ has elements of order 4 and hence B contains involutions from at least one of the classes (2_2) and (2_4) of G. Let $L_4^\varepsilon(q)$ denote $L_4(q)$ if $\varepsilon = +$ and $U_4(q)$ if $\varepsilon = -$. Then any subgroup $A \cong L_4^\varepsilon(q)$ of G contains involutions from each of the classes (2_2) and (2_4). Indeed, A has two conjugacy classes of involutions, central and noncentral, which are squares and so they belong to (2_2) or (2_4). The central involutions in A have centralizers of order divisible by $(q-\varepsilon 1)^2$, whence they are from the class (2_4). But $L_4^\varepsilon(q)$ has an elementary Abelian subgroup of order q^4. As G has no elementary Abelian subgroup of order q^4 with involutions only from (2_4), it follows that A contains involutions also from (2_2). Thus A and B necessarily have a common involution, which implies $2 \mid |A \cap B|$, a contradiction.

Now we prove the existence of the factorizations in the remaining three cases 4), 5), 6).

Case 4). Here we use the following symplectic realization of the group $L_2(q^3)$ in $PSp_6(q)$, $q = 2^n$. Let K be a field extension of $k = GF(q)$ of degree 3. There is an element ω of K such that 1, ω, ω^2 form a basis of K over k, and set $\omega^3 = p_0 + p_1\omega + p_2\omega^2$, where $p_i \in k$ $(i = 0, 1, 2)$. Further, let

$$S = \begin{pmatrix} a_0 + a_1\omega + a_2\omega^2 & b_0 + b_1\omega + b_2\omega^2 \\ c_0 + c_1\omega + c_2\omega^2 & d_0 + d_1\omega + d_2\omega^2 \end{pmatrix}, \quad a_i, b_i, c_i, d_i \in k \ \ (i = 0, 1, 2)$$

be any matrix with $\det S = 1$. Then the following matrices form a subgroup A of G isomorphic to $L_2(q^3)$:

$$W = \begin{pmatrix} a_0 & p_0a_2 & w_{13} & b_0 & b_1 & b_2 \\ a_1 & a_0 + p_1a_2 & w_{23} & b_1 & b_2 + p_0^{-1}(p_2b_0 + p_1b_1) & p_0^{-1}(b_0 + p_1b_2) \\ a_2 & a_1 + p_2a_2 & w_{33} & b_2 & p_0^{-1}(b_0 + p_1b_2) & p_0^{-1}(b_1 + p_2b_2) \\ c_0 & p_0c_2 & w_{61} & d_0 & d_1 & d_2 \\ p_0c_2 & w_{61} & w_{62} & p_0d_2 & d_0 + p_1d_2 & d_1 + p_2d_2 \\ w_{61} & w_{62} & w_{63} & p_0d_1 + p_0p_2d_2 & w_{65} & w_{66} \end{pmatrix},$$

where

$$w_{13} = p_0a_1 + p_0p_2a_2, \quad w_{23} = p_1a_1 + (p_1p_2 + p_0)a_2, \quad w_{33} = a_0 + p_2a_1 + (p_2^2 + p_1)a_2,$$

$$w_{61} = p_0c_1 + p_0p_2c_2, \quad w_{62} = p_0c_0 + p_0p_2c_1 + p_0c_2(p_2^2 + p_1),$$

$$w_{63} = p_0p_2c_0 + p_0c_1(p_2^2 + p_1) + p_0c_2(p_2^3 + p_0), \quad w_{65} = p_1d_1 + d_2(p_0 + p_1p_2),$$

$$w_{66} = d_0 + p_2d_1 + d_2(p_2^2 + p_1).$$

The isomorphism is given by the map $S \mapsto W$.

Now let $B \cong L_4^\varepsilon(q)$ be a subgroup of G. Then $|A \cap B| \geqq q^3 - \varepsilon 1$ and, using the subgroup list of A, we see that $|A \cap B| \leqq 2(q^3 - \varepsilon 1)$. The above realization of A shows that the involutions in A are from the class (2_3) of G. However (as shown above), any $L_4^\varepsilon(q)$ subgroup of G contains involutions only from the classes (2_2) and (2_4). Thus $2 \nmid |A \cap B|$ and then $|A \cap B| = q^3 - \varepsilon 1$, which implies $G = AB$. This is (3) of the theorem.

Case 5). We use the following two realizations of the group $G = PSp_6(q)$, $q = 2^n$:

(i)
$$PSp_6(q) = \{X \in GL_6(q) \mid X^t H X = H\},$$

where $H = \mathrm{diag}(J, J, J)$, $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(ii)
$$PSp_6(q) = \{Y \in GL_6(q) \mid Y^t I Y = I\},$$

where $I = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$.

Let $X, Y \in GL_6(q)$ and $Y = T_0^{-1} X T_0$, where

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = T_0^{-1}.$$

Then $Y^t I Y = I$ if and only if $X^t H X = H$.

Now, with respect to (i), we have

$$\left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & \begin{matrix} 1 & \\ & 1 \end{matrix} \end{array} \right) \in PSp_6(q) \right\} \cong PSp_4(q).$$

On the other hand (see [4]), with respect to (ii), a $G_2(q)$ subgroup of $PSp_6(q)$ is generated by the matrices $X_{\pm r}(t)$, $r \in \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$, $t \in GF(q)$, where

$$X_a(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_b(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{2a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{3a+2b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix $X_{-r}(t)$ is the transpose of $X_r(t)$. Now a direct computation shows that the common elements of the above $PSp_4(q)$ and $G_2(q)$ subgroups are exactly as follows:

$$T_0 \begin{pmatrix} w & 0 & 0 & u^2+vw & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & u+vw \\ u^{-1} & 0 & 0 & u^{-1}v & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & u^{-2} & 0 & 0 & u^{-2}v \end{pmatrix} T_0 \quad (u \in GF(q)^*, \ v, w \in GF(q)),$$

$$T_0 \begin{pmatrix} u^2 & 0 & 0 & v & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & v \\ 0 & 0 & 0 & u^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u^{-2} \end{pmatrix} T_0 \quad (u \in GF(q)^*, \ v \in GF(q)).$$

Hence $|PSp_4(q) \cap G_2(q)| = q(q^2-1)$ (in fact,

$$PSp_4(q) \cap G_2(q) = T_0 \langle X_{3a+b}(t), X_{-(3a+b)}(t) \rangle T_0 \cong L_2(q)).$$

Now order consideration imply

$$PSp_6(q) = G_2(q).PSp_4(q).$$

This is the first factorization in (4) of the theorem.

Now let $A \cong G_2(q)$ be the subgroup of G described in the above paragraph, and $B_1 \cong O_6^\epsilon(q) \cong L_4^\epsilon(q).2$ be a subgroup of G. Then $G = AB_1$ and $A \cap B_1 \cong SL_3^\epsilon(q).2$ (see [8]). Let B be the $L_4^\epsilon(q)$ subgroup in B_1. Now A has two conjugacy classes of involutions — central and non-central. It is not difficult to see that these involutions are from the classes (2_4) and (2_3) of G, respectively. Further, A has a single class of $SL_3^\epsilon(q).2$ subgroups (cf. [4]) and every such subgroup contains non-central involutions. Consequently, every $SL_3^\epsilon(q).2$ subgroup of A contains involutions from (2_3). But (as we have seen) B has no involutions from (2_3). Thus $A \cap B$ is a proper subgroup of $SL_3^\epsilon(q).2$. Then (by order considerations) $G = AB$; in particular, $A \cap B \cong SL_3^\epsilon(q)$. This gives the remaining two factorizations in (4) of the theorem.

Case 6). Now $q = 4$. In case 5) we proved that $G = AB$, where $A \cong G_2(4)$, $B \cong U_4(4)$, and $D = A \cap B \cong U_3(4)$. Take a subgroup $C \cong J_2$ of A. Then (as shown in [9]) $A = CD$. It follows that $|B \cap C| = |D \cap C| = 150$. This implies $G = BC$, the factorization in (2) of the theorem.

This completes the proof of the theorem.

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FACTORIZATIONS OF THE GROUPS $PSU_6(q)$ *

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Цанко Генчев, Еленка Генчева. ФАКТОРИЗАЦИИ ГРУПП $PSU_6(q)$

Доказан следующий результат:
Пусть $G = PSU_6(q)$ и $G = AB$, где A, B — собственные неабелевые простые подгруппы G. Тогда имеет место одно из следующих:
(1) $q = 2$ и $A \cong M_{22}$, $B \cong PSU_5(2)$;
(2) $q = 2$ и $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;
(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ и $A \cong PSU_5(q)$, $B \cong G_2(q)$;
(4) $q \not\equiv -1 \pmod 5$ и $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

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The following result is proved.
Let $G = PSU_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:
(1) $q = 2$ and $A \cong M_{22}$, $B \cong PSU_5(2)$;
(2) $q = 2$ and $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;
(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ and $A \cong PSU_5(q)$, $B \cong G_2(q)$;
(4) $q \not\equiv -1 \pmod 5$ and $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

1. INTRODUCTION

In [4], the first author determined all the factorizations (with two proper simple subgroups) of the groups of Lie type of Lie rank 1 or 2. In the present paper we extend this investigation to groups of Lie type of Lie rank 3. Let $PSU_6(q)$ be

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the simple group of Lie type $({}^2A_5)$ over the finite field $GF(q^2)$. Assuming the classification of the finite simple groups, we prove the following result.

Theorem. *Let $G = PSU_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 2$ *and* $A \cong M_{22}$, $B \cong PSU_5(2)$;

(2) $q = 2$ *and* $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;

(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ *and* $A \cong PSU_5(q)$, $B \cong G_2(q)$;

(4) $q \not\equiv -1 \pmod 5$ *and* $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

The factorizations (1) – (4) *exist.*

The factorizations of the groups $PSU_6(q)$ into the product of two maximal subgroups have been determined in [9].

Throughout this paper we use standard group-theoretic notation. Simple group means non-Abelian simple group. $|G|_p$ denotes the order of a Sylow p-subgroup of a group G and $M(G)$ denotes the Schur multiplier of G. Next, A_n is the alternating group of degree n and $L_n(q)$, $U_n(q)$ stand for $PSL_n(q)$, respectively $PSU_n(q)$. Notation and basic information of the (known) simple groups can be found in [1], [2].

The factorizations of the groups $U_6(2)$ and $U_6(3)$ are determined in [5]. This gives (1), (2) and (4) (with $q = 2, 3$) in the theorem. Thus we can assume that $q \geqq 4$.

The following lemmas are needed in the proof of the theorem.

