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RELATIVE SET GENERICITY

VERA BOUTCHKOVA

A set of natural numbers is generic relatively a set B if and only if it is the preimage of some set A using a B-generic B-regular enumeration such that both A and its complement are e-reducible to B.

Keywords: genericity, e-reducibility, enumerations

Mathematics Subject Classification 2000: 03D30

0. INTRODUCTION

The genericity and set genericity, as defined by Copestake in [2], are widely explored and have an important role in studying the structure of the enumeration degrees.

In this paper we consider the genericity relative a set of natural numbers, which is in fact a *set n-genericity.* We refer to some well-known facts in this area, most of which can be found in [2] and [1] and can be used to prove similar properties for the relative genericity.

Further we provide some results concerning *regular enumerations* of the set of natural numbers that we use to prove a characterization theorem. Concerning the regular enumerations, the used notions and results are taken mostly from Soskov's course on Recursion Theory and the author's Master's Thesis.

Basic notions and definitions

By ω we denote the set of all natural numbers, 2ω denoting the set of all even and $2\omega + 1$ — the set of all odd natural numbers; by $[0..n-1]$, where $n \in \omega$, we denote the set $\{x \in \omega \mid x < n\}$. We use N to denote an arbitrary denumerable set.

We use bijective recursive coding of pairs of natural numbers $\langle\cdot,\cdot\rangle$, the notation $\langle x_1, x_2, \dots, x_k\rangle$ meaning $\langle x_1, \langle x_2, \dots, x_k\rangle\rangle$, and of finite sets, where D_v denotes the finite set with code v. By φ, ψ, ... we denote partial functions from ω into ω and let $Gr(\varphi) = \{\langle x, y\rangle \mid \varphi(x) = y\}$ be the graph of the function φ. The notation $\varphi(x)\downarrow$ means $x \in Dom(\varphi)$, and $\varphi(x)\uparrow$ means $x \notin Dom(\varphi)$. The notation $\subseteq$ is used to denote *inclusion* between sets, *extension* between functions, ω-strings or 0-1-strings, considered as finite functions.

By C_A we denote the semicharacteristic function of a set $A \subseteq \omega$, and by χ_A — its characteristic function, where

$$\chi_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{if } x \notin A. \end{cases}$$

If each of P and Q denotes some property of natural numbers, we use the following abbreviation:

$$\mu y_{\in\omega}[Q(y)][P(y)] \simeq \begin{cases} \mu y_{\in\omega}\big[Q(y)\&P(y)\big], & \text{if } \exists y\big(P(y)\&Q(y)\big), \\ \mu y_{\in\omega}\big[Q(y)\big], & \text{if } \exists y\big(Q(y)\big) \text{ and } \neg\big(P(y)\&Q(y)\big), \\ \uparrow, & \text{if } \forall y(\neg Q(y)), \end{cases}$$

where $\mu y_{\in\omega}[Q(y)]$ is the least y having the property Q.

Let A, B and C... be sets of natural numbers. We use the following standard definitions and notations:

$A \leq_e B$ if and only if $A = \Psi_a(B)$ for some e-operator Ψ_a, defined as $\Psi_a(B) = \{x \mid \exists v\big(\langle x, v\rangle \in W_a \,\&\, D_v \subseteq B\big)\}$, where W_a is the recursively enumerable set with Gödel code a. $A \equiv_e B$ if and only if $A \leq_e B$ and $B \leq_e A$. The enumeration degree (e-degree) of the set A is the equivalence class $Deg_e(A) = \{B \subseteq \omega \mid A \equiv_e B\}$. We denote the e-degrees by a, b, c ...

We use the standard *join* operation of two sets $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$ having the property that $Deg_e(A \oplus B)$ is the least upper bound of $Deg_e(A)$ and $Deg_e(B)$.

A set of natural numbers C is said to be *total* if its complement is e-reducible to C, i. e. $\overline{C} \leq_e C$ (which is equivalent to $C \equiv_e C^+$, where we define $C^+ = C \oplus \overline{C}$, and thus for every set $C^+ \equiv_e Gr(\chi_C)$).

1. B-GENERIC SETS

Definition 1.1. *ω-string* is a finite function from ω into ω with domain an initial segment of ω. $\varnothing_\omega$ denotes the nowhere defined function, considered as an *empty ω-string*; note that *length* of σ_ω is $lh(\sigma_\omega) = \mu x\big[\neg\exists y(\sigma_\omega(x) = y)\big]$.

0-1-*string* (or 2-*valued string*) is an ω-string α_ω such that $Rng(\alpha_\omega) \subseteq \{0, 1\}$. For every 0-1-string α_ω we define the set $\alpha_\omega^+ = \{x \mid \alpha_\omega(x) \simeq 0\}$.

Definition 1.2. The set A is *B-generic*, for $B \subseteq \omega$, if and only if for every set S, such that S is a set of 0-1-strings and $S \leq_e B$,

$$\exists \alpha_\omega \subseteq \chi_A\big(\alpha_\omega \in S \vee \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin S)\big).$$

The set A is *quasi-minimal over B* if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and

(2) if C is a total set such that $C \leq_e A$, then $C \leq_e B$.

The set A is *minimal-like over B* if and only if

(1) $B \leq_e A$, but $A \not\leq_e B$; and

(2) for every partial function φ such that $\varphi \leq_e A$, there exists a partial function ψ such that $\varphi \subseteq \psi$ and $\psi \leq_e B$.

In analogue to the definitions in [1], an e-degree containing such set is said to be strongly minimal-like over B.

Here we mention some of the properties of the B-generic sets that we will need later: A is B-generic if and only if $\overline{A}$ is B-generic; if A is B-generic, there is no infinite e-reducible to B subset of A; every B-generic set A is infinite and not e-reducible to B.

Concerning the existence of a B-generic set, a minimal-like set over any set B and the existence of a quasi-minimal set over any set B, see [1, 2], it is proven that for an arbitrary B-generic set A, the set $A \oplus B$ is minimal-like and quasi-minimal over B.

Theorem 1.3. *Let $B_0, B_1, \dots, B_n, \dots$ be a sequence of sets of natural numbers. There exists a set of natural numbers A, which is minimal-like over this sequence, i. e. such that the next two conditions hold:*

1) $\forall n(B_n \leq_e A)$;

2) *For every partial function φ such that $\varphi \leq_e A$, there exist a partial function ψ and a natural number n such that $\varphi \subseteq \psi$ and $\psi \leq_e B_0 \oplus \dots \oplus B_n$.*

Proof. In the following proof the notation $\overset{\infty}{\forall} x\, P(x)$ is equivalent to $\exists y \forall x(x \geq y \Rightarrow P(x))$. We define a set A, satisfying two requirements:

(a) $\forall n \overset{\infty}{\forall} x\big(\langle x, n\rangle \in A \Leftrightarrow x \in B_n\big)$, and

(b) $\forall e\Big(\Psi_e(A)$ is a function $\Rightarrow \exists \psi\big(\Psi_e(A) \subseteq \psi \,\&\, \psi \leq_e B_0 \oplus \dots \oplus B_{2e+1}\big)\Big)$, and

build finite sets $A_0 \subseteq \dots \subseteq A_s \subseteq \dots$, having the property:

$\forall s\big(\langle x, m\rangle \in A_{s+1} \setminus A_s \,\&\, m \leq s \Rightarrow x \in B\big)$ for all x and m.

Stage 0. Let $A_0 = \varnothing$.

Stage $2e+1$. A_s is built, where $s = 2e$. We have two cases:

Case 1. There exists $\langle x, n\rangle$ such that $x \in B_n$ and $\langle x, n\rangle \notin A_s$. Then we can define $A_{s+1} = A_s \cup \{\langle x, n\rangle\}$ for the first such $\langle x, n\rangle = \mu\langle x, n\rangle$.

Case 2. Otherwise, define $A_{s+1} = A_s$.

Stage $2e+2$. A_s is built, where $s = 2e+1$. Again we have two cases:

Case 1. There exists a finite set D_v such that $A_s \subseteq D_v$ and $\Psi_e(D_v)$ is not a function (i. e. $\exists x \exists y \exists z$ such that $y \neq z \,\&\, \langle x, y\rangle \in \Psi_e(D_v) \,\&\, \langle x, z\rangle \in \Psi_e(D_v)$) and such that $\forall t \forall m(\langle t, m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)$.

Define A_{s+1} to be the least D_v (i. e. having the least code v) with this property.

Case 2. Otherwise, define $A_{s+1} = A_s$.

End.

Finally, define $A = \bigcup_{s=0}^{\infty} A_s$.

For this set we can prove the properties (a) and (b), from which our theorem follows.

The interesting direction of the proof of (a) is ($\Rightarrow$). We can prove that $\forall n \overset{\infty}{\forall} x \,(\langle x, n\rangle \in A \Rightarrow x \in B_n)$. Assume it is not true, i. e. there exist n and infinitely many $x_0 < \ldots < x_i < \ldots$ such that $\langle x_i, n\rangle \in A$ and $x_i \notin B_n$. Therefore $\forall x_i \exists s_i(\langle x_i, n\rangle \in A_{s_i+1} \setminus A_{s_i})$. But at every stage s the set $A_{s+1} \setminus A_s$ is finite, then there exist infinitely many $x_{s_0}, \ldots, x_{s_i}, \ldots$ from this sequence such that at stages $s_0 < \ldots < s_i < \ldots$ we have $\langle x_{s_i}, n\rangle \in A_{s_i+1} \setminus A_{s_i}$. But $x_{s_i} \notin B_n$ and then the stages $s_i + 1$ must be even (i. e. $s_i + 1 = 2e_i + 2$), and we have Case 1, i. e. $A_{s_i+1} = D_v$, where $D_v \supseteq A_{s_i}$ and $\forall t \forall m(\langle t, m\rangle \in D_v \setminus A_{s_i} \,\&\, m \leq s_i \Rightarrow t \in B_m)$. Therefore for every $s_i \geq n$ if $\langle x_{s_i}, n\rangle \in A_{s_i+1} \setminus A_{s_i}$, then $x_{s_i} \in B_n$, which is a contradiction.

The proof of (b) consists in the following: supposing $\Psi_e(A)$ to be a graph of some function, at Stage $2e+2$, for $s = 2e+1$ we have Case 2. Define the set $G_\psi = \{\langle x, y\rangle \mid \exists D_v(D_v \supseteq A_s \,\&\, \langle x, y\rangle \in \Psi_e(D_v) \,\&\, \forall \langle t, m\rangle(\langle t, m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m))\}$. Therefore the following conditions hold:

- $G_\psi \leq_e B_0 \oplus \ldots \oplus B_s$;
- $G_\psi = Gr(\psi)$, i. e. G_ψ is a graph of some function ψ, since assuming it is not true, there exist x and $y_1 \neq y_2$ such that $\langle x, y_1\rangle \in G_\psi$ and $\langle x, y_1\rangle \in G_\psi$. Therefore there exist finite sets D_{v_1} and D_{v_2}, both extending A, such that $\langle x, y_1\rangle \in \Psi_e(D_{v_i})$ and $\forall \langle t, m\rangle(\langle t, m\rangle \in D_{v_i} \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)$. Then for $D_v = D_{v_1} \cup D_{v_2}$, $\Psi_e(D_v)$ is not a function and $\forall \langle t, m\rangle(\langle t, m\rangle \in D_v \setminus A_s \,\&\, m \leq s \Rightarrow t \in B_m)$, which is a contradiction with Case 2;
- $\Psi_e(A) \subseteq G_\psi$, since assuming there is $\langle x, y\rangle \in \Psi_e(A) \setminus G_\psi$, there exists $A_{s+p} \supseteq A_s$ such that $\langle x, y\rangle \in \Psi_e(A_{s+p})$ and $\exists \langle t, m\rangle$ $(\langle t, m\rangle \in A_{s+p} \setminus A_s \,\&\, m \leq s \,\&\, t \notin B_m)$. It follows that there is i, such that $0 \leq i < p$ and $\langle t, m\rangle \in A_{s+i+1} \setminus A_{s+i}$, and therefore $m \leq s+i$. Since $A_{s+i+1} \setminus A_{s+i} \neq \varnothing$, we have Case 1 at Stage $s+i = 2e_i + 1$ or Case 1 at Stage $s+i = 2e_i$. But in both cases it follows that $t \in B_m$, which is a contradiction.

This proves our proposition. □

As a corollary of the above theorem we obtain the existence of strongly minimal-like e-degree over an infinite ascending sequence of e-degrees.

2. B-GENERIC REGULAR ENUMERATIONS

In this section we illustrate briefly some results obtained using the relative generic regular enumerations and many of the proofs will be only sketched.

Definition 2.1. Let $B \subseteq \omega$ be a non-empty set of natural numbers.

1) The total and surjective function $f : \omega \to \omega$ is called B-regular ω-enumeration if $f(2\omega) = B$, where $f(2\omega) = \{f(2x) \mid x \in \omega\}$.

2) An ω-string τ_ω is B-regular if $\tau_\omega(2\omega) \subseteq B$, where $\tau_\omega(2\omega) = \{y \mid \exists x\ (\tau_\omega(2x) = y)\}$.

3) The B-regular ω-enumeration f is called B-generic if for every e-reducible to B set of ω-strings F the following holds:

$$\exists \sigma_\omega \subseteq f\big(\sigma_\omega \in F \vee \forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin F)\big).$$

For every non-empty set B one can iteratively build a B-generic B-regular enumeration f at stages, using ω-strings to satisfy the requirements in the definition of f.

It is true that $f \not\leq_e B$ for every B-generic B-regular enumeration f. This can be proved assuming $f \leq_e B$ and defining the e-reducible to B set of ω-strings $S = \{\tau_\omega \mid \tau_\omega(2\omega) \subseteq B \,\&\, \tau_\omega \not\subseteq f\}$, that will lead to the contradiction.

Proposition 2.2. *For every B-generic B-regular enumeration f, for every set R such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$, the set $f^{-1}(R)$ is B-generic.*

Proof. Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, we have that $\chi_{f^{-1}(R)} = \chi_R \circ f$. Assume $f^{-1}(R)$ is not B-generic, i. e. there is an e-reducible to B set of ω-strings such that

$$\forall \alpha_\omega\big(\alpha_\omega \subseteq \chi_{f^{-1}(R)} \Rightarrow \alpha_\omega \notin F \,\&\, \exists \beta_\omega(\beta_\omega \supseteq \alpha_\omega \,\&\, \beta_\omega \in F)\big). \tag{1}$$

Define $S = \{\sigma_\omega \mid \exists \alpha_\omega(\alpha_\omega \in F \,\&\, \chi_R \circ \sigma_\omega = \alpha_\omega)\}$, where $\chi_R \circ \sigma_\omega = \alpha_\omega$ if and only if $\big(lh(\alpha_\omega) = lh(\sigma_\omega) \,\&\, \forall x < lh(\alpha_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big)\big)$, therefore S is a set of B-regular ω-strings and $S \leq_e B$. But f is a B-generic B-regular enumeration, so there is $\sigma_\omega \subseteq f$ such that either $\sigma_\omega \in S$ or $\forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin S)$.

Assuming $\sigma_\omega \in S$, there is $\alpha_\omega \in F$ such that $\chi_R \circ \sigma_\omega = \alpha_\omega$, but $\sigma_\omega \subseteq f$ and then $\chi_R \circ f \supseteq \alpha_\omega$, i. e. $\alpha_\omega \subseteq \chi_{f^{-1}(R)}$, which is a contradiction with (1). Therefore for that σ_ω the following holds:

$$\forall \tau_\omega \supseteq \sigma_\omega(\tau_\omega \notin S). \tag{2}$$

Define $\alpha_\omega = \chi_R \circ \sigma_\omega$. Since $\sigma_\omega \subseteq f$, then $\alpha_\omega \subseteq \chi_R \circ f = \chi_{f^{-1}(R)}$, and from (1) it follows that there exists β_ω such that $\beta_\omega \supseteq \alpha_\omega$ and $\beta_\omega \in F$. Therefore $\beta_\omega \supseteq \chi_R \circ \sigma_\omega = \alpha_\omega$ and $lh(\beta_\omega) \geq lh(\alpha_\omega)$. If we fix two elements of B — $a \in R \cap B$ and $b \in \overline{R} \cap B$, we can define an ω-string τ_ω such that $\tau_\omega \supseteq \sigma_\omega$, $lh(\tau_\omega) = lh(\beta_\omega)$ and $\forall x\big(lh(\sigma_\omega) \leq x \leq lh(\tau_\omega) \Rightarrow (\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)\big)$, i. e. $\beta_\omega = \chi_R \circ \tau_\omega \supseteq$

$\chi_R \circ \sigma_\omega = \alpha_\omega$. Since $\beta_\omega \in F$ and $\chi_R \circ \tau_\omega = \beta_\omega$, then $\tau_\omega \in S$, which is a contradiction with (b). Therefore $f^{-1}(R)$ is not B-generic set. $\square$

The next corollary follows directly from Proposition 2.2 and the properties of relative generic sets in Section 1.

Corollary 2.3. *For every B-generic B-regular enumeration f, for every set R such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$, the set $f^{-1}(R) \oplus B$ is quasi-minimal over B.*

Lemma 2.4. *Let A be B-generic. Let $R \subseteq \omega$ such that $R \leq_e B$, $\overline{R} \leq_e B$, $R \cap B \neq \varnothing$ and $\overline{R} \cap B \neq \varnothing$. Let δ_ω be an ω-string, having the properties* (1) *and* (2). *Then:*

(1) δ_ω *is B-regular;*

(2) $\forall x < lh(\delta_\omega)\ (x \in A \Leftrightarrow \delta_\omega(x) \in R)$.

For every S such that S is an e-reducible to B set of ω-strings, there exists an ω-string σ_ω, having the properties (a)–(d):

(a) $\sigma_\omega \supseteq \delta_\omega$;

(b) σ_ω *is B-regular;*

(c) $\forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)$;

(d) $\sigma_\omega \in S \ \vee\ \forall \tau_\omega(\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin S)$.

Proof. Let us denote by $\alpha_\omega \sim_R \sigma_\omega$ the property

$$\forall x \in Dom(\sigma_\omega)\big(\alpha_\omega(x) = 0 \Leftrightarrow \sigma_\omega(x) \in R\big),$$

where α_ω is a 0-1-string, σ_ω is an ω-string and $R \subseteq \omega$.

Define the set $P = \{\alpha_\omega \mid \exists \sigma_\omega\big(\sigma_\omega \in S \ \&\ \sigma_\omega \supseteq \delta_\omega \ \&\ \sigma_\omega(2\omega) \subseteq B \ \&\ lh(\alpha_\omega) = lh(\sigma_\omega) \ \&\ \alpha_\omega \sim_R \sigma_\omega\big)\}$ that is e-reducible to B. Since A is B-generic, we have two possibilities:

Case 1. $\exists \alpha_\omega \subseteq \chi_A\ (\alpha_\omega \in P)$. In this case there exists σ_ω — a B-regular extension of δ_ω in S with the same length as α_ω, such that $\alpha_\omega \sim_R \sigma_\omega$. But $\alpha_\omega \subseteq \chi_A$, then

$$\forall x < lh(\sigma_\omega)\big(x \in A \Leftrightarrow \sigma_\omega(x) \in R\big),$$

i. e. σ_ω has the properties (a)–(d).

Case 2. $\exists \alpha_\omega \subseteq \chi_a \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin P)$. In this case

$$\exists \alpha_\omega \subseteq \chi_A\big(lh(\delta_\omega) \leq lh(\alpha_\omega) \ \&\ \forall \beta_\omega \supseteq \alpha_\omega(\beta_\omega \notin S)\big).$$

Fix two elements: a in $R \cap B \neq \varnothing$ and b in $\overline{R} \cap B \neq \varnothing$. Now we can define an ω-string σ_ω such that $\sigma_\omega \supseteq \delta_\omega$ and $lh(\sigma_\omega) = lh(\alpha_\omega)$ and for the arguments x, where $lh(\delta_\omega) \leq x < lh(\alpha_\omega)$, we have $\sigma_\omega(x) \simeq a$ if $\alpha_\omega(x) = 0$; and $\sigma_\omega(x) \simeq b$ if $\alpha_\omega(x) = 1$. Since δ_ω is B-regular, then σ_ω is B-regular, too. And from (2) and $\alpha_\omega \subseteq \chi_A$ follows that $\forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)$. So, σ_ω has the properties (a)–(c). It remains to verify (d).

First, notice that $\alpha_\omega \sim_R \sigma_\omega$. Assume that there exists τ_ω such that $\tau_\omega \supseteq \sigma_\omega \supseteq \delta_\omega$ and $\tau_\omega \in S$ (then τ_ω is B-regular). Therefore there exists a 0-1-string β_ω such that $\beta_\omega \supseteq \alpha_\omega$ and $lh(\beta_\omega) = lh(\tau_\omega)$, and for the arguments $lh(\alpha_\omega) \leq x < lh(\tau_\omega)$ we have $\beta_\omega(x) \simeq 0$ if $\tau_\omega(x) \in R$, and $\beta_\omega(x) \simeq 1$ if $\tau_\omega(x) \notin R$. Since $\alpha_\omega \sim_R \sigma_\omega$ for this β, it follows that $\forall x < lh(\beta_\omega)$ $(\beta_\omega(x) = 0 \Leftrightarrow \tau_\omega(x) \in R)$, i. e. $\beta_\omega \sim_R \tau_\omega$, and therefore $\beta_\omega \in P$, which is a contradiction with Case 2. Then the property (d) holds.

In both cases we have found an ω-string satisfying (a)–(d). □

Proposition 2.5. *Let A be B-generic and R be such that $R \cap B \neq \varnothing$, $\overline{R} \cap B \neq \varnothing$, $R \leq_e B$ and $\overline{R} \leq_e B$. There exists a B-generic B-regular enumeration f such that $A = f^{-1}(R)$.*

Proof. Since $f^{-1}(R) = \{x \mid f(x) \in R\}$, $A = f^{-1}(R)$ is equivalent to $\forall x(x \in A \Leftrightarrow f(x) \in R)$.

We build a sequence of ω-strings $\sigma^0_\omega \subseteq \sigma^1_\omega \subseteq \ldots \sigma^q_\omega \subseteq \ldots$ such that each σ^q_ω has the properties (1) and (2):

(1) σ^q_ω is B-regular, i. e. $\sigma^q_\omega(2\omega) \subseteq B$;

(2) $\forall x < lh(\sigma^q_\omega)$ $(x \in A \Leftrightarrow \sigma^q_\omega(x) \in R)$.

If (1) holds for all σ^q_ω, then $f(2\omega) \subseteq B$. If (2) holds for each σ^q_ω, then from (3) it follows that $A = f^{-1}(R)$.

At *Stage* $(2e+1)$ we insure f to be total, surjective and $f(2\omega) \subseteq B$, i. e.

(3) $\forall q = 2e+1$ $\big(lh(\sigma^{q+1}_\omega) > lh(\sigma^q_\omega)\big)$;

(4) $\forall x \in \omega\ \exists q = 2e+1$ $\big(x \in Rng(\sigma^q_\omega)\big)$;

(5) $\forall x \in B\ \exists q = 2e+1$ $\big(x \in \sigma^q_\omega(2\omega)\big)$.

At *Stage* $(2e+2)$ we insure f to be B-generic, i. e.

(6) $\forall q = 2e+2$ $\Big($ if $\Psi_e(B)$ is a set of B-regular ω-strings, then

$\big(\sigma^q_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma^q_\omega(\tau_\omega \notin \Psi_e(B))\big)\Big)$.

Stage 0. Define $\sigma^0_\omega = \varnothing_\omega$.

Stage $2e+1$. At this stage σ^q_ω is built with $q = 2e$.

Let x_0, x_1, x_2 and x_3 be the first numbers, greater or equal to $lh(\sigma^q_\omega)$, that belong to $2\omega \cap A$, $(2\omega+1) \cap A$, $2\omega \cap \overline{A}$ and $(2\omega+1) \cap \overline{A}$, respectively. Such x_i exist, because assuming, for example, $\forall x$ $(x \geq lh(\sigma^q_\omega)$ & $x \in 2\omega \Rightarrow x \notin A)$, the set $C_0 = \{x \mid x \geq lh(\sigma^q_\omega)$ & $x \in 2\omega\}$ is infinite and recursively enumerable and $C_0 \subseteq \overline{A}$, which is a contradiction with the properties of the B-generic sets.

Let $m = \max\{x_0, x_1, x_2, x_3\}$. Define σ^{q+1}_ω such that $\sigma^{q+1}_\omega \supseteq \sigma^q_\omega$ and $lh(\sigma^{q+1}_\omega) = m+1 > lh(\sigma^q_\omega)$, and for the arguments $lh(\sigma^q_\omega) \leq x \leq m$ define as follows:

$$\sigma_\omega^{q+1}(x) \simeq \begin{cases} \mu y[y \in R \cap B][y \notin Rng(\sigma_\omega^q)], & x \in 2\omega \,\&\, x \in A, \\ \mu y[y \in \overline{R} \cap B][y \notin Rng(\sigma_\omega^q)], & x \in 2\omega \,\&\, x \notin A, \\ \mu y[y \in R][y \notin Rng(\sigma_\omega^q)], & x \notin 2\omega \,\&\, x \in A, \\ \mu y[y \in \overline{R}][y \notin Rng(\sigma_\omega^q)], & x \notin 2\omega \,\&\, x \notin A. \end{cases}$$

Stage $2e+2$. At this stage σ_ω^q is built with $q = 2e+2$.

Define $G = \{\sigma_\omega \mid \sigma_\omega(2\omega) \subseteq B \;\&\; \forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)\}$, i. e. $G = \{\sigma_\omega \mid$ for σ_ω (1) and (2) hold true$\}$. We have two possibilities:

Case 1. $\exists \sigma_\omega \supseteq \sigma_\omega^q \Big(\sigma_\omega \in G \;\&\; \big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega \supseteq \sigma_\omega (\tau_\omega \notin \Psi_e(B))\big)\Big)$.
Define σ_ω^{q+1} to be the least such σ_ω.

Case 2. $\forall \sigma_\omega \supseteq \sigma_\omega^q \Big(\sigma_\omega \in G \Rightarrow \big(\sigma_\omega \notin \Psi_e(B) \;\&\; \exists \tau_\omega \supseteq \sigma_\omega\ (\tau_\omega \in \Psi_e(B))\big)\Big)$.
Define $\sigma_\omega^{q+1} = \sigma_\omega^q$.

End.

Define $f = \bigcup_{q=0}^{\infty} \sigma_\omega^q$.

Using an induction on q, one can prove that for each σ_ω^q the conditions (1) and (2) hold. At Stage $2e+1$ we satisfy the requirements (3)–(5). It follows that f is a B-regular enumeration and $A = f^{-1}(R)$.

From (1) and (2) for σ_ω it follows that for every $e \in \omega$, if $\Psi_e(B)$ is a set of B-regular ω-strings, then there exists σ_ω, having the properties (a)–(d) of Lemma 2.4, i. e. $\sigma_\omega \supseteq \sigma_\omega^q$, σ_ω is B-regular, $\forall x < lh(\sigma_\omega)\ (x \in A \Leftrightarrow \sigma_\omega(x) \in R)$ and $\big(\sigma_\omega \in \Psi_e(B) \vee \forall \tau_\omega\ (\tau_\omega \supseteq \sigma_\omega \Rightarrow \tau_\omega \notin \Psi_e(B))\big)$. This means that if $\Psi_e(B)$ is a set of B-regular ω-strings, at Stage $2e+1$, we never have Case 2, i. e. the requirement (6) is satisfied.

Therefore our f is a B-generic B-regular enumeration such that $A = f^{-1}(R)$.

□

Theorem 2.6. *Let B be a non-empty set of natural numbers. Any set $A \subseteq \omega$ is B-generic if and only if there exist a set R and a B-generic B-regular enumeration f such that $R \leq_e B$ and $\overline{R} \leq_e B$, and $A = f^{-1}(R)$.*

Proof. ($\Leftarrow$) The Proposition 2.2.

($\Rightarrow$) If A is B-generic and there exist at least two different elements in B (otherwise B is recursively enumerable and therefore e-equivalent to a set containing at least two different elements) $a \neq b$. Then for $R = \{a\}$ the conditions in Proposition 2.5 hold and therefore there exists a B-generic B-regular enumeration f such that $A = f^{-1}(R)$, and for the existence of B-generic B-regular enumeration we need only $B \neq \varnothing$. □

REFERENCES

1. Cooper, B. Enumeration reducibility, nondeterministic computations and relative computability of partial functions. Recursion Theory Week, Oberwolfach 1989, Springer-Verlag, Berlin–Heidelberg–New York, 1990, 57–110.
2. Copestake, K. 1-Genericity in the enumeration degrees. *J. of Symbolic Logic*, **53**, 1988, 878–887.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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ON A CLASS OF VERTEX FOLKMAN GRAPHS

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Let $a_1, \ldots, a_r$ be positive integers and $m = \sum_{i=1}^{r}(a_i - 1) + 1$. For a graph G the symbol $G \to (a_1, \ldots, a_r)$ means that in every r-colouring of the vertices of G there exists a monochromatic a_i-clique of colour i, for some i, $1 \leq i \leq r$. In this paper we consider the graphs $G \to (a_1, \ldots, a_r)$ (vertex Folkman graphs) with $\mathrm{cl}(G) < m - 1$.

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1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We say that G is an n-vertex graph, when $|V(G)| = n$. For $v \in V(G)$ we denote by $\mathrm{Ad}(v)$ the set of all vertices adjacent to v. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$ (the clique number of G). A set of vertices in a graph G is said to be independent if no two of them are adjacent. The cardinality of any largest independent set of vertices in G is written as $\alpha(G)$ (the independence number of G).