Lemma 1.1 ([7]). *If q is odd, then $SL_6(q)$ does not contain an elementary Abelian subgroup of order* 8 *such that its involutions are conjugate.*

Lemma 1.2. *The group $U_6(q)$ contains a subgroup isomorphic to $U_5(q)$ if and only if $q \not\equiv -1 \pmod 5$.*

Proof. If $q \equiv -1 \pmod 5$, the group $U_5(q)$ has an elementary Abelian subgroup of order 25 all of whose non-identity elements are conjugate. On the other hand, it is easily checked that this is impossible in $U_6(q)$, so $U_6(q)$ cannot contain $U_5(q)$. If $q \not\equiv -1 \pmod 5$, the statement is clear.

If a, b are positive integers and $(a, b) = 1$, then $\mathrm{Ord}_a(b)$ denotes the multiplicative order of b modulo a (i. e. the least positive integer s with $b^s \equiv 1 \pmod a$).

Lemma 1.3 (see [8]). *Let q be a prime power and s a positive integer. Then there exists a prime r such that $\mathrm{Ord}_r(q) = s$ unless $s = 6$ and $q = 2$ or $s = 2$ and q a Mersenne prime.*

2. PROOF OF THE THEOREM

The group $G = PSU_6(q)$, $q = p^n$, p any prime, has order

$$q^{15}(q^2 - 1)(q^3 + 1)(q^4 - 1)(q^5 + 1)(q^6 - 1)/(6, q + 1).$$

Using Lemma 1.3, choose a prime r such that $\mathrm{Ord}_r(p) = 10n$. Then $r = 10nt + 1$ for some $t \geqq 1$. Now $r \mid q^5 + 1$ and hence $r \mid |G|$. We shall suppose that $r \nmid |A|$. We next discuss the possibilities for A.

Let $A \cong A_l$ ($l \geqq 5$). Then $l \geqq r \geqq 11$. Hence A contains a subgroup X isomorphic to $L_2(8)$. As $M(L_2(8)) = 1$, X must embed into $SU_6(q^2)$ and hence into $SL_6(q^2)$. This contradicts Lemma 1.1 if p is odd. Thus $p = 2$. Set $n = 3^\alpha n_1$, where $3 \nmid n_1$. It is directly checked that if $3^\nu \mid |G|$ then $\nu \leqq 5\alpha + 7$. On the other hand, $|A|$ is a multiple of 3^μ, where

$$\mu \geqq \left[\frac{l}{3}\right] + \left[\frac{l}{9}\right] \geqq \left[\frac{10nt+1}{3}\right] + \left[\frac{10nt+1}{9}\right] \geqq 4.3^\alpha.n_1t \geqq 4(2\alpha+2) > 5\alpha+7$$

unless $n = t = 1$ or $n = 3$, $t = 1$. This forces $G = U_6(2)$ and $A \cong A_l$ ($l \geqq 11$) or $G = U_6(8)$ and $A \cong A_l$ ($l \geqq 31$). But $|A_{11}| \nmid |U_6(2)|$ and $|A_{31}| \nmid |U_6(8)|$, whence $|A| \nmid |G|$. This contradiction shows that $A \ncong A_l$.

Let A be a sporadic group (${}^2F_4(2)'$ is excluded by $r \equiv 1 \pmod{10}$). If $p = 2$ then the choice of r implies $r = 11$ (and $n = 1$), $r = 41$ (and $n = 2$), $r = 61$ (and $n = 6$), or $r > 71$. Now it is easily verified that there is no sporadic group of order divisible by at least one of these primes and dividing the order of the corresponding $U_6(2^n)$ (recall $q = 2^n > 2$). So $p > 2$. Now if $A \ncong M_{11}, M_{12}, M_{22}, HS, McL, Suz$ then (recall $|A|$ is divisible by a prime $r \equiv 1 \pmod{10}$). A contains a subgroup Y (possibly $Y = A$) with $|M(Y)|$ prime to 6 and Y contains an E_8 subgroup all of whose involutions are conjugate in Y (see [2]). This contradicts Lemma 1.1. Thus, we have proved that if A is sporadic then $A \cong M_{11}, M_{12}, M_{22}, HS, McL$ or Suz; hence $r = 11$ and consequently $n = 1$, that is, $G = U_6(p)$ for odd p.

Further, let A be a group of Lie type of characteristic $\neq p$. Now [6] leads to $A \cong L_2(11)$. Hence $r = 11$, so $G = U_6(p)$.

Finally, let A be a group of Lie type over the field $GF(q')$, $q' = p^m$ (see [1] for the orders of these groups). If $A \cong A_l(q')$ ($\cong L_{l+1}(q')$), $l \geqq 1$, then $r \mid |A|$ yields $r \mid q'^k - 1$ for some k, $2 \leqq k \leqq l+1$. Our choice of r then implies $10n \mid mk$. Choose a prime r_1 with $\mathrm{Ord}_{r_1}(p) = mk$. This is possible by Lemma 1.3 and $r_1 \mid |G|$ (as obviously $r_1 \mid |A|$). Since $mk \geqq 10n$, necessarily $r_1 \mid q^5 + 1$, i.e. $r_1 \mid q^{10} - 1$ and then the choice of r_1 leads to $mk \mid 10n$. Thus $10n = mk$, $2 \leqq k \leqq l+1$. Now as $|A|_p \leqq |G|_p = p^{15n}$, we have $ml(l+1)/2 \leqq 15n = 3mk/2 \leqq 3m(l+1)/2$. It follows that $l = 1$, $m = 5n$, or $l = 2$, $m = 5n$ or $m = 10n/3$ ($3 \mid n$), or $l = 3$, $m = 5n/2$ ($2 \mid n$), that is, $A \cong L_2(q^5)$, $L_3(q^5)$, $L_3(q^{10/3})$, or $L_4(q^{5/2})$. However, then $|A| \nmid |G|$.

If $A \cong B_l(q')$ ($\cong P\Omega_{2l+1}(q')$) or $A \cong C_l(q')$ ($\cong PSp_{2l}(q')$), $l \geqq 2$, similar arguments produce $A \cong PSp_4(q^{5/2})$ ($2 \mid n$), $PSp_6(q^{5/3})$, or $P\Omega_7(q^{5/3})$ ($3 \mid n$) and again $|A| \nmid |G|$.

Let $A \cong D_l(q')$ or ${}^2D_l(q')$, $l \geqq 4$. It follows (just as above) that $10n = mk$ for some k, $2 \leqq k \leqq 2l$. But then $ml(l-1) \leqq 15n = 3mk/2 \leqq 3ml$ (as $|A|_p \leqq |G|_p$) which forces $l = 4$, $k = 8$, i.e. $A \cong D_4(q^{5/4})$ or ${}^2D_4(q^{5/4})$ ($4 \mid n$) and $|A| \nmid |G|$.

If $A \cong G_2(q')$, we have similarly $6m \leqq 15n$ and $|A|$ then shows that $r \mid q'^6 - 1$ which yields $10n \mid 6m$. Thus $m = 5n/3$ ($3 \mid n$) and $A \cong G_2(q^{5/3})$. But then $|A| \nmid |G|$.

If $A \cong Sz(q')$ (m odd > 1), we have $2m \leqq 15n$ and hence r must divide $q'^2 + 1$ which yields $10n \mid 4m$. Thus $m = 5n$, $5n/2$ or $15n/2$ ($2 \mid n$), that is, $A \cong Sz(q^5)$, $Sz(q^{5/2})$ or $Sz(q^{15/2})$ and again $|A| \nmid |G|$.

Let $A \cong {}^3D_4(q')$ or $A \cong {}^2F_4(q')$ (and m is odd > 1). Then $12m \leqq 15n$ and the choice of r implies that $r \mid q'^8 + q'^4 + 1$ if $A \cong {}^3D_4(q')$ and $r \mid q'^6 + 1$ if $A \cong {}^2F_4(q')$. In either case $r \mid q'^{12} - 1$ whence $10n \mid 12m$. Thus $m = 5n/6$ $(6 \mid n)$ and $A \cong {}^3D_4(q^{5/6})$ or ${}^2F_4(q^{5/6})$. But then $|A| \nmid |G|$.

Let $A \cong {}^2G_2(q')$ $(p = 3,\ m > 1,\ 2 \nmid m)$. Then $3m \leqq 15n$ and hence r must divide $q'^3 + 1$ which yields $10n \mid 6m$. Thus $m = 5n$ and $A \cong {}^2G_2(q^5)$ or $m = 5n/3$ $(3 \mid n)$ and $A \cong {}^2G_2(q^{5/3})$. However, then $|A| \nmid |G|$.

Let $A \cong F_4(q')$, $E_6(q')$, $E_7(q')$, $E_8(q')$, or ${}^2E_6(q')$. Then $|A|_p \leqq p^{15n}$ means $15n \geqq 24m$, $36m$, $63m$, $120m$, or $36m$, respectively. But $r \mid |A|$ implies $r \mid q'^k - 1$ for some k, where $k \leqq 12$, 12, 18, 30, or 18, respectively. This leads to $10n \leqq km$, i. e. $15n \leqq 3mk/2 \leqq 18m$, $18m$, $27m$, $45m$, or $27m$, respectively, a contradiction.

Let $A \cong {}^2A_l(q')$ $(\cong U_{l+1}(q'))$, $l \geqq 2$. Now $r \mid q'^k + (-1)^{k-1}$ for some k, $2 \leqq k \leqq l+1$. If k is even then $r \mid q'^k - 1$ which leads to $10n \mid mk$. As in the case $A \cong A_l(q')$, we have also $mk \mid 10n$. Now $ml(l+1)/2 \leqq 15n = 3mk/2 \leqq 3m(l+1)/2$ whence necessarily $l = 2$, or $l = 3$ and $k = 2$ or $k = 4$, respectively. But then $A \cong U_3(q^5)$ or $A \cong U_4(q^{5/2})$ $(2 \mid n)$ and again $|A| \nmid |G|$. Thus k is odd and $r \mid q'^k + 1$ whence $r \mid q'^{2k} - 1$. This leads to $2mk = 10n$ and then $ml(l+1)/2 \leqq 15n = 3mk \leqq 3m(l+1)$ yields $l = 2$, $m = 5n/3$ $(3 \mid n)$, or $l = 3$, $m = 5n/3$ $(3 \mid n)$, or $l = 4$, $m = n$, or $l = 5$, $m = n$, or $l = 6$, $m = 5n/7$ $(7 \mid n)$. Accordingly, $A \cong U_3(q^{5/3})$, $U_4(q^{5/3})$, $U_5(q)$, $U_6(q)$, or $U_7(q^{5/7})$. In each case either $|A| \nmid |G|$, or $A = G$, or $A \cong U_5(q)$.

Thus we are reduced to the possibilities $G = U_6(p)$, p odd, $A \cong L_2(11)$, M_{11}, M_{12}, M_{22}, HS, McL, or Suz and $G = U_6(q)$, $A \cong U_5(q)$. In the first case $\mathrm{Ord}_{11}(p) = 10$ implies that $p \neq 3, 5, 11$. It follows that in any case p^{14} divides $|B|$. We shall prove that there is no possibility for B.

Indeed, $B \ncong A_l$ as otherwise l must be too large and then we reach a contradiction just as in the case $A \cong A_l$ above. B is not isomorphic to a sporadic group because if p is odd, $p \neq 3, 5, 11$, then p^{14} does not divide the order of any sporadic group. Further, [6] shows that B cannot be a group of Lie type of characteristic $\neq p$. Finally, if B is of Lie type of characteristic p then (checking the orders of these groups) we conclude that $B \cong L_2(p^{14})$, $L_2(p^{15})$, $L_3(p^5)$, $L_6(p)$, $U_3(p^5)$, or $U_6(p)$. However, then either $|B| \nmid |G|$ or $B = G$, an impossibility.

In the second case $A \cong U_5(q)$. Choose a prime r_1 such that $\mathrm{Ord}_{r_1}(p) = 6n$ (the choice is possible as $q \geqq 4$). As $r_1 \mid q^6 - 1$ and as $q^6 - 1$ divides $|B : A \cap B| = |G : A| = q^5(q^6-1).(5, q+1)/(6, q+1)$, it follows that $r_1 \mid |B|$. Now we again consider the possibilities for B (as for A above) taking also into account that $q^5(q^6 - 1) \mid |B|$. This leads to $B \cong L_3(q^2)$, $PSp_6(q)$, $P\Omega_7(q)$ $(2 \nmid q)$, or $G_2(q)$.

Note that if q is odd then G does not contain a subgroup isomorphic to $G_2(q)$ and consequently a subgroup isomorphic to $P\Omega_7(q)$ (as $P\Omega_7(q)$ contains $G_2(q)$). Indeed, this follows from Lemma 1.1, as $B \cong G_2(q)$ (q odd) has Schur multiplier of order prime to 6, 2-rank three and only one conjugacy class of involutions.

Now, if $B \cong G_2(q)$ $(q = 2^n > 2)$ or $B \cong PSp_6(q)$ we reach (3) or (4) of the theorem.

Lastly, let $A \cong U_5(q)$, $B \cong L_3(q^2)$. Denote $D = A \cap B$; then $|D| = q(q^4-1).(6,q+1)/(3,q^2-1)$ (recall $(5,q+1)=1$). By the known subgroup structure of $L_3(q^2)$, it follows that D is contained in a subgroup of B isomorphic to

$$H = \left\{ \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a,b,c \in GF(q^2);\ \ A \in GL_2(q^2),\ \ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle,$$

where ω is an element of order $(3,q^2-1)$ in $GF(q^2)$. Further, $H = FK$ and $F \triangleleft H$, $F \cap K = 1$ where

$$F = \left\{ \left(\begin{array}{c|cc} 1 & b & c \\ \hline 0 & & \\ & & E \\ 0 & & \end{array} \right) \mid b,c \in GF(q^2) \right\} \cong E_{q^4},$$

$$K = \left\{ \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a \in GF(q^2);\ \ A \in GL_2(q^2),\ \ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle$$

$$\cong GL_2(q^2)/\mathbb{Z}_{(3,q^2-1)}.$$

Suppose that $T = D \cap F \neq 1$. Then $T \triangleleft D$ and $T \cong E_{p^k}$, where $p \leqq p^k \leqq q$. The centralizer of any non-identity p-element in $L_3(q^2)$ has order dividing $q^6(q^2-1)$. Hence $|C_D(T)|$ divides $q(q^2-1).(6,q+1)/(3,q^2-1)$. Then $|D/C_D(T)|$ is divisible by q^2+1. However, $D/C_D(T)$ is a subgroup of $\mathrm{Aut}(T) \cong GL_k(p)$, so

$$|GL_k(p)| = p^{k(k-1)/2}(p-1)\ldots(p^k-1)$$

must be divisible by q^2+1 which (in view of $p^k \leqq q$) contradicts Lemma 1.3.

Thus $D \cap F = 1$ and hence D is isomorphic to a subgroup of $H/F \cong K$. Of course, K contains a subgroup $L \cong SL_2(q^2)$ of index $(q^2-1)/(3,q^2-1)$ and then $D \cap L$ is a proper subgroup of L of order divisible by $q(q^2+1).(6,q+1)$. It follows that $L_2(q^2)$ has a proper subgroup of order divisible by $q(q^2+1)$ which (for $q \geqq 4$) contradicts the structure of $L_2(q^2)$. ∎

It remains to show that the factorizations in (3) and (4) actually exist. From [7, Proposition 3.3] we have

$$SU_6(q^2) = SU_5(q^2).Sp_6(q)$$

with "natural" embeddings of $SU_5(q^2)$ and $Sp_6(q)$ in $SU_6(q^2)$. Factoring out by $Z(SU_6(q^2))$, we obtain the factorization in (4), as $SU_5(q^2) \equiv U_5(q)$ (by Lemma 1.2).

Now we prove the existence of the factorization in (3) of the theorem. We use the following two realizations of the group $SU_6(q^2)$:

(i) $$SU_6(q^2) = \left\{X \in GL_6(q^2) \mid \overline{X}^t I X = I,\ \det X = 1\right\},$$

$$I = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix};$$

(ii) $$SU_6(q^2) = \left\{Y \in GL_6(q^2) \mid \overline{Y}^t Y = E,\ \det Y = 1\right\}.$$

(Here, if $D = (d_{ij})$ is a matrix with entries in $GF(q^2)$ then $\overline{D} = (d_{ij}^q)$ and D^t is the transpose of D.)

Let $X, Y \in GL_6(q^2)$ and $Y = T^{-1}XT$, where

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 & 0 & t^q \\ 0 & t & 0 & 0 & t^q & 0 \\ 0 & 0 & t & t^q & 0 & 0 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 0 & 0 & t^q & 1 & 0 & 0 \\ 0 & t^q & 0 & 0 & 1 & 0 \\ t^q & 0 & 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 & 0 & 1 \\ 0 & t & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 1 & 0 & 0 \end{pmatrix}$$

$$(t \in GF(q^2),\ t + t^q = 1).$$

Then $\overline{Y}^t Y = E$ if and only if $\overline{X}^t I X = I$.

Now, with respect to (ii), we have

$$\left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & 1 \end{array} \right) \in SU_6(q^2) \right\} \cong SU_5(q^2).$$

On the other hand (see [3]), with respect to (i), a $G_2(q)$ subgroup of $SU_6(q^2)$ is generated by the matrices $X_{\pm r}(t)$, $r \in \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$, $t \in GF(q)$, where

$$X_a(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_b(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{2a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{3a+2b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix $X_{-r}(t)$ is the transpose of $X_r(t)$. Now a direct computation shows that the common elements of the above $SU_5(q^2)$ and $G_2(q)$ subgroups are exactly as follows:

$$T^{-1}.\begin{pmatrix} v & 0 & 0 & 0 & v^{-1}s & 0 \\ 0 & v & 0 & 0 & 0 & v^{-1}s \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & v^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & v^{-1} \end{pmatrix}.T \quad (v \in GF(q)^*,\ s \in GF(q)),$$

$$T^{-1}.\begin{pmatrix} u^{-1}l & 0 & 0 & 0 & u+u^{-1}lk & 0 \\ 0 & u^{-1}l & 0 & 0 & 0 & u+u^{-1}lk \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ u^{-1} & 0 & 0 & 0 & u^{-1}k & 0 \\ 0 & u^{-1} & 0 & 0 & 0 & u^{-1}k \end{pmatrix}.T$$

$$(u \in GF(q)^*;\ l,k \in GF(q)).$$

Hence $|SU_5(q^2) \cap G_2(q)| = q(q^2-1)$ (in fact, $SU_5(q^2) \cap G_2(q) \cong L_2(q)$). Now order considerations imply

$$SU_6(q^2) = SU_5(q^2).G_2(q)$$

whence (again by Lemma 1.2) the factorization in (3) follows.

This completes the proof of the theorem.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
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Tome 86, 1992

О (3,4)-КРАТНОСТЯХ РАМСЕЯ

НИКОЛАЙ ХАДЖИИВАНОВ, ИВАН ПАШОВ

Николай Хаджииванов, Иван Пашов. О (3,4)-КРАТНОСТЯХ РАМСЕЯ

Через $M_n(3,4)$ обозначена минимальная сумма чисел 3-клик и 4-антиклик для n-вершинного графа. Доказано, что $M_n(3,4) \sim m(3,4)\binom{n}{3}$, где $\frac{1}{30} \leqq m(3,4) \leqq \frac{1}{9}$.

Nikolay Khadzhiivanov, Ivan Pashov. ON THE RAMSEY (3,4)-MULTIPLICITY

Let $M_n(3,4)$ be the minimal sum of the 3-clique's and 4-anticlique's numbers in an arbitrary n-vertex graph. The asymptotic formula $M_n(3,4) \sim m(3,4)\binom{n}{3}$ with $\frac{1}{30} \leqq m(3,4) \leqq \frac{1}{9}$ is proved.

1. ВВЕДЕНИЕ

Рассматриваются только обыкновенные графы. Множество, состоящее из p вершин графа, называется p-кликой (соотв. p-антикликой), если любые две из этих вершин смежны (соотв. несмежны). Через $c_p(G)$ и $a_q(G)$ будем обозначать соответственно число p-клик и число q-антиклик графа G. Сумма $c_p(G) + a_q(G)$ называется (p,q)-кратностью Рамсея графа G и обозначается через $M(G;p,q)$. Если зафиксировать n, p и q, минимум $M(G;p,q)$ по всевозможным n-вершинным графам G обозначается через $M_n(p,q)$. Легко сообразить, что $M_n(p,q) = M_n(q,p)$, так как $M(G;p,q) = M(\overline{G};q,p)$, где $\overline{G}$ — дополнительный граф графа G. Очевидно $M_n(2,2) = \binom{n}{2}$. Гудман [1] определил $M_n(3,3)$. Другие значения $M_n(p,q)$ при $p \geqq 2$,

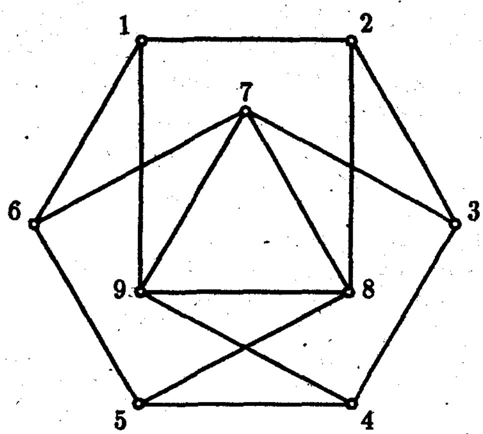

Рис. 1

$q \geqq 2$ пока неизвестны.