If $W \subseteq V(G)$, then: $G[W]$ is the subgraph induced by W and $G - W$ is the subgraph induced by $V(G) \setminus W$.

In this paper we shall use also the following notations:

$\chi(G)$ — the chromatic number of G;
$\pi(G)$ — the maximum number of independent edges in G
(the matching number of G);

$\overline{G}$ — the complement of graph G;
K_n — the complete graph of n vertices;
P_n — the path of n vertices;
C_n — the simple cycle of n vertices.

By $C_n = v_1, \ldots, v_n$ we denote that

$$V(C_n) = \{v_1, \ldots, v_n\} \quad \text{and} \quad E(C_n) = \{[v_i, v_{i+1}],\ i = 1, \ldots, n-1,\ [v_1, v_n]\}.$$

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[v_1, v_2],\ v_1 \in V(G_1),\ v_2 \in v(G_2)\}$

2. THE VERTEX FOLKMAN GRAPHS

Definition. Let G be a graph, $a_1, \ldots, a_r$, $r \geq 2$, be positive integers and

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \varnothing,\ i \neq j,$$

be an r-colouring of the vertices of G. This r-coloring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the set V_i contains no a_i-clique. The symbol $G \rightarrow (a_1, \ldots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

It is obvious that:

Proposition 1. *If* $m = \sum_{r=1}^{r} (a_i - 1) + 1$, *then* $K_m \rightarrow (a_1, \ldots, a_r)$.

Proposition 2. *For any* $r \geq 2$

$$G \rightarrow \underbrace{(2, \ldots, 2)}_{r} \iff \chi(G) \geq r + 1.$$

Proposition 3. *Let* $G \rightarrow (a_1, \ldots, a_r)$ *and* $\{b_1, \ldots, b_t\} \subseteq \{a_1, \ldots, a_r\}$. *Then* $G \rightarrow (b_1, \ldots, b_t)$.

Proposition 4. *Let* $A \subseteq V(G)$ *be an independent set of* G *and* $G_1 = G - A$. *Let also* $G \rightarrow (a_1, \ldots, a_r)$, *where* $a_i \geq 2$ *for some* $i \in \{1, \ldots, r\}$. *Then* $G_1 \rightarrow (a_1, \ldots, a_i - 1, \ldots a_r)$.

Proof. Assume the opposite and let $V_1 \cup \ldots \cup V_r$ be an $(a_1, \ldots, a_i - 1, \ldots, a_r)$-free r-colouring of $V(G_1)$. Then $V_1 \cup \ldots \cup (V_i \cup A) \cup \ldots \cup V_r$ is an $(a_1, \ldots, a_i, \ldots, a_r)$-free r-colouring of $V(G)$, which is a contradiction.

Proposition 5. *For any permutation* φ *of the symmetric group* S_r *we have*

$$G \rightarrow (a_1, \ldots, a_r) \iff G \rightarrow (a_{\varphi(1)}, \ldots, a_{\varphi(r)}).$$

Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers. Then we put

$$m = \sum_{i=1}^{r} (a_i - 1) + 1 \quad \text{and} \quad a = \max\{a_1, \ldots, a_r\}. \tag{1}$$

We put also

$$H(a_1,\dots,a_r;q) = \{G : G \to (a_1,\dots,a_r) \text{ and } \mathrm{cl}(G) < q\}$$

$$F(a_1,\dots,a_r;q) = \min\{|V(G)| : G \in H(a_1,\dots,a_r;q)\}$$

It is clear that if $\mathrm{cl}(G) < a$, then there exists an $(a_1,\dots,a_r)$-free r-colouring of $V(G)$. Folkman has proved in [3] that if $q \geq a+1$, then $H(a_1,\dots,a_r;q) \neq \varnothing$. The graphs of $H(a_1,\dots,a_r;q)$, $q \geq a+1$, will be called the vertex Folkman graphs. The numbers $F(a_1,\dots,a_r;q)$ are called vertex Folkman numbers.

It is clear that K_{m-1} has an $(a_1,\dots,a_r)$-free r-colouring of $V(K_{m-1})$. It is clear also that from $\chi(G) \leq m-1$ it follows that G has an $(a_1,\dots,a_r)$-free vertex r-colouring. Therefore we have the following:

Proposition 6. *If $G \to (a_1,\dots,a_r)$, then $\chi(G) \geq m$.*

Since $K_m \to (a_1,\dots,a_r)$ and $K_{m-1} \not\to (a_1,\dots,a_r)$, if $q \geq m+1$, we have $F(a_1,\dots,a_r;q) = m$.

For the numbers $F(a_1,\dots,a_r;m)$ the following facts are known:

Theorem A ([6]). *Let a_1, a_2, ..., a_r, $r \geq 2$, be positive integers and m and a satisfy* (1). *If $m \geq a+1$, then $F(a_1,\dots,a_r;m) = m+a$.*

Theorem B ([7]). *Let a_1, a_2, ..., a_r, $r \geq 2$, be positive integers and m and a satisfy* (1). *If $m \geq a+1$, $G \in H(a_1,\dots,a_r;m)$ and $|V(G)| = m+a$, then $G = K_{m-a-1} + \overline{C}_{2a+1}$.*

In the present paper we consider the vertex Folkman numbers $F(a_1,\dots,a_r;m-1)$, $m \geq a+2$.

From Proposition 5 follows that $F(a_1,\dots,a_r;q)$ is a symmetric function and thus we may assume that $a_1 \leq a_2 \leq \dots \leq a_r$. Note that if $a_1 = 1$, then $F(a_1,\dots,a_r;q) = F(a_2,\dots,a_r;q)$. Hence we may assume also $a_i \geq 2$, $i = 1,\dots,r$.

Theorem A yields $F(2,2;3) = 5$.

In the special case $a_1 = \dots = a_r = 2$, $r \geq 3$, we have:

Theorem C. *For any $r \geq 3$ it is true that*

$$F(\underbrace{2,\dots,2}_{r};r) = \begin{cases} 11, & r = 3 \text{ or } r = 4 \\ r+5, & r \geq 5. \end{cases}$$

Mycielski, [8], presented an 11-vertex graph $G \in H(2,2,2;3)$, proving that $F(2,2,2;3) \leq 11$. Chvatal, [2], show that Mycielski's graph is the unique 11-vertex graph in the class $H(2,2,2;3)$ and hence $F(2,2,2;3) = 11$. The inequality $F(2,2,2,2;4) \geq 11$ is proved in [11] and the inequality $F(2,2,2,2,;4) \leq 11$ is proved in [10] and [15] (see also [4] and [12]). The equality $F(\underbrace{2,\dots,2}_{r};r) = r+5$, $r \geq 5$, is proved in [10], [15] and later in [7]. If $r \geq 5$, then $K_{r-5} + C_5 + C_5$ is the unique $(r+5)$-vertex graph in $H(\underbrace{2,\ldots,2}_{r};r)$, [10]. The class $H(2,2,2,2;4)$ contains 56 11-vertex graphs, [4]. In [4] it is proved also that $F(2,2,2,2;3)=22$. This is the unique known vertex Folkman number $F(a_1,\ldots,a_r;q)$ for which $q\le m-2$.

3. A LOWER BOUND ON THE VERTEX FOLKMAN NUMBERS $F(a_1,\ldots,a_r;m-1)$

Theorem 1. *Let $a_1, \ldots, a_r$ be positive integers. Let a and m satisfy* (1) *and $m\ge a+2$. Then*

$$F(a_1,\ldots,a_r;m-1)\ge m+a+2.$$

Proof. According to Proposition 5 we may assume that $a_1\le a_2\le\cdots\le a_r=a$. Let $G\in H(a_1,\ldots,a_r;m-1)$. Let also A be an independent set of G, $|A|=\alpha(G)$ and $G_1=C-A$. It follows from $m\ge a+2$ that $a_{r-1}\ge 2$. According to Proposition 4, $G_1\in H(a_1,\ldots,a_{r-1}-1,a_r;m-1)$. According to Theorem A, $|V(G_1)|\ge m+a-1$, i.e. $|V(G)|\ge m+a-1+\alpha(G)$. Since $\alpha(G)\ge 2$, it follows that $|V(G)|\ge m+a+1$. We prove that $|V(G)|\ne m+a+1$. Assume the opposite. Then $|V(G_1)|=m+a-1$ and $\alpha(G)=2$. According to Theorem B, $G_1=K_{m-a-2}+\overline{C}_{2a+1}$. Let $A=\{u_1,u_2\}$, $V(K_{m-a-2})=\{z_1,\ldots,z_{m-a-2}\}$ and $C_{2a+1}=v_1,v_2,\ldots,v_{2a+1}$.

Case 1. $\mathrm{Ad}(u_i)\not\supseteq V(K_{m-a-2})$, $i=1,2$. In this case $\chi(G)=\chi(G_1)=m-1$, which contradicts Proposition 6.

Case 2. $\mathrm{Ad}(u_1)\not\supseteq V(K_{m-a-2})$ and $\mathrm{Ad}(u_2)\supseteq V(K_{m-a-2})$. Let u_1 and z_1 be not adjacent. It follows from $\mathrm{cl}(G)\le m-2$ that $\mathrm{Ad}(u_2)\not\supseteq V(\overline{C}_{2a+1})$. Hence we may assume that u_2 and v_1 are not adjacent. The equality

$$V(G)=\{z_1,u_1\}\cup\{z_2\}\cup\ldots\cup\{z_{m-a-2}\}\cup\{u_2,v_1\}\cup\{v_2,v_3\}\cup\ldots\cup\{v_{2n},v_{2n+1}\}$$

implies $\chi(G)\le m-1$, which contradicts Proposition 6.

Case 3. $\mathrm{Ad}(u_i)\supseteq V(K_{m-a-2})$, $i=1,2$. We put

$$M=\{v_{2i-1}: i=1,\ldots,p-1\}\subseteq V(\overline{C}_{2a+1}).$$

It is clear that M is an $(a-1)$-clique. We prove that

$$M\not\subseteq \mathrm{Ad}(u_i),\quad i=1,2. \tag{2}$$

Assume the opposite and let $M\subseteq\mathrm{Ad}(u_1)$. From $\mathrm{cl}(G)\le m-2$, $\{u_1,v_{2a-1},v_{2a}\}$ is an independent set, which contradicts $\alpha(G)=2$.

We put

$$V'=V(K_{m-a-2})\cup\{v_{2a+1},v_{2a},v_{2a-1},v_{2a-2}\},\quad G'=G[V'],$$

$$V_r'=V(\overline{C}_{2a+1})-\{v_{2a+1},v_{2a},v_{2a-1},v_{2a-2}\},\quad V_r=V_r'\cup\{u_1,u_2\}.$$

Obviously, $\chi(G')=m-a=\sum_{i=1}^{r-1}(a_i-1)$. This equality implies that there exists an $(a_1,\ldots,a_{r-1})$-free $(r-1)$-colouring $V_1\cup\ldots\cup V_{r-1}$ of $V(G')$. Since M is the unique $(a-1)$-clique in V_r', from (2) follows that V_r contains no a-cliques. Hence $V_1 \cup \ldots \cup V_r$ is an $(a_1, \ldots, a_r)$-free r-colouring of $V(G)$, which is a contradiction.

Corollary. $F(4,4;6) \geq 13$.

In [13] it is proved that $F(4,4;6) \leq 14$, but the exact value of $F(4,4;6)$ is unknown.

4. ON THE NUMBERS $F(3,p;p+1)$ AND $F(2,2,p;p+1)$

Lemma 1. *Let $V' \subseteq V(\overline{C}_{2p+1})$, $|V'| = m$ and $G = \overline{C}_{2p+1}[V']$. If $m < 2p+1$, then* $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$.

Proof. It follows from $m < 2p+1$ that $\overline{G}$ is a subgraph of the graph P_{2p} (the path of $2p$ vertices). Hence $\chi(\overline{G}) \leq 2$. Let $V(\overline{G}) = V_1 \cup V_2$, where V_1 and V_2 are independent sets of $\overline{G}$. Then $\alpha(\overline{G}) \geq \max\{|V_1|, |V_2|\}$. Hence $\alpha(\overline{G}) \geq \left\lceil \frac{m}{2} \right\rceil$, i.e. $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$.

Let $C_{2p+1} = v_1, v_2, \ldots, v_{2p+1}$, $p \geq 3$, and $M_1 = V(C_{2p+1}) - \{v_1, v_{2p-1}, v_{2p-2}\}$. The map σ defined by $\sigma(v_i) = v_{i+1}$, $i = 1, \ldots, 2p$, and $\sigma(v_{2p+1}) = v_1$ is obviously an automorphism of $\overline{C}_{2p+1}$. We put $M_i = \sigma^{i-1}(M_1)$, $i = 1, \ldots, 2p+1$. We denote by Γ_p the extension of $\overline{C}_{2p+1}$ by adding the new vertices $u_1, \ldots, u_{2p+1}$, each two of which are not adjacent and such that $\mathrm{Ad}(u_i) = M_i$, $i = 1, \ldots, 2p+1$. The graph Γ_3 is given on Fig. 1. This graph is published in [9].

Theorem 2. *For any $p \geq 3$ we have $\Gamma_p \in H(3,p;p+1)$.*

Proof. Since $\overline{C}_{2p+1}[M_i] = \overline{K}_2 + \overline{P}_{2p-4}$, $\mathrm{cl}(\overline{C}_{2p+1}[M_i]) = p-1$. Hence $\mathrm{cl}(\Gamma_p) = p$.

Let $V_1 \cup V_2$ be the 2-colouring of $V(\Gamma_p)$. We put $V_i' = V(\overline{C}_{2p+1}) \cap V_i$, $G_i = \overline{C}_{2p+1}[V_i']$, $i = 1, 2$. Assume that $\mathrm{cl}(G_1) < 3$ and $\mathrm{cl}(G_2) < p$. Lemma 1 and $\mathrm{cl}(G_1) < 3$ imply $|V_1'| \leq 4$. Lemma 1 and $\mathrm{cl}(G_2) < p$ yield $|V_2'| \leq 2p-2$, i.e. $|V_1'| \geq 3$. So, $|V_1'| = 3$ or $|V_1'| = 4$.

Case 1. $|V_1'| = 3$. Since $\mathrm{cl}(G_1) < 3$, V_1' contains two non adjacent vertices. Hence we may assume that $v_1, v_2 \in V_1'$. We put $w = V_1' - \{v_1, v_2\}$ and $Q = \{v_{2k+1} : k = 1, \ldots, p\}$. Since Q is a p-clique and $\mathrm{cl}(G_2) < p$, we have $w \in Q$.

Subcase 1a. $w \in Q - \{v_{2p-1}, v_{2p+1}\}$. If $u_2 \in V_1$, then $\{u_2, v_1, w\}$ is a 3-clique in V_1. Let $u_2 \in V_2$. We put $Q' = \{v_{2k} : k = 2, \ldots, p-1\}$. It is clear that Q' is a $(p-2)$-clique. Since $Q' \subseteq \mathrm{Ad}(u_2)$ and $v_{2p+1} \in \mathrm{Ad}(u_2)$, $Q' \cup \{v_{2p+1}, u_2\}$ is a p-clique in V_2.

Subcase 1b. $w = v_{2p-1}$. If $u_{2p} \in V_1$, then $\{v_1, u_{2p}, v_{2p-1}\}$ is a 3-clique in V_1. Let $u_{2p} \in V_2$. We put $Q'' = Q - \{v_{2p-1}, v_{2p-3}\}$. Since $Q'' \cup \{v_{2p-2}\} \subseteq \mathrm{Ad}(u_{2p})$, $Q'' \cup \{v_{2p-2}, u_{2p}\}$ is a p-clique in V_2.

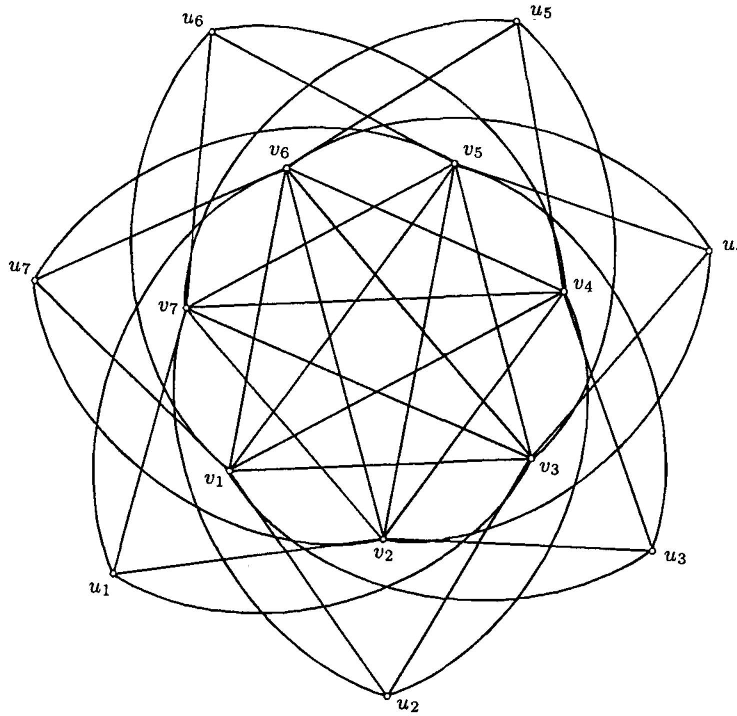

Fig. 1. Graph Γ_3

Subcase 1c. $w = v_{2p+1}$. This subcase is equivalent to $w = v_3$ in subcase 1a.

Case 2. $|V_1'| = 4$. From $\mathrm{cl}(G_1) < 3$ and $\alpha(G_1) = 2$ it follows that $E(\overline{G}_1)$ contains two edges e_1, e_2 without common vertex. Hence, we may assume that $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_i, v_{i+1}\}$ for some $i \in \{3, \ldots, 2p\}$.

Subcase 2a. $i = 2k$, $2 \le k \le p$. Let $u_4 \in V_1$. If $k = 2$, then $\{u_4, v_2, v_5\}$ is a 3-clique in V_1. If $3 \le k \le p$, then $\{u_4, v_{2k}, v_2\}$ is a 3-clique in V_1. Let $u_4 \in V_2$. We put $Q_1 = \{v_{2l+1} : l = 1, \ldots, k-1\}$ and $Q_2 = \{v_{2l} : l = k+1, \ldots, p\}$, $k < p$. If $k = p$, then $Q_2 = \varnothing$. It is clear that $\widetilde{Q} = Q_1 \cup Q_2$ is a $(p-1)$-clique. Since $\widetilde{Q} \subseteq \mathrm{Ad}(u_4)$, $\widetilde{Q} \cup \{u_4\}$ is a p-clique in V_2.

Subcase 2b. $i = 2k-1$, $2 \le k \le p$. Let $m = 2p - 2k + 3$. Then $\sigma^m(v_{2k-1}) = v_1$, $\sigma^m(v_{2k}) = v_2$, $\sigma^m(v_1) = v_{m+1}$, $\sigma^m(v_2) = v_{m+2}$ (the map σ is defined above). Since m is an odd number, the subcase 2b is equivalent to the subcase 2a.

Theorem 3. *If $p \ge 3$, then*

$$2p + 4 \le F(3, p; p+1) \le 4p + 2. \tag{3}$$

Proof. From Theorem 2, $F(3, p; p+1) \le |V(\Gamma_p)| = 4p + 2$. The lower bound in (3) has been proved in Theorem 1.

The inequality $F(3,3;4) \le 14$ is proved in [9]. The work [16] provides a computer proof of the inequality $F(3,3;4) \ge 14$ and thus $F(3,3;4) = 14$. In [13] it is proved that $F(3,4;5)=13$. The exact value of $F(3,p;p+1)$, $p \geq 5$, is unknown.

Theorem 4. *If $p \geq 3$, then*

$$2p+4 \leq F(2,2,p;p+1) \leq 4p+2. \tag{4}$$

Proof. The lower bound in (4) has been proved in Theorem 1. Since $\Gamma_p \to (3,p)$ implies $\Gamma_p \to (2,2,p)$, we have $F(2,2,p;p+1) \leq 4p+2$.

In [14] it is proved that $F(2,2,4;5)=13$. From Theorem 3, $10 \leq F(2,2,3;4) \leq 14$. The exact value of $F(2,2,3;4)$ is unknown.

5. ON THE NUMBERS $F(\underbrace{2,\ldots,2}_{r},p;p+r-1)$

We put

$$F(\underbrace{2,\ldots,2}_{r},p;p+r-1) = F_r(2,p),$$

$$H(\underbrace{2,\ldots,2}_{r},p;p+r-1) = H_r(2,p).$$

Theorem 5. *Let $G \in H(2,2,p;p+1) = H_2(2,p)$. Then for any $r \geq 2$, $K_{r-2}+G \in H_r(2,p)$.*

Proof. It follows from $\mathrm{cl}(G) < p+1$ that $\mathrm{cl}(K_{r-2}+G) < p+r-1$. We prove that

$$K_{r-2}+G \to (\underbrace{2,\ldots,2}_{r},p) \tag{5}$$

by induction on r. The base $r=2$ is clear, since $G \in H_2(2,p)$. Assume that $r \geq 3$ and

$$K_{r-3}+G \to (\underbrace{2,\ldots,2}_{r-1},p). \tag{6}$$

Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-colouring of $V(K_{r-2}+G)$. Let $w \in V(K_{r-2})$ and $K_{r-2}+G = \{w\}+(K_{r-3}+G)$. If $V_i \cap V(K_{r-3}+G) = \varnothing$ for some i, then from (6) it follows that $V_1 \cup \ldots \cup V_{r+1}$ is not $(2,\ldots,2,p)$-free. Let

$$V_i \cap V(K_{r-3}+G) \neq \varnothing, \quad i=1,\ldots,r+1. \tag{7}$$

Assume that V_i, $i=1,\ldots,r$, are independent sets. From (7), $w \notin V_i$, $i=1,\ldots,r$. Hence $w \in V_{r+1}$. Let $V'_{r+1} = V_{r+1} \setminus \{w\}$. Then $V_1 \cup \ldots \cup V_{r-1} \cup (V_r \cup V'_{r+1})$ is an r-colouring of $V(K_{r-3}+G)$. From (6), $V_r \cup V'_{r+1}$ contains a p-clique. Since V_r is an independent set, V'_{r+1} contains a $(p-1)$-clique. Hence V_{r+1} contains a p-clique. Thus, (5) holds.

Theorem 6. *For any $p \geq 3$ and $r \geq 2$, one has*

$$2p+r+2 \leq F_r(2,p) \leq 4p+r. \tag{8}$$

Proof. Theorem 2 yields $\Gamma_p \in H(2,2,p;p+1)$. From Theorem 5 it follows that $K_{r-2} + \Gamma_p \in H_r(2,p)$. Hence $F_r(2,p) \leq 4p + r$. The lower bound in (8) follows from Theorem 1.

Theorem 7. *For any $r \geq 2$, one has*

$$r + 10 \leq F_r(2,4) \leq r + 11. \tag{9}$$

Proof. Consider the 13-vertex graph Q (the complementary graph $\overline{Q}$ is given on Fig. 2). It is proved in [14] that $Q \in H(2,2,4;5)$. From Theorem 5, $K_{r-2}+Q \in H_r(2,4)$. Hence $F_r(2,4) \leq r+11$. The lower bound in (9) follows from Theorem 1.

6. ON THE NUMBERS $F(\underbrace{3,\ldots,3}_{r},p;2r+p-1)$

We put

$$F(\underbrace{3,\ldots,3}_{r},p;2r+p-1) = F_r(3,p),$$

$$H(\underbrace{3,\ldots,3}_{r},p;2r+p-1) = H_r(3,p).$$

Theorem 8. *Let $G \in H(3,p;p+1) = H_1(3,p)$. Then for any $r \geq 1$, $K_{2r-2} + G \in H_r(3,p)$.*

Proof. From $\mathrm{cl}(G) < p+1$ we have $\mathrm{cl}(K_{2r-2}+G) < 2r+p-1$. We prove

$$K_{2r-2} + G \to (\underbrace{3,\ldots,3}_{r},p) \tag{10}$$

by induction on r. The base $r = 1$ is clear, since $G \in H_1(3,p)$. Assume that $r \geq 2$ and

$$K_{2r-4} + G \to (\underbrace{3,\ldots,3}_{r-1},p). \tag{11}$$

Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-colouring of $V(K_{2r-2}+G)$ and suppose that

$$\text{each } V_i,\ i=1,\ldots,r, \text{ contains no 3-cliques.} \tag{12}$$

Let $K_{2r-2}+G = K_2+(K_{2r-4}+G)$, where $V(K_2) = \{a,b\}$. If $V_i \cap V(K_{2r-4}+G) = \varnothing$ for some i, then from (11) it follows that $V_1 \cup \ldots \cup V_{r+1}$ is not $(3,\ldots,3,p)$-free. Suppose that

$$V_i \cap V(K_{2r-4}+G) \neq \varnothing, \quad i = 1,\ldots,r+1. \tag{13}$$

Case 1. $a,b \in V_i$ for some $i \in \{1,\ldots,r\}$. It follows from (13) that V_i contains a 3-clique, which contradicts (12).

Case 2. $a \in V_i$, $b \in V_j$, $i \neq j$, $i,j \in \{1,\ldots,r\}$. We may assume that $a \in V_1$, $b \in V_2$. We put $V_1' = V_1 - \{a\}$, $V_2' = V_2 - \{b\}$. From (12), V_1' and V_2' are independent sets. Hence $V_1' \cup V_2'$ contains no 3-cliques. Consider an r-colouring

$(V_1' \cup V_2') \cup V_3 \cup \ldots \cup V_{r+1}$ of $V(K_{2r-4}+G)$. It follows from (11) and (12) that V_{r+1} contains a p-clique.

Case 3. $a \in V_i$, $i \neq r+1$ and $b \in V_{r+1}$. We may assume that $a \in V_r$. We put $V_r' = V_r - a$, $V_{r+1}' = V_{r+1} - \{b\}$. From (12), V_r' is an independent set. Consider an r-colouring $V_1 \cup \ldots \cup V_{r-1} \cup (V_r' \cup V_{r+1}')$ of $V(K_{2r-4}+G)$. By (11) and (12), $V_r' \cup V_{r+1}'$ contains a p-clique. Since V_r' is independent, V_{r+1}' contains a $(p-1)$-clique. Hence V_{r+1} contains a p-clique.

Case 4. $a, b \in V_{r+1}$. We put $V_{r+1}' = V_{r+1} - \{a, b\}$. Consider an r-colouring $V_1 \cup \ldots \cup (V_r \cup V_{r+1}')$. From (11) and (12), $V_r \cup V_{r+1}'$ contains a p-clique. By (12), V_{r+1}' contains a $(p-2)$-clique. Hence V_{r+1} contains a p-clique. Thus, (10) holds.

Theorem 9. *Let $p \geq 3$ and $r \geq 1$. Then*

$$2p + 2r + 2 \leq F_r(3,p) \leq 4p + 2r. \tag{14}$$

Proof. By Theorem 2 and Theorem 8, $K_{2r-2} + \Gamma_p \in H_r(3,p)$. Hence $F_r(3,p) \leq 4p + 2r$. The lower bound in (14) follows from Theorem 1.

Theorem 10. *There holds*

$$2r + 10 \leq F_r(3,4) \leq 2r + 11, \quad r \geq 1. \tag{15}$$

Proof. The lower bound in (15) follows from Theorem 1. Consider the 13-vertex graph Q (see Fig. 2). It is proved in [13] that $Q \in H_1(3,4)$. According to Theorem 8, $K_{2r-2} + Q \in H_r(3,4)$. Hence $F_r(3,4) \leq 2r + 11$.

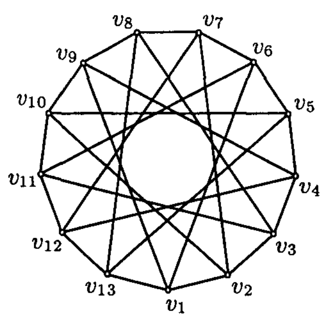

Fig. 2. Graph $\overline{Q}$

Theorem 11. *Let $r \geq 2$. Then*

$$F_r(3,3) = F(\underbrace{3,\ldots,3}_{r+1}; 2r+2) \leq 2r + 10.$$

Proof. We consider the graph Q, which complementary graph $\overline{Q}$ is given on Fig. 2. Obviously, $\alpha(Q) = 2$ and it is true that $\mathrm{cl}(Q) = 4$, [18]. We prove $K_1 + Q \in H(3,3,3;6) = H_2(3,3)$. From $\mathrm{cl}(Q) = 4$ it follows that $\mathrm{cl}(K_1 + Q) = 5$. Let $V_1 \cup V_2 \cup V_3$ be a 3-colouring of $V(K_1 + Q)$ and $V(K_1) = \{w\}$. We may assume $w \in V_1$. Assume also that V_1 contains no 3-cliques. Then $V_1' = V_1 - \{w\}$ is an independent set of Q. From $\alpha(Q) = 2$ it follows $|V_1'| \leq 2$. Hence either $|V_2| \geq 6$ or $|V_3| \geq 6$. Let $|V_2| \geq 6$ and $G = Q[V_2]$. It is clear that $\alpha(G) = 2$. From $\alpha(G) = 2$ and $|V_2| \geq 6$ it follows $\mathrm{cl}(G) \geq 3$, [18], i.e. V_2 contains a 3-clique.

So, $K_1 + Q \to (3,3,3)$ and $\mathrm{cl}(K_1 + Q) = 5$. Hence $K_1 + Q \in H_2(3,3)$. By induction on r it follows $K_{2r-4} + (K_1 + Q) \in H_r(3,3)$ (see the proof of Theorem 8). Hence $F_r(3,3) \leq 2r + 10$.