Грийнвуд и Глиссон [2] установили, что $M_n(3,4) > 0$ только тогда, когда $n \geqq 9$. Н. Ненов и Н. Хаджииванов [3, 4] доказали, что $M_9(3,4) = 1$ и установили, что имеются только два графа (см. графы G_1 и G_2 на рис. 1 и 2), для которых $M(G;3,4) = 1$. Они же доказали [5], что если 10-вершинный граф G не имеет 3-клик, тогда $M(G;3,4) \geqq 5$, а совместно с Ив. Пашовым [6] — что если 10-вершинный граф G не имеет 4-антиклик, тогда $M(G;3,4) \geqq 4$.

В настоящей заметке докажем, что $M_{10}(3,4) = 4$ и

$$M_n(3,4) \sim m(3,4)\binom{n}{3}, \quad \text{где } \frac{1}{30} \leqq m(3,4) \leqq \frac{1}{9}.$$

2. ЕСЛИ В G ИМЕЕТСЯ 3-КЛИКА И 4-АНТИКЛИКА БЕЗ ОБЩИХ ВЕРШИН, ТОГДА $M(G;3,4) \geqq 4$

Пусть G — граф, в котором имеются 3-клика t и 4-антиклика q и $t \cap q = \varnothing$. Докажем, что $M(G;3,4) \geqq 4$.

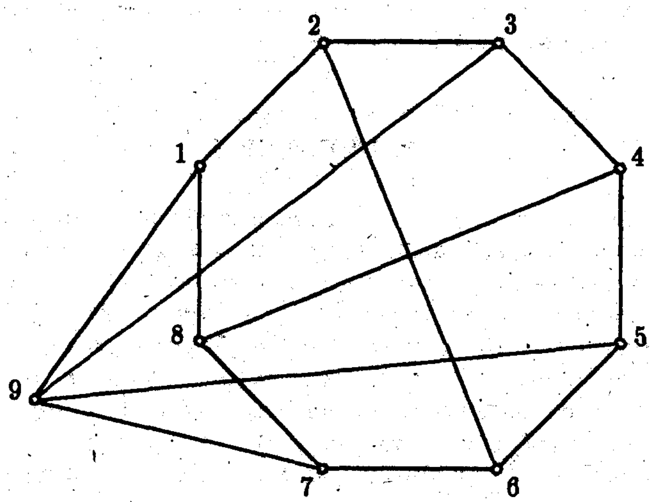

Рис. 2

Неравенство выполняется, если некоторая вершина из q смежна всем трем вершинам из t, потому что тогда $c_3(G) \geqq 4$.

Неравенство имеет место и если существуют такие две вершины из q, любая из которых смежна двум вершинам из t. В этом случае $c_3(G) \geqq 3$ и следовательно $M(G;3,4) \geqq 4$.

Неравенство выполнено и если в q имеется одна вершина u, которая смежна ровно двум вершинам из t, а любая из остальных трех вершин из q смежна не более одной вершине из t. В этом случае число промежуточных ребер, соединяющих q с t, не более чем 5 и поэтому существует вершина v из t, из которой выходит самое более одно промежуточное ребро. Сейчас u является вершиной 3-клики, отличной от t, а v — вершиной 4-антиклики, отличной от q. Это показывает, что $c_3(G) \geqq 2$ и $a_4(G) \geqq 2$, так что $M(G;3,4) \geqq 4$.

Если в t имеется вершина, которая не смежна ни одной вершине из q, тогда она является вершиной 5-антиклики, так что $a_4(G) \geqq 5$ и неравенство снова выполнено.

Чтобы окончательно доказать искомое неравенство, осталось рассмотреть случай, когда любая вершина из t смежна некоторой вершине из q и одновременно любая вершина из q смежна не более одной вершине из t. Теперь число промежуточных ребер не больше 4 и имеются такие две вершины из t, любая из которых имеет ровно одну смежную в q. Упомянутые две вершины являются вершинами двух 4-антиклик, отличных от q и между собой. Следовательно $a_4(G) \geqq 3$ и $M(G;3,4) \geqq 4$.

Доказательство неравенства завершено.

3. $M_{10}(3, 4) \geqq 4$

Надо доказать, что если G — 10-вершинный граф, тогда $M(G;3,4) \geqq 4$. Это утверждение верно, если в G нет 3-клик или 4-антиклик (см. п. 1). Оно верно и тогда, когда в G можно найти 3-клику и 4-антиклику без общих вершин (см. п. 2). Поэтому в дальнейшем будем считать, что в G имеется хотя бы одна 3-клика и хотя бы одна 4-антиклика и притом всякая 3-клика имеет общую вершину с произвольной 4-антикликой.

Через t обозначим некоторую 3-клику, через q — 4-антиклику, а через v — их общую вершину. Рассмотрим 9-вершинный граф $G - v$, который получается из G удалением вершины v и, разумеется, всех ребер, инцидентных с v. Если $M(G - v;3,4) \geqq 2$, тогда очевидно $M(G;3,4) \geqq 4$. Поэтому в дальнейшем будем считать, что $M(G - v;3,4) \leqq 1$. Так как $G - v$ — 9-вершинный граф, то $M(G - v;3,4) = 1$ (см. п. 1).

Искомое неравенство верно очевидно и если v является вершиной некоторой 3-клики, отличной от t, или некоторой 4-антиклики, отличной от q. Поэтому будем считать, что это не так.

Возникают две возможности $G - v = G_1$ или $G - v = G_2$ (см. п. 1), каждую из которых исследуем в отдельности.

I. $G - v = G_1$

Множество $q \setminus v$ является 3-антикликой в G_1. 4-антиклика q должна пересекать 3-клику $[7,8,9]$; без ограничения общности можно считать, что их общая вершина 7. Остальные две вершины множества $q \setminus v$ находятся в множествах $\{1,2\}$ и $\{4,5\}$. Из-за симметрии достаточно рассмотреть только два случая: $q \setminus v = \{7,1,5\}$ или $q \setminus v = \{7,1,4\}$.

Пусть $q \setminus v = \{7,1,5\}$, т. е. $q = \{v,7,1,5\}$. Вершина v смежна вершине 2, иначе $\{v,7,5,2\}$ будет вторая 4-антиклика, содержащая v. Вершина v смежна вершине 3, иначе $\{v,1,5,3\}$ будет вторая 4-антиклика, содержащая v. Вершина v смежна и вершине 4, иначе $\{v,1,7,4\}$ будет вторая 4-антиклика, содержащая v. Тогда $[v,2,3]$ и $[v,3,4]$ будут две 3-клики, содержащие вершину v, что является противоречием.

Пусть теперь $q \setminus v = \{7,1,4\}$, т. е. $q = \{v,7,1,4\}$. Сейчас v смежна вершине 2, иначе $\{v,7,4,2\}$ будет вторая 4-антиклика, содержащая v. Вершина v смежна вершине 8, иначе $\{v,1,4,8\}$ будет вторая 4-антиклика, содержащая v. Наконец, вершина v смежна вершине 5, иначе $\{v,1,7,5\}$ будет вторая 4-антиклика, содержащая v. Таким образом $[v,2,8]$ и $[v,8,5]$ — 3-клики, содержащие v, что является противоречием.

Таким образом доказано, что первая возможность на самом деле не представляется.

II. $G - v = G_2$

Пусть $t = [v,a,b]$, где $[a,b]$ — ребро графа G_2. Из-за симметрии этого графа имеет место один из следующих трех случаев:

1) $[a,b] = [9,1]$.

Теперь вершина v несмежна вершинам 2, 8, 5, 7, так как иначе будем иметь вторую 3-клику, содержащую v. Тогда $\{v,2,5,8\}$ и $\{v,2,5,7\}$ — две 4-антиклики, содержащие v, что является противоречием.

2) $[a,b] = [1,2]$.

Вершина v не смежна вершинам 8, 3, 9, 6 и следовательно $\{v,8,6,9\}$, $\{v,8,6,3\}$ — 4-антиклики, содержащие v — противоречие.

3) $[a,b] = [2,6]$.

Сейчас v не смежна вершинам 1, 3, 5, 7 и следовательно $\{v,1,3,5,7\}$ — 5-антиклика, так что v содержится не только в одной 4-антиклике — противоречие.

Таким образом доказано, что и вторая возможность на самом деле не представляется.

Окончательно, неравенство $M(G;3,4) \geq 4$ доказано.

4. ПРИМЕР 10-ВЕРШИННОГО ГРАФА G, $c_3(G) = a_4(G) = 2$

На рис. 3 изображен 8-вершинный граф без 3-клик и 4-антиклик. Присоединяя к нему новую вершину a, объявляя ее смежной вершинам 1, 3 и 5, получим 9-вершинный граф с одной 3-кликой и одной 4-антикликой (рис. 4). Присоединяя к последнему графу новую вершину b, объявляя ее смежной вершинам 2, 4 и 6, получим 10-вершинный граф G, изображенный на рис. 5, для которого $c_3(G) = 2$ и $a_4(G) = 2$, так что $M(G; 3, 4) = 4$.

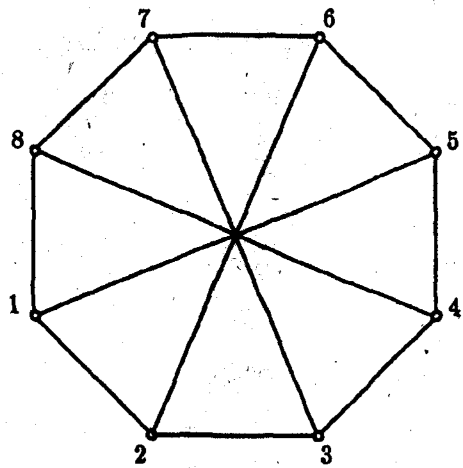

Рис. 3

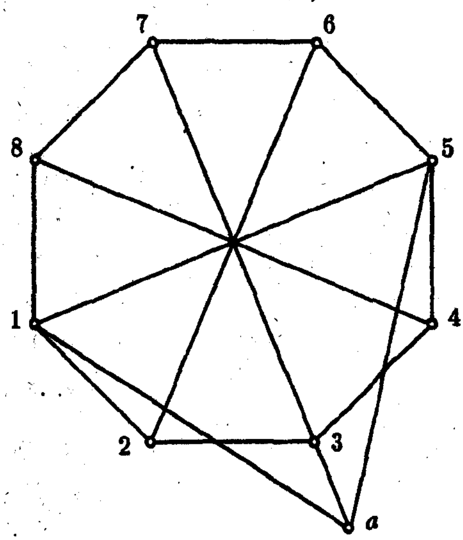

Рис. 4

С помощью этого примера и утверждения, доказанного в п. 3, нами получена следующая:

Теорема 1. $M_{10}(3, 4) = 4$.

5. ПОСЛЕДОВАТЕЛЬНОСТЬ $M_n(3,4)/\binom{n}{3}$ НЕ УБЫВАЕТ

Это утверждение очевидно эквивалентно следующему неравенству:

(1) $(n-3)M_n(3,4) \geqq nM_{n-1}(3,4)$.

Пусть G — n-вершинный граф. Легко сообразить, что:

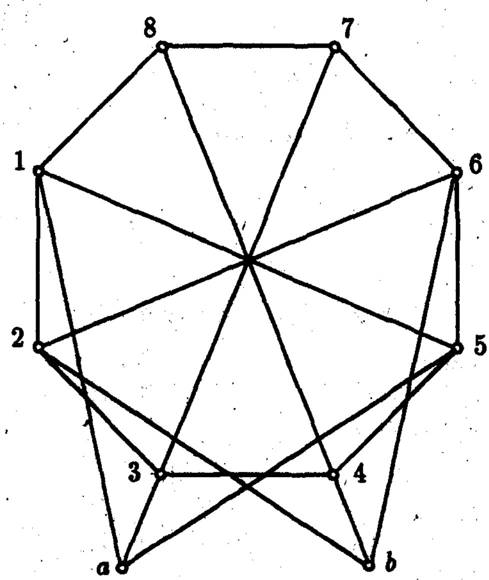

Рис. 5

(2) $$\sum_v c_3(G-v) = (n-3)c_3(G),$$

(3) $$\sum_v a_4(G-v) = (n-4)a_4(G).$$

Почленным суммированием, получаем

(4) $$\sum_v M(G-v;3,4) = (n-3)c_3(G) + (n-4)a_4(G) \leqq (n-3)M(G;3,4).$$

Так как $G-v$ — $(n-1)$-вершинный граф, то

(5) $$M_{n-1}(3,4) \leqq M(G-v;3,4).$$

Из (4) и (5) вытекает

(6) $$nM_{n-1}(3,4) \leqq (n-3)M(G;3,4).$$

Неравенство (6), которое можно переписать следующим образом:

$$M(G;3,4) \geqq \frac{n}{n-3} M_{n-1}(3,4),$$

имеет место для любого n-вершинного графа G и поэтому

$$M_n(3,4) \geqq \frac{n}{n-3} M_{n-1}(3,4),$$

что и требовалось доказать.

Из (1) следует, что

$$\frac{M_n(3,4)}{\binom{n}{3}} \geqq \frac{M_{10}(3,4)}{\binom{10}{3}}, \quad n \geqq 10,$$

и так как $M_{10}(3,4) = 4$, получаем

Теорема 2. $M_n(3,4) \geqq \frac{1}{30}\binom{n}{3}$ *при* $n \geqq 10$.

6. ОЦЕНКА СВЕРХУ ДЛЯ $M_n(3,4)$

При фиксированном n сумма $\binom{n_1}{3} + \binom{n_2}{3} + \binom{n_3}{3}$, где натуральные числа n_i подчинены условию $n_1 + n_2 + n_3 = n$, принимает свое наименьшее значение $h(n)$ точно тогда, когда n_i почти равны, т. е. $|n_i - n_j| \leqq 1$. Очевидно

$$h(n) = \begin{cases} 3\binom{n/3}{3}, & \text{если } n \equiv 0 \pmod 3; \\ 2\binom{(n-1)/3}{3} + \binom{(n+2)/3}{3}, & \text{если } n \equiv 1 \pmod 3; \\ 2\binom{(n+1)/3}{3} + \binom{(n-2)/3}{3}, & \text{если } n \equiv 2 \pmod 3, \end{cases}$$

т. е.

$$h(n) = \begin{cases} \dfrac{n(n-3)(n-6)}{54}, & \text{если } n \equiv 0 \pmod 3; \\ \dfrac{(n-1)(n-4)^2}{54}, & \text{если } n \equiv 1 \pmod 3; \\ \dfrac{(n-2)^2(n-5)}{54}, & \text{если } n \equiv 2 \pmod 3. \end{cases}$$

Рассмотрим теперь n-вершинный граф G_n, который является объединением трех дизъюнктных полных графов с почти равным количеством элементов. Очевидно $a_4(G_n) = 0$ и $c_3(G_n) = h(n)$, так что $M(G_n; 3, 4) = h(n)$. Таким образом нами доказано следующее предложение:

Теорема 3. $M_n(3,4) \leqq h(n)$.

7. АСИМПТОТИКА ФУНКЦИИ $M_n(3,4)$

Из теорем 2 и 3 следует, что

$$\frac{1}{30} \leqq \frac{M_n(3,4)}{\binom{n}{3}} \leqq \frac{h(n)}{\binom{n}{3}} \leqq \frac{1}{9} \quad \text{при } n \geqq 10.$$

Так как последовательность $\dfrac{M_n(3,4)}{\binom{n}{3}}$ — неубывающая и ограниченная, она сходится. Обозначим ее предел через $m(3,4)$. Доказана следующая

Теорема 4. $M_n(3,4) \sim m(3,4)\binom{n}{3}$, *где* $\dfrac{1}{30} \leqq m(3,4) \leqq \dfrac{1}{9}$.

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A COINCIDENCE THEOREM FOR ORTHOGONAL MAPS

SIMEON STEFANOV

Симеон Стефанов. ТЕОРЕМА О СОВПАДЕНИИ ДЛЯ ОРТОГОНАЛЬНЫХ ОТОБРАЖЕНИЙ

Получена теорема типа теоремы Борсук–Улама для ортогональных отображений в конечномерных эвклидовых пространствах. Этот результат эквивалентен факту, что $\mathbb{Z}$ является группой Борсука–Улама относительно ортогональных представлений. Следствием доказано несуществование полусопряженности между некоторыми стандартными линейными динамическими системами на сферах. Наконец показано, что каждая группа вида $G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k$, где A — конечная абелева группа, является группой Борсука– Улама относительно ортогональных представлений.

Simeon Stefanov. A COINCIDENCE THEOREM FOR ORTHOGONAL MAPS

A Borsuk–Ulam type theorem for orthogonal maps acting in finite-dimensional Euclidean spaces is obtained. This result is equivalent to the fact that $\mathbb{Z}$ is a Borsuk–Ulam group with respect to orthogonal representations. As a corollary, the nonexistence of a semiconjugacy between some standard linear dynamical systems on spheres is proved. Finally, it is shown that every group of the form $G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k$, where A is a finite Abelian group, is a Borsuk–Ulam group with respect to orthogonal representations.

1. INTRODUCTION

Various theorems generalizing the Borsuk–Ulam theorem in different directions have been obtained (see for example [3, 5, 6, 9, 11, 12]). All these generalizations usually replace the antipodal map in the sphere by the action of some finite group, or by a compact Lie group action. However, nothing is known about the action of a noncompact group (say $\mathbb{Z}$), as far as we know, even if this action is orthogonal or unitary.

We shall prove, in this article, some Borsuk–Ulam type theorems for orthogonal maps in Euclidean spaces. Each map generates an action of the group $\mathbb{Z}$ in the corresponding space. The main result is the following

Theorem 1. *Let E and F be finite-dimensional Euclidean spaces and $U : E \to E$, $V : F \to F$ be orthogonal maps such that*

$$\dim E - \dim E_U > \dim F - \dim F_V,$$

where $E_U = \{x \in E \mid Ux = x\}$, $F_V = \{x \in F \mid Vx = x\}$.

Furthermore, let $f : E \to F$ be a continuous map such that

$$fU(x) = Vf(x) \text{ for each } x \in E.$$

Then for any open bounded set $\Omega \subset E$ with $\mathbb{O} \in \Omega$ there exist $x \in \partial\Omega$ and $k \in \mathbb{Z}$ such that $U^k x \neq x$ but $f(U^k x) = f(x)$.

It is easy to see that if U and V are the antipodal maps and $\partial\Omega = S(E)$ is the unit sphere in E, we obtain the classical Borsuk–Ulam theorem, which asserts (in our notation), that if $\dim E > \dim F$, then there are no odd maps $f : S(E) \to S(F)$.

The map $U : E \to E$ is called *free* if $U^k x = x$ and $k \neq 0$ imply $x = \mathbb{O}$. For such maps we prove a stronger result:

Theorem 2. *Let $U : E \to E$ and $V : F \to F$ be free orthogonal maps and $\dim E > \dim F$. Let $f : E \to F$ be such that $fU = Vf$.*

Then for any open bounded $\Omega \subset E$ with $\mathbb{O} \in \Omega$ there exists $x \in \partial\Omega$ such that $f(x) = \mathbb{O}$.

This result yields that if $m > n$, then there is no $f : S^m \to S^n$ such that $fU = Vf$, where U and V are free orthogonal maps in S^m and S^n, respectively. In the context of discrete time dynamical systems (cf. [7]) it means that no two such systems are semiconjugated, so the dynamics of $U : S^m \to S^m$ is essentially more complex than the dynamics of $V : S^n \to S^n$. An analogue of this result for flows is also valid.

Using the terminology of [12], we may restate the main theorem to say that $\mathbb{Z}$ is a Borsuk–Ulam group with respect to orthogonal representations (cf. Section 4 for the definition). Combining this with other known results, we prove that every group of the form

$$G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k,$$

where A is a finite Abelian group, is a Borsuk–Ulam group with respect to orthogonal representations. In [12] it is proved for compact Abelian Lie groups.

Naturally, all the results remain valid for unitary maps and representations.

One may ask whether we can take $k = 1$ in Theorem 1, i. e. whether the equation $f(Ux) = f(x)$ has solutions on $\partial\Omega$. In Section 5 we show that this is not always true and answer, meanwhile, a question of Wasserman about the existence of equivariant maps between spheres.

The proof of the main theorem relies heavily on a recent result of Rabier [8], generalizing the classical Hopf–Rueff theorem [4].

We shall recall some well-known results about the rational dependance of real numbers, related to the Kronecker theorem.

Let $\theta_1, \ldots, \theta_n$ be nonzero real numbers and

$$\sum m_j\theta_j = p, \tag{1}$$

where $m_j, p \in \mathbb{Z}$. We shall write briefly $(m, \theta) = p$.