7. A NEW PROOF OF THEOREM B

B. Toft has conjectured that if G is a $(2p+1)$-vertex graph, $\alpha(G) = p$ and $\alpha(G - \{u,v\}) = \alpha(G)$ for all $u, v \in V(G)$, then $G = C_{2p+1}$. This conjecture is verified in [17] and [5] (see problem 8.26, p. 58). The proof in [5], actually, establishes the following stronger statement:

Theorem D. *Let G be a $(2p+1)$-vertex graph, $\alpha(G) = p$, $\alpha(G - v) = \alpha(G)$ for all $v \in V(G)$, and $\alpha(G - \{u,v\}) = \alpha(G)$ for any pair u, v adjacent vertices. Then $G = C_{2p+1}$.*

Theorem D is proved also in [7]. In the proof of Theorem B we shall use the following:

Lemma 2. *Let the graph G be such that* $\mathrm{cl}(G - v) = \mathrm{cl}(G)$ *for all $v \in V(G)$. Then $\pi(\overline{G}) \geq \mathrm{cl}(G)$.*

This lemma is proved in [17] (see also problem 8, p. 302, in [1]).

The proof of Theorem B. According to Proposition 5 we may assume that $a_1 \leq \cdots \leq a_r = a$. We prove Theorem B by induction on m. By the inequality $m \geq a+1$, the minimal admissible value of m is $a+1$. The base of the induction is then $m = a+1$. From $m = a+1$ it follows that $a_1 = \cdots = a_{r-2} = 1$, $a_{r-1} = 2$, and $\mathrm{cl}(G) = a$. According to Proposition 3, $G \to (2, a)$. By $G \to (2, a)$, $\mathrm{cl}(G) = \mathrm{cl}(G - v)$ $\forall v \in V(G)$ and $\mathrm{cl}(G - \{u,v\}) = \mathrm{cl}(G)$ for each pair u, v non adjacent vertices, i.e. the graph $\overline{G}$ satisfies the conditions of Theorem D. Hence $\overline{G} = C_{2a+1}$, i.e. $G = \overline{C}_{2a+1}$.

Let $m \geq a+2$. Let L be a graph such that $V(L) = V(G)$, $E(L) \supseteq E(G)$ and $\mathrm{cl}(L) = m-1$. It is clear that $L \to (a_1, \ldots, a_r)$. We prove that $\mathrm{cl}(L - v_0) < \mathrm{cl}(L)$ for some $v_0 \in V(L)$. Assume the opposite. According to Lemma 2, we have $\pi(\overline{L}) \geq m-1$. Hence

$$\chi(L) \leq m - 1 + \big(|V(L)| - 2(m-1)\big) = a + 1.$$

From $m \geq a+2$ it follows $\chi(L) \leq m-1$. This contradicts Proposition 6.

So, $\exists v_0 \in V(L)$ such that $\mathrm{cl}(L - v_0) < \mathrm{cl}(L) = m-1$. By $m \geq a+2$, $a_{r-1} \geq 2$. According to Proposition 4, $L - v_0 \to (a_1 \ldots, a_{r-1} - 1, a_r)$. Hence $L - v_0 \in H(a_1, \ldots, a_{r-1} - 1, a_r; m-1)$. By the inductive hypothesis, $L - v_0 = K_{m-a-2} + \overline{C}_{2a+1}$. The vertex v_0 is adjacent to each vertex of $V(K_{m-a-2} + \overline{C}_{2a+1})$ (otherwise, $\chi(L) < m$, which contradicts Proposition 6). Therefore, $L = K_{m-a-1} + \overline{C}_{2a+1}$. Since each proper subgraph of $K_{m-a-1} + \overline{C}_{2a+1}$ has an $(a_1, \ldots, a_r)$-free r-colouring of the vertices (see [7], Proposition 3), we have $G = K_{m-a-1} + \overline{C}_{2a+1}$.

The proof of Theorem B is complete.

REFERENCES

1. Berg, C. Graphs and Hypergraphs. *Mathematical Library,* **6**, North-Holland, 1976.
2. Chvatal, V. The minimality of the Mycielski graph. In: *Lecture Notes Math.*, **406**, 1974, 243–246.
3. Folkman, J. Graphs with monochromatic complete subgraph in every edge coloring. *SIAM J. Appl. Math.*, **18**, 1970, 19–24.
4. Jensen, T., G. Royle. Small graphs with chromatic number 5: a computer search. *J. Graph Theory,* **19**, 1995, 107–116.
5. Lovasz, L. Combinatorial problems and exercises. Akademiai Kiado, Budapest, 1979.
6. Luczak, T., S. Urbanski. A note on the restricted vertex Ramsey numbers. *Periodica Math. Hungarica,* **33**, 1996, 101–103.
7. Luczak, T., A. Rucinski, S. Urbanski. On the minimal vertex Folkman graphs. *Discrete Math.*, **236**, 2001, 245–262.
8. Mycielski, J. Sur le coloriage des graphes. *Coll. Math.*, **3**, 1955, 161–162.
9. Nenov, N. An example of a 15-vertex (3,3)-Ramsey graph with clique number 4. *C. R. Acad. Bulg. Sci.*, **34**, 1981, 1487–1489 (in Russian).
10. Nenov, N. On the Zykov numbers and some its applications to Ramsey theory. *Serdica Bulgaricae math. publicationes,* **9**, 1983, 161–167 (in Russian).
11. Nenov, N. The chromatic number of any 10-vertex graph without 4-cliques is at most 4. *C. R. Acad. Bulg. Sci.*, **37**, 1984, 301–304 (in Russian).
12. Nenov, N. On the small graphs with chromatic number 5 without 4-cliques. *Discrete Math.*, **188**, 1998, 297–298.
13. Nenov, N. On the vertex Folkman number $F(3,4)$. *C. R. Acad. Bulg. Sci.*, **54**, 2001, No. 2, 23–26.
14. Nenov, N. On the 3-colouring vertex Folkman number $F(2,2,4)$. *Serdica, Math. Journal,* **27**, 2001, 131–136.
15. Nenov, N. Ramsey graphs and some constants related to them. Ph.D. Thesis at University of Sofia, Sofia, 1980.
16. Piwakowski, K., S. Radziszowski, S. Urbanski. Computation of the Folkman number $F_e(3,3;5)$. *J. Graph Theory,* **32**, 1999, 41–49.
17. Vizing, V., L. Melnikov. Solution of a Toft problem. *Diskret. Analiz,* **19**, 1971, 11–14 (in Russian).
18. Greenwood, R., A. Gleason. Combinatorial relation and chromatic graphs. *Canad. J. Math.*, **7**, 1955, 1–7.

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ON THE DURATION DOMAINS FOR THE INTERVAL TEMPORAL LOGIC

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The duration domains for the Interval Temporal Logic are characterized as the positive cones of the right-ordered groups.

Keywords: duration domain, interval temporal logic, right-ordered group, positive cone

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In 1985 B. Moszkowski [7] introduced a logical system called Interval Temporal Logic. Its semantics, proposed by B. Dutertre [2] in 1995, uses a kind of structures called *duration domains.* The same kind of structures have been used later also by D. Guelev for the semantics of other logical systems (cf., for example, [4, 5]). The structures in question can be defined as triples $(D, +, 0)$, where D is a set, $+$ is a binary operation in D, 0 is an element of D and the following five axioms are identically satisfied in D:

(D1) $(x + y) + z = x + (y + z)$,
(D2) $x + 0 = 0 + x = x$,
(D3) $x + z = y + z \Rightarrow x = y, \quad z + x = z + y \Rightarrow x = y$,
(D4) $x + y = 0 \Rightarrow x = y = 0$,
(D5) $\exists z(x + z = y \vee y + z = x), \quad \exists z(z + x = y \vee z + y = x)$.

The aim of the present paper is to characterize the duration domains as the positive cones of the right-ordered groups. This will be done by proving theorems 1 and 2 below.

A *right-ordered group* (cf. [1, 6]) is a structure $(G, +, 0, -, \geq)$, where $(G, +, 0, -)$ is a group (not necessarily abelian), $+$, 0, $-$ being respectively the binary group operation, the neutral element of the group and the unary operation of constructing the inverse element, and $\geq$ is a linear ordering in G such that for all x, y, z in G the following implication holds:

$$x \geq y \Rightarrow x + z \geq y + z$$

(as in [6], we assume the orderings reflexive, although the orderings in [1] are assumed to be irreflexive). *The positive cone* of such a structure is the set of all elements x of G that satisfy the condition $x \geq 0$. If P is the positive cone of a right-ordered group $(G, +, 0, -, \geq)$, then the following three conditions are satisfied for all x and y in G:

(P1) $x \in P \wedge -x \in P \Rightarrow x = 0$,
(P2) $x \in P \wedge y \in P \Rightarrow x + y \in P$,
(P3) $x \in P \vee -x \in P$.

Conversely, whenever $(G, +, 0, -)$ is a group and P is a subset of G with the properties (P1)–(P3), then a binary relation $\geq$ in G exists such that $(G, +, 0, -, \geq)$ is a right-ordered group with positive cone P.

Theorem 1. *Let $(G, +, 0, -, \geq)$ be a right-ordered group and P be its positive cone. Let $+_P$ be the restriction of the operation $+$ to P^2. Then $(P, +_P, 0)$ is a duration domain.*

Proof. The element 0 of G belongs to P by (P3), hence, taking into account also (P2), we may consider the structure $(P, +_P, 0)$. This structure obviously satisfies the axioms (D1)–(D3), and (D4) follows immediately from the property (P1). To verify (D5), suppose x and y are some elements of P. If we set $u = (-x) + y$, then the equalities $x + u = y$ and $y + (-u) = x$ hold, and, since some of the elements u and $-u$ belongs to P by (P3), this establishes the first statement of (D5). The second one can be established in a similar way. □

Remark. Under the assumptions of the above theorem, if the considered group is not abelian, then the operation $+_P$ is not commutative.[1] In fact, let x and y be elements of G such that $x + y \neq y + x$. By (P3) some of the elements x and $-x$ belongs to P and also some of the elements y and $-y$ belongs to P. Therefore it is sufficient to establish the inequalities

$$x + (-y) \neq (-y) + x, \quad (-x) + y \neq y + (-x), \quad (-x) + (-y) \neq (-y) + (-x).$$

To prove the first one, we suppose the equality $x + (-y) = (-y) + x$ and get $y + (x + (-y)) + y = y + ((-y) + x) + y$, i.e. $y + x = x + y$. In a similar way we

[1] Since there are non-abelian right-ordered groups (examples of such groups can be found, for instance, in [1] and [3, ch. 2]), this implies the existence of a duration domain with non-commutative addition operation.

show the impossibility of the equality $(-x)+y=y+(-x)$. Finally, if we suppose that $(-x)+(-y)=(-y)+(-x)$, then we get $-((-x)+(-y))=-((-y)+(-x))$, and this leads again to the contradictory equality $y+x=x+y$.

Theorem 2. *Any duration domain can be obtained in the way from* Theorem 1 *at a convenient choice of some right-ordered group* $(G,+,0,-,\geq)$.

Proof. Let $(D,+,0)$ be a duration domain. To each element s of $D\setminus\{0\}$ we make to correspond an object $\overline{s}$ not belonging to D in such a way that $\overline{s}\neq\overline{t}$ whenever s and t are distinct elements of $D\setminus\{0\}$. Then we set

$$G=D\cup\{\overline{s}\mid s\in D\setminus\{0\}\},$$

and we define the inverse element of any element of G by setting $-0=0$ and

$$-s=\overline{s},\quad -\overline{s}=s$$

for any s in $D\setminus\{0\}$. We extend the binary operation $+$ from D to G by stipulating the equalities

$$(z+s)+\overline{s}=z,\quad x+\overline{t+x}=\overline{t},\quad \overline{s}+(s+z)=z,\quad \overline{y+t}+y=\overline{t},\quad \overline{s}+\overline{t}=\overline{t+s}$$

for all x,y,z in D and all s,t in $D\setminus\{0\}$.[2] It follows immediately that

$$0+\overline{t}=\overline{t}+0=\overline{t},\quad s+\overline{s}=\overline{s}+s=0$$

for all s,t in $D\setminus\{0\}$, hence

$$0+u=u+0=u,\quad u+(-u)=(-u)+u=0$$

for all u in G. If we denote the set D by P, then the properties (P1)–(P3) will be obviously present. Therefore the proof will be completed if we show that the operation $+$ in G is associative. This reduces to showing that for all p,q,r in D the following seven implications hold:

(A1) $r\neq 0\Rightarrow (p+q)+\overline{r}=p+(q+\overline{r})$,
(A2) $q\neq 0\Rightarrow (p+\overline{q})+r=p+(\overline{q}+r)$,
(A3) $q\neq 0\wedge r\neq 0\Rightarrow (p+\overline{q})+\overline{r}=p+(\overline{q}+\overline{r})$,
(A4) $p\neq 0\Rightarrow (\overline{p}+q)+r=\overline{p}+(q+r)$,
(A5) $p\neq 0\wedge r\neq 0\Rightarrow (\overline{p}+q)+\overline{r}=\overline{p}+(q+\overline{r})$,
(A6) $p\neq 0\wedge q\neq 0\Rightarrow (\overline{p}+\overline{q})+r=\overline{p}+(\overline{q}+r)$,
(A7) $p\neq 0\wedge q\neq 0\wedge r\neq 0\Rightarrow (\overline{p}+\overline{q})+\overline{r}=\overline{p}+(\overline{q}+\overline{r})$.

Thus the remaining part of the proof decomposes into the verifications of (A1)–(A7), where p, q, r are arbitrary elements of D.

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[2] To show that the above definition is a legitimate one, we use all axioms (D1)–(D5); in particular, the axiom (D4) is used for showing that it is not possible to have simultaneously two equalities $z+s=x$, $s=t+x$ or two equalities $s=y+t$, $s+z=y$, where $x,y,z\in D$, $s,t\in D\setminus\{0\}$, and the axiom (D5) is used for showing that the extension is defined for any pair of elements of G.

Verification of (A1). Let $r \neq 0$. By axiom (D5), there is some element z of D such that $q = z + r$ or $r = z + q$. We choose such a z and we could assume that $z \neq 0$ in the second case, since if $z = 0$, then the second case is covered by the first one. If $q = z + r$, then

$$p + (q + \overline{r}) = p + ((z + r) + \overline{r}) = p + z, \quad (p + q) + \overline{r} = ((p + z) + r) + \overline{r} = p + z.$$

Consider now the case when $r = z + q$ and $z \neq 0$. Then

$$p + (q + \overline{r}) = p + (q + \overline{z + q}) = p + \overline{z}$$

and it is natural to apply the axiom (D5) again for choosing an element z' of G such that either $p = z' + z$ or $z = z' + p$, z' being distinct from 0 in the second case. If $p = z' + z$, then

$$(p + q) + \overline{r} = (z' + z + q) + \overline{r} = (z' + r) + \overline{r} = z', \quad p + (q + \overline{r}) = (z' + z) + \overline{z} = z'.$$

Otherwise, i.e. when $z = z' + p$ and $z' \neq 0$, we have

$$(p + q) + \overline{r} = (p + q) + \overline{z' + (p + q)} = \overline{z'}, \quad p + (q + \overline{r}) = p + \overline{z' + p} = \overline{z'}.$$

Verification of (A2). Let $q \neq 0$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. Choosing such a z, we shall have

$$(p + \overline{q}) + r = ((z + q) + \overline{q}) + r = z + r$$

in the first case and

$$(p + \overline{q}) + r = (p + \overline{z + p}) + r = \overline{z} + r$$

in the second one. By the same axiom, there is an element z' of D such that either $r = q + z'$ or $q = r + z'$, z' being distinct from 0 in the second case. Choosing such a z', we shall have

$$p + (\overline{q} + r) = p + (\overline{q} + (q + z')) = p + z'$$

in the first case and

$$p + (\overline{q} + r) = p + (\overline{r + z'} + r) = p + \overline{z'}$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $p = z + q$, $r = q + z'$. Then

$$(p + \overline{q}) + r = z + q + z', \quad p + (\overline{q} + r) = z + q + z'.$$

Combination 1.2: $p = z + q$, $q = r + z'$, $z' \neq 0$. Then

$$p + (\overline{q} + r) = ((z + r) + z') + \overline{z'} = z + r = (p + \overline{q}) + r.$$

Combination 2.1: $q = z + p$, $z \neq 0$, $r = q + z'$. Then

$$(p + \overline{q}) + r = \overline{z} + (z + (p + z')) = p + z' = p + (\overline{q} + r).$$

Combination 2.2: $q = z + p$, $z \neq 0$, $q = r + z'$, $z' \neq 0$. Then $z + p = r + z'$. By axiom (D5), there is an element z'' of D such that either $r = z + z''$ or $z = r + z''$, z'' being distinct from 0 in the second case. In the first case we get

$$p = z'' + z', \;\; (p + \overline{q}) + r = \overline{z} + (z + z'') = z'', \;\; p + (\overline{q} + r) = (z'' + z') + \overline{z'} = z''.$$

In the second one we have

$$z'' + p = z', \;\; (p + \overline{q}) + r = \overline{r + z''} + r = \overline{z''}, \;\; p + (\overline{q} + r) = p + \overline{z'' + p} = \overline{z''}.$$

Verification of (A3). Let $q \neq 0$, $r \neq 0$. Then $p + (\overline{q} + \overline{r}) = p + \overline{r + q}$. By Axiom (D5), there is an element z of D such that either $p = z + q$ or $q = z + p$, z being distinct from 0 in the second case. In the first case we get

$$(p + \overline{q}) + \overline{r} = ((z + q) + \overline{q}) + \overline{r} = z + \overline{r}, \;\; p + (\overline{q} + \overline{r}) = z + (q + \overline{r + q}) = z + \overline{r}.$$

In the second one we have

$$(p + \overline{q}) + \overline{r} = (p + \overline{z + p}) + \overline{r} = \overline{z} + \overline{r} = \overline{r + z}, \;\; p + (\overline{q} + \overline{r}) = p + \overline{(r + z) + p} = \overline{r + z}.$$

Verification of (A4). Similar to the verification of (A1).

Verification of (A5). Let $p \neq 0$, $r \neq 0$. By Axiom (D5), there is an element z of D such that either $q = p + z$ or $p = q + z$, z being distinct from 0 in the second case. Choosing such a z, we shall have

$$(\overline{p} + q) + \overline{r} = (\overline{p} + (p + z)) + \overline{r} = z + \overline{r}$$

in the first case and

$$(\overline{p} + q) + \overline{r} = (\overline{q + z} + q) + \overline{r} = \overline{z} + \overline{r} = \overline{r + z}$$

in the second one. By the same axiom, there is an element z' of D such that either $q = z' + r$ or $r = z' + q$, z' being distinct from 0 in the second case. Choosing such a z', we shall have

$$\overline{p} + (q + \overline{r}) = \overline{p} + ((z' + r) + \overline{r}) = \overline{p} + z'$$

in the first case and

$$\overline{p} + (q + \overline{r}) = \overline{p} + (q + \overline{z' + q}) = \overline{p} + \overline{z'} = \overline{z' + p}$$

in the second one. The four combinations of cases below have to be considered.

Combination 1.1: $q = p + z$, $q = z' + r$. Then $p + z = z' + r$. By Axiom (D5), there is an element z'' of D such that either $z = z'' + r$ or $r = z'' + z$, z'' being distinct from 0 in the second case. In the first case we get

$$p + z'' = z', \;\; (\overline{p} + q) + \overline{r} = (z'' + r) + \overline{r} = z'', \;\; \overline{p} + (q + \overline{r}) = \overline{p} + (p + z'') = z''.$$

In the second one we have

$$p = z' + z'', \;\; (\overline{p} + q) + \overline{r} = z + \overline{z'' + z} = \overline{z''}, \;\; \overline{p} + (q + \overline{r}) = \overline{z' + z''} + z' = \overline{z''}.$$

Combination 1.2: $q = p + z,\ r = z' + q,\ z' \neq 0$. Then

$$(\overline{p} + q) + \overline{r} = z + \overline{(z' + p) + z} = \overline{z' + p} = \overline{p} + (q + \overline{r}).$$

Combination 2.1: $p = q + z,\ z \neq 0,\ q = z' + r$. Then

$$\overline{p} + (q + \overline{r}) = \overline{z' + (r + z)} + z' = \overline{r + z} = (\overline{p} + q) + \overline{r}.$$

Combination 2.2: $p = q + z,\ z \neq 0,\ r = z' + q,\ z' \neq 0$. Then

$$(\overline{p} + q) + \overline{r} = \overline{z' + q + z},\ \ \overline{p} + (q + \overline{r}) = \overline{z' + q + z}.$$

Verification of (A6). Similar to the verification of (A3).

Verification of (A7). Let $p \neq 0,\ q \neq 0,\ r \neq 0$. Then

$$(\overline{p} + \overline{q}) + \overline{r} = \overline{q + p} + \overline{r} = \overline{r + q + p},\ \ \overline{p} + (\overline{q} + \overline{r}) = \overline{p} + \overline{r + q} = \overline{r + q + p}. \square$$

APPENDIX

The proof of Theorem 2 makes use of the existence of some set that has the same cardinality as $D \setminus \{0\}$ and does not meet D. The existence of such a set can be obtained as a particular case of the statement that for any sets A and B there is a set having the same cardinality as A and not meeting B. This statement follows immediately from certain facts of the cardinal arithmetic, but some of them in the final analysis are based on the Axiom of Choice. Here is a direct proof of the statement without using that axiom. Let

$$C = (A \times \mathcal{P}(B)) \cap B,$$

where $\mathcal{P}(B)$ is the set of the subsets of B. Let f be the projection mapping of C into $\mathcal{P}(B)$ defined by the equality

$$f(x, Y) = Y.$$

Since C is a subset of B, the range of f is a proper subset of $\mathcal{P}(B)$ (as the well-known diagonal argument shows, the set $\{z \in C \mid z \notin f(z)\}$ is an element of $\mathcal{P}(B)$ not belonging to the range of f). If Y_0 is an element of $\mathcal{P}(B) \setminus \mathrm{range}(f)$, then the set $A \times \{Y_0\}$ does not meet B, and clearly $A \times \{Y_0\}$ *has the same cardinality as* A.

REFERENCES

1. Conrad, P. Right-ordered groups. *Mich. Math. J.*, **6**, 1959, 267–275.
2. Dutertre, B. On First Order Interval Temporal Logic. Technical Report CSD-TR-94-3, Royal Holloway, Univ. of London, Egham, Dept. of Computer Science, 1995.
3. Fuchs, L. Partially Ordered Algebraic Systems. Pergamon Press, Oxford–London–New York–Paris, 1963.

4. Guelev, D. P. A Calculus of Durations on Abstract Domains: Completeness and Extensions. UNU/IIST Report, No. 139, Macau, 1998.
5. Guelev, D. P. Iteration of Simple Formulas in Duration Calculus. UNU/IIST Report, No. 141, Macau, 1998.
6. Matsushita, S. On the foundation of orders in groups. *J. Inst. of Polytechnics, Osaka City Univ.*, **2**, 1951–52, Ser. A, 19–22.
7. Moszkowski, B. Temporal logic for multilevel reasoning about hardware. *IEEE Computer*, **18**(2), 1985 10–19.

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VARIATIONAL METHODS FOR THE TOTALLY MONOTONIC FUNCTIONS AND APPLICATIONS

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We derive new variational methods and formulas for the totally monotonic functions and apply them for finding the sharp estimates for the coefficients of the inverse functions of the examined ones. Two conjectures for these coefficients are stated.

Keywords: variational methods, totally monotonic functions, maximum and minimum of the coefficients of the inverse functions of totally monotonic functions

Mathematics Subject Classification 2000: Primary 30C45, 30C50; Secondary 12D10, 26C10, 30B10

1. INTRODUCTION

Let N denote the class of Nevanlinna analytic functions

$$w = f(z) = \int_0^1 \frac{\mu(t)}{z-t} = \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \quad z \notin [0,1], \tag{1}$$

where $\mu(t)$ is a probability measure on $[0,1]$, i.e. $\mu(t)$ is a nondecreasing function on $[0,1]$ with $\mu(0) = 0$ and $\mu(1) = 1$, and

$$a_n = \int_0^1 t^{n-1}\, d\mu(t), \quad n = 1, 2, \ldots, \quad a_1 = 1. \tag{2}$$

If we replace z by $1/z$ in (1), we obtain the class T of analytic functions

$$w = \varphi(z) \equiv f\left(\frac{1}{z}\right) = \int_0^1 \frac{z\,d\mu(t)}{1-tz} = \sum_{n=1}^{\infty} a_n z^n, \quad z \notin [1, +\infty], \tag{3}$$

with totally monotonic Taylor coefficients, which has been introduced by Hausdorff [1]. According to the Thale theorem [2, pp. 234–235, Theorem 2.3] (see also Goodman [3, pp. 183–184, Section 8]) the disk $\{z : |z-(1/2)| > (1/2)\}$ is the maximal domain of univalence for the class N. Hence the half-plane $\{z : \operatorname{Re} z < 1\}$ is the maximal domain of univalence for the class T. Wirths [4, p. 512, Corollary 2.3] has found the Koebe domain of the class T with respect to the unit disk $|z| < 1$. In [5] it is noted that the Koebe domains of the classes N and T with respect to the disks $|z| > 1$ and $|z| < 1$ are one and the same, respectively. Therefore we need to study only the class T in the unit disk $|z| < 1$. It follows from the Wirths result (see also [5, p. 345, Corollary 2]) that the largest common region of convergence of all Taylor series at the point $w = 0$ of the inverse functions $z = \psi(w)$ of the functions (3) in $|z| < 1$ is the disk $|w| < 1/2$. Let

$$z = \psi(w) = \sum_{n=1}^{\infty} b_n w^n, \quad |w| < \frac{1}{2}, \quad b_1 = 1, \tag{4}$$

be such series, where the coefficients b_n are determined by the coefficients a_n in (2) with the help of Theorem 3 below.

In this paper we derive variational methods which yield more precise information in comparison with the Wirths result [4, p. 513, Theorem 2.3] for the extremal functions of a given bounded real-valued continuous functional in the class T. As an application of these methods we find the minimum and the maximum of the coefficients b_2, b_3 and b_4 and state two conjectures for the extrema of all coefficients b_n, $n = 2, 3, 4, \ldots$, in (4).

2. VARIATIONAL FORMULAS FOR THE CLASS T

The variational methods and results represented by Theorems 1 and 2 below are new.

Theorem 1. *Let ε with $-1 < \varepsilon < 1$, $\varepsilon \neq 0$, be an arbitrary number and let the function $\varphi(z)$ belong to the class T. Then the varied function*

$$\varphi_*(z) = \int_0^1 \frac{z\,d\mu(t)}{1 - \dfrac{(1-\varepsilon)t}{1+\varepsilon-2\varepsilon t} z}, \qquad z \notin [1, +\infty], \tag{5}$$

also belongs to the class T and it has the asymptotic representation

$$\varphi_*(z) = \varphi(z) - 2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2}\,d\mu(t) + O(\varepsilon^2), \quad |z| < 1, \tag{6}$$

where $O(\varepsilon^2)$ denotes a magnitude, the ratio of which to ε^2 is uniformly bounded for z lying in an arbitrary closed set of the disk $|z| < 1$.

Proof. The linear fractional function

$$\tau = \frac{(1-\varepsilon)t}{1+\varepsilon-2\varepsilon t}, \qquad 0 \le t \le 1, \quad -1 < \varepsilon < 1, \quad \varepsilon \ne 0, \tag{7}$$

for fixed ε, increases with t from 0 to 1. This property of (7) permits us to substitute $(1-\varepsilon)t/(1+\varepsilon-2\varepsilon t)$ for t in (3) to obtain (5). The function (5) belongs to the class T with the probability measure

$$\nu(\tau) := \mu\left(\frac{(1+\varepsilon)\tau}{1-\varepsilon+2\varepsilon\tau}\right), \qquad 0 \le \tau \le 1.$$

The difference between (5) and (3) is

$$\varphi_*(z) - \varphi(z) = -2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2} \frac{d\mu(t)}{1-\varepsilon\dfrac{2t-1-tz}{1-tz}} \tag{8}$$

$$= -2\varepsilon z^2 \int_0^1 \frac{t(1-t)}{(1-tz)^2} \sum_{\nu=0}^{\infty} \varepsilon^\nu \left(\frac{2t-1-tz}{1-tz}\right)^\nu d\mu(t), \qquad |z| < 1,$$

since $|2t-1-tz| \le |1-tz|$ for $0 \le t \le 1$ and $|z| < 1$. Thus from (8) we obtain (6), which completes the proof of Theorem 1.

Theorem 2. *For given point z of the disk $|z| < 1$ and a given analytic function $\Phi(u_0, u_1, \ldots, u_n; z)$, $n \ge 0$, on the set $\bigcup_T \{\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\}$, the minimum (maximum) of the functional*

$$\operatorname{Re}\Phi\left(\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\right) \tag{9}$$

in the class T is attained only either in the subclass $T_1 \subset T$ of functions

$$\varphi(z) = cz + (1-c)\frac{z}{1-z} \in T_1, \quad 0 \le c \le 1, \tag{10}$$

or in the subclass $T_2 \subset T$ of functions

$$\varphi(z) = \sum_{k=1}^{p} \frac{c_k z}{1-t_k z} \in T_2 \tag{11}$$

with

$$1 \le p \le n+2, \quad 0 \le t_1 \le t_2 \le \cdots \le t_p \le 1, \quad 0 \le c_k \le 1, \quad \sum_{k=1}^{p} c_k = 1, \tag{12}$$

where t_1, t_2, ..., t_p are among the numbers 0 *and* 1, *and the roots in the interval* $0 \le t \le 1$ *of the equation*

$$\operatorname{Re}\left\{ \left[\frac{\partial\Phi[\varphi(z)]}{\partial u_0} z^2(1-tz)^n \right.\right. \tag{13}$$

$$\left.\left. + \sum_{s=1}^{n} \frac{\partial\Phi[\varphi(z)]}{\partial u_s} s! t^{s-2}(s-1+2tz)(1-tz)^{n-s} \right] (1-t\overline{z})^{n+2} \right\} = 0,$$

where we assume that at the extremum of the functional (9) *the equation* (13) *is not an identity for all t in the interval* $0 \le t \le 1$,

$$\Phi[\varphi(z)] \equiv \Phi\left(\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\right),$$

and the empty sum for $n = 0$ is zero by convention.