Definition. We say that the range of the system $\theta_1, \ldots, \theta_n$ equals r, if the space

$$\{m \in \mathbb{Z}^n \mid (m, \theta) \in \mathbb{Z}\}$$

is $(n-r)$-dimensional over $\mathbb{Z}$. Then we write

$$\operatorname{rank}(\theta_1, \ldots, \theta_n) = r.$$

In particular, the equality $\operatorname{rank}(\theta_1, \ldots, \theta_n) = n$ means that the numbers $1, \theta_1, \ldots, \theta_n$ are rationally independent, so (1) implies $m_1 = \cdots = m_n = p = 0$.

The following is a well-known geometrical fact (cf. [1, 2]).

Proposition 1. *Let θ_j be real numbers with* $\operatorname{rank}(\theta_1, \ldots, \theta_n) = r$. *Consider the following subset of the n-torus $\mathbb{T}^n$:*

$$A = \left\{ \left(e^{2k\pi i\theta_1}, \ldots, e^{2k\pi i\theta_n}\right) \mid k \in \mathbb{Z} \right\}. \tag{2}$$

Then the closure $\overline{A}$ is homeomorphic with the union of some (nonintersecting) copies of the r-torus $\mathbb{T}^r$. If $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are the co-ordinates modulo 1 in $\mathbb{T}^n$, then each such copy is a linear torus represented by the n-plane

$$\sum_{j=1}^{n} m_{ij}x_j = c_i, \quad i = 1, \ldots, n-r.$$

Here $m_{ij} \in \mathbb{Z}$, the range of the matrix (m_{ij}) equals $n-r$, and $\sum m_{ij}\theta_j \in \mathbb{Z}$.

This proposition yields the following generalization of the Kronecker theorem:

Proposition 2. *Let* $\operatorname{rank}(\theta_1, \ldots, \theta_n) = r$ *and μ_0 be such that*

$$\operatorname{rank}(\mu_0, \theta_1, \ldots, \theta_n) = r + 1.$$

Let, furthermore, A be defined by (2), $\left(e^{2\pi i y_1}, \ldots, e^{2\pi i y_n}\right) \in \overline{A}$ and $x_0 \in \mathbb{R}$. Then for any $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ such that

$$|k_m\mu_0 + p_0 - x_0| < \frac{1}{m}, \quad |k_m\theta_j + p_j - y_j| < \frac{1}{m}, \qquad j = 1, \ldots, n,$$

for some integers p_0, p_j.

Proof. Consider the set

$$B = \left\{ \left(e^{2k\pi i\mu_0}, e^{2k\pi i\theta_1}, \ldots, e^{2k\pi i\theta_n}\right) \mid k \in \mathbb{Z} \right\}.$$

According to Proposition 1, $\overline{B}$ is an union of $(r+1)$-dimensional tori in $\mathbb{T}^{n+1}$, though the projection of $\overline{B}$ over $\mathbb{T}^n$ is $\overline{A}$, which is an union of r-tori. Therefore the projection of $\overline{B}$ over the first factor of $\mathbb{T}^{n+1}$ is the whole circle S^1. Then

$(e^{2\pi i x_0}, e^{2\pi i y_1}, \ldots, e^{2\pi i y_n}) \in \overline{B}$, that implies the needed property (passing to co-ordinates modulo 1).

3. SOME LEMMAS

All the maps are assumed to be continuous.

Given some θ_j and $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, we shall furthermore use the notation

$$\operatorname{traj}(z) = \left\{ \left(e^{2k\pi i\theta_1} z_1, \ldots, e^{2k\pi i\theta_n} z_n\right) \mid k \in \mathbf{Z} \right\}.$$

This is in fact the trajectory of z with respect to the unitary map in $\mathbf{C}^n$ with eigen values $e^{2\pi i\theta_j}$. Then, as following from Proposition 1, the closure $\overline{\operatorname{traj}(z)}$ is a (finite) union of tori.

Lemma 1. *Let $z_0 \in \mathbf{C}^n$, $z_0 \neq 0$, and the map $\varphi : \overline{\operatorname{traj}(z_0)} \to S^1$ be such that*

$$\varphi\left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_n} z_n\right) = e^{2\pi i\mu_0}\varphi(z_1, \ldots, z_n) \tag{3}$$

for any $z = (z_1, \ldots, z_n) \in \overline{\operatorname{traj}(z_0)}$, where θ_j, μ_0 are nonzero.

Then

$$m\mu_0 = \sum m_j\theta_j + p$$

for some integer m, m_j, p, where $m \neq 0$.

Proof. We may assume that $z_j \neq 0$ for any j, since otherwise we simply ignore the zero co-ordinates.

Suppose the contrary. It means that

$$\operatorname{rank}(\mu_0, \theta_1, \ldots, \theta_n) > \operatorname{rank}(\theta_1, \ldots, \theta_n).$$

Choose $y_1, \ldots, y_n$ so that $\left(e^{2\pi i y_1}, \ldots, e^{2\pi i y_n}\right) \in \overline{A}$, where A is defined by (2), and an arbitrary $x_0 \in \mathbf{R}$. Then, according to Proposition 2, there is a sequence of integers $k_m \to \infty$ such that

$$e^{2k_m\pi i\theta_j} \to e^{2\pi i y_j}, \quad e^{2k_m\pi i\mu_0} \to e^{2\pi i x_0}$$

as $m \to \infty$. The condition (3) then gives

$$\varphi\left(e^{2k_m\pi i\theta_1} z_1, \ldots, e^{2k_m\pi i\theta_n} z_n\right) = e^{2k_m\pi i\mu_0}\varphi(z_1, \ldots, z_n),$$

and taking the limit as $m \to \infty$,

$$\varphi\left(e^{2\pi i y_1} z_1, \ldots, e^{2\pi i y_n} z_n\right) = e^{2\pi i x_0}\varphi(z_1, \ldots, z_n).$$

It turns out that the last equality is true for an arbitrary $x_0 \in \mathbf{R}$, which is impossible.

Lemma 2. *Let $z_0 \in \mathbf{C}^n$, $z_0 \neq 0$, the map $\varphi : \overline{\operatorname{traj}(z_0)} \to S^1$ satisfies (3), and*

$$m\mu_0 = \sum m_j\theta_j + p, \quad m \neq 0; \quad m, m_j, p \in \mathbf{Z}.$$

Suppose that $\varphi(z_0) = 1$ and $z_0 = (v_1, \ldots, v_n)$. For $z \in \overline{\operatorname{traj}(z_0)}$ consider the function

$$\Phi(z) = v_1^{-m_1} \ldots v_n^{-m_n} z_1^{m_1} \ldots z_n^{m_n}$$

Then $\varphi^m(z) = \Phi(z)$ for any $z \in \overline{\operatorname{traj}(z_0)}$.

Proof. Let us note, first, the following: if $\psi(z)$ is another function satisfying (3) and $\psi(z_0) = \varphi(z_0) = 1$, then $\psi(z) = \varphi(z)$ for any $z \in \overline{\mathrm{traj}(z_0)}$. This is due to the fact that φ is uniquely defined on $\mathrm{traj}(z_0)$ by property (3) and the value $\varphi(z_0)$, so it is uniquely defined on $\overline{\mathrm{traj}(z_0)}$.

Compare now the functions $\Phi(z)$ and $\varphi^m(z)$. We have

$$\Phi\left(e^{2\pi i\theta_1}z_1, \ldots, e^{2\pi i\theta_n}z_n\right) = v_1^{-m_1}\ldots v_n^{-m_n} e^{2\pi i\sum m_j\theta_j} z_1^{m_1}\ldots z_n^{m_n}$$
$$= e^{2\pi i m\mu_0}\Phi(z_1, \ldots, z_n),$$

so both $\Phi(z)$ and $\varphi^m(z)$ satisfy (3) with constants $\theta_1, \ldots, \theta_n, m\mu_0$. Moreover, $\Phi(z_0) = \varphi^m(z_0) = 1$, thus $\Phi(z) = \varphi^m(z)$ for any $z \in \overline{\mathrm{traj}(z_0)}$.

The following lemma is the main one in the article.

Lemma 3. *Let $\theta_1, \ldots, \theta_n, \mu_1, \ldots, \mu_{n-1}$ be irrational numbers and $\varphi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ be such that*

$$\varphi_k\left(e^{2\pi i\theta_1}z_1, \ldots, e^{2\pi i\theta_n}z_n\right) = e^{2\pi i\mu_k}\varphi_k(z_1, \ldots, z_n) \tag{4}$$

for $k = 1, \ldots, n-1$, and each $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, where $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$.

Then for any open bounded $\Omega \subset \mathbf{C}^n$ with $\mathbf{0} \in \Omega$ there exists $z \in \partial\Omega$ such that $\varphi(z) = \mathbf{0}$.

Proof. We shall reduce this proposition to a recent result of Rabier [8], which generalizes the classical Hopf–Rueff theorem [4].

Let $\mathrm{rank}(\theta_1, \ldots, \theta_n) = r$. Then by definition

$$\sum_{j=1}^{n} n_{ij}\theta_j + q_i = 0, \quad i = 1, \ldots, n-r, \tag{5}$$

where the range of the matrix (n_{ij}) equals $n-r$ (and all the coefficients are integers). Recall that if

$$A = \left\{\left(e^{2s\pi i\theta_1}, \ldots, e^{2s\pi i\theta_n}\right) \mid s \in \mathbf{Z}\right\},$$

then $\overline{A}$ is the union of r-tori, which are represented in co-ordinates $(x_1, \ldots, x_n) \in \mathbf{R}^n$ modulo 1 by some parallel r-planes

$$\sum_{j=1}^{n} n_{ij}x_j + c_i^{(m)} = 0, \quad i = 1, \ldots, n-r, \ m = 1, \ldots, m_0, \tag{6}$$

where m_0 is the number of these tori (Proposition 1). Note that some of the planes (6) pass through the origin $\mathbf{0}$, since for $s = 0$, $(1, 1, \ldots, 1) \in A$, but $(1, \ldots, 1) = (0, \ldots, 0)$ modulo 1. Denote the corresponding plane by α,

$$\alpha : \sum_{j=1}^{n} n_{ij}x_j = 0, \quad i = 1, \ldots, n-r. \tag{7}$$

It is easy to see that the rational points are dense in α. Indeed, if $\det(n_{ij}) \neq 0$ for $i = 1, \ldots, n-r$, $j = r+1, \ldots, n$, then α is parametrized by the variables $x_1, \ldots, x_r$, so giving them rational values we obtain rational solutions $x_{r+1}, \ldots, x_n$ of (7).

We may suppose that $\varphi_k(z) \neq 0$ for some $z \in \mathbf{C}^n$, since otherwise we ignore the k-th component of φ and keep on the same reasoning. So, the functions $\psi_k(z) = \varphi_k(z)/\|\varphi_k(z)\|$ are well-defined on $\overline{\mathrm{traj}(z)}$ and satisfy (4). Then, according to Lemma 1,

$$(8) \qquad m_k\mu_k = \sum_{j=1}^{n} m_{jk}\theta_j + p_k, \quad k = 1, \ldots, n-1,$$

where $m_k \neq 0$.