Proof. The extremal functions $\varphi(z) \in T$ exist since the functional (9) is continuous and bounded on T and the class T is normal and compact in $|z| < 1$. If we set

$$u_s = \varphi^{(s)}(z), \quad u_s^* = \varphi_*^{(s)}(z) \quad (0 \le s \le n), \tag{14}$$

then the increments by the asymptotic formula (6) are

$$du_s = u_s^* - u_s = -2\varepsilon s! \int_0^1 t(1-t)I_s(t,z)\, d\mu(t) + O(\varepsilon^2) \quad (0 \le s \le n), \tag{15}$$

where

$$I_s(t,z) = \left(\frac{t}{1-tz}\right)^s \left[\binom{s+1}{1}\left(\frac{z}{1-tz}\right)^2 + 2\binom{s}{1}\frac{z}{(1-tz)t} + \binom{s-1}{1}\frac{1}{t^2}\right] \tag{16}$$

for $0 \le s \le n$, and $\binom{m}{1} = m$ for $m = 1, 2, \ldots$ and $\binom{m}{1} = 0$ for $m = 0, -1$. Further we introduce the abridged notations

$$\Phi \equiv \Phi(u_0, u_1, \ldots, u_n; z), \quad \Phi^* \equiv \Phi(u_0^*, u_1^*, \ldots, u_n^*; z), \tag{17}$$

where u_s and u_s^* $(0 \le s \le n)$ are given by (14). Then for sufficiently small $|\varepsilon|$ we have the Taylor series

$$\Phi^* = \Phi + \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\sum_{s=0}^{n} \frac{\partial}{\partial u_s} du_s\right)^{\nu} \Phi \tag{18}$$

for the functions (17). From (18) and (15)–(16) we obtain

$$\Phi^* = \Phi - 2\varepsilon \sum_{s=0}^{n} s! \frac{\partial\Phi}{\partial u_s} \int_0^1 t(1-t)I_s(t,z)\, d\mu(t) + O(\varepsilon^2). \tag{19}$$

It follows from (19) that

$$\operatorname{Re}\Phi^* = \operatorname{Re}\Phi - 2\varepsilon \int_0^1 t(1-t) \operatorname{Re}\left[\sum_{s=0}^{n} s! \frac{\partial \Phi}{\partial u_s} I_s(t,z)\right] d\mu(t) + O(\varepsilon^2). \tag{20}$$

The extremality of the function $\varphi(z)$ in the class T and the arbitrariness of ε imply that the coefficient of ε in (20) vanishes, i.e.

$$\int_0^1 t(1-t) \operatorname{Re}\left[\sum_{s=0}^{n} s! \frac{\partial \Phi}{\partial u_s} I_s(t,z)\right] d\mu(t) = 0. \tag{21}$$

If the equation

$$P(t) \equiv \operatorname{Re}\left[\sum_{s=0}^{n} s! \frac{\partial \Phi}{\partial u_s} I_s(t,z)\right] = 0 \tag{22}$$

is not an identity for all t in the interval $0 \le t \le 1$, i.e. if the conditions for the equation (13) hold, then the equation of the extremality (21) is fulfilled if and only if the measure $\mu(t)$ is a step function with points of discontinuity at 0, 1 and the roots of the equation (22) in t in the closed interval $[0,1]$, i.e. the roots of the equation (13) in $t \in [0,1]$, where the sum of the corresponding jumps equals to unit. In fact, this is evident if $\mu(t)$ is a corresponding step function. Conversely, it follows from the Goluzin variational formula applied to the class T (see, for example, [6, p. 93, formula (19)]) that $\mu(t)$ is a constant between any two adjacent roots of the equation (22) for the extremal function $\varphi(z)$ (see the comments for formulas (27)–(28) in [6, pp. 94–95]). Hence, the extremal functions $\varphi(z)$ belong to the subclasses $T_1 \subset T$ and $T_2 \subset T$ of functions (10) and (11)–(12), respectively, where the upper bound of the number p is determined in the following manner.

Let the real number ε be with a sufficiently small $|\varepsilon|$. If the extremal function $\varphi(z) \in T_2$ and in (11)–(12) we substitute $c_k + \varepsilon$ and $c_{k+1} - \varepsilon$ for c_k and c_{k+1}, respectively, then the varied function

$$\varphi_{**}(z) = \varphi(z) + \varepsilon \left[\frac{z}{1 - t_k z} - \frac{z}{1 - t_{k+1} z}\right] \tag{23}$$

also belongs to the subclass T_2. If we set analogously

$$u_s = \varphi^{(s)}(z), \quad u_s^{**} = \varphi_{**}^{(s)}(z) \quad (0 \le s \le n), \tag{24}$$

then by formula (23) the increments are

$$du_s = u_s^{**} - u_s = \varepsilon \left[\frac{\partial^s}{\partial z^s} \frac{z}{1 - t_k z} - \frac{\partial^s}{\partial z^s} \frac{z}{1 - t_{k+1} z}\right] \quad (0 \le s \le n). \tag{25}$$

For brevity, we again denote

$$\Phi \equiv \Phi(u_0, u_1, \ldots, u_n; z), \qquad \Phi^{**} \equiv \Phi(u_0^{**}, u_1^{**}, \ldots, u_n^{**}; z), \tag{26}$$

where u_s and u_s^{**} $(0 \le s \le n)$ are given by (24). Then the corresponding Taylor series (18) for the functions (26) and (25) yield

$$\operatorname{Re}\Phi^{**} = \operatorname{Re}\Phi + \varepsilon \operatorname{Re}\left\{\sum_{s=0}^{n} \frac{\partial \Phi}{\partial u_s} \left[\frac{\partial^s}{\partial z^s} \frac{z}{1 - t_k z} - \frac{\partial^s}{\partial z^s} \frac{z}{1 - t_{k+1} z}\right]\right\} + O(\varepsilon^2). \tag{27}$$

In addition, from the conditions for the equation (13) (or (22)) it follows that we have the inequality

$$\frac{\partial\Phi[\varphi(z)]}{\partial u_s} \equiv \frac{\partial}{\partial u_s}\Phi\left(\varphi(z), \varphi'(z), \ldots, \varphi^{(n)}(z); z\right) \neq 0 \tag{28}$$

at least for one $s \in \{0, 1, \ldots, n\}$. Then the extremality of $\varphi(z)$ in (27), the arbitrariness of ε and the inequality (28) imply the condition

$$\operatorname{Re}\left\{\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\left[\frac{\partial^s}{\partial z^s}\frac{z}{1-t_k z} - \frac{\partial^s}{\partial z^s}\frac{z}{1-t_{k+1}z}\right]\right\} = 0. \tag{29}$$

The condition (29) shows that the function

$$Q(t) \equiv \operatorname{Re}\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\frac{z}{1-tz}\right], \quad 0 \leq t \leq 1, \tag{30}$$

has equal values at any two adjacent points of discontinuity t_k and t_{k+1} of the measure $\mu(t)$ for the subclass T_2, i.e. $Q(t)$ has equal values at all the points of discontinuity of the measure $\mu(t)$ for the subclass T_2. Hence, the derivative $Q'(t)$ vanishes at least at one point inside the intervals between any two adjacent points of discontinuity of $\mu(t)$ in $0 \leq t \leq 1$. But from (30) and (22), having in mind (16), we conclude that

$$Q'(t) = \operatorname{Re}\left[\sum_{s=0}^{n}\frac{\partial\Phi}{\partial u_s}\frac{\partial^s}{\partial z^s}\frac{z^2}{(1-tz)^2}\right] = \operatorname{Re}\left[\sum_{s=0}^{n} s!\frac{\partial\Phi}{\partial u_s}I_s(t,z)\right] = P(t). \tag{31}$$

The equation (22) or the equivalent algebraic equation (13) have no more than $2n+2$ roots in t. Taking into account the endpoints 0 and 1, we conclude that the step measure $\mu(t)$ has no more than $2n+4$ points of discontinuity in the interval $0 \leq t \leq 1$. It follows from (31) that if the points of discontinuity of $\mu(t)$ in $0 \leq t \leq 1$ are more than $n+2$, then the equation (22) (or (13)) will have more than $2n+2$ roots in $0 \leq t \leq 1$, which is impossible. Hence, the number p satisfies the inequalities in (12). If the extremal function $\varphi(z) \in T_1$, the corresponding assertions are established in the same way.

This completes the proof of Theorem 2.

3. APPLICATION TO THE COEFFICIENT PROBLEM OF THE INVERSE FUNCTIONS IN THE CLASS T

We need the following

Theorem 3. *In terms of the coefficients a_n in (2), the coefficients b_n in (4) have the following simplest explicit form:*

$$b_n = \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k\binom{n+k-1}{k}D_{n-1,k}(a_2, a_3, \ldots, a_{n-k+1}), \quad n \geq 2, \tag{32}$$

where

$$D_{n-1,k}(a_2, a_3, \dots, a_{n-k+1}) \equiv \sum \frac{k!(a_2)^{\nu_1}(a_3)^{\nu_2}\dots(a_{n-k+1})^{\nu_{n-k}}}{\nu_1!\nu_2!\dots\nu_{n-k}!} \tag{33}$$

for $1 \le k \le n-1$, $n \ge 2$, *are the ordinary Bell polynomials in* a_2, a_3, $\dots$, a_{n-k+1}, *and the sum is taken over all nonnegative integers* $\nu_1, \nu_2, \dots, \nu_{n-k}$ *satisfying*

$$\begin{aligned} &\nu_1 + \nu_2 + \dots + \nu_{n-k} = k, \\ &\nu_1 + 2\nu_2 + \dots + (n-k)\nu_{n-k} = n-1, \quad 1 \le k \le n-1,\ n \ge 2. \end{aligned} \tag{34}$$

Proof. We use the method in [6, pp. 91–93, Theorem 1], which is applicable to each analytic function $F(z)$ in $|z| < 1$ normalized by the requirements $F(0) = F'(0) - 1 = 0$ (see in [6] a recurrence relation for the polynomials (33) and tables for the polynomials (33) and the coefficients (32)).

Theorem 4. *The minimum (maximum) of the coefficients* b_n, $n \ge 2$, *from* (32) *in the class* T *is attained only either in the subclass* $T_1 \subset T$ *of functions* (10) *or in the subclass* $T_2 \subset T$ *of functions* (11)–(12) *with:*

(i) $1 \le p \le m$ *if* $n = 2m$, $m = 1, 2 \dots$,

(ii) $1 \le p \le m+1$ *if* $n = 2m+1$, $m = 1, 2 \dots$,

where in (12) *the points* $t_1, t_2, \dots, t_p$ *are among the numbers* 0 *and* 1 *and the roots in the interval* $0 \le t \le 1$ *of the equation*

$$P(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s}(s-1)t^{s-2} = 0, \quad n \ge 2 \tag{35}$$

(*for* $n = 2$ *this equation is impossible — see below Corollary* 1), *and the function*

$$Q(t) = \sum_{s=2}^{n} \frac{\partial b_n}{\partial a_s} t^{s-1}, \quad Q'(t) = P(t), \quad n \ge 2, \tag{36}$$

has equal values at any two adjacent points of the sequence $t_1, t_2, \dots, t_p$.

Proof. We apply Theorem 2 for $z = 0$ and the function

$$\begin{aligned} b_n &\equiv \Phi(u_0, u_1, \dots, u_n; 0) \\ &\equiv \frac{1}{n}\sum_{k=1}^{n-1}(-1)^k \binom{n+k-1}{k} D_{n-1,k}\left(\frac{u_2}{2!}, \frac{u_3}{3!}, \dots, \frac{u_{n-k+1}}{(n-k+1)!}\right) \end{aligned} \tag{37}$$

on the set $\bigcup_T \{\varphi(0), \varphi'(0), \dots, \varphi^{(n)}(0); 0\}$, where $n \ge 2$, having in mind (32)–(34). For the function (37), the equation (22) (or (13)) and the function (30) for the condition (29) are reduced to (35) and (36), respectively, where

$$\frac{\partial b_n}{\partial a_s} = s!\frac{\partial \Phi}{\partial u_s}, \quad a_s = \frac{\varphi^{(s)}(0)}{s!} \equiv \frac{u_s}{s!}, \quad 2 \le s \le n. \tag{38}$$

It is clear from (38) and (32)–(34) that for the function (37) the equation (35) is not an identity in t in the interval $0 \le t \le 1$ since, for example, $\partial b_n/\partial a_n = -1 \ne 0$, $n \ge 2$. Further:

(i) For $n = 2m$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (3) for the extremum of (37) has not more than $2m$ points of discontinuity among the roots of the equation (35) and the points 0 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than one, the function (36) has equal values. If $\mu(t)$ has more than m ($m > 1$) points of discontinuity in $0 \leq t \leq 1$, then the equation (35) will have more than $2m - 2$ roots in $0 \leq t \leq 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $0 \leq t \leq 1$ are not more than m ($m \geq 1$). Therefore, the interval of the integer p in (12) is contracted to $1 \leq p \leq m$.

(ii) For $n = 2m + 1$, $m = 1, 2, \ldots$, the function $\mu(t)$ in (3) for the extremum of (37) has not more than $2m + 1$ points of discontinuity among the roots of the equation (35) and the points 0 and 1. For all the points of discontinuity of the extremal step function $\mu(t)$, if they are more than one, the function (36) has equal values. If $\mu(t)$ has more than $m + 1$ points of discontinuity in $0 \leq t \leq 1$, then the equation (35) will have more than $2m - 1$ roots in $0 \leq t \leq 1$, which is impossible. Hence, the points of discontinuity of $\mu(t)$ in $0 \leq t \leq 1$ are not more than $m + 1$. Therefore, the interval of the integer p in (12) is contracted to $1 \leq p \leq m + 1$.

This completes the proof of Theorem 4.

Corollary 1. *The coefficient b_2 from* (32) *satisfies the sharp inequalities*

$$-1 \leq b_2 \leq 0, \tag{39}$$

where the equalities hold only for the following extremal functions:

— *on the left-hand side of* (39), *for the function*

$$\psi(w) = \frac{w}{1+w} = \sum_{n=1}^{\infty}(-1)^{n-1}w^n, \tag{40}$$

inverse of the function

$$\varphi(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \in T_1; \tag{41}$$

— *on the right-hand side of* (39), *for the function*

$$\psi(w) = w, \tag{42}$$

inverse of the function

$$\varphi(z) = z \in T_1. \tag{43}$$

Proof. For $n = 2$, Theorem 4(i) yields $p = 1$. For $n = 2$, from (32)–(34) and (35) we obtain

$$b_2 = -a_2, \quad \frac{\partial b_2}{\partial a_2} = -1, \tag{44}$$

and

$$P(t) = -1 \neq 0, \tag{45}$$

respectively. It follows from (45) that the point of discontinuity of $\mu(t)$ can be either $t_1 = 0$ or $t_1 = 1$ with the corresponding jumps $c_1 = 1$ and $c_1 = 1$. Therefore, we obtain the two extremal functions (41) and (43) of the form (10), the inverse ones of which (40) and (42) supply the equalities in (39), respectively.

Remark 1. The inequalities (39) and the extremal functions (41) (or (40)) and (43) (or (42)) follow from (44) and (2) for $n = 2$ as well.

Corollary 2. *The coefficient b_3 from* (32) *satisfies the sharp inequalities*

$$-\frac{1}{8} \leq b_3 \leq 1, \tag{46}$$

where the equalities hold only for the following extremal functions:
— on the left-hand side of (46), *for the function*

$$\begin{aligned}\psi(w) &= \frac{2}{3}\left(1 + w - \sqrt{1 - w + w^2}\right) \\ &= w + \frac{2}{3}\sum_{n=2}^{\infty}(-1)^{n-1}w^n \sum_{n/2\leq\nu\leq n}\binom{1/2}{\nu}\binom{\nu}{n-\nu},\end{aligned} \tag{47}$$

where $\sqrt{1} = 1$ and the inner sum is taken over all integers ν satisfying $n/2 \leq \nu \leq n$, inverse of the function

$$\varphi(z) = \frac{3}{4}z + \frac{1}{4}\frac{z}{1-z} = z + \frac{1}{4}\sum_{n=2}^{\infty} z^n \in T_1; \tag{48}$$

— on the right-hand side of (46), *for the function* (40), *inverse of the function* (41), *respectively.*

Proof. For $n = 3$, Theorem 4(ii) yields $p = 1, 2$. For $n = 3$, from (32)–(34), (35), (36) and (38) we obtain

$$b_3 = -a_3 + 2a_2^2, \quad a_2 = \frac{\varphi''(0)}{2}, \quad a_3 = \frac{\varphi'''(0)}{6}, \tag{49}$$

$$\frac{1}{2}P(t) = 2a_2 - t = 0, \tag{50}$$

and

$$Q(t) = 4a_2t - t^2, \quad Q'(t) = P(t), \tag{51}$$

respectively.

If $p = 1$, then (11)–(12) are reduced to

$$\varphi(z) = \frac{z}{1 - tz} \in T_2, \tag{52}$$

where t can be either the root $t = 2a_2$ of (50) or any of the points 0 and 1. From (52) we obtain $\varphi''(0) = 2t$. On the other hand, $\varphi''(0) = 2a_2 = t$, and hence $t = 0$. Then (52) takes the form $\varphi(z) = z$, the inverse one of which is $\psi(w) = w$. It is clear that the identity is not an extremal function. If $t = 1$ in (52), then we obtain that $\varphi''(0) = 2$, $\varphi'''(0) = 6$ and the equations (49) yield $b_3 = 1$. Thus for the function (40), inverse of the extremal function (41), the bound 1 in (46) is attained.

If $p = 2$, then (11) has two terms, corresponding to the condition $Q(t_1) = Q(t_2)$, where $Q(t)$ is determined by (51) and t_1 and t_2 are among the numbers 0, $t = 2a_2$ and 1. According to this condition and the Rolle theorem, the equation (50) has an odd number of roots between t_1 and t_2. This is possible only if $t_1 = 0$ and $t_2 = 1$. Hence, the extremal function $\varphi(z)$ is

$$\varphi(z) = cz + (1-c)\frac{z}{1-z} \in T_2, \quad 0 < c < 1. \tag{53}$$

Further, the condition $Q(0) = Q(1)$, where $Q(t)$ is given by (51), yields $a_2 = \varphi''(0)/2 = 1/4$. On the other hand, from (53) we obtain $\varphi''(0) = 2(1-c)$, and hence $c = 3/4$. For $c = 3/4$, from (53) we obtain the extremal function (48) and its inverse function (47) for which the bound $-1/8$ in (46) is attained.

Remark 2. The second sharp inequality in (46) and the extremal function (41) (or (40)) can be obtained in another way. With the help of the Cauchy inequality and (2) we obtain that

$$a_2^2 = \left(\int_0^1 1.t\,d\mu(t)\right)^2 \leq \int_0^1 1^2\,d\mu(t). \int_0^1 t^2\,d\mu(t) = a_3. \tag{54}$$

Now from (54) and the first equation in (49) we obtain the sharp inequalities

$$b_3 \leq a_3 \leq 1$$

with the unique extremal function (41) (or (40)).

Corollary 3. *The coefficient b_4 from* (32) *satisfies the sharp inequalities*

$$-1 \leq b_4 \leq \frac{5+4\sqrt{10}}{135} = 0.13073415\ldots, \tag{55}$$

where the equalities hold only for the following extremal functions:

— *on the left-hand side of* (55), *for the function* (40), *inverse of the function* (41);

— *on the right-hand side of* (55), *for the inverse function of the function* (53) *for*

$$c = \frac{10-\sqrt{10}}{15} = 0.45584816\ldots\,. \tag{56}$$

Proof. For $n = 4$, Theorem 4(i) yields $p = 1, 2$. For $n = 4$, from (32)–(34), (35), (36) and (38) we obtain

$$b_4 = -a_4 + 5a_2a_3 - 5a_2^3, \tag{57}$$

$$P(t) = 5a_3 - 15a_2^2 + 10a_2t - 3t^2 = 0, \tag{58}$$

$$Q(t) = \left(5a_3 - 15a_2^2\right)t + 5a_2t^2 - t^3, \quad Q'(t) = P(t), \tag{59}$$

where $a_{2,3,4}$ are the coefficients of the extremal functions $\varphi(z) \in T_2$, i.e.

$$a_2 = \frac{\varphi''(0)}{2}, \quad a_3 = \frac{\varphi'''(0)}{6}, \quad a_4 = \frac{\varphi^{(IV)}(0)}{24}, \tag{60}$$

respectively.

If $p = 1$, then (11)–(12) are reduced to (52), where t can be either any root of (58) in $0 \le t \le 1$ or any of the points 0 and 1. Converting (52) or by means of (60), (52) and (57), we obtain

$$b_4 = -t^3, \quad 0 \le t \le 1. \tag{61}$$

If $p = 2$, then (11), having in mind (12), can have the following forms:

$$\varphi(z) = cz + (1-c)\frac{z}{1-tz} \in T_2, \quad 0 < c < 1, \quad 0 < t < 1, \tag{62}$$

$$\varphi(z) = c\frac{z}{1-tz} + (1-c)\frac{z}{1-z} \in T_2, \quad 0 < c < 1, \quad 0 < t < 1, \tag{63}$$

and (53), where t (in general different for each function) is a real root of (58) and the other root of (58) has to lie in the open intervals $(0,t)$, $(t,1)$ and $(0,1)$ in accordance to (59) and the conditions

$$Q(0) = Q(t), \quad Q(t) = Q(1), \quad Q(0) = Q(1), \tag{64}$$

respectively.

(a) From the first equation of (64) and (59) we obtain the corresponding equation for (62), namely,

$$5a_3 - 15a_2^2 + 5a_2 t - t^2 = 0. \tag{65}$$

It follows from (58) and (65) that

$$a_2 = \frac{2t}{5}, \quad a_3 = \frac{7t^2}{25}. \tag{66}$$

On the other hand, from (60) and (62) we get

$$a_2 = (1-c)t, \quad a_3 = (1-c)t^2, \quad a_4 = (1-c)t^3. \tag{67}$$

The equations (66) and (67) yield the different values $c = 3/5$ and $c = 18/25$, respectively, i.e. the extremal function $\varphi(z)$ is not of the form (62).

(b) From the second equation of (64) and (59) we obtain the corresponding equation for (63), namely,

$$5a_3 - 15a_2^2 + 5a_2(t+1) - t^2 - t - 1 = 0. \tag{68}$$

It follows from (58) and (68) that

$$a_2 = \frac{2t+1}{5}, \quad a_3 = \frac{7t^2+2t+3}{25}. \tag{69}$$

On the other hand, from (60) and (63) we get

$$a_2 = ct + 1 - c, \quad a_3 = ct^2 + 1 - c, \quad a_4 = ct^3 + 1 - c. \tag{70}$$

The equations (69) and (70) yield the equations

$$c = \frac{2(t-2)}{5(t-1)}, \quad 3t^2 - 12t + 2 = 0, \quad t = \frac{6-\sqrt{30}}{3} = 0.1742581\ldots \tag{71}$$

(the other root of the second equation is not in the open interval $(0,1)$). From (71), (69) and (70) we obtain

$$c = \frac{2\,(10+\sqrt{30})}{35}, \quad a_2 = \frac{15-2\sqrt{30}}{15},$$
$$a_3 = \frac{35-6\sqrt{30}}{15}, \quad a_4 = \frac{345-62\sqrt{30}}{45}. \tag{72}$$

By the values of $a_{2,3}$ from (72) the equation (58) becomes

$$9t^2 - 2\left(15-2\sqrt{30}\right)t + 2\left(17-3\sqrt{30}\right) = 0. \tag{73}$$

Really, for the roots of (73) we have

$$0 < \frac{6-\sqrt{30}}{3} < \frac{12-\sqrt{30}}{9} < 1.$$

Finally, by the values of $a_{2,3,4}$ from (72) and (57) we obtain

$$b_4 = -\frac{45-8\sqrt{30}}{45} = -0.026271\ldots\,. \tag{74}$$

(c) From the third equation of (64) and (59) we obtain the corresponding equation for (53), namely,

$$5a_3 - 15a_2^2 + 5a_2 - 1 = 0. \tag{75}$$

It follows from (60) and (53) that

$$a_2 = 1-c, \quad a_3 = 1-c, \quad a_4 = 1-c. \tag{76}$$

From (75)–(76) we get the values

$$c_{1,2} = \frac{10 \pm \sqrt{10}}{15}. \tag{77}$$

The equations (76) and (77) yield

$$a_2 = \frac{5 \mp \sqrt{10}}{15}, \quad a_3 = \frac{5 \mp \sqrt{10}}{15}, \quad a_4 = \frac{5 \mp \sqrt{10}}{15}, \tag{78}$$

respectively. By the values of $a_{2,3}$ from (78) the equation (58) becomes

$$9t^2 - 2\left(5 \mp \sqrt{10}\right)t + 2 \mp \sqrt{10} = 0. \tag{79}$$

Really, each equation of (79) has one root in $0 < t < 1$, respectively. Finally, from (78) and (57) we find

$$b_4 = \frac{5-4\sqrt{10}}{135} = -0.05666\ldots, \quad b_4 = \frac{5+4\sqrt{10}}{135} = 0.13073415\ldots, \tag{80}$$

respectively. Now the comparison of (61), (74) and (80) leads us to (55) and (56), which completes the proof of Corollary 3.

Remark 3. The first inequality in (55) and the extremal function (41) (or (40)) follow also from (57), (54) and (2), namely,

$$b_4 = -a_4 + 5a_2\left(a_3 - a_2^2\right) \geq -a_4 \geq -1.$$

For the coefficients b_5, b_6, ..., we can proceed in the same way.

Conjecture 1. *In the class T each coefficient b_n, $n = 2, 3, \ldots$, from* (32) *attains its minimum (maximum) only for the rational functions of the form* (10).

Conjecture 2. *In the class T each coefficient b_n, $n = 2, 3, \ldots$, from* (32) *satisfies the sharp inequalities*

$$b_{2m} \geq -1, \quad m = 1, 2, \ldots,$$

and

$$b_{2m+1} \leq 1, \quad m = 1, 2, \ldots,$$

where the equalities hold only for the extremal function (40)*, inverse of the function* (41).

For $n = 2, 3, 4$ these conjectures are proved in the above corollaries 1–3.

REFERENCES

1. Hausdorff, F. Summationsmethoden und Momentfolgen I, II. *Math. Z.*, **9**, 1921, 74–109, 280–299.
2. Thale, J. S. Univalence of continued fractions and Stieltjes transforms. *Proc. Amer. Math. Soc.*, **7**, 1956, 232–244.
3. Goodman, A. W. An invitation to the study of univalent and multivalent functions. *Int. J. Math. and Math. Sci.*, **2**, No. 2, 1979, 163–186.
4. Wirths, K. J. Über totalmonotone Zahlenfolgen. *Arch. Math.*, **26**, No. 5, 1975, 508–517.
5. Todorov, P. G., and M. O. Reade. The Koebe domain of the classes $N_1(a)$ and $N_2(a)$ of Nevanlinna analytic functions. *Complex Variables Theory Appl.*, **7**, 1987, 343–348.
6. Todorov, P. G. Sharp estimates for the coefficients of the inverse functions of the Nevanlinna univalent functions of the classes N_1 and N_2. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, **68**, 1998, 91–102.

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FIBERED SURFACES*

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The fibered surfaces are shown to be finite branched coverings of products of algebraic curves. As a consequence, the fundamental group of a finite surface turns to be commensurable with a product of the fundamental groups of Riemann surfaces.

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The compact Kähler surface S is said to be fibered if there is a surjective holomorphic map $S \to C_g$ with connected fibers onto a curve C_g of genus $g \geq 2$. The work focuses on some properties of fibered surfaces S. The first section exhibits S as a finite ramified covering $S \to C_g \times C_h$, $g+h = h^{1,0}(S)$ of products of curves. As a consequence, the second section shows the commensurability of the fundamental group $\pi_1(S)$ of a fibered surface with the product $\pi_1(C_m) \times \pi_1(C_n)$ of fundamental groups of appropriate Riemann surfaces.

1. STRUCTURE RESULT

Proposition 1. *Any fibered surface $f_1 : S \to C_g$, $g \geq 2$, with non-isotropic $H^{1,0}(S)$ is a finite ramified covering $f = (f_1, f_2) : S \to C_g \times C_h$, $g + h \leq h^{1,0}(S)$.*

According to the Theorem of Castelnuovo de Franchis (cf. [1]), for any fibered surface $f_1 : S \to C_g$ the subspace $f_1^* H^{1,0}(C_g) \subset H^{1,0}(S)$ is isotropic, which means

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that the wedge product of any two forms from $f_1^* H^{1,0}(C_g)$ is zero. Let us start with the following

Lemma 2. *Let X be a compact complex manifold with functionally independent $\varphi_1, \varphi_2 \in H^{1,0}(X)$ and $\mathbb{C}$-linearly independent $\psi_1, \dots, \psi_m \in H^{1,0}(X)$. Suppose that with respect to some coordinate covering $X = \cup_{\alpha \in A} W^{(\alpha)}$ there hold $\psi_j = \sum_{i=1}^{k} \lambda_{ji}^{(\alpha)} \varphi_i$ for some local meromorphic functions $\lambda_{ji}^{(\alpha)} : W^{(\alpha)} \to \mathbf{P}_1$, $1 \leq j \leq m$, and $k = 1$ or 2. Then there exist global holomorphic functions $b_i, d_j, f_j, g_j : X \to \mathbb{C}$, such that $\omega_i := \frac{\varphi_i}{b_i}$, $i = 1,2$, are global holomorphic $(1,0)$-forms, as well as $\psi_j = f_j \omega_1$ in the case of $k = 1$ and $d_j \psi_j = f_j \omega_1 + g_j \omega_2$ in the case of $k = 2$.*

Proof. The local rings $\mathcal{O}_{W^{(\alpha)}}$ of the holomorphic functions on $W^{(\alpha)}$ are factorial. Their fraction fields $\mathcal{M}_{W^{(\alpha)}}$ consist of the meromorphic functions on $W^{(\alpha)}$. That allows to represent uniquely $\lambda_{ji}^{(\alpha)} = \frac{a_{ji}^{(\alpha)}}{c^{(\alpha)} ji}$ as ratios of relatively prime $a_{ji}^{(\alpha)}, c_{ji}^{(\alpha)} \in \mathcal{O}_{W^{(\alpha)}}$. Since φ_i and ψ_j are globally defined, at $x \in W^{(\alpha)} \cap W^{(\beta)}$, one has $0 \equiv \psi_j^{(\alpha)}(x) - \psi_j^{(\beta)}(x) = \sum_{i=1}^{k} (\lambda_{ji}^{(\alpha)} - \lambda_{ji}^{(\beta)}) \varphi_i(x)$, which implies $\lambda_{ji}^{(\alpha)} = \lambda_{ji}^{(\beta)}$ due to the functional independence of φ_i.

One can represent the global meromorphic functions $\lambda_{ji} : X \to \mathbb{C}$ by global holomorphic numerators and denominators. Indeed, on $W^{(\alpha)} \cap W^{(\beta)}$ the relation $a_{ji}^{(\alpha)} c_{ji}^{(\beta)} = a_{ji}^{(\beta)} c_{ji}^{(\alpha)}$ requires a_{ji}^{β} to be divisible by $a_{ji}^{(\alpha)}$, according to $GCD(a_{ji}^{(\alpha)}, c_{ji}^{(\alpha)}) = 1$. Exchanging α with β, one obtains

$$a_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{(\beta)}} = u_{ji}^{(\alpha\beta)} a_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{\beta}}, \quad c_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{\beta}} = u_{ji}^{(\alpha\beta)} c_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{\beta}}$$

for some locally invertible $u_{ji}^{(\alpha\beta)}$. Due to the compactness of X, one can choose a finite coordinate covering and adjust all $u^{(\alpha\beta)} ji = 1$. After fixing some $a_{ji}^{(\alpha)}$, one puts $a_{ji}^{(\beta)}|_{W^{(\alpha)} \cap W^{(\beta)}} = a_{ji}^{(\alpha)}|_{W^{(\alpha)} \cap W^{(\beta)}}$ for all $\beta \in \{\beta_1, \dots, \beta_k\}$ with $W^{(\alpha)} \cap W^{(\beta)} \neq \varnothing$ and extends holomorphically $a_{ji}^{(\beta)}$ over the simply connected $W^{(\beta)}$. The same procedure is applied to all β with $W^{(\beta)} \cap W^{(\beta_i)} \neq \varnothing$, $1 \leq i \leq k$, etc.

In the case $k = 2$ let us consider the greatest common divisors $d_j := GCD(c_{j1}, c_{j2})$ and introduce $b_{ji} := \frac{c_{ji}}{d_j}$ for all $1 \leq j \leq m$, $i = 1,2$. Then $\theta_j := d_j \psi_j = \sum_{i=1}^{2} \frac{a_{ji}}{b_{ji}} \varphi_i$. For future convenience let us put $b_{j1} := c_{j1}$, $\theta_j := \psi_j = \frac{a_{j1}}{b_{ji}} \varphi_1$ for $k = 1$.

Multiplying θ_j by $b_{j,3-i}$ for $i = 1,2$ and bearing in mind that $GCD(a_{ji}, b_{ji}) = 1$, $GCD(b_{j,3-i}, b_{ji}) = 1$, one concludes that b_{ji} divide φ_i, i.e., $\frac{\varphi_i}{b_{ji}}$ are global holomorphic $(1,0)$-forms. The same holds if $k = 1$. Then the least common multiples

$b_i := LCM(b_{ji}|1 \leq j \leq m)$ divide φ_i and allow to define the global holomorphic $\omega_i := \frac{\varphi_i}{b_i}$. As a result, one obtains the representations $\theta_j = \sum_{i=1}^{k} a_{ji}\frac{b_i}{b_{ji}}\omega_i$ as $\mathcal{O}_X$-linear combinations of ω_i, Q.E.D.

Proof of Proposition 1. The subspace $U := f_1^*H^{1,0}(C_g) \subsetneq H^{1,0}(S)$ is maximal isotropic, according to the connectedness of the fibers of f_1. Therefore, any $\mathbb{C}$-basis $u_1,\ldots,u_g$ of U is of the form $u_i = \lambda_i^{(\alpha)}u_1$, $2 \leq i \leq g$, for some local meromorphic functions $\lambda_i^{(\alpha)} : W^{(\alpha)} \to \mathbf{P}_1$ on the coordinate charts $W^{(\alpha)} \subset S$. According to Lemma 2, there exist global holomorphic functions $\xi_1,\ldots,\xi_g$ and a global holomorphic $(1,0)$-form $\omega_1 \in U$ such that $u_i = \xi_i\omega_1$, $1 \leq i \leq g$.

For a non-ruled fibered surface, $U := f_1^*H^{1,0}(C_g)$ is a proper subspace of $H^{1,0}(S)$. Any complement of U has a basis $v_1,\ldots,v_k$, $k = h^{1,0}(S) - g$ with $\omega_1 \wedge v_j \neq 0$ for all $1 \leq j \leq k$. The functionally independent ω_1, v_1 on the surface S generate $H^{1,0}(S)$ over the fields $\mathcal{M}_{W^{(\alpha)}}$ of local meromorphic functions. That allows to represent $v_i = \sigma_i^{(\alpha)}\omega_1 + \tau_i^{(\alpha)}v_1$ on $W^{(\alpha)}$, $\sigma_i^{(\alpha)},\tau_i^{(\alpha)} \in \mathcal{M}_{W^{(\alpha)}}$. The application of Lemma 2 yields global holomorphic functions $b_1, b_2, d_j, \lambda_j, \mu_j$, $1 \leq j \leq k$, such that $\widetilde{\omega_1} := \frac{\omega_1}{b_1}$, $\omega_2 := \frac{v_1}{b_2}$ are global holomorphic $(1,0)$-forms and $d_jv_j = \lambda_j\widetilde{\omega_1} + \mu_j\omega_2$, $2 \leq j \leq k$. Let V_0 be the $\mathbb{C}$-span of $\varphi_1 = v_1 = b_2\omega_2 = \mu_1\omega_2$, $\varphi_j = d_jv_j - \lambda_j\widetilde{\omega_1} = \mu_2\omega_2$, $2 \leq j \leq k$, and V be a maximal isotropic subspace of $H^{1,0}(S)$, containing V_0. Wedging by v_1 an arbitrary $v = \sum_{i=1}^{g} c_i\xi_i\omega_1 \in V \cap U$ and bearing in mind that $\omega_1 \wedge v_1 \neq 0$, one infers $\sum_{i=1}^{g} c_i\xi_i = 0$. As far as $\xi_1\omega_1,\ldots,\xi_g\omega_1$ are $\mathbb{C}$-linearly independent, there follows $c_i = 0$ for all $1 \leq i \leq g$. In other words, $U \cap V = 0$ and there exist maximal isotropic subspaces U, V with $U \oplus V \subseteq H^{1,0}(S)$.

If $\dim_{\mathbb{C}} V \geq 2$, Castelnuovo-de Franchis' Theorem implies that there is a surjective holomorphic map $f_2 : S \to C_h$ with connected fibers, such that $f_2^*H^{1,0}(C_h) = V$. The holomorphic map $f = (f_1, f_2) : S \to C_g \times C_h$ is generically of $rank_{\mathbb{C}}df = 2$ since

$$f^* = f_1^* \oplus f_2^* : H^{1,0}(C_g \times C_h) = H^{1,0}(C_G) \oplus H^{1,0}(C_h) \to U \oplus V \subseteq H^{1,0}(S)$$

and f_1^*, f_2^* are injective. According to Remmert's Proper Mapping Theorem, $f(S)$ is a 2-dimensional complex analytic subspace of $C_g \times C_h$. Therefore $f(S) = C_g \times C_h$. The generic fiber of f is a compact complex analytic 0-dimensional subspace of S, i.e., finite number of points.

In the case of $V = Span_{\mathbb{C}}(v_1)$, let us consider the dual $V^* \subset H_1(S,\mathbb{C})$ and its quotient $E := V^*/V^* \cap H_1(S,\mathbb{Z})_{free}$ modulo the free part of $H_1(S,\mathbb{Z})$. As a closed subtorus of the compact Albanese variety $Alb(S) = H^{1,0}(S)^*/H_1(S,\mathbb{Z})_{free}$, E is an elliptic curve. For any fixed $s_0 \in S$ the holomorphic map $f_2' : S \to E$, $f_2'(S) := \int_{s_0}^{s} v_1 modulo H_1(S,\mathbb{Z})_{free}$ is of $rank_{\mathbb{C}}df_2' = 1$, whereas surjective. Since the fibers of f'_2 can be disconnected, we pass to Stein factorization $f_2 : S \to C_h$, $h \geq 1$. Then apply the rest of the proof for $\dim_{\mathbb{C}} V \geq 2$, Q.E.D.

Remark. *Generalization of Proposition* 1 *to higher dimensional compact Kähler manifolds.* Catanese has generalized in [3] the theorem of Castelnuovo-de Franchis. Let us say that the normal Kähler variety Y is of Albanese general type if the irregularity $h^{1,0}(Y) > \dim_{\mathbb{C}} Y$ and the image of Albanese map $\alpha : Y \to Alb(Y)$ is of $\dim_{\mathbb{C}} \alpha(Y) = \dim_{\mathbb{C}} Y$. The compact Kähler manifold X_n of $\dim_{\mathbb{C}} X_n = n$ is Albanese general type k-fibration if it admits a surjective holomorphic map $f_1 : X_n \to Y_k$ with connected fibers onto a normal k-dimensional Kähler variety of Albanese general type. Catanese has shown that a necessary and sufficient condition for the existence of an Albanese general type k-fibration $f_1 : X_n \to Y_k$ is the presence of a maximal subspace $U \subset H^{1,0}(X_n)$ with $\Lambda^{k+1} U = 0$, containing a subspace $U_0 \subseteq U$ of $\dim_{\mathbb{C}} U_0 \geq k+1$, whose k-wedge $\Lambda^k U_0$ is embedded in $H^{k,0}(X_n)$. A slight modification of the proof of Proposition 1 establishes that if a compact Kähler n-dimensional manifold X_n admits an Albanese general type $(n-1)$-fibration $f_1 : X_n \to X_{n-1}$, whose generic fibers are different from $\mathbf{P}_1$, then X_n is a finite ramified covering $f : X_n \to X_{n-1} \times X_1$ of the product of X_{n-1} and a Riemann surface X_1 of genus ≥ 1. The study of the complements of Albanese general k-fibrations $f_1 : X_n \to X_k$ with an arbitrary k is obstructed by the condition $\Lambda^{n-k} U_0 \hookrightarrow H^{n-k,0}(X_n)$, which is not easy to be understood.

2. THE FUNDAMENTAL GROUP

Corollary 3. *If the surface S is a finite ramified covering $f = (f_1, f_2) : S \to C_g \times C_h$, $g \geq 2$, $h \geq 2$, then its fundamental group $\pi_1(S)$ is commensurable with $\pi_1(C_m) \times \pi_1(C_n)$ for some $m \geq g$, $n \geq h$.*

Proof. Campana has shown in [2] that for any surjective holomorphic map $X \to C$ of a compact Kähler manifold X onto a Riemann surface C there is a finite etale cover $r : \widetilde{X} \to X$ such that the Stein factorization $\widetilde{f} : \widetilde{X} \to \widetilde{C}$ of $fr : \widetilde{X} \to C$ has no multiple fibers and there is a finite map $\rho : \widetilde{C} \to C$ with $\rho \widetilde{f} = fr$. The application of this result to $f_1 : S \to C_g$ yields a finite etale cover $r_1 : S_1 \to S$, a surjective holomorphic map $f'_1 : S_1 \to C_m$, $m \geq g$, without multiple fibers, and a finite map $\rho_1 : C_m \to C_g$ such that $f_1 r_1 = \rho_1 f'_1$. The subsequent application of the aforementioned result to $f_2 r_1 : S_1 \to C_h$ provides a finite etale cover $r_2 : Z \to S_1$, a holomorphic surjection $\varphi_2 : Z \to C_n$, $n \geq h$, without multiple fibers, and a finite map $\rho_2 : C_n \to C_h$ with $f_2 r_1 r_2 = \rho_2 \varphi_2$. Consequently, the composition $\varphi_1 := f'_1 r_2 : Z \to C_m$ of the unramified r_2 and f'_1 has no multiple fibers. The Cartesian product $\varphi = (\varphi_1, \varphi_2) : Z \to C_m \times C_n$ is a finite covering, as far as $r_1 r_2 : Z \to S$ is a finite etale, $f = (f_1, f_2) : S \to C_g \times C_h$ is finite and there is a projection $(\rho_1, \rho_2) : C_m \times C_n \to C_g \times C_h$. We claim that φ is unramified since the generic fibers of $\varphi_1 : Z \to C_m$ and $\varphi_2 : Z \to C_n$ have no self-intersections. Indeed, for apropriate ramified coverings $p_1 : C_m \to \mathbf{P}_1$ and $p_2 : C_n \to \mathbf{P}_1$ one obtains linear pencils of divisors $p_1 \varphi_1 : Z \to \mathbf{P}_1$ and $p_2 \varphi_2 : Z \to \mathbf{P}_1$. According to Bertini's theorem, the generic fibers $(p_i\varphi_i)^{-1}(x) = \varphi_i^{-1}(p_i^{-1}(x))$, $i = 1, 2$, have no singularities outside the base locus. Thus, $Z \to C_m \times C_n$ is a finite unramified covering and $\pi_1(Z)$ is a finite index subgroup of $\pi_1(C_m) \times \pi_1(C_n)$. On the other hand, $r_1 r_2 : Z \to S$ is finite and unramified, so that $\pi_1(Z)$ is a finite index subgroup of $\pi_1(S)$. That justifies the commensurability of $\pi_1(S)$ and $\pi_1(C_m)\times\pi_1(C_n)$, Q.E.D.

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REFERENCES

1. Amor'os, J., M. Burger, K. Corlette, D. Kotschick, D. Toledo. Fundamental Groups of Compact Kähler Manifolds. *Math. Surv. Mon.*, **44**, Providence, 1996.
2. Campana, F. Negativity of compact curves in infinite covers of projective surfaces. *J. Alg. Geom.*, **7**, 1998, 673–693.
3. Catanese, F. Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. *Inv. Math.*, **104**, 1991, 263–289.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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EXISTENCE OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR THE EQUATION $f(t, x, x', x'') = 0$ WITH FULLY NONLINEAR BOUNDARY CONDITIONS

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Existence results for two-point boundary value problems are established, the equation is not solved with respect to the high derivative, and the boundary conditions are nonlinear and full. The proofs are based on a variant of a basic theorem of Granas, Guenther and Lee. The a priori bounds needed for its application are obtained by the barrier strips technique.

Keywords: boundary value problems, nonlinear boundary conditions, existence, barrier strips

Mathematics Subject Classification 2000: 34B15

1. INTRODUCTION

In this paper we study the solvability of boundary value problems (BVPs) of the form

$$\begin{cases} f(t,x,x',x'') &= 0, \quad t \in [0,1], \\ W_i(x) &= V_i(x), \quad i = 1,2. \end{cases} \tag{1.1}$$

Here the scalar function $f(t, x, p, q)$ is continuous and has continuous first derivatives only on suitable subsets of $[0, 1] \times R^3$,

$$V_1(x) = \varphi\left(x(0), x'(0), x(1), x'(1)\right), \quad V_2(x) = \psi\left(x(0), x'(0), x(1), x'(1)\right),$$

$\phi, \psi : R^4 \to R$ are continuous, and $(W_1(x), W_2(x))$ are of the type

$$(M_1)\left(x(0), x'(1)\right), \quad (M_2)\left(x'(0), x(1)\right) \quad \text{or} \quad (D)\left(x(0), x(1)\right).$$

Further, we will write as (M_1), (M_2) and (D) the BVP (1.1) in the cases (M_1), (M_2) and (D), respectively.

The solvability of BVPs for the equation $x'' = f(t, x, x')$ with various nonlinear boundary conditions is studied in [1–8], for example, under various conditions on $f(t, x, p)$.

The results [9–14], see also [15], guarantee the existence of $C^2[0,1]$-solutions to BVPs for the equation $x'' = f(t, x, x', x'') - y(t)$. Moreover, the solutions satisfy mixed boundary conditions (M_1) or (M_2) in [9], periodic ones in [10], Neumann ones in [9, 11], Dirichlet or periodic ones in [9, 12, 13], and either Dirichlet, Neumann, Sturm-Liouvile, periodic or antiperiodic ones in [14]; in the last work uniqueness results are also obtained. Moreover, the growth of $f(t, x, p, q)$ is linear with respect to x, p and q in [10–12], semilinear in [13], quadratic with respect to p and linear with respect to q in [14]. In addition f satisfies various further conditions. The results [16] guarantee an existence and an uniqueness of $C^2[0,1]$-solution to the BVP for the equation $x'' = f(t, x, x', x'')$ with boundary conditions of the form

$$\mathbf{a}_{i1}x(0) + \mathbf{a}_{i2}x'(0) + \mathbf{a}_{i3}x(1) + \mathbf{a}_{i4}x'(1) = 0, i = 1, 2.$$

In [16] $f(t, x, p, q)$ satisfies a growth condition, which is a Nagumo one with respect to p and a linear one with respect to q, and some further conditions. The approach [10–14, 16] relies on the topological transversality [8] or similar arguments. The existence results [17] guarantee $W^{2,\infty}[0,1]$-solutions or $C^2[0,1]$-solutions to the Dirichlet BVP for the equation (1.1). The function $f(t, x, p, q)$ is defined on $[0,1] \times R^n \times R^n \times Y$, where Y is a non-empty closed connected and locally connected subset of R^n. Growth conditions on f are not used. The approach [17] follows that introduced in [18] with regard to the Cauchy problem. The results [19] guarantee an existence of $C^2[0,1]$-solutions to the BVP for the equation $x'' + g(t, x, x', x'') = y(t)$ with either Dirichlet, Neumann or mixed boundary conditions. The authors use conditions of Lipschitz type on g and barrier strips [20].

In this paper we also do not use assumptions on the growth of f. Using again the barrier strips technique [20], see also [19] for similar conditions, we obtain some uniformely a priori bounds for x', x and x'' (in this order) for the eventual solutions $x(t) \in C^2[0,1]$ to the family of BVPs

$$\begin{cases} Kx'' = \lambda(Kx'' + f(t, x, x', x'')), & t \in [0,1], \\ W_i(x) = \lambda V_i(x), & i = 1, 2, \end{cases} \qquad (1.1)_\lambda$$

where $\lambda \in [0,1]$, and K is a suitable positive constant; further, we will write as $(M_1)_\lambda$, $(M_2)_\lambda$ and $(D)_\lambda$ the family $(1.1)_\lambda$ in the cases (M_1), (M_2) and (D), respectively. Then the solvability of the problems considered follows by a basic existence result (Theorem 4.1) proved by an application of the topological transversality theorem [8].

2. HYPOTHESES

We will say that $(\mathbf{A}_1)$ holds for the constants F and L if:

$(\mathbf{A}_1)$ $L \geq F$ and there are functions $F_i^-(t), L_i^+(t) \in C[0,1]$, $i = 1,2$, such that

$$L_1^+(1) \geq L, F \geq F_1^-(1),$$

$$L_1^+(t) \text{ is nonincreasing and } F_1^-(t) \text{ is nondecreasing on } [0,1],$$

$$L_2^+(t) > L_1^+(t) \quad \text{and} \quad F_1^-(t) > F_2^-(t) \quad \text{for} \quad t \in [0,1],$$

and there is a constant $K > 0$ for which

$$f(t,x,p,q) \geq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (-\infty, 0), t \in [0,1] \text{ and } L_1^+(t) \leq p \leq L_2^+(t)\Big\}$,

$$f(t,x,p,q) \leq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (0,\infty), t \in [0,1] \text{ and } F_2^-(t) \leq p \leq F_1^-(t)\Big\}$.

We will say that $(\mathbf{A}_2)$ holds for the constants F and L if:

$(\mathbf{A}_2)$ $L \geq F$ and there are functions $F_i^+(t), L_i^-(t) \in C[0,1]$, $i = 1,2$, such that

$$L_1^-(0) \geq L, F \geq F_1^+(0),$$

$$L_1^-(t) \text{ is nondecreasing and } F_1^+(t) \text{ is nonincreasing on } [0,1],$$

$$L_2^-(t) > L_1^-(t) \quad \text{and} \quad F_1^+(t) > F_2^+(t) \quad \text{for} \quad t \in [0,1],$$

and there is a constant $K > 0$ for which

$$f(t,x,p,q) \leq -Kq$$

on $\Big\{(t,x,p,q) : x \in R,\ q \in (0,\infty),\ t \in [0,1] \text{ and } L_1^-(t) \leq p \leq L_2^-(t)\Big\}$,

$$f(t,x,p,q) \geq -Kq$$

on $\Big\{(t,x,p,q) : x \in R, q \in (-\infty, 0), t \in [0,1] \text{ and } F_2^+(t) \leq p \leq F_1^+(t)\Big\}$.

Remark. The constant K from $(\mathbf{A}_1)$ and the constant K from $(\mathbf{A}_2)$ could be different.

Lemma 2.1. *Let the condition $(\mathbf{A}_1)$ hold for some F and L and $x(t) \in C^2[0,1]$ be a solution to $(1.1)_\lambda$ (with the constant K from $(\mathbf{A}_1)$). Suppose there is an interval $T_1 \subseteq [0,1]$ such that*

$$L_1^+(t) \leq x'(t) \leq L_2^+(t) \quad \text{for} \quad t \in T_1. \tag{2.1}$$

Then $x''(t) \geq 0$ for $t \in T_1$. If there is an interval $T_2 \subseteq [0,1]$ such that

$$F_2^-(t) \leq x'(t) \leq F_1^-(t) \quad \text{for} \quad t \in T_2,$$

then $x''(t) \le 0$ *for* $t \in T_2$.

Proof. We will show only that (2.1) yields $x''(t) \ge 0$ for $t \in T_1$. The assertion is true for $\lambda = 0$. Now let $\lambda \in (0,1]$. Assume there is $t_0 \in T_1$ such that $x''(t_0) < 0$. Then

$$0 > Kx''(t_0) = \lambda\,[Kx''(t_0) + f(t_0, x(t_0), x'(t_0), x''(t_0))] \ge 0.$$

The contradiction obtained yields the assertion. □

Lemma 2.2. *Let the condition* $(\mathbf{A}_2)$ *hold for some* F *and* L *and* $x(t) \in C^2[0,1]$ *be a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_2)$*). Suppose there is an interval* $T_1 \subseteq [0,1]$ *such that*

$$L_1^-(t) \le x'(t) \le L_2^-(t) \quad for \quad t \in T_1.$$

Then $x''(t) \le 0$ *for* $t \in T_1$. *If there is an interval* $T_2 \subseteq [0,1]$ *such that* $F_2^+(t) \le x'(t) \le F_1^+(t)$ *for* $t \in T_2$, *then* $x''(t) \ge 0$ *for* $t \in T_2$.

Proof. The proof is the same as for Lemma 2.1 except for a few inessential changes in the details. □

Denote

$$\max u(t) := \max_{[0,1]} u(t), \quad \min u(t) := \min_{[0,1]} u(t), \quad \text{and} \quad \|u\|_0 := \max|u(t)|.$$

Let M, $Q \in R^+$ be some constants, and $L(t), F(t) \in C[0,1]$ be some functions such that $L(t) \ge F(t)$ on $[0,1]$. Let the functions $G_i^-(t), G_i^+(t), H_i^-(t), H_i^+(t) \in C[0,1], i = 1,2$, be such that for

$$C = \max\{\|F\|_0, \|L\|_0\} \tag{2.2}$$

we have

$$\begin{cases} G_1^+(t) \ge 2C, G_1^-(t) \ge 2C \text{ for } t \in [0,1], \\ H_1^+(t) \le -2C, H_1^-(t) \le -2C \text{ for } t \in [0,1], \\ G_1^+(t) \text{ and } H_1^+(t) \text{ are nonincreasing on } [0,1], \\ G_1^-(t) \text{ and } H_1^-(t) \text{ are nondecreasing on } [0,1], \\ G_2^+(t) > G_1^+(t), G_2^-(t) > G_1^-(t), \text{ for } t \in [0,1], \\ H_1^+(t) > H_2^+(t), H_1^-(t) > H_2^-(t) \text{ for } t \in [0,1]. \end{cases} \tag{2.3}$$

Replace

$$Y_1 := \Big\{ (t,x,p,q) \;:\; |x| \le M + \varepsilon,\ t \in [0,1],\ p \in [F(t) - \varepsilon,\ L(t) + \varepsilon] \quad \text{and}$$

$$q \in \Big[\min\Big\{\min H_2^+(t), \min H_2^-(t)\Big\} - \varepsilon,\ \max\Big\{\max G_2^+(t), \max G_2^-(t)\Big\} + \varepsilon\Big]\Big\},$$

where

$$\begin{cases} \varepsilon > 0 \text{ is small enough and such that } H_1^+(t) > H_2^+(t) + \varepsilon, \\ H_1^-(t) > H_2^-(t) + \varepsilon,\ G_1^+(t) > G_2^+(t) + \varepsilon,\ G_1^-(t) > G_2^-(t) + \varepsilon, \end{cases} \tag{2.4}$$

$$Y_2 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [\min\{H_2^+(t), H_2^-(t)\},\ \max\{G_2^+(t), G_2^-(t)\}] \Big\}$$

$$Y_3 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [H_2^+(t), H_1^+(t)] \cup [G_1^+(t), G_2^+(t)] \Big\}$$

$$Y_4 := \Big\{ (t,x,p,q) : x \in [-M,M], \text{ and } (t,p,q) \text{ is such that}$$

$$t \in [0,1],\ p \in [F(t),L(t)],\ q \in [H_2^-(t), H_1^-(t)] \cup [G_1^-(t), G_2^-(t)] \Big\}$$

$$Y_5 := \Big\{ (\lambda,t,x,p) : \lambda \in [0,1],\ x \in [-Q,Q],\ t \in [0,1],\ p \in [F(t),L(t)] \Big\}.$$

We will say that **(B)** holds for the functions $L(t), F(t) \in C[0,1]$ and the constant $M \in R^+$ if:

(B) There are functions $G_i^-(t)$, $G_i^+(t)$, $H_i^-(t)$, $H_i^+(t) \in C[0,1]$, $(i = 1,2)$, which satisfy (2.3) and are such that

$$\begin{cases} f(t,x,p,q) \text{ and } f_q(t,x,p,q) \text{ are continuous on } Y_1 \\ \qquad \text{and } f_q(t,x,p,q) < 0 \quad \text{on } Y_1, \end{cases} \tag{2.5}$$

$$f_t(t,x,p,q),\ f_x(t,x,p,q) \text{ and } f_q(t,x,p,q) \text{ are continuous on } Y_2,$$

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \geq 0 \text{ on } Y_3,$$

and

$$f_t(t,x,p,q) + f_x(t,x,p,q)p + f_p(t,x,p,q)q \leq 0 \text{ on } Y_4.$$

We will say that **(C)** holds for the functions $L(t), F(t) \in C[0,1]$ and for the constants $Q \in R^+$, Q_1, Q_2 if:

(C) $$F(\lambda,t,x,p,Q_1)F(\lambda,t,x,p,Q_2) \leq 0 \text{ for } (\lambda,t,x,p) \in Y_5,$$

where $F(\lambda,t,x,p,q) = (\lambda-1)Kq + \lambda f(t,x,p,q)$, and K is the constant from $(1.1)_\lambda$.

3. TOPOLOGICAL PRELIMINARIES

For the sake of completeness, we give the topological transversality theorem which will be used later; moreover, we follow [8].

Let X be a metric space, and Y be a convex subset of a Banach space E. The continuous map $F : X \to Y$ is called compact if $F(X)$ is a compact subset of Y. The continuous map $F : X \to Y$ is completely continuous if it maps bounded subsets in X into compact subsets of Y.

Theorem 3.1 (Shauder's fixed point theorem). *Let Y be a convex subset of E, and $F : Y \to Y$ be a compact map. Then there exists a point $x_0 \in Y$ such that $F(x_0) = x_0$.*

We say that the homotopy $\{H_\lambda : X \to Y\}$, $0 \leq \lambda \leq 1$, is compact if the map $H(x, \lambda) : X \times [0,1] \to Y$ given by $H(x,\lambda) \equiv H_\lambda(x)$ for $(x, \lambda) \in X \times [0,1]$ is compact.

Let $U \subset Y$ be open in Y, ∂U be the boundary of$U$ in Y, and $\overline{U} = \partial U \cup U$. The compact map $F : \overline{U} \to Y$ is called admissible if it is fixed point free on ∂U. We denote the set of all such maps by $\mathbf{L}_{\partial U}(\overline{U}, Y)$.