Now we shall prove that there exist integers $A_1, \ldots, A_n$ such that:

i) $(A_1, \ldots, A_n) \in \alpha$,

ii) $A_j \neq 0$ for any $j = 1, \ldots, n$,

iii) $\sum\limits_{j=1}^{n} m_{jk}A_j \neq 0$ for any $k = 1, \ldots, n-1$.

We shall show first that the plane α is not contained in a hyperplane $\mathbf{R}_j^{n-1} = \{x \in \mathbf{R}^n \mid x_j = 0\}$. Suppose the contrary: $\alpha \subset \mathbf{R}_j^{n-1}$. Then the plane α' : $\sum n_{ij}x_j + q_i = 0$, $i = 1, \ldots, n-r$, is parallel to α, therefore $x_j = \mathrm{const}$ in α'. But $(\theta_1, \ldots, \theta_n) \in \alpha'$ (see (5)), thus $x_j = \theta_j$ in α'. On the other hand, the rational points are dense in α' (as well as in α), consequently $\theta_j \in \mathbf{Q}$, which is a contradiction.

Consider now the linear map $M : \mathbf{R}^n \to \mathbf{R}^{n-1}$ with a matrix $M = (m_{ij})$. Let $\mathbf{R}_k^{n-2} = \{x \in \mathbf{R}^{n-1} \mid x_k = 0\}$. We shall prove that α is not contained in some $M^{-1}(\mathbf{R}_k^{n-2})$. Really, suppose the contrary, then $M(\alpha) \subset \mathbf{R}_k^{n-2}$, so for $x \in \alpha$ we have $\big(M(x)\big)_k = 0$. Hence the equalities $\sum\limits_{j=1}^{n} n_{ij}x_j = 0$, $i = 1, \ldots, n-r$, imply $\sum\limits_{j=1}^{n} m_{jk}x_j = 0$. Then the vector $(m_{1k}, \ldots, m_{nk})$ is a linear combination of the vectors $(n_{i1}, \ldots, n_{in})$, $i = 1, \ldots, n-r$. So $m_{jk} = \sum\limits_{i=1}^{n-r} \beta_i n_{ij}$, $j = 1, \ldots, n$, where $\beta_i \in \mathbf{Q}$. The last equality together with (8) and (5) gives

$$m_k\mu_k = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n-r}\beta_i n_{ij}\Big)\theta_j + p_k = \sum_{i=1}^{n-r}\Big(\sum_{j=1}^{n} n_{ij}\theta_j\Big)\beta_i + p_k = \sum_{i=1}^{n-r}(-q_i)\beta_i + p_k,$$

which is a contradiction, since the right-hand side is rational, though μ_k is irrational (and $m_k \neq 0$).

So, we conclude that α is not contained in the union of the linear spaces $\mathbf{R}_j^{n-1}$, $M^{-1}(\mathbf{R}_k^{n-2})$. Therefore there exists a rational point $(A_1/B, \ldots, A_n/B)$ in α which is not contained in this union. But then, clearly, $(A_1, \ldots, A_n) \in \alpha$, $A_j \neq 0$, and $\sum\limits_{j=1}^{n} m_{jk}A_j \neq 0$, hence the conditions i) — iii) are fulfiled.

Consider now the following flow defined in $\mathbf{C}^n$ by the formula

$$(9) \qquad tz = \left(e^{2\pi i A_1 t}z_1, \ldots, e^{2\pi i A_n t}z_n\right), \quad t \in \mathbf{R}.$$

It has periodic trajectories, since $A_j \in \mathbb{Z}$. It is easy to see that the trajectory of some z with respect to the flow is contained in the set $\overline{\operatorname{traj}(z)}$:

$$(10) \qquad \bigcup\{tz \mid t \in \mathbf{R}\} \subset \overline{\operatorname{traj}(z)}.$$

Indeed, suppose first that $z_j \neq 0$ for any j and consider the point $z' = (z_1/\|z_1\|, \ldots, z_n/\|z_n\|) \in \mathbb{T}^n$. Passing as above in x-co-ordinates modulo 1, the trajectory of z' with respect to the flow is represented by the line

$$x_j = A_j t, \quad j = 1, \ldots, n; \; t \in \mathbf{R}.$$

But this line obviously lies in α, for α contains two of its points — $(0, \ldots, 0)$ and $(A_1, \ldots, A_n)$. Therefore

$$\bigcup\{tz' \mid t \in \mathbf{R}\} \subset \overline{\operatorname{traj}(z')},$$

that implies, of course, (10). To obtain the inclusion (10) for arbitrary $z \in \mathbf{C}^n_*$, one has to find a sequence $z_m \to z$, where all the co-ordinates of z_m are nonzero, and then to take limit as $m \to \infty$.

Consider now the sets

$$V_k = \{z \in \mathbf{C}^n \mid \varphi_k(z) \neq 0\}$$

and let, as above, $\psi_k(z) = \varphi(z)/\|\varphi_k(z)\|$ for $z \in V_k$. Clearly, the functions ψ_k also satisfy (4). Lemma 2 then implies that over each $\overline{\operatorname{traj}(z)}$ we have

$$\psi_k^{m_k}(z) = \Phi_k(z),$$

where

$$\Phi_k(z) = v_1^{-m_{1k}} \ldots v_n^{-m_{nk}} z_1^{m_{1k}} \ldots z_n^{m_{nk}},$$

and $z_0 = (v_1, \ldots, v_n)$ is a point of $\overline{\operatorname{traj}(z)}$ such that $\psi_k(z_0) = 1$. Since ψ_k is defined and continuous in V_k, then Φ_k is also continuous in V_k.

Furthermore, if $\zeta = e^{2\pi i t} \in S^1$, then the point $(\zeta^{A_1} z_1, \ldots, \zeta^{A_n} z_n)$ also belongs to $\overline{\operatorname{traj}(z)}$. It follows from (10) and (9). Consequently,

$$\Phi_k\left(\zeta^{A_1} z_1, \ldots, \zeta^{A_n} z_n\right) = \zeta^{\sum m_{jk} A_j} \Phi_k(z_1, \ldots, z_n).$$

This is the crucial property we shall make use of.

Let now $\Omega \subset \mathbf{C}^n$ be an open bounded set with $0 \in \Omega$. Suppose that the lemma is false, i. e. that $\varphi(z) \neq 0$ for any $z \in \partial\Omega$. Then $\partial\Omega \subset \bigcup_{k=1}^{n-1} V_k$. Take real functions $t_k : \mathbf{C}^n \to \mathbf{R}$ defined by

$$t_k(z) = \operatorname{dist}(z, \mathbf{C}^n \setminus V_k).$$

Note that $t_k\left(\zeta^{A_1} z_1, \ldots, \zeta^{A_n} z_n\right) = t_k(z_1, \ldots, z_n)$. It is due to the fact that the set V_k is invariant with respect to the flow (9), for φ_k satisfy (4) and the flow has the property (10). Set

$$\Phi(z) = \left(t_1(z)\Phi_1(z), \ldots, t_{n-1}(z)\Phi_{n-1}(z)\right).$$

This is a well-defined map $\Phi : \mathbf{C}^n \to \mathbf{C}^{n-1}$. Moreover, its k-th co-ordinate $\Phi_{(k)}$ has the property

$$\Phi_{(k)}\left(\zeta^{A_1}z_1, \ldots, \zeta^{A_n}z_n\right) = \zeta^{\Sigma m_{jk}A_j}\Phi_{(k)}(z_1, \ldots, z_n)$$

for any $\zeta \in S^1$. All the powers of ζ are nonzero integers, as following from i) — iii).

Now we refer to a theorem of Rabier, who proved in [8] that for any map $\Phi : \mathbf{C}^n \to \mathbf{C}^{n-1}$ with the above property and for any open bounded $\Omega \subset \mathbf{C}^n$ with $\mathbb{O} \in \Omega$ there exists $z_0 \in \partial\Omega$ such that $\Phi(z_0) = \mathbb{O}$.

Let $z_0 \in V_k$. But then $t_k(z_0) \neq 0$ and $\Phi_k(z_0) \neq 0$, so $\Phi(z_0) \neq \mathbb{O}$, which is a contradiction.

Lemma 3 is proved.

4. THE MAIN THEOREMS

We shall prove first some propositions concerning periodic orthogonal maps, essentially following Wasserman [12] (with some insignificant modifications).

Let E and F be finite-dimensional Euclidean spaces with given (linear) representations of a group G. We say, for brevity, that E and F are representations of G. A map $f : E \to F$ is *equivariant*, if $f(gx) = gf(x)$ for any $g \in G$, $x \in E$. It is said to be *isovariant*, if it is equivariant and $f(gx) = f(x)$ implies $gx = x$. An isovariant map is one-to-one on each orbit. Denote, as usual,

$$E_G = \{x \in E \mid gx = x \text{ for any } g \in G\}.$$

Definition (Wasserman [12]). The group G is a *Borsuk–Ulam group* if for any two representations E and F with a given isovariant map $f : E \to F$ we have

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

It is shown in [12] that every finite Abelian group is a Borsuk–Ulam group. We shall prove here a stronger version of this result — namely the following

Lemma 4. *Let E and F be representations of the finite Abelian group G, $\Omega \subset E$ be an open bounded subset with $\mathbb{O} \in \Omega$, and $f : E \to F$ be an equivariant map, which is isovariant on $\partial\Omega$. Then*

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

(The map f is isovariant on $\partial\Omega$ if $f(gx) = f(x)$ implies $gx = x$ for any $x \in \partial\Omega$).

Definition. The group G is a *strong* Borsuk–Ulam group if for any two representations E, F with a given equivariant map $f : E \to F$, which is isovariant on the boundary $\partial\Omega$ of some open bounded $\Omega \subset E$ with $\mathbb{O} \in \Omega$, we have

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

Lemma 4 may be then restated as follows:

Lemma 4′. *Every finite Abelian group is a strong Borsuk–Ulam group.*

We shall suppose furthermore that all representations are orthogonal, since any linear representation of a finite group is equivalent to an orthogonal one.