Definition 3.1. The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if there is a fixed point free compact map $G : \overline{U} \to Y$ such that $G|\partial U = F|\partial U$. The map F in $\mathbf{L}_{\partial U}(\overline{U}, Y)$, which is not inessential, is called essential.

Theorem 3.2. *Let $p \in U$ be arbitrary and $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ be the constant map $F(x) = p$ for $x \in \overline{U}$. Then F is essential.*

Proof. Let $G : \overline{U} \to Y$ be a compact map such that $G|\partial U = F|\partial U$. Define the map $H : Y \to Y$ by

$$H(x) = p \quad \text{for} \quad x \in Y \backslash \overline{U},$$

$$H(x) = G(x) \quad \text{for} \quad x \in \overline{U}.$$

Clearly, $H : Y \to Y$ is a compact map. By Shauder's theorem H has a fixed point $x_0 \in Y$, i. e. $H(x_0) = x_0$. By definition of H we have $x_0 \in U$. Thus, $G(x_0) = x_0$ since H equals G on U. So every compact map from $\overline{U}$ into Y, which agrees with F on ∂U, has a fixed point. That is, F is essential. □

Definition 3.2. The maps $F, G \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ are called homotopic ($F \sim G$) if there is a compact homotopy $H_\lambda : \overline{U} \to Y$ such that H_λ is admissible for each $\lambda \in [0,1]$ and $G = H_0$, $F = H_1$.

Lemma 3.1. *The map $F \in \mathbf{L}_{\partial U}(\overline{U}, Y)$ is inessential if and only if it is homotopic to a fixed point free map.*

Proof. Let F be inessential and $G : \overline{U} \to Y$ be a compact fixed point free map such that $G|\partial U = F|\partial U$. Then the homotopy $H_\lambda : \overline{U} \to Y$, defined by

$$H_\lambda(x) = \lambda F(x) + (1 - \lambda) G(x), \quad \lambda \in [0,1],$$

is compact, admissible and such that $G = H_0$, $F = H_1$.

Now let $H_0 : \overline{U} \to Y$ be a compact fixed point free map, and $H_\lambda : \overline{U} \to Y$ be an admissible homotopy joining H_0 and F. To show that $H_\lambda, \lambda \in [0,1]$, is an inessential map, consider the map $H : \overline{U} \times [0,1] \to Y$ such that $H(x,\lambda) \equiv H_\lambda(x)$ for each $x \in \overline{U}$ and $\lambda \in [0,1]$ and define the set $B \subset \overline{U}$ by

$$B = \{x \in \overline{U} : H_\lambda(x) \equiv H(x,\lambda) = x \quad \text{for some} \quad \lambda \in [0,1]\}.$$

If B is empty, then $H_1 = F$ has no fixed point which means that F is inessential. So we may assume that B is non-empty. In addition, B is closed and such that $B \cap \partial U = \varnothing$ since $H_\lambda, \lambda \in [0,1]$, is an admissible map. Now consider the Urysohn function $\theta : \overline{U} \to [0,1]$ with

$$\theta(x) = 1 \text{ for } x \in \partial U \quad \text{and} \quad \theta(x) = 0 \text{ for } x \in B$$

and define the homotopy $H^*_\lambda : \overline{U} \to Y, \lambda \in [0,1]$, by

$$H^*_\lambda = H(x, \theta(x)\lambda) \quad \text{for} \quad (x,\lambda) \in \overline{U} \times [0,1].$$

It is easy to see that $H^*_\lambda : \overline{U} \to Y$ is inessential. In particular, $H_1 = F$ is inessential, too. The proof is completed. □

As a consequence of Lemma 3.1 we have:

Theorem 3.3 (Topological transversality theorem). *Let $F, G \in \mathrm{L}_{\partial U}(\overline{U}, Y)$ be homotopics maps. Then one of these maps is essential if and only if the other one is.*

Theorem 3.3 is used in the following equivalent form:

Theorem 3.4 (Topological transversality theorem). *Let Y be a convex subset of a Banach space E, and $U \subset Y$ be open. Suppose:*

(i) *$F, G : \overline{U} \to Y$ are compact maps;*
(ii) *$G \in \mathrm{L}_{\partial U}(\overline{U}, Y)$ is essential;*
(iii) *$H_\lambda(x)$, $\lambda \in [0,1]$, is a compact homotopy joining F and G, i. e. $H_0(x) = G(x)$, $H_1(x) = F(x)$;*
(iv) *$H_\lambda(x)$, $\lambda \in [0,1]$, is a fixed point free on ∂U.*

Then H_λ, $\lambda \in [0,1]$, has at least one fixed point $x_0 \in U$, and, in particular, there is an $x_0 \in U$ such that $x_0 = F(x_0)$.

4. A BASIC EXISTENCE RESULT, ANCILLARY RESULTS

The following theorem is a modification of [8, Chapter II, Theorem 6.1].

Theorem 4.1. *Let $\varphi, \psi : R^4 \to R$ be continuous. Assume there are constants Q, Q_1, Q_2 (independent of λ) and functions $L(t), F(t) \in C[0,1]$ (independent of λ) such that:*

(i) $|x(t)| < Q$, $F(t) < x'(t) < L(t)$, $Q_1 < x''(t) < Q_2$, $t \in [0,1]$, *for each solution $x(t) \in C^2[0,1]$ to $(1.1)_\lambda$ (with fixed $K > 0$) and for $\lambda \in [0,1]$;*

(ii) *$f(t,x,p,q)$ and $f_q(t,x,p,q)$ are continuous, and $f_q(t,x,p,q) < 0$ on*

$$\Big\{(t,x,p,q) : x \in [-Q,Q],\ q \in [Q_1,Q_2],\ t \in [0,1] \text{ and } p \in [F(t),L(t)]\Big\}.$$

(iii) *$F(\lambda,t,x,p,Q_1)F(\lambda,t,x,p,Q_2) \le 0$ for $(\lambda,t,x,p) \in \Lambda =$*

$$\Big\{(\lambda,t,x,p) : \lambda \in [0,1],\ x \in [-Q,Q],\ t \in [0,1] \text{ and } p \in [F(t),L(t)]\Big\};$$

Then the BVP (1.1) has at least one $C^2[0,1]$-solution.

Proof. From (ii) and (iii) it follows that there is an unique function $G(\lambda,t,x,p)$ continuous on Λ and such that

$$q = G(\lambda,t,x,p) \text{ for } (\lambda,t,x,p) \in \Lambda$$

is equivalent to the equation $F(\lambda,t,x,p,q) = 0$ on $\Lambda \times [Q_1,Q_2]$. Thus, the family $(1.1)_\lambda$ is equivalent to the family of BVPs

$$\begin{cases} x'' = G(\lambda,t,x,x'), & t \in [0,1], \\ W_i(x) = \lambda V_i(x), & i = 1,2, \end{cases} \tag{4.1}$$

$\lambda \in [0,1]$. Note that $F \equiv -Kq$ for $\lambda = 0$ and it yields

$$G(0,t,x,p) = 0 \text{for} (t,x,p) \in \Omega, \tag{4.2}$$

where $\Omega = \Big\{(t,x,p) : x \in [-Q,Q], t \in [0,1], \text{ and } p \in [F(t),L(t)]\Big\}$.

Define the map

$$L_1 : C^2[0,1] \to C[0,1] \times R^2 \quad \text{by} \quad L_1 x = \Big(x'',\ W_1(x),\ W_2(x)\Big)$$

and the maps

$$G_\lambda : C^1[0,1] \to C[0,1] \times R^2 \quad \text{by}$$

$$G_\lambda(x) = \Big(G(\lambda,t,x,x'),\ \lambda V_1(x),\ \lambda V_2(x)\Big) \quad \text{for} \quad \lambda \in [0,1].$$

It is easy to see that L_1 is a continuous, linear, one-to-one map of $C^2[0,1]$ onto $C[0,1] \times R^2$. So L_1 has a continuous inverse L_1^{-1}. Finally, define $j : C^2[0,1] \to C^1[0,1]$ by $jx = x$, which is a completely continuous embeding.

Now define the set

$$U = \Big\{x \in C^2[0,1] : \text{for } t \in [0,1],\ |x(t)| < M,\ F(t) < x'(t) < L(t),\ Q_1 < x''(t) < Q_2\Big\}$$

and consider the homotopy

$$H : \overline{U} \times [0,1] \to C^2[0,1] \text{ defined by } H(x,\lambda) \equiv H_\lambda(x) \equiv L_1^{-1} \circ G_\lambda \circ j(x).$$

This homotopy is compact since $j(\overline{U})$ is a compact subset of $C^1[0,1]$, and G_λ, $\lambda \in [0,1]$, and L_1^{-1} are continuous on $j(\overline{U})$ and $G_\lambda\big(j(\overline{U})\big)$, respectively. In addition, the equation

$$L_1^{-1} \circ G_\lambda \circ j(x) = x \quad \text{yields} \quad L_1(x) = G_\lambda(x),$$

which is the BVPs (4.1). Then it follows from **(i)** that $H_\lambda(x)$ is a fixed point free on ∂U, i. e. $H_\lambda(x)$ is an admissible map for all $\lambda \in [0,1]$. Finally, $H_0 = 0$ by (4.2).

So H_0 is an essential map by [8, Chapter I, Theorem 2.2]. Now we are in a position to apply Theorem 3.4. It implies that $H_1 = L_1^{-1} \circ G_1 \circ j$ is essential, too, which means that the original problem (1.1) has a solution in $C^2[0,1]$. □

The next results prepare the application of Theorem 4.1. They guarantee the a priori bounds from (i) of Theorem 4.1.

Lemma 4.1.A. *Let* $(\mathbf{A}_1)$ *hold for some constants* M_1 *and* M_2 *(i. e.* $(\mathbf{A}_1)$ *holds for* $F(t) \equiv M_1$ *and* $L(t) \equiv M_2$, $t \in [0,1]$*). Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_1)$*) such that* $M_1 \le x'(1) \le M_2$. *Then*

$$F_1^-(t) \le x'(t) \le L_1^+(t) \quad \textit{for} \quad t \in [0,1].$$

Proof. Suppose the set

$$S_0 = \{t \in [0,1] : L_1^+(t) < x'(t) \le L_2^+(t)\}$$

or

$$S_1 = \{t \in [0,1] : F_2^-(t) \le x'(t) < F_1^-(t)\}$$

is not empty. The continuity of $x'(t)$ and the inequalities $F_1^-(1) \le x'(1) \le L_1^+(1)$ imply that there are closed intervals

$$[t_0, t_0'] \subseteq S_0 \quad \text{or} \quad [t_1, t_1'] \subseteq S_1$$

such that

$$x'(t_0) > x'(t_0') \quad \text{or} \quad x'(t_1) < x'(t_1'). \tag{4.3}$$

On the other hand, by Lemma 2.1, we have

$$x''(t) \ge 0 \text{for} t \in [t_0, t_0'] \quad \text{or} \quad x''(t) \le 0 \text{ for } t \in [t_1, t_1'].$$

Consequently,

$$x'(t_0) \le x'(t_0') \quad \text{or} \quad x'(t_1) \ge x'(t_1').$$

The contradiction to (4.3) shows that S_0 and S_1 are empty, which yields the lemma. □

Lemma 4.1.B. *Let* $(\mathbf{A}_1)$ *hold for some constants* M_1 *and* M_2 *and* $(\mathbf{B})$ *hold for* $L(t) \equiv L_1^+(t), F(t) \equiv F_1^-(t), t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant (2.2),* N *is some constant, and the functions* $L_1^+(t)$ *and* $F_1^-(t)$ *are from the condition* $(\mathbf{A}_1)$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* K *from* $\mathbf{A}_1$*)) such that*

$$|x(0)| \le N, t \in [0,1], \quad \textit{and} \quad M_1 \le x'(1) \le M_2.$$

Then

$$|x(t)| \le M \quad \textit{for} \quad t \in [0,1] \tag{4.4}$$

and

$$\min\Big\{H_1^+(1),\, H_1^-(0)\Big\} \le x''(t) \le \max\Big\{G_1^+(0), G_1^-(1)\Big\},\ t \in [0,1]. \tag{4.5}$$

Proof. In fact $C = \max\{\|L_1^+\|_0, \|F_1^-\|_0\}$. By the mean value theorem there is $d \in (0,1)$ such that $x''(d) = x'(1) - x'(0)$. Lemma 4.1.A implies

$$F_1^-(t) \le x'(t) \le L_1^+(t) \quad \text{for} \quad t \in [0,1],$$

i. e. $|x'(t)| \le C$ for $t \in [0,1]$. So

$$x''(d) \le 2C \le G_1^+(t) \quad \text{for} \quad t \in [0,d]. \tag{4.6}$$

On the other hand, for each $t \in (0,d]$ there is $c \in (0,t)$ such that

$$x(t) - x(0) = x'(c)t,$$

which yields

$$|x(t)| \le M \text{ for } t \in [0,d].$$

Now suppose the set

$$S = \{t \in [0,d] : G_1^+(t) < x''(t) \le G_2^+(t)\}$$

is not empty. The continuity of $x''(t)$ and (4.6) imply that there is a closed interval

$$[t_0, t_0'] \subseteq S \quad \text{such that} \quad x''(t_0) > x''(t_0'). \tag{4.7}$$

Since for $t \in [t_0, t_0']$

$$-M \le x(t) \le M, \quad F_1^-(t) \le x'(t) \le L_1^+(t), \quad G_1^+(t) < x''(t) \le G_2^+(t),$$

we have

$$f_q(t, x(t), x'(t), x''(t)) < 0, \ t \in [t_0, t_0'], \tag{4.8}$$

and

$$f_t(t, x(t), x'(t), x''(t)) + f_x(t, x(t), x'(t), x''(t))x' + f_p(t, x(t), x'(t), x''(t))x'' \ge 0$$

for $t \in [t_0, t_0']$. From the differential equation $(1.1)_\lambda$ for $t \in [t_0, t_0']$ we obtain

$$\begin{cases} [K(1-\lambda) - \lambda f_q(t, x(t), x'(t), q_h)]\,[x''(t+h) - x''(t)] \\ = hf_t(P_{1h}) + f_x(P_{2h})[x(t+h) - x(t)] + f_p(P_{3h})[x'(t+h) - x'(t)] \\ \to f_t(P) + f_x(P)x'(t) + f_p(P)x''(t), \end{cases} \tag{4.9}$$

where $(t, x(t), x'(t), x''(t))$ and the points P_{1h}, P_{2h} and P_{3h} tend to P. Because of (4.8) it follows from (4.9) that $x'''(t)$ exists and

$$x''' = \lambda\,(f_t + f_x x' + f_p x'')\,/\,[K(1-\lambda) - \lambda f_q]\,, \tag{4.10}$$

which yields

$$x'''(t) \ge 0 \quad \text{for} \quad t \in [t_0, t_0'].$$

Then

$$x''(t_0) \le x''(t_0'),$$

a contradiction to (4.7). Consequently,

$$x''(t) \le G_1^+(t) \quad \text{for} \quad t \in [0,d].$$

The inequality

$$H_1^-(t) \le x''(t), \quad t \in [0,d],$$

may be obtained in a similar way.

Similarly, the inequalities

$$|x(t)| \leq M \quad \text{and} \quad H_1^+(t) \leq x''(t) \leq G_1^-(t), \quad t \in [d, 1],$$

may be established. □

Lemma 4.2.A. *Let* $(\mathbf{A}_2)$ *hold for some constants* M_3 *and* M_4. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with the constant* K *from* $(\mathbf{A}_2)$*) such that* $M_3 \leq x'(0) \leq M_4$. *Then*

$$F_1^+(t) \leq x'(t) \leq L_1^-(t) \quad \textit{for} \quad t \in [0,1].$$

Proof. The lemma can be obtained by using Lemma 2.2 and following the proof of Lemma 4.2.A. □

Lemma 4.2.B. *Let* $(\mathbf{A}_2)$ *hold for some constants* M_3 *and* M_4, *and* $(\mathbf{B})$ *hold for* $L(t) \equiv L_1^-(t), F(t) \equiv F_1^+(t)$, $t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant* (2.2), N *is some constant, and the functions* $L_1^-(t)$ *and* $F_1^+(t)$ *are from the condition* $(\mathbf{A}_2)$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* K *from* $(\mathbf{A}_2)$*) such that*

$$|x(1)| \leq N, \ t \in [0,1], \quad \textit{and} \quad M_3 \leq x'(0) \leq M_4.$$

Then (4.4) *and* (4.5) *hold with current notations.*

Proof. It is not too different from the proof of Lemma 4.1.B. □

Lemma 4.3.A. *Let* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$, *where* $M_i \in R$, $i = \overline{1,4}$. *Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* $K = \min\{K_1, K_2\}$, *where* K_i *is the "value" of the constant* K *from* $(\mathbf{A}_i)$, $i = 1, 2$*) such that*

$$M_1 \leq x(0) \leq M_2 \quad \textit{and} \quad M_3 \leq x(1) \leq M_4.$$

Then for $t \in [0,1]$

$$\min\left\{F_1^-(0), F_1^+(1)\right\} \leq x'(t) \leq \max\left\{L_1^+(0), L_1^-(1)\right\}.$$

Proof. There is $d \in (0,1)$ such that

$$x'(d) = x(1) - x(0) = \lambda[\psi(x(0), x'(0), x(1), x'(1)) - \varphi(x(0), x'(0), x(1), x'(1))],$$

from where it follows

$$\min\{0, M_3 - M_2\} \leq \lambda(M_3 - M_2) \leq x'(d)$$

and

$$x'(d) \leq \lambda(M_4 - M_1) \leq \max\{0, M_4 - M_1\}.$$

For $t \in [0, d]$ we have

$$F_1^-(t) \leq x'(t) \leq L_1^+(t), \quad \text{by Lemma 4.1.A},$$

and for $t \in [d,1]$ we have

$$F_1^+(t) \le x'(t) \le L_1^-(t), \quad \text{by Lemma 4.2.A,}$$

and the assertion follows. □

Lemma 4.3.B. *Let* $(\mathbf{A}_1)$ *and* $\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$*, where* $M_i \in R$*,* $i = \overline{1,4}$*, and* $(\mathbf{B})$ *hold for*

$$L(t) \equiv \max\left\{L_1^+(0), L_1^-(1)\right\}, \qquad F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\},$$

and $M = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\}$*, where* C *is the constant* (2.2)*. Suppose* $x(t) \in C^2[0,1]$ *is a solution to* $(1.1)_\lambda$ *(with* $K = \min\{K_1, K_2\}$*, where* K_i *is the "value" of the constant* K *from* $(\mathbf{A}_i)$*,* $i = 1,2)$ *such that*

$$M_1 \le x(0) \le M_2 \quad \textit{and} \quad M_3 \le x(1) \le M_4.$$

Then (4.4) *and* (4.5) *hold with a current notations.*

Proof. It is not too different from the proof of Lemma 4.1.B. □

5. EXISTENCE RESULTS

Theorem 5.1. *Let* $\varphi, \psi : R^4 \to R$ *be continuous. Suppose there are constants* M_i*,* $i = 1,2$*, and* N *such that:*

(i) $M_1 \le \psi(s_1, s_2, s_3, s_4) \le M_2$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_1)$ *holds for* M_1 *and* M_2;

(iii) $|\varphi(s_1, s_2, s_3, s_4)| \le N$ *for* $(s_1, s_2, s_3, s_4) \in R \times \left[F_1^-(0), L_1^+(0)\right] \times R \times [M_1, M_2]$;

(iv) $(\mathbf{B})$ *holds for* $L(t) \equiv L_1^+(t)$*,* $F(t) \equiv F_1^-(t)$*,* $t \in [0,1]$*, and* $M = C + N$*, where* C *is the constant* (2.2);

(v) $(\mathbf{C})$ *holds for* $L(t) \equiv L_1^+(t) + \varepsilon$*,* $F(t) \equiv F_1^-(t) - \varepsilon$*,* $t \in [0,1]$*, for*

$$Q = C+N+\varepsilon,\ Q_1 = \min\Big\{H_1^+(1), H_1^-(0)\Big\} - \varepsilon,\ Q_2 = \max\Big\{G_1^+(0), G_1^-(1)\Big\} + \varepsilon,$$

where C *is the constant* (2.2)*, and* ε *satisfies* (2.4).

Then the mixed BVP (M_1) *has a* $C^2[0,1]$*-solution.*

Proof. Let $x(t) \in C^2[0,1]$ be a solution to $(\mathrm{M}_1)_\lambda$. Then

$$F_1^-(t) - \varepsilon < x'(t) < L_1^+(t) + \varepsilon \quad \text{for} \quad t \in [0,1], \text{by Lemma 4.1.A,}$$

and Lemma 4.1.B yields the bounds

$$|x(t)| < Q \quad \text{for} \quad t \in [0,1],$$

$$Q_1 < x''(t) < Q_2 \quad \text{for} \quad t \in [0,1].$$

Then the condition (i) of Theorem 4.1 holds. From (2.5) it follows that the condition (ii) of Theorem 4.1 holds. Finally, (v) implies that the condition (iii) of Theorem

4.1 holds. So we can apply Theorem 4.1 to conclude that the problem (M_1) has a solution in $C^2[0,1]$. □

Theorem 5.2. *Let $\varphi, \psi : R^4 \to R$ be continuous. Suppose there are constant M_i, $i = 3, 4$, and N such that:*

(i) $M_3 \le \varphi(s_1, s_2, s_3, s_4) \le M_4$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_2)$ *holds for* M_3 *and* M_4;

(iii) $|\psi(s_1, s_2, s_3, s_4)| \le N$ *for* $(s_1, s_2, s_3, s_4) \in R \times [M_3, M_4] \times R \times [F_1^+(1), L_1^-(1)]$;

(iv) $(\mathbf{B})$ *holds for* $L(t) \equiv L_1^-(t)$, $F(t) \equiv F_1^+(t)$, $t \in [0,1]$, *and* $M = C + N$, *where* C *is the constant* (2.2);

(v) $(\mathbf{C})$ *holds for* $L(t) \equiv L_1^-(t) + \varepsilon$, $F(t) \equiv F_1^+(t) - \varepsilon$, $t \in [0,1]$, *for*

$$Q = C + N + \varepsilon,\ Q_1 = \min\left\{H_1^+(1), H_1^-(0)\right\} - \varepsilon,\ Q_2 = \max\left\{G_1^+(0), G_1^-(1)\right\} + \varepsilon,$$

where C *is the constant* (2.2), *and* ε *satisfies* (2.4).

Then the mixed BVP (M_2) *has a* $C^2[0,1]$*-solution.*

Proof. It is not too different from the proof of Theorem 5.1. Consider $(M_2)_\lambda$. Now Lemma 4.2.A guarantees the a priori bound for x', and Lemma 4.2.B guarantees the a priori bounds for x and x''. □

Theorem 5.3. *Let $\varphi, \psi : R^4 \to R$ be continuous. Suppose there are constants M_i, $i = \overline{1,4}$, such that:*

(i) $M_1 \le \varphi(s_1, s_2, s_3, s_4) \le M_2$ *and* $M_3 \le \psi(s_1, s_2, s_3, s_4) \le M_4$ *for* $(s_1, s_2, s_3, s_4) \in R_4$;

(ii) $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *hold for* $F = \min\{0, M_3 - M_2\}$ *and* $L = \max\{0, M_4 - M_1\}$;

(iii) $(\mathbf{B})$ *holds for* $L(t) \equiv \max\left\{L_1^+(0), L_1^-(1)\right\}$, $F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\}$ *and* $M = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\}$, *where* C *is the constant* (2.2);

(iv) $(\mathbf{C})$ *holds for the functions*

$$L(t) \equiv \max\left\{L_1^+(0), L_1^-(1)\right\} + \varepsilon, \quad F(t) \equiv \min\left\{F_1^-(0), F_1^+(1)\right\} - \varepsilon,$$

$$Q = C + \max\{|M_1|, |M_2|, |M_3|, |M_4|\} + \varepsilon, \quad Q_1 = \min\left\{H_1^+(1), H_1^-(0)\right\} - \varepsilon,$$

$$Q_2 = \max\left\{G_1^+(0), G_1^-(1)\right\} + \varepsilon,$$

where C *is the constant* (2.2), *and* ε *satisfies* (2.4).

Then the Dirichlet BVP (D) *has a* $C^2[0,1]$*-solution.*

Proof. It is not too different from the proof of Theorem 5.1. Now consider the family $(D)_\lambda$. The a priori bound for x' follows by Lemma 4.3.A, and the a priori bounds for x and x'' follow by Lemma 4.3.B. □

Example 6.1. Consider the boundary value problem

$$-(2-t)x'' - tx''^3 + \sin(x' - 0.2) = 0, \quad t \in [0,1],$$
$$x(0) = 0, \quad x'(1) = 0.15.$$

For $L = F = 0.15$ $(\mathbf{A}_1)$ holds. Moreover, we can choose

$$L_1^+(t) = 0.25,\ L_2^+(t) = 0.3,\ F_1^-(t) = 0.1,\ F_2^-(t) = 0.05,\ t \in [0,1],$$

and K is sufficiently small; to say $K = 10^{-10}$. It is easy to see that $f_q = -(2-t)q - 3tq^2 < 0$ for $t \in [0,1]$ and each q. This fact allows us to conclude that $(\mathbf{B})$ holds for $L(t) = 0.25$, $F(t) = 0.1$, $t \in [0,1]$, and $M = 0.25$. Moreover, we can choose

$$G_1^+(t) = 0.9,\ G_2^+(t) = 1,\ G_1^-(t) = 2,\ G_2^-(t) = 3,$$
$$H_1^-(t) = -0.9,\ H_2^-(t) = -1,\ H_1^+(t) = -2,\ H_2^+(t) = -3,\ t \in [0,1].$$

Finally, from

$$2.01(\lambda - 1)K + \lambda\left[-2.01(2-t) - (2.01)^3 t + \sin(p - 0.2)\right] \leq 0$$

and

$$-2.01(\lambda - 1)K + \lambda\left[-(-2.01)(2-t) - (-2.01)^3 t + \sin(p - 0.2)\right] \geq 0$$

for $\lambda, t \in [0,1]$ and each p we conclude that $(\mathbf{C})$ holds for $Q_1 = -2.01$ and $Q_2 = 2.01$. *Thus the problem considered has a* $C^2[0,1]$*-solution by Theorem 5.1.*

Example 6.2. Consider the boundary value problem

$$x'^2 - 4 - 50(2-t)x'' - tx''^5 = 0, \quad t \in [0,1],$$
$$x(0) = [x^2(0) + x'^2(0) + x^2(1) + x'^2(1) + 1]^{-1}, \quad x(1) = \sin^2 x'(1).$$

For $L = 1$ and $F = -1$ $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ hold. Moreover, we can choose

$$L_1^+(t) = 2.1,\ L_2^+(t) = 2.2,\ F_1^-(t) = -1.1,\ F_2^-(t) = -1.2,$$
$$L_1^-(t) = 1.1,\ L_2^-(t) = 1.2,\ F_1^+(t) = -2.1,\ F_2^+(t) = -2.2,\ t \in [0,1],$$

and K is sufficiently small; to say $K = 10^{-10}$. It is easy to see that $f_q = -50(2-t) - 5q^4 t < 0$ for $t \in [0,1]$ and each q. Thus $(\mathbf{B})$ holds for $L(t) = 2.1$, $F(t) = -2.1$, $t \in [0,1]$ and, $M = 3.1$. Moreover, we can choose

$$G_1^+(t) = 6.5,\ G_2^+(t) = 6.6,\ G_1^-(t) = 10,\ G_2^-(t) = 11,$$
$$H_1^-(t) = -6.5,\ H_2^-(t) = -6.6,\ H_1^+(t) = -10,\ H_2^+(t) = -11,\ t \in [0,1].$$

Finally, from

$$10.01(\lambda - 1)K + \lambda\left[p^2 - 4 - 50(2-t)10.01 - (10.01)^5 t\right] \leq 0$$

and

$$-10.01(\lambda - 1)K + \lambda\left[p^2 - 4 - 50(2-t)(-10.01) - (-10.01)^5 t\right] \geq 0$$

for $\lambda, t \in [0,1]$ and $p \in [-3.11, 3.11]$ we conclude that (**C**) holds for $L(t) = 2.11$, $F(t) = -2.11$, $t \in [0,1]$, $Q = 3.11$, $Q_1 = -10.01$ and $Q_2 = 10.01$; $\varepsilon = 0.01$. Thus the problem considered has a $C^2[0,1]$-solution by Theorem 5.3.

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REFERENCES

1. Erbe, L. H. Existence of solutions to boundary value problems for second order differential equations. *Nonlinear Analysis*, **6**, 1982, 1155–1162.
2. Vasilev, N., I. Klokov. Bases of the Theory of Boundary Value Problems for Ordinary Differential Equations. Zinatne, Riga, 1978 (in Russian).
3. Rachuncova, I. Topological degree methods in BVPs with nonlinear condiions. *Ann. Math. Sil.*, **10**, 1996, 103–111.
4. Jackson, L. K., P. K. Palamides. An existence theorem for a nonlinear two point boundary value problem. *J. Differential Equations*, **53**, 1984, 48–66.
5. Baxly, J. V., S. E. Brown. Existence and uniqueness for two-point boundary value problems. *Proc. Roy. Soc. Edinburg*, **88A**, 1981, 219–234.
6. Fabry, C., P. Habets. Upper and lower solutions for second order boundary value problems with nonlinear boundary conditions. In: Seminaire de mathematique, Louvain-La-Neuve, Belgique, 1984.
7. Thompson, H. B. Second order ordinary differential equations with fully nonlinear two-point boundary conditions. II. *Pacific J. Math.*, **172**, 1996, 279–297.
8. Granas, A., R. B. Guenther, J. W. Lee. Nonlinear boundary value problems for ordinary differential equations. Dissnes Math., Warszawa, 1985.
9. Mao, Y., J. Lee. Two point boundary value problems for nolinear differential equations. *Rocky Mount. J. Math.*, **10**, 1980, 35–58.
10. Petryshyn, W. V., Z. S. Yu. Periodic solutions of nonlinear second-order differential equations which are not solvable for the highest-order derivative. *J. Math. Anal. Appl.*, **89**, 1982, 462–488.
11. Petryshyn, W. V., Z. S. Yu. Solvability of Neumann BV problems for nonlinear second order ODE's which need not be solvable for the highest order derivative. *J. Math. Anal. Appl.*, **91**, 1983, 244–253.
12. Fitzpatrick, P. M., W. V. Petryshyn. Galerkin method in the constructive solvability of nonlinear Hammerstein equations with applications to differential equations. *Trans. Amer. Math. Soc.*, **238**, 1978, 321–340.
13. Fitzpatrick, P. M. Existence results for equations involving noncompact perturbation of Fredholm mappings with applications to differential equations. *J. Math. Anal. Appl.*, **66**, 1978, 151–177.
14. Petryshyn, W. V. Solvability of various boundary value problems for the equation $x'' = f(t, x, x', x'') - y$. *Pacific J. Math.*, **122**, 1986, 169–195.
15. Petryshyn, W. V. Approximation-solvability of nonlinear functional and differential equations. Marcel Dekker, Inc., New York, 1993.
16. Tineo, A. Existence of solutions for a class of boundary value problems for the equation $x'' = F(t, x, x', x'')$. *Comment. Math. Univ. Carolin*, **29**, 1988, 285–291.

17. Marano, S. A. On a boundary value problem for the differential equation $f(t, x, x', x'') = 0$. *J. Math. Anal. Appl.*, **182**, 1994, 309–319.
18. Ricceri, B. On the Cauchy problem for the differential equation $f(t, x, x', \dots, x^{(k)}) = 0$. *Glasgow Math. J.*, **33**, 1991, 343–348.
19. Mao, Y., J. Lee. Two point boundary value problems for nonlinear differential equations. *Rocky Maunt. J. Math.*, **26**, 1996, 1499–1515.
20. Kelevedjiev, P. Existence results for singular inilial value problems for first order differential equations. *Nonlinear Analysis*, **39**, 2000, 65–77.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

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GALERKIN SPECTRAL METHOD FOR HIGHER-ORDER BOUNDARY VALUE PROBLEMS ARISING IN THERMAL CONVECTION

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In the present work we develop a Galerkin spectral technique for solving coupled higher-order boundary value problems arising in continuum mechanics. The set of the so-called beam functions are used as a basis together with the harmonic functions. As featuring examples we solve two fourth-order boundary value problems related to the convective flow of viscous liquid in a vertical slot and a coupled convective problem. We show that the rate of convergence of the series is fifth-order algebraic both for linear and nonlinear problems of fourth order. The coupled problem exhibits fourth- and fifth-order convergence for the different unknown functions. Though algebraic, the fourth order rate of convergence is fully adequate for the generic problems under consideration, which makes the new technique a useful tool in numerical approaches to convective problems.

Keywords: spectral methods, beam functions, natural convection

MSC 2000: 37L65, 74S25, 76M22, 76E06, 76R10

1. INTRODUCTION

Fourth-order boundary value problems are the standard model in continuum mechanics arising both in elasticity and in viscous liquid dynamics. The simplified 1D models are respectively the beam equations and Poiseuille flow. The method developed here can be applied to both elasticity and fluid dynamics. For the sake of definiteness we will focus our attention on thermal convection in a vertical slot, which is a generalization of the Poiseuille flow.

There is a compelling need to develop fast spectral methods that will lead to more efficient algorithms. Such algorithms would allow a rapid interrogation of parameter space in order to discover and understand mechanisms of flow and instability. The performance of a spectral method depends heavily on the type of the basis system. Naturally, a basis system of functions which does not satisfy all of the boundary conditions, such as Fourier functions, would exhibit very poor convergence near the boundaries, where the solution is supposed to satisfy four boundary conditions. An elucidating discussion on the performance of different set of functions can be found in the encyclopedic book of Boyd [2]. In the present work we embark on developing spectral techniques involving the so-called *beam functions* introduced first by Lord Rayleigh, see [10]. Along these lines we will investigate also in a future work the performance of Galerkin techniques with a basis derived from Chebyshev polynomials — something that goes, however, beyond the scope of the present work.

The application of the beam-Galerkin method to Poiseuille flow is at present well developed, see [9, 4]. We go a step further here and consider the generic boundary value problem for convective flows of viscous liquids. These are rather complex ones, hence geometrically simplified situations are considered in order to identify the physical mechanisms, e.g. straight ducts and/or slots. These mechanisms are often operative in more complicated situations. Even for the simplest geometries with plane parallel flows, the mathematical models are represented by higher-order boundary value problems in one and two dimensions and analytical solutions are not available. In the same time the parametric space of physical interest and significance is enormous (4–5 dimensionless parameters to vary). The Rayleigh number and modulation frequency can take on very high values, signaling the occurrence of boundary or internal layers of steep profiles of the field variables. This makes the development of effective numerical approaches a must.

2. THERMAL CONVECTION IN A VERTICAL SLOT

Consider the 2D flow in a vertical slot with a linear vertical temperature gradient, differentially heated walls, and subject to modulation of gravity in the vertical direction. The problem definition is well-described in the literature (refer to [1, 6] and Fig. 1 for a definition sketch), and the notation we use is standard:

$$x = \frac{x^*}{L} - 1, \quad y = \frac{y^*}{L}, \quad \omega = \omega^* \frac{L^2}{\kappa},$$
$$t = t^* \omega^*, \quad \psi = \frac{\psi^*}{\nu}, \quad \theta = \frac{T^*}{\delta T} + x - \tau_B y,$$

where ν is the kinematic viscosity, κ — the thermal diffusivity, $2L$ — the width of the slot, and δT — the horizontal temperature difference. The asterisk denotes dimensional variables, while the same notation without an asterisk stands for the respective dimensionless quantity. Note that the field $\theta(x, y, t)$ is the departure from the linear vertical and horizontal stratification. Hence one can seek solutions which are periodic in the vertical dimension.

The dimensionless boundary value problem under consideration reads

$$\frac{1}{Pr}\left(\omega\frac{\partial\Delta\psi}{\partial t}+\frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}\right)$$
$$= - Ra\left(\frac{\partial\theta}{\partial x}-1\right)[1+\varepsilon\cos(t)]+\Delta^2\psi, \tag{1}$$

$$\omega\frac{\partial\theta}{\partial t}+\left(\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right)=\frac{\partial\psi}{\partial y}+\tau_B\frac{\partial\psi}{\partial x}+\Delta\theta \tag{2}$$

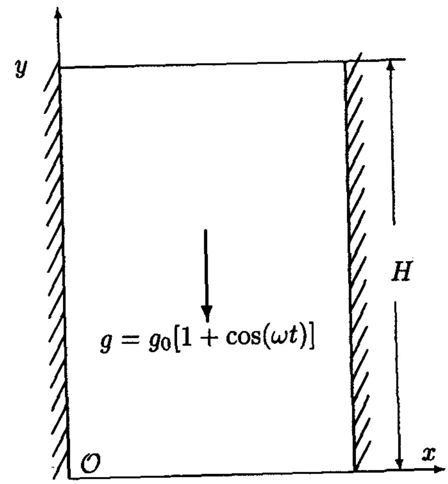

Fig. 1. Flow geometry

with boundary conditions

$$\psi=\frac{\partial\psi}{\partial x}=\theta=0 \quad \text{for} \quad x=\pm1\,, \tag{3}$$

and periodic conditions in vertical direction

$$\begin{aligned}
\psi(x,0,t) &= \psi(x,H,t),\\
\psi_y(x,0,t) &= \psi_y(x,H,t),\\
\psi_{yy}(x,0,t) &= \psi_{yy}(x,H,t),\\
\psi_{yyy}(x,0,t) &= \psi_{yyy}(x,H,t),\\
\theta(x,0,t) &= \theta(x,H,t),\\
\theta_y(x,0,t) &= \theta_y(x,H,t),
\end{aligned} \tag{4}$$

where $H = H^*/L = 2\pi/\alpha$ is the dimensionless height of the vertical box: equivalently, α is the dimensionless vertical wave number of the periodic solutions.

The Rayleigh number Ra, the Prandtl number Pr, and stratifications parameter, γ, are defined as:

$$Ra = \frac{\beta g_0 \delta T L^3}{\nu\kappa}, \quad Pr = \frac{\nu}{\kappa}, \quad 4\gamma^4 = \tau_B Ra,$$

where β is the coefficient of thermal expansion of the liquid, g_0 — the mean gravity, ε — the dimensionless amplitude of gravity modulations, ω — the dimensionless frequency, and τ_B is the dimensionless vertical temperature gradient. Using a difference approximation and an operator splitting, the 2D flow is investigated numerically in [5]. We focus our attention on the 1D case for the purposes of developing the new numerical technique.

Under the selected boundary conditions the problem also admits a plane--parallel solution of the form $\Psi(x,t)$, $\Theta(x,t)$ for which the governing system reduces to the following:

$$\frac{\omega}{Pr}\frac{\partial^3\Psi}{\partial t\partial x^2} = -Ra\left[1+\frac{\partial\Theta}{\partial x}\right][1+\varepsilon\cos(t)] + \frac{\partial^4\Psi}{\partial x^4}, \tag{5}$$

$$\omega\frac{\partial\Theta}{\partial t} = \tau_B\frac{\partial\Psi}{\partial x} + \frac{\partial^2\Theta}{\partial x^2}, \tag{6}$$

with the same boundary conditions (3).

The 1D flow was first treated in [6], where different régimes of flow were studied. The parametric bifurcation of the 1D solutions was studied in detail in [5] by means of a fully implicit difference scheme and a related 1D problem in [12].

A way out of these difficulties is to use spectral decomposition with respect to complete orthonormal (CON) systems in x-direction. The performance of a spectral method depends heavily on the type of the basis system of functions. The scope of this paper is to implement these ideas for the one-dimensional in space and time-dependent problem (5), (6), (3).

In order to assess the approximation, convergence rate and truncation error, it is enough to consider a model ODE which contains all of the different terms of the time dependent system. A simplified first step is to consider just one ODE of fourth order and to compile the rest of the technique.

To this end we consider the following three boundary value problems (b.v.p.):

1. B.v.p. containing both fourth and second-order derivatives:

$$\frac{d^4u}{dx^4} + 2\frac{d^2u}{dx^2} + u = 1, \quad u(-1) = u(1) = 0, \quad u'(-1) = u'(1) = 0, \tag{7}$$

which possesses an analytical solution:

$$u(x) = 1 - \frac{2\cos x[\cos 1 + \sin 1] - 2x\sin 1\sin x}{2+\sin 2}. \tag{8}$$

2. A nonlinear version of the above b.v.p.:

$$\frac{d^4u}{dx^4}+2\frac{d^2u}{dx^2}+u=1-100u^2(x),$$
$$u(-1)=u(1)=0, \quad u'(-1)=u'(1)=0, \tag{9}$$

where the large coefficient 100, multiplying the nonlinear term, is selected for the sake of making the nonlinearity more appreciable.

3. The higher-order coupled b.v.p. for an ODE system, which retains all of the important terms in the full-fledged unsteady problem for the thermal convection in a vertical slot:

$$\frac{d^4\Psi}{dx^4}=Ra\left[-1+\frac{d\Theta}{dx}\right]+\frac{1}{Pr}\frac{\partial^2\Psi}{\partial x^2},$$
$$\Theta-\frac{d\Psi}{dx}=\frac{d^2\Theta}{dx^2}, \quad \Psi=\Psi_x=\Theta=0, \quad \text{for} \quad x=\pm 1. \tag{10}$$

We find the above system generically representative of the problem under consideration, because it retains the second spatial derivatives. In a sense, it can be considered as a simplification of an Euler time-stepping scheme with time increment equal to one.

3. THE SPECTRAL TECHNIQUE

The expansion in x direction is nontrivial because of the higher-order boundary value problem for the stream function. The right CON system for a fourth-order problem was introduced by Lord Rayleigh for the problem of vibration of elastic beams. For the specific boundary conditions arising in viscous liquid dynamics the system and its completeness were discussed in [3]. The product formulas as well as the expansion formulas for the derivatives of different orders were derived in a preceding authors work [4]. The product formula is essential for the application to a nonlinear problem.

3.1. BEAM FUNCTIONS

Consider the Sturm-Liouville problem

$$\frac{d^4u}{dy^4}=\lambda^4 u\,, \quad u=\frac{du}{dy}=0\,, \quad \text{for} \quad x=\pm 1\,. \tag{11}$$

The nontrivial solutions (eigen-functions) of this problem are given by

$$s_m=\frac{1}{\sqrt{2}}\left[\frac{\sinh\lambda_m x}{\sinh\lambda_m}-\frac{\sin\lambda_m x}{\sin\lambda_m}\right], \qquad \operatorname{cotanh}\lambda_m-\operatorname{cotan}\lambda_m=0\,, \tag{12}$$

$$c_m=\frac{1}{\sqrt{2}}\left[\frac{\cosh\kappa_m x}{\cosh\kappa_m}-\frac{\cos\kappa_m x}{\cos\kappa_m}\right], \qquad \tanh\kappa_m+\tan\kappa_m=0\,. \tag{13}$$

These functions have been introduced by Lord Rayleigh to solve problems arising in beam theory and they are sometimes called beam functions. A major step in the advancement of the application of the beam functions to fluid-dynamics problems was made by Poots [9]. The magnitudes of the different eigenvalues can be found in most of the above cited works from the literature.

Chandrasekhar [3] derived their counterparts for problems with cylindrical symmetry. For applications to stability problems, see also [7, 11].

The expressions for developing the nonlinear terms into series with respect to the system appeared simultaneously in [8] and [4] though in different form. We stick here to the notations of [4] as more explicit and easier to verify.

3.2. EXPANSIONS FOR THE DERIVATIVES

The different derivatives can be expressed in series with respect to the system as follows:

$$c'_n = \sum_{m=1}^{\infty} a_{nm} s_m , \qquad a_{nm} = \frac{4\kappa_n^2 \lambda_m^2}{\kappa_n^4 - \lambda_m^4} , \tag{14}$$

$$s'_n = \sum_{m=1}^{\infty} \bar{a}_{nm} c_m , \qquad \bar{a}_{nm} = \frac{4\kappa_m^2 \lambda_n^2}{-\kappa_m^4 + \lambda_n^4} , \tag{15}$$

$$c''_n = \sum_{m=1}^{\infty} \beta_{nm} c_m, \quad s''_n = \sum_{m=1}^{\infty} \bar{\beta}_{nm} s_m, \tag{16}$$

$$\beta_{nm} = \begin{cases} \dfrac{4\kappa_n^2 \kappa_m^2}{\kappa_m^4 - \kappa_n^4} \left(\kappa_m \tanh \kappa_m - \kappa_n \tanh \kappa_n\right), & m \neq n, \\ \kappa_n \tanh \kappa_n - \left(\kappa_n \tanh \kappa_n\right)^2 , & m = n, \end{cases} \tag{17}$$

$$\bar{\beta}_{nm} = \begin{cases} \dfrac{4\lambda_n^2 \lambda_m^2}{\lambda_n^4 - \lambda_m^4} \left(\lambda_n \operatorname{cotanh} \lambda_n - \lambda_m \operatorname{cotanh} \lambda_m\right), & m \neq n, \\ \lambda_n \operatorname{cotanh} \lambda_n - \left(\lambda_n \operatorname{cotanh} \lambda_n\right)^2 , & m = n, \end{cases} \tag{18}$$

$$c'''_n = \sum_{m=1}^{\infty} d_{nm} s_m , \qquad d_{nm} = \frac{4\kappa_n^3 \lambda_m^3}{-\kappa_n^4 + \lambda_m^4} \tanh \kappa_n \operatorname{cotanh} \lambda_m, \tag{19}$$

$$s'''_n = \sum_{m=1}^{\infty} \bar{d}_{nm} c_m , \qquad \bar{d}_{nm} = \frac{4\kappa_m^3 \lambda_n^3}{-\kappa_m^4 + \lambda_n^4} \tanh \kappa_m \operatorname{cotanh} \lambda_n. \tag{20}$$

3.3. PRODUCTS OF BEAM FUNCTIONS

The most important for the present work are the product formulae

$$c_n(x)c_m(x) = \sum_{k=1}^{\infty} h_k^{nm} c_k(x), \quad \sqrt{2}h_k^{nm} = \sqrt{2}\int_{-1}^{1} c_n(x)c_m(x)c_k(x)\mathrm{d}x \tag{21}$$

$$= \frac{-(\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)-\kappa_n\tanh\kappa_n}{(\kappa_m+\kappa_k)^2-\kappa_n^2}$$
$$+\frac{-(\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{-(\kappa_m-\kappa_k)^2+\kappa_n^2}$$
$$+\frac{-(\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{(\kappa_m+\kappa_k)^2+\kappa_n^2}$$
$$+\frac{-(\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)+\kappa_n\tanh\kappa_n}{(\kappa_m-\kappa_k)^2+\kappa_n^2}$$
$$+\frac{-(\kappa_n+\kappa_k)(\tanh\kappa_n+\tanh\kappa_k)+\kappa_m\tanh\kappa_m}{(\kappa_n+\kappa_k)^2+\kappa_m^2}$$
$$+\frac{-(\kappa_n-\kappa_k)(\tanh\kappa_n-\tanh\kappa_k)+\kappa_m\tanh\kappa_m}{(\kappa_n-\kappa_k)^2+\kappa_m^2}$$
$$+\frac{-(\kappa_n+\kappa_m)(\tanh\kappa_n+\tanh\kappa_m)+\kappa_k\tanh\kappa_k}{(\kappa_n+\kappa_m)^2+\kappa_k^2}$$
$$+\frac{-(\kappa_n-\kappa_m)(\tanh\kappa_n-\tanh\kappa_m)+\kappa_k\tanh\kappa_k}{(\kappa_n-\kappa_m)^2+\kappa_m^2},$$

$$s_n c_m = \sum_{k=1}^{\infty} f_k^{nm} s_k, \; s_n s_m = \sum_{k=1}^{\infty} f_m^{nk} c_k, \quad \sqrt{2}f_k^{nm} = \sqrt{2}\int_{-1}^{1} s_n c_m s_k \mathrm{d}x \tag{22}$$

$$= \frac{(\lambda_n+\lambda_k)(\coth\lambda_n+\coth\lambda_k)-\kappa_m\tanh\kappa_m}{(\lambda_k+\lambda_n)^2-\kappa_m^2}$$
$$+\frac{-(\lambda_k-\lambda_n)(\coth\lambda_k-\coth\lambda_n)+\kappa_m\tanh\kappa_m}{(\lambda_k-\lambda_n)^2-\kappa_m^2}$$
$$+\frac{-(\lambda_k+\kappa_m)(\coth\lambda_k+\tanh\kappa_m)+\lambda_n\coth\lambda_n}{(\lambda_k+\kappa_m)^2+\lambda_n^2}$$
$$+\frac{-(\lambda_k-\kappa_m)(\coth\lambda_k-\tanh\kappa_m)+\lambda_n\coth\lambda_n}{(\lambda_k-\kappa_m)^2+\lambda_n^2}$$
$$+\frac{-(\lambda_n+\kappa_m)(\coth\lambda_n+\tanh\kappa_m)+\lambda_k\coth\lambda_k}{(\lambda_n+\kappa_m)^2+\lambda_k^2}$$
$$+\frac{-(\lambda_n-\kappa_m)(\coth\lambda_n-\tanh\kappa_m)+\lambda_k\coth\lambda_k}{(\lambda_n-\kappa_m)^2+\lambda_k^2}$$
$$+\frac{-(\lambda_n+\lambda_k)(\coth\lambda_n+\coth\lambda_k)+\kappa_m\tanh\kappa_m}{(\lambda_k+\lambda_n)^2+\kappa_m^2}$$
$$+\frac{(\lambda_k-\lambda_n)(\coth\lambda_n-\coth\lambda_k)+\kappa_m\tanh\kappa_m}{(\lambda_k-\lambda_n)^2+\kappa_m^2}.$$

The most obvious test to verify the correctness and consistency of the above derived formulas for the products is to take the product of some two particular functions c_n and c_m and to compare pointwise the products $c_n c_m$ and $s_n c_m$ with their Galerkin expansions into c_k and s_k, respectively. For the products of even functions this comparison is shown in Fig. 2.

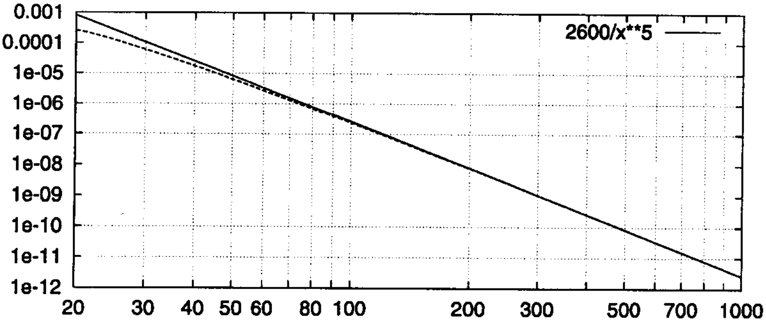

Fig. 2. The convergence of the series for the product $c_6 c_3$. Solid line: h_i^{63}; dashed line: the best fit curve $h_i^{63} = 2600\, i^{-5}$

Our numerical experiments with different products of beam functions invariably led us to the fifth-order convergence

$$f_k^{mn} \sim \hat{f}(m,n)k^{-5}, \qquad h_k^{mn} \sim \hat{h}(m,n)k^{-5}.$$

Thus a conjecture is in order that the fifth order of convergence of the series for a quadratic nonlinear term is a general property of the system of beam functions.

3.4. EXPANSION OF UNITY

We also expanded the unity into a c_m series as follows:

$$1 = \sum_{k=1}^{\infty} h_k c_k(x), \quad h_k = \int_{-1}^{1} c_k(x)\mathrm{d}x = \frac{2\sqrt{2}\tanh \kappa_k}{\kappa_k}. \tag{23}$$

The convergence of this expansion is algebraic of first order. This is due to the fact that the unity does not satisfy the boundary conditions for the beam functions and as a result a strong Gibbs effect is observed near the boundaries.

Yet the overall rate of convergence of the method is fifth-order algebraic, because in the left-hand side of the problems under consideration the fourth power of the respective eigen-value appears as a multiplier.

For the convective problem under consideration the difficulties arise from the fact that the boundary value problem for temperature function is of second order, which means that the system of beam functions is not suitable for expanding the temperature field. It is clear that the best suited to the task system are the trigonometric *sines* and *cosines*. Hence we need to develop expressions for expanding the beam functions into trigonometric functions and vice versa:

$$\sin l\pi x = \sum_{k=1}^{\infty} \sigma_{lk} s_k(x), \qquad \sigma_{lk} = \frac{2\sqrt{2} l\pi (\lambda_k)^2 (-1)^l}{l^4\pi^4 - \lambda_k^4}, \tag{24}$$

$$\cos l\pi x = \sum_{k=1}^{\infty} \chi_{lk} c_k(x), \qquad \chi_{lk} = \frac{2\sqrt{2}\kappa_k^3 (-1)^{l+1} \tanh \kappa_k}{l^4\pi^4 - \kappa_k^4}, \tag{25}$$

$$c_n(x) = \sum_{l=1}^{\infty} \hat{\chi}_{nl} \cos l\pi x, \qquad \hat{\chi}_{nl} = \frac{2\sqrt{2}\kappa_n^3 (-1)^{l+1} \tanh \kappa_n}{l^4\pi^4 - \kappa_n^4}, \tag{26}$$

$$s_n(x) = \sum_{l=1}^{\infty} \hat{\sigma}_{nl} \sin l\pi x, \qquad \hat{\sigma}_{nl} = \frac{2\sqrt{2} l\pi (\lambda_n)^2 (-1)^l}{l^4\pi^4 - \lambda_n^4}. \tag{27}$$

Once again we point out that the convergence when expanding $\cos(l\pi x)$ into c_k series is first order k^{-1} (see (25)) due to the fact that it does not satisfy both b.c. for the beam functions. It satisfies the condition on the derivatives but fails to satisfy the conditions on the function itself. Clearly, the situation with the $\sin(l\pi x)$ is better and the rate of convergence is of second order k^{-2} (see (24)), because the *sine* functions satisfy the boundary conditions on the functions and the disagreement is more subtle since the conditions on the first derivative are not satisfied. The situation with the expansions of s_x and c_k in Fourier series is reversed. The order of convergence for c_k is l^{-4} (see (26)), and for s_x is l^{-3} (see (27)). As it will be shown in what follows, this property is of crucial importance for the overall rate of convergence.

4. THE GALERKIN METHOD

In this section we present the numerical tests and verifications of the Galerkin technique using as featuring examples the three boundary value problems outlined in Section 2.

4.1. SOLVING THE MODEL FOURTH-ORDER PROBLEM

We solve (7) numerically using the developed here beam-Galerkin expansion with respect to the complete orthonormal (CON) system of functions $c_n(x)$, $s_n(x)$. Because of the nature of the boundary conditions, we can constrain ourselves to the subset of even functions c_n (a fact verified also by the analytic solution) and expand the sought function into series with respect to $c_n(x)$:

$$u(x) = \sum_{1}^{N} b_n c_n(x). \tag{28}$$

Making use of the above compiled formulas we obtain for the coefficients b_n the following linear algebraic system of N equations with N unknowns:

$$(1+\kappa_i^4)b_i + 2\sum_{j=1}^{N} b_j \beta_{ij} = \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i}, \qquad i = 1,\ldots,N, \tag{29}$$

with β_{ij} defined in (17).

The last system is solved by means of LAPACK routine *dgesv*.

We found that the coefficients b_i decay with the number of the term i as i^{-5}, which is clearly seen in Fig. 3.

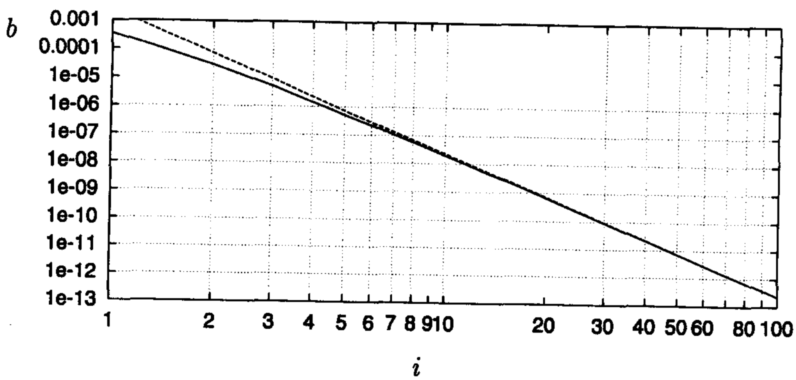

Fig. 3. Convergence of the beam-Galerkin series for the model equation (7). Solid line: b_i; dashed line: the best fit curve $b_i = 0.0023\, i^{-5}$

The obtained spectral solution is compared to the analytical one and the overall truncation error is estimated. As it is to be expected for a series with fifth-order algebraic convergence, the truncation error for $N = 100$ is of order of $O(10^{-10})$.

4.2. THE NONLINEAR MODEL PROBLEM

The nonlinear problem (9) results into the following nonlinear algebraic system:

$$(1+\kappa_i^4)b_i + 2\sum_{j=1}^{N} b_j \beta_{ij} = \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i} - 100\sum_{m=1}^{N}\sum_{n=1}^{N} b_m b_n h_i^{mn}, \qquad i = 1,\ldots,N, \tag{30}$$

where h_i^{mn} is defined in formula (21). We solve the latter with semi-implicit method and iterations.

The results about the convergence of the spectral solution are shown in Fig. 4. The convergence is once again algebraic of fifth order.

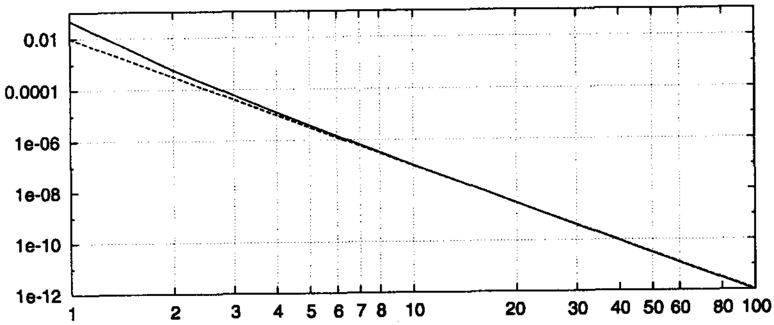

Fig. 4. Rate of convergence for the solution of the nonlinear equation. Solid line: b_i; dashed line: $b_i = 0.01\, i^{-5}$

4.3. THE COUPLED SYSTEM

In this case we consider the coupled system of one fourth-order equation for ψ and one second-order equation for θ (10). Because of the obvious symmetry of the boundary value problem under consideration, we can seek a solution in which the stream function is even and the temperature is an odd function. Acknowledging the symmetry of the problem, we develop the sought function into the series

$$\Psi(x,t) = \sum_{k=1}^{K} p_k c_k(x), \quad \Theta(x,t) = \sum_{k=1}^{K} d_k \sin(k\pi x). \tag{31}$$

Upon introducing these expansions into (5), (6) and making use of the above compiled formulae, an algebraic system for the coefficients d_k and p_k is derived:

$$\begin{aligned} &-\kappa_i^4 p_i + \frac{1}{Pr}\sum_{j=1}^{N} p_j \beta_{ij} \\ &= -Ra\left[\sum_{m=1}^{N} d_m \frac{m\pi 2\sqrt{2}(-1)^{m+1}\kappa_i^3 \tanh\kappa_i}{m^4\pi^4 - \kappa_i^4} - \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i}\right], \\ &\qquad i = 1,\dots,N, \end{aligned} \tag{32}$$

$$\begin{aligned} &(1 + l^2\pi^2) d_l = \tau_B \sum_{n=1}^{N}\sum_{m=1}^{N} p_n \frac{8\sqrt{2}\kappa_n^2\kappa_m^2 l\pi(-1)^l}{(\kappa_n^4 - \kappa_m^4)(l^4\pi^4 - \kappa_m^4)}, \\ &\qquad l = 1,\dots,N. \end{aligned} \tag{33}$$

The results for the coefficients p_i and d_l are presented in Fig. 5. The peculiar finding is that the rate of convergence for Θ is algebraic of fifth order, while the rate for Ψ is one order lower (fourth-order). The analytical explanation of this phenomena will be the object of a separate study. Here it will suffice to mention that the off-diagonal elements in (32) can degrade the rate of convergence, while in the equation (33) for Θ no off-diagonal elements are present and the convergence is of fifth order as in the previous examples.