Lemma 5. *The group $G = \mathbf{Z}_p$, for p prime, is a strong Borsuk–Ulam group.*

Proof. Suppose the contrary, i.e. that $\dim E - \dim E_G > \dim F - \dim F_G$ and $f : E \to F$ is isovariant on $\partial\Omega$. Decompose $E = E_G \oplus E'$, $F = F_G \oplus F'$, then $\dim E' > \dim F'$. Let $\pi : F \to F'$ denote the projection over the second factor, $S(F')$ be the unit sphere in F', and $r : F' \setminus \{0\} \to S(F')$ be the radial projection. Consider the set

$$\widetilde{\Omega} = \{gx \mid g \in G, \ \ x \in \Omega\},$$

which is an invariant partition in E between 0 and ∞ (in other terms $E \setminus \widetilde{\Omega} = E_0 \cup E_1$, where E_0, E_1 are open invariant and nonempty, $E_0 \ni 0$ is bounded). It is clear that $G = \mathbf{Z}_p$ acts freely on $\widetilde{\Omega} \cap E'$, as well as on $S(F')$, and the map

$$r\pi f : \widetilde{\Omega} \cap E' \to S(F')$$

is $\mathbf{Z}_p$-equivariant. But no such maps exist (for $\dim E' > \dim F'$), as shown for example in [9].

The following lemma is a reproduction of a proposition of [12] in the context of strong Borsuk–Ulam groups.

Lemma 6. *Let $1 \to H \to G \to K \to 1$ be an exact sequence of finite groups and H, K are strong Borsuk–Ulam groups. Then G is also a strong Borsuk–Ulam group.*

(In [12] it is proved for ordinary Borsuk–Ulam groups).

Proof. Let E and F be representations of G and $f : E \to F$ be an equivariant map, which is isovariant on $\partial\Omega$, where $\Omega \subset E$ is an open bounded set with $0 \in \Omega$. Since f is also H-isovariant on $\partial\Omega$ and H is a strong Borsuk–Ulam group,

$$\dim E - \dim E_H \leqq \dim F - \dim F_H.$$

On the other hand, E_H and F_H are representation spaces for $K \approx G/H$, moreover $f|_{E_H} : E_H \to F_H$ is K-isovariant on $\partial\Omega \cap E_H$. Therefore $\dim E_H - \dim(E_H)_K \leqq \dim F_H - \dim(F_H)_K$. Clearly, $(E_H)_K \approx E_G$, $(F_H)_K \approx F_G$, thus

$$\dim E_H - \dim E_G \leqq \dim F_H - \dim F_G.$$

Consequently

$$\dim E - \dim E_G \leqq \dim F - \dim F_G.$$

Lemma 4′ is now an immediate consequence of Lemmas 5 and 6.

Pass now to the main theorem.

Hereafter E, F are finite-dimensional Euclidean spaces. For a given orthogonal map $U : E \to E$ we shall denote by E_U the subspace

$$E_U = \{x \in E \mid Ux = x\}.$$

Theorem 1. *Let $U : E \to E$ and $V : F \to F$ be orthogonal maps and $f : E \to F$ be such that*

$$fU(x) = Vf(x) \quad \textit{for any} \ \ x \in E.$$

Suppose that $\dim E - \dim E_U > \dim F - \dim F_V$.

Then for any open bounded set $\Omega \subset E$ with $0 \in \Omega$ there exist $x \in \partial\Omega$ and $k \in \mathbf{Z}$ such that $U^k x \neq x$ but

$$f\left(U^k x\right) = f(x).$$

Proof. Let $E_{\text{per}} = \{x \in E \mid U^k x = x \text{ for some } k \neq 0\}$. Clearly, E_{per} is a linear subspace of E. Moreover, $f\left(E_{\text{per}}\right) \subset F_{\text{per}}$. (Where F_{per} is appropriately defined.) Let $m \in \mathbf{Z}$ be such that $U^m x = x$ for any $x \in E_{\text{per}}$ and $V^m x = x$ for any $x \in F_{\text{per}}$. Then a $\mathbf{Z}_m$-action is defined in E_{per} and F_{per} as follows: if ω is the formant of $\mathbf{Z}_m$, let $\omega x = Ux$ in E_{per} and $\omega x = Vx$ in F_{per}. Obviously, $f\big|_{E_{\text{per}}}\colon\ E_{\text{per}} \to F_{\text{per}}$ is $\mathbf{Z}_m$-equivariant, since $fU = Vf$. If $f\big|_{E_{\text{per}}}$ is not isovariant on $\partial\Omega \cap E_{\text{per}}$, then for some $x \in \partial\Omega \cap E_{\text{per}}$ and some $k \in \mathbf{Z}$ we have $U^k x \neq x$ and $f\left(U^k x\right) = f(x)$, so the theorem is proved. Suppose now that $f\big|_{E_{\text{per}}}$ is isovariant on $\partial\Omega \cap E_{\text{per}}$. Then, as following from Lemma 4,

$$\dim E_{\text{per}} - \dim E_U \leqq \dim F_{\text{per}} - \dim F_V.$$

Consider the orthogonal decompositions

$$E = E_{\text{per}} \oplus E', \quad F = F_{\text{per}} \oplus F'.$$

By the above inequality and the condition of the theorem we have $\dim E' > \dim F'$. Let $\pi : F \to F'$ be the projection over the second factor. Consider the map $f' = \pi \circ f\big|_{E'}\colon E' \to F'$, which commutes, clearly, with U and V $(f'U = Vf')$. Note that the restrictions $U' = U\big|_{E'}$, $V' = V\big|_{F'}$ have no periodic points different from 0, thus E' and F' are even-dimensional spaces. Then one may diagonalize U' and V' with an appropriate change of co-ordinates, so that in complex notation we have $E' = \mathbf{C}^m$, $F' = \mathbf{C}^n$ and

$$U'(z_1, \ldots, z_m) = \left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_m} z_m\right),$$

$$V'(z_1, \ldots, z_n) = \left(e^{2\pi i\mu_1} z_1, \ldots, e^{2\pi i\mu_n} z_n\right),$$

where θ_j, μ_r are irrational numbers (for U' and V' have no periodic points different from 0). Let $f' = (\varphi_1, \ldots, \varphi_n) : \mathbf{C}^m \to \mathbf{C}^n$. The property $f'U' = V'f'$ is written then in the form

$$\varphi_r\left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_m} z_m\right) = e^{2\pi i\mu_r} \varphi_r(z_1, \ldots, z_m)$$

for $r = 1, \ldots, n$. But $m > n$ and Lemma 3 implies that $f'(z) = 0$ for some $z \in \partial\Omega \cap E'$. Then $\pi f(z) = f'(z) = 0$, thus $f(z) \in F_{\text{per}}$. Let $k \neq 0$ be such that $V^k f(z) = f(z)$. Then $U^k z \neq z$, since $z \in E'$, though

$$f\left(U^k z\right) = V^k f(z) = f(z).$$

The theorem is proved.

We shall give, in the next section, an example showing that we cannot claim the existence of $x \in \partial\Omega$ such that $f(Ux) = f(x)$, hence the presence of the integer k in the theorem is inavoidable. However, in case of free U, V a stronger result is valid.

Recall that $U : E \to E$ is called *free*, if $U^k x = x$ and $k \neq 0$ imply $x = 0$.

Theorem 2. *Let $U : E \to E$ and $V : F \to F$ be free orthogonal maps, and $f : E \to F$ be such that $fU = Vf$. Suppose that* $\dim E > \dim F$.

Then for any open bounded $\Omega \subset E$ with $0 \in \Omega$ there exists $x \in \partial\Omega$ such that $f(x) = 0$.

Proof. We have $E_{\text{per}} = \{0\}$, $F_{\text{per}} = \{0\}$, hence, following the proof of Theorem 1 we find some $x \in \partial\Omega$ such that $f(x) = 0$.

Corollary. *Let $m > n$ and $U : S^m \to S^m$, $V : S^n \to S^n$ be free orthogonal maps. Then there is no map $f : S^m \to S^n$ such that $fU = Vf$.*

This proposition may be interpreted in the context of dynamical systems. Indeed, U and V define discrete time dynamical systems in S^m and S^n, respectively, and a map $f : S^m \to S^n$ such that $fU = Vf$ is a semiconjugacy between them (cf. [7]). Then the corollary claims that no two systems of that type are semiconjugated for $m > n$. So, the first system is, in some sence, essentially more complex than the second one.

Theorem 3. *$\mathbb{Z}$ is a (strong) Borsuk–Ulam group with respect to orthogonal representations.*

This theorem is an immediate consequence of Theorem 1 and the definition of strong Borsuk–Ulam group.

Corollary. *$\mathbb{R}$ is a (strong) Borsuk–Ulam group with respect to orthogonal representations.*

Proof. Consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1.$$

It is shown in [3] that (in our terminology) the circle S^1 is a strong Borsuk–Ulam group. Then Lemma 6 and Theorem 3 imply that $\mathbb{R}$ is also such a group.

As above, we may restate the last corollary in terms of nonexistence of a semiconjugacy between linear flows on spheres. This result partially intersects with a theorem in [10] concerning such flows.

Another corollary of Lemma 6 is that the direct sum, $G_1 \oplus G_2$, of two strong Borsuk–Ulam groups is also such a group. We may formulate then the most general result of this type.

Theorem 4. *Every group of the form*

$$G = A \oplus \mathbb{Z}^m \oplus \mathbb{R}^n \oplus \mathbb{T}^k,$$

where A is a finite Abelian group, is a (strong) Borsuk–Ulam group with respect to orthogonal representations.

The proof follows from Lemma 4′, Theorem 3 and the previous remarks.

5. AN EXAMPLE

In this section we show that, in the setting of Theorem 1, the equation $f(Ux) = f(x)$ may not have nonzero solutions. This example answers, meanwhile, a question of Wasserman [12].

Let $E = \mathbf{R}^4$, $F = \mathbf{R}^3$, and

$$U(a,b,c,d) = (-d,-c,b,a), \quad V(a,b,c) = (-a,-b,-c).$$

Then, obviously, $E_U = \{0\}$, $F_V = \{0\}$. Define $f : E \to F$ by

$$f(a,b,c,d) = (a^2 + b^2 - c^2 - d^2, ac + bd, bc - ad). \tag{11}$$

This is in fact the Hopf fibration when restricted to S^3. One easily checks that $fU = Vf$ and that the equality $f(Ux) = f(x)$ implies $x = 0$.

Therefore we cannot take $k = 1$ in Theorem 1.

Let $S(E)$ denotes the unit sphere in E.

In his paper [12] Wasserman asked whether there exist a group G, representations E and F of G, such that $\dim E > \dim F$, $F_G = \{0\}$ and a G-equivariant map $f : S(E) \to S(F)$. Our example answers affirmatively this questions for $G = \mathbf{Z}_4$, since $U^4 = \mathrm{id}_E$, $V^4 = \mathrm{id}_F$. It is easy to see then that the map $f : E \to F$, defined by (11), transforms $S(E)$ into $S(F)$ and is $\mathbf{Z}_4$-equivariant. Note, finally, that $F_G = F_V = \{0\}$.

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