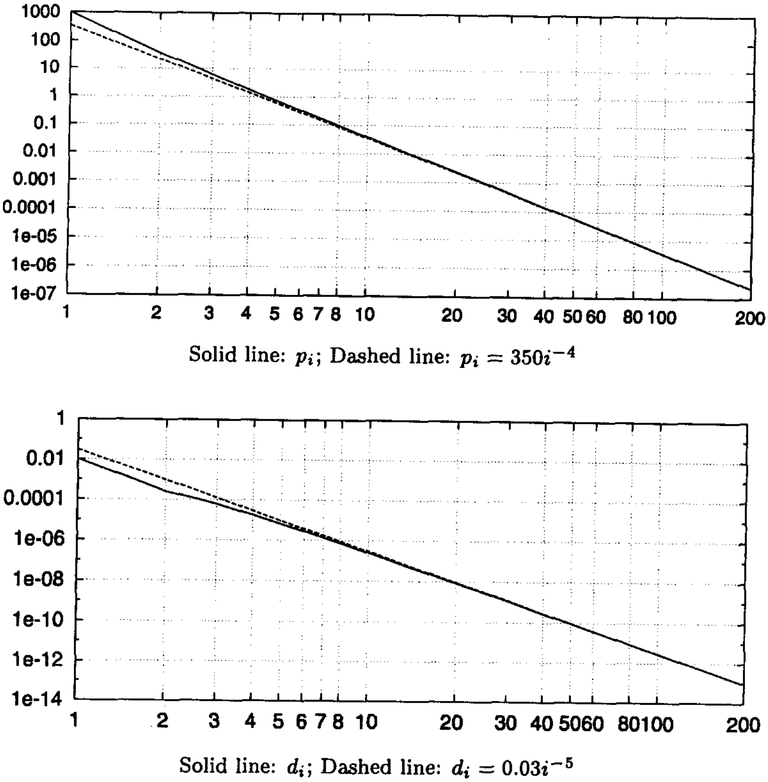

Fig. 5. The rate of convergence for the coupled system for $Ra = 6000$, $Pr = 1$ and $\tau_B = 0.001$. The upper panel shows the spectral coefficient for the function Psi; the lower panel shows Θ

The fourth order for the rate of convergence means that a number of terms $N = 100$ is fully adequate to obtain results with a very high precision 10^{-8}.

5. CONCLUSIONS

In the present work a new Galerkin technique is developed for coupled thermo-convective flows in a vertical slot. The well-known beam functions are used as basis set together with the trigonometric functions. The formulas for the cross expansion of the two systems are not available from the literature and are derived here. The construction of the numerical algorithms is also presented.

Three generic model problems are considered. The spectral solutions exhibit a fifth-order algebraic convergence except for the case of the coupled system pertinent to the convection in a vertical slot, where the rate is of fourth order for one of the functions. The fourth or fifth order means that although algebraic, the convergence is fast enough for all practical purposes. The theoretical and numerical findings are illustrated graphically.

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REFERENCES

1. Bergholz, R. F. Instability of steady natural convection in a vertical fluid layer. *J. Fluid Mech.*, **84**, 1978, 743–768.
2. Boyd, J. P. Fourier and Chebyshev Spectral Methods. Dover, New York, 2000.
3. Chandrasekhar, S. Hydrodynamic and Hydromagnetic Instability. Oxford University Press, Clarendon, London, 1961. Appendix V.
4. Christov, C. I. A method for treating the stochastic bifurcation of plane Poiseuille flow. *Ann. Univ. Sof., Fac. Math. Mech.*, **76 (1982)**, l. 2 – Mecanique, 1987, 87–113.
5. Christov, C. I., G. M. Homsy. Nonlinear dynamics of two dimensional convection in a vertically stratified slot with and without gravity modulation. *J. Fluid Mech.*, 2001 (to appear).
6. Farooq, A., G. M. Homsy. Linear and nonlinear dynamics of a differentially heated slot under gravity modulation. *J. Fluid Mech.*, **313**, 1996, 1–38.
7. Harris, D. L., W. H. Reid. On orthogonal functions which satisfy four boundary conditions. I. Tables to use in Fourier-type expansions. *Abstract of Astrophysical Journal, Supplement Series*, **III(33)**, 1968, 450–450.
8. Jhaveri, B. S., F. Rosenberger. Exact triple integrals of beam functions. *J. Comput. Phys.*, **45**, 1982, 300–302.
9. Poots, G. Heat transfer by laminar free convection in enclosed plane gas layers. *Quart. J. Mech. Appl. Math.*, **11**, 1958, 357–273.
10. Lord Rayleigh. Theory of Sound. Dover, New York, 1945.
11. Reid, W. L., D. L. Harris. On orthogonal functions which satisfy four boundary conditions. II. Integrals to use in Fourier-type expansions. *Abstract of Astrophysical Journal, Supplement Series*, **III(33)**, 1968, 450–450.
12. Vinod, S. A., C. I. Christov, G. M. Homsy. Resonant thermocapillary and buoyant flows with finite frequency gravity modulation. *Phys. Fluids*, **11**, 1999, 2565–2576.

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ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"

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LH^ω-SMOOTH BUMP FUNCTIONS IN WEIGHTED ORLICZ SEQUENCE SPACES

B. ZLATANOV[1]

An exact estimate is given for the modulus of smoothness in weighted Orlicz sequence spaces and the best order of LH^ω-smoothness of bump functions is found for $\alpha \leq 2$.

Keywords: weighted Orlicz sequence space, smooth bump function

Mathematics Subject Classification 2000: 46B45, 46E39

1. INTRODUCTION

For many problems in the Geometry of Banach spaces and Nonlinear analysis in Banach spaces, the existence of bump functions with prescribed order of smoothness or with derivatives sharing properties of Hölder's type is of essential importance.

As a Frechet smooth norm immediately produces a bump function of the same smoothness, all negative results about bump functions are negative results about the smoothness in the class of all equivalent norms.

The question of finding an upper bound of the order of Frechet differentiability of bump functions in arbitrary Orlicz space is solved in [10]. Recently, in [11] Ruiz has proved that for a given Orlicz function M all weighted Orlicz sequence spaces $\ell_M(w)$, generated by weight sequence $w = \{w_j\}_{j=1}^{\infty}$, verifying the condition

$$\lim_{k\to\infty} w_{j_k} = 0, \qquad \sum_{k=1}^{\infty} w_{j_k} = \infty, \tag{1}$$

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for some subsequence $\{w_{j_k}\}_{k=1}^{\infty}$, are mutually isomorphic. This result raises the question whether the best possible ω_1-Hölder properties of the first derivatives of bump functions in $\ell_M(w)$ depend on the sequence $w = \{w_j\}_{j=1}^{\infty}$, verifying (1).

For the proof of the main result we shall need an estimate from below of the modulus of smoothness in weighted Orlicz sequence space $\ell_M(w)$. Maleev and Troyanski [9] have found an upper estimate for the modulus of smoothness of an arbitrary Orlicz space. Figiel in [3] has shown that this estimate is exact up to equivalent renorming in Orlicz spaces. Using the method of Figiel, we will show in Section 4, Lemma 1, that the estimates found in [9] are exact up to equivalent renorming also in weighted Orlicz sequence space.

2. PRELIMINARY RESULTS

We denote by X a Banach space, X^* its dual one, S_X the unit shpere in X, $\mathbb{N}$ the naturals, and $\mathbb{R}$ the reals. Everywhere differentiabilily is understood as Frechet differentiabilily.

An Orlicz function M is an even, continuous, convex and monotone in $[0, \infty)$ function with $M(0) = 0$, $M(t) > 0$ for any $t \neq 0$. The Orlicz function M is said to have the property Δ_2 if there exists a constant C such that $M(2t) \leq CM(t)$ for every $t \in [0, \infty)$.

To every Orlicz function M the following numbers are associated:

$$\alpha_M^0 = \sup\{p : \sup_{0<\lambda,\, t\leq 1} M(\lambda t)/(M(\lambda)t^p) < \infty\},$$

$$\alpha_M^\infty = \sup\{p : \sup_{1\leq\lambda,\, t<\infty} M(\lambda)t^p/M(\lambda t) < \infty\},$$

$\alpha_M = \min\left\{\alpha_M^0, \alpha_M^\infty\right\}$ (see, e.g., [5], p. 143, and [6], p. 382).

Let (S, Σ, μ) be a positive measure space. The Orlicz space $L_M(\mu)$ is defined as the set of all equivalence classes of μ-measurable scalar functions x on S such that

$$\widetilde{M}(x/\lambda) = \int_S M(x(t)/\lambda)\, d\mu(t) < \infty$$

for some $\lambda > 0$, equipped with the Luxemburg's norm

$$\|x\| = \inf\left\{\lambda > 0 : \widetilde{M}(x/\lambda) \leq 1\right\}. \tag{2}$$

For $S = \mathbb{N}$ and $w = \{w_j\}_{j=1}^{\infty} = \{\mu(j)\}_{j=1}^{\infty}$ we get the weighted Orlicz sequence spaces $\ell_M(w)$. In this case we have $x = \{x_j\}_{j=1}^{\infty} \in \ell_M(w)$ iff there exists $\lambda > 0$:

$$\widetilde{M}(x/\lambda) = \sum_{j=1}^{\infty} w_j M(x_j/\lambda) < \infty.$$

Clearly, the unit vector sequence is an unconditional basis in $\ell_M(w)$. When $w_j = 1$ for each $j \in \mathbb{N}$, we obtain the usual Orlicz sequence space and denote it by ℓ_M instead of $\ell_M(w)$.

Let $w = \{w_j\}_{j=1}^{\infty}$ be a sequence and $w_j > 0$ for every $j \in \mathbb{N}$. By $w \in \Lambda$ we mean that there exists a subsequence $\{w_{j_k}\}_{k=1}^{\infty}$, verifying conditions (1).

We call modulus of smoothness of X the function

$$\rho_X(\tau) = \frac{1}{2}\sup\{\|x+\tau y\| + \|x-\tau y\| - 2 : x, y \in S_X\}, \quad \tau > 0.$$

We introduce the following function necessary for the estimation of $\rho_{(\ell_M(w),|\cdot|)}(\tau)$ with respect to an appropriate equivalent norm $|\cdot|$ in $\ell_M(w)$:

$$G_M(\tau) = \tau^2 \sup\left\{\frac{M(uv)}{u^2M(v)} : u \in [\tau, 1],\, v > 0\right\}, \quad \tau \in (0,1].$$

The function $f : X \to \mathbb{R}$ is said to be differentiable at $x \in X$ if there exists $z_x^* \in X^*$ such that

$$f(x+ty) = f(x) + tz_x^*(y) + r(x,y,t), \tag{3}$$

where $\lim\limits_{t\to 0} t^{-1}\sup\{r(x,y,t) : y \in S_X\} = 0$. The functional z_x^* is called derivative of f at x and is denoted by $f'(x)$.

In the applications often are considered functions, which are not only differentiable, but their derivatives share properties of Hölder's type.

By Ω we denote the class of all functions $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t) = o(t)$ and $\omega_1(t) = \omega(t)/t$ is a nondecreasing function, satisfying the condition $\omega_1(\lambda t) \leq \lambda\omega_1(t)$ for every $\lambda \geq 1$.

We say that $f : X \to \mathbb{R}$ is locally H^ω-smooth in the open subset $V \subset X$ if f is continuously differentiable in V and for every $x \in V$ there exist $\delta = \delta(x) > 0$ and $A = A(x) > 0$ such that

$$\|f'(y) - f'(z)\| \leq A\omega_1(\|y-z\|) = A\frac{\omega(\|y-z\|)}{\|y-z\|} \tag{4}$$

for every $y, z \in B(x;\delta) \subset V$ (see [1]).

If there exists $A > 0$ such that (4) is fulfilled for arbitrary $y, z \in V$, the function f is called H^ω-smooth in V. The class of all H^ω-smooth (locally H^ω-smooth) functions in V is denoted by $H^\omega(V)$ $(LH^\omega(V))$, respectively.

We say that $b : X \to \mathbb{R}$ is a bump function iff $\operatorname{supp} b = \overline{\{x \in X;\, b(x) \neq 0\}}^{\|\cdot\|}$ is a bounded non empty set.

It is easy to observe that if there exists $H^\omega(X)$ $(LH^\omega(X))$ — a smooth equivalent norm, then there exists $H^\omega(X)$ $(LH^\omega(X))$ — a smooth bump (see, e.g., [2], p. 9). The converse is not true. Haydon [4] gives an example of a space with C^∞-smooth bump, which has not even a Gâteaux differentiable equivalent norm.

3. MAIN RESULT

Theorem 1. *Let $X = \ell_M(w)$, where M is an Orlicz function, satisfying the Δ_2-condition at 0 and at ∞, $\alpha_M \in (1,2)$, $w \in \Lambda$ and $\omega \in \Omega$. If b is an LH^ω-smooth bump function in $\ell_M(w)$, then*

$$G_M(\tau) = O(\omega(\tau)).$$

In the proof of the next Lemma 1 we shall use the following result:

Proposition 1 ([3], [6]). *There exists a positive absolute constant L such that* $\rho_X(\sigma)/\sigma^2 \le L\rho_X(\tau)/\tau^2$, *whenever* $0 < \tau < \sigma$.

Lemma 1. *Let* $X = \ell_M(w)$, *where* M *is an Orlicz function, satisfying the* Δ_2*-condition, and* $w \in \Lambda$. *Then for every equivalent norm* $|\cdot|$ *on* X *there exists a constant* $K = K_{|\cdot|} > 0$ *such that*

$$\rho_{(X,|\cdot|)}(\tau) \ge KG_M(\tau), \qquad \tau \in (0,1].$$

Proof. We can assume WLOG that $|x| \le \|x\| \le b|x|$ for every $x \in X$, where $\|\cdot\|$ is the Luxemburg norm (2). As the norm $|\cdot|$ is fixed, we can denote $\rho(\tau) = \rho_{(X,|\cdot|)}(\tau)$ and we shall denote the subsequence $\{w_{j_k}\}_{k=1}^{\infty}$ fulfilling (1) again by $\{w_j\}_{j=1}^{\infty}$ just for simplicity.

Observe first that from the equivalence of the norms it follows that $\sum\limits_{i=1}^{n} \rho(|x_i|) \le 1$, provided $\sum\limits_{i=1}^{n} \rho(\|x_i\|) \le 1$. Hence by Lindenstrauss' theorem (in the Figiel's form [3]) there exist signs $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, so that $\left|\sum\limits_{i=1}^{n} \varepsilon_i x_i\right| \le 1 + \sqrt{3}$, which gives us that

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \le \left(1 + \sqrt{3}\right) b = d, \tag{5}$$

whenever $\sum\limits_{i=1}^{n} \rho(\|x_i\|) \le 1$.

For every given $\tau \in (0,1]$, $u \in [\tau, 1]$, $v \in (0, \infty)$ we put $n = [1/\rho(u)]$, $c = uv$, where by $[a]$ we denote the largest integer not greater than a.

For every v we can choose a sequence of integers $\{m_k\}_{k=1}^{\infty}$:

$$\frac{1}{2M(v)} < \sum_{j=m_k+1}^{m_{k+1}} w_j < \frac{1}{M(v)},$$

because $w \in \Lambda$. Let $x_k \in X$, $k = 1, \ldots, n$, be disjointly supported vectors such that

$$x_k = c \sum_{j=m_k+1}^{m_{k+1}} e_j,$$

where $\{e_j\}_{j=1}^{\infty}$ is the unit vector basis in X. Obviously,

$$1 = \sum_{j=m_k+1}^{m_{k+1}} w_j M(c/\|x_k\|) = M(c/\|x_k\|) \sum_{j=m_k+1}^{m_{k+1}} w_j < \frac{M(c/\|x_k\|)}{M(v)}.$$

So we obtain that $\|x_k\| \le u$, which yields the inequalities $\sum\limits_{i=1}^{n} \rho(\|x_i\|) \le n\rho(u) \le 1$.

Using (5), we obtain immediately the inequalities

$$1 \geq \sum_{k=1}^{n} \sum_{j=m_k+1}^{m_{k+1}} w_j M(c/d) = M(c/d) \sum_{k=1}^{n} \sum_{j=m_k+1}^{m_{k+1}} w_j \geq M(c/d) n \frac{1}{2M(v)} \,.$$

Hence

$$\frac{M(c/d)}{M(v)} \leq \frac{2}{n} \leq 2\frac{2}{n+1} \leq 4\rho(u).$$

Since $c = uv$, then there exists a constant α, depending only on d and the Δ_2-condition, such that $M(uv) \leq \alpha M(c/d)$. Finally, we obtain

$$\frac{M(uv)}{M(v)} \leq \alpha \frac{M(c/d)}{M(v)} \leq 4\alpha\rho(u). \tag{6}$$

To finish the proof, we need only to apply (6) and Proposition 1. Indeed,

$$G_M(\tau) = \tau^2 \sup_{u\in[\tau,1],\, v>0} \frac{M(uv)}{u^2 M(v)} \leq \tau^2 \sup_{u\in[\tau,1],\, v>0} 4\alpha \frac{\rho(u)}{u^2} \leq \tau^2 4\alpha L \frac{\rho(\tau)}{\tau^2} \,.$$

Combining the result in [9] with Lemma 1, we find that the estimate of the modulus of smoothness in weighted Orlicz sequence spaces is exact up to an equivalent renorming.

5. PROOF OF THE MAIN RESULT

In the proof of Theorem 1 we shall need a variant of known theorems (see, e.g., [2], p. 199). As the proofs are literally the same, we shall omit them.

Theorem 2 (see, e.g., [2], 5.3.1). *Assume that a Banach space $X \not\supset c_0$. Suppose that X admits a bump function $b(x) \in LH^\omega(X)$. Then X admits a bump function $f(x) \in H^\omega(X)$.*

Theorem 3 (see, e.g., [2], 5.3.2). *Assume that a Banach space admits a bump function $b(x) \in H^\omega(X)$. Then X admits an equivalent norm $|\cdot| \in H^\omega(S_X)$.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let f be an LH^ω-smooth bump function in $X = \ell_M(w)$. Since there is no isomorphic copy of c_0 in X, then according to Theorem 2 there is a H^ω-smooth bump function in X. According to Theorem 3, there is an equivalent $H^\omega(S_X)$-smooth norm $|||\cdot|||$ such that

$$\rho_{X,|||\cdot|||}(t) \leq K\omega(t), \quad t \geq 0,\ K > 0. \tag{7}$$

On the other hand, we have just proved that the best order of the modulus of smoothness of any equivalent renorming of X is $G_M(t)$, i.e.

$$\rho_{X,|||\cdot|||}(t) \geq c\,G_M(t), \quad c = c_{|||\cdot|||} > 0 \tag{8}$$

for every equivalent norm $|||\cdot|||$ in X. Combining (7) and (8), we obtain

$$G_M(t) \leq \frac{K}{c}\,\omega(t).$$

Remark. If $\alpha_M = 2$ and M, satisfying the condition

$$\sup\{M(uv)/u^{\alpha_M}M(v)\ :\ u,v \in (0,\infty)\} < \infty,$$

is solved in [8].

Remark. If $M \sim t^2$, then there exists an equivalent, infinitly many times Frechet differentiable norm, and it is seen right away that $G_M(\tau) = \tau^2$, so there is nothing to be proved.

REFERENCES

1. Deville, R. A characterization of C^∞-smooth Banach spaces. *Bull. London Math. Soc.*, **22**, 1990, 13–17.
2. Deville, R., G. Godefroy, V. Zizler. Smoothness and renorming in Banach spaces. Pitman Monographs and Survey in Pure and Applied Mathematics, Longmann Scientific & Technical, Longman House, Brint Mill, Harlow, 1993.
3. Figiel, T. On the moduli of convexity and smoothness. *Studia Math.*, **56**, 1976, 121–155.
4. Haydon, R. Trees in renorming theory. *Proc. London Math. Soc.*, **78**, 1999, 541–584.
5. Lindenstrauss, J., L. Tzafriri. Classical Banach spaces I. Springer, New York, 1977.
6. Lindenstrauss, J., L. Tzafriri. Classical Banach spaces II. Springer, New York, 1979.
7. Luxemburg, W. A. Banach function spaces. Doctoral Thesis, Delft, 1955.
8. Maleev, R. Norms of best smoothness in Orlicz spaces. *Zeitschrift für Analysis und ihre Anwendungen*, **12**, 1993, 123–135.
9. Maleev, R., S. Troyanski. On the moduli of convexity and smoothness in Orlicz spaces. *Studia Math.*, **54**, 1975, 131–141.
10. Maleev, R., S. Troyanski. Smooth functions in Orlicz spaces. *Contemporary Math.*, **85**, 1989, 355–370.
11. Ruis, C. On the structure of Rosenthal's space X_φ in Orlicz function spaces. *Rocky Mountain J. Math.*, **27**, 1997, 1223–1238.
12. Sundaresan, K., S. Swaminathan. Geometry and nonlinear analysis in Banach spaces. *Lecture Notes in Math.*, **1131**, Berlin–Heidelberg–New York, Springer, 1985.

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ON SIX–DIMENSIONAL HERMITIAN SUBMANIFOLDS OF A CAYLEY ALGEBRA SATISFYING THE G-COSYMPLECTIC HYPERSURFACES AXIOM

M. BANARU

It is proved that the six-dimensional Hermitian submanifolds of Cayley algebra, satisfying the g-cosymplectic hypersurfaces axiom, are Kählerian manifolds.

Keywords: Hermitian manifold, almost contact metric structure, cosymplectic structure, g-cosymplectic hypersurfaces axiom

Mathematics Subject Classification 2000: 53C10, 58C05

1. INTRODUCTION

One of the most important properties of a hypersurface of an almost Hermitian manifold is the existence on a such hypersurface of an almost contact metric structure, determined in a natural way. As it seems, this is the reason for the great importance of the almost Hermitian manifolds in differential geometry and in the modern theoretical physics. This structure has been studied mainly in the case of Kählerian [1, 2] and quasi-Kählerian [3, 4] manifolds. In the case when the embedding manifold is Hermitian, however, comparatively little is known about the geometry of its hypersurfaces. In the present work a certain result is obtained in this direction by using the Cartan structure equations of such hypersurfaces.

Let $\mathbf{O} \equiv R^8$ be the Cayley algebra. As it is well-known [5], two non-isomorphic 3-vector cross products are defined on it by means of the relations

$$P_1(X, Y, Z) = -X(\overline{Y}Z) + \langle X, Y\rangle Z + \langle Y, Z\rangle X - \langle Z, X\rangle Y,$$

$$P_2(X,Y,Z) = -(X\overline{Y})Z + \langle X,Y\rangle Z + \langle Y,Z\rangle X - \langle Z,X\rangle Y,$$

where $X, Y, Z \in \mathbf{O}$, $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbf{O}$ and $X \to \overline{X}$ is the conjugation operator. Moreover, any other 3-vector cross product in the octave algebra is isomorphic to one of the above-mentioned two.

If $M^6 \subset \mathbf{O}$ is a six-dimensional oriented submanifold, then the induced almost Hermitian structure $\{J_\alpha, g = \langle\cdot,\cdot\rangle\}$ is determined by the relation

$$J_\alpha(X) = P_\alpha(X, e_1, e_2), \quad \alpha = 1, 2,$$

where $\{e_1, e_2\}$ is an arbitrary orthonormal basis of the normal space of M^6 at a point p, $X \in T_p(M^6)$ [5]. The submanifold $M^6 \subset \mathbf{O}$ is called Hermitian if the almost Hermitian structure induced on it is integrable. The point $p \in M^6$ is called general [6] if

$$e_0 \notin T_p(M^6) \quad \text{and} \quad T_p(M^6) \subseteq L(e_0)^\perp,$$

where e_0 is the unit of Cayley algebra and $L(e_0)^\perp$ is its orthogonal supplement. A submanifold $M^6 \subset \mathbf{O}$, consisting only of general points, is called a general-type submanifold [6]. In what follows, all submanifolds M^6 that will be considered are assumed to be of general type.

2. COSYMPLECTIC HYPERSURFACES OF HERMITIAN $M^6 \subset \mathbf{O}$

Let N be an oriented hypersurface of a Hermitian $M^6 \subset \mathbf{O}$ and let σ be the second fundamental form of the immersion of N into M^6. As it is well-known [2, 4], the almost Hermitian structure on M^6 induces an almost contact metric structure on N. We recall [3, 4] that an almost contact metric structure on the manifold N is defined by the system $\{\Phi, \xi, \eta, g\}$ of tensor fields on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of a type $(1,1)$, and g is a Riemannian metric on N such that

$$\eta(\xi) = 1, \quad \Phi(\xi) = 0, \quad \eta \circ \Phi = 0, \quad \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y), \quad X, Y \in \aleph(N).$$

The almost contact metric structure is called cosymplectic [4] if

$$\nabla\eta = \nabla\Phi = 0.$$

(Here ∇ is the Riemannian connection of the metric g.) The first group of the Cartan structure equations of a hypersurface of a Hermitian manifold looks as follows [8]:

$$
\begin{aligned}
d\omega^a =& \omega^a_b \wedge \omega^b + B^{ab}{}_c \omega^c \wedge \omega_b \\
&+ \left(\sqrt{2}B^{a3}{}_b + i\sigma^a_b\right)\omega^b \wedge \omega + \left(-\frac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^{ab}\right)\omega_b \wedge \omega, \\
d\omega_a =& -\omega^b_a \wedge \omega_b + B_{ab}{}^c \omega_c \wedge \omega^b \\
&+ \left(\sqrt{2}B_{a3}{}^b - i\sigma^b_a\right)\omega_b \wedge \omega + \left(-\frac{1}{\sqrt{2}}B_{ab}{}^3 - i\sigma_{ab}\right)\omega^b \wedge \omega, \\
d\omega =& \left(\sqrt{2}B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b\right)\omega^b \wedge \omega_a \\
&+ \left(B_{3b}{}^3 + i\sigma_{3b}\right)\omega \wedge \omega^b + \left(\left(B^{3b}{}_3 - i\sigma^b_3\right)\omega \wedge \omega_b.\right.
\end{aligned}
\tag{1}
$$

Here the B's are Kirichencko structure tensors of the Hermitian manifold [9]; $a, b, c = 1, 2$; $\widehat{a} = a + 3$; $i = \sqrt{-1}$. Taking into account that the first group of the Cartan structure equations of the cosymplectic structure must look as follows [10]:

$$
\begin{aligned}
d\omega^a &= \omega^a_b \wedge \omega^b, \\
d\omega_a &= -\omega^b_a \wedge \omega_b, \\
d\omega &= 0,
\end{aligned}
\tag{2}
$$

we get the conditions whose simultaneous fulfilment is a criterion for the hypersurface N to be cosymplectic:

$$
\begin{gathered}
1)\ B^{ab}{}_c = 0, \quad 2)\ \sqrt{2}B^{a3}{}_b + i\sigma^a_b = 0, \quad 3)\ -\frac{1}{\sqrt{2}}B^{ab}{}_3 + i\sigma^a_b = 0, \\
4)\ B^{3a}{}_b - \sqrt{2}B_{3b}{}^a - 2i\sigma^a_b = 0, \quad 5)\ B^{3b}{}_3 - i\sigma^b_3 = 0,
\end{gathered}
\tag{3}
$$

and the formulae, obtained by complex conjugation (no need to write them down explicitly).

Now, let us analyze the obtained conditions. From $(3)_3$ it follows that

$$\sigma^{ab} = -\frac{1}{\sqrt{2}}B^{ab}{}_3.$$

By alternating of this relation we have

$$0 = \sigma^{[ab]} = -\frac{i}{\sqrt{2}}B^{[ab]}{}_3 = -\frac{i}{2\sqrt{2}}(B^{ab}{}_3 - B^{ba}{}_3) = -\frac{i}{\sqrt{2}}B^{ab}{}_3.$$

Therefore $B^{ab}{}_3 = 0$ and consequently $\sigma^{ab} = 0$. From $(3)_2$ we get that $B^{3a}{}_b = \frac{i}{\sqrt{2}}\sigma^a_b$. We substitute this value in $(3)_4$. As a result we have

$$\sigma^a_b = i\sqrt{2}B_{3b}{}^a.$$

Now, we use the relation for the Kirichenko structure tensors of six-dimensional Hermitian submanifolds of Cayley algebra [9]:

$$B^{\alpha\beta}{}_{\gamma} = \frac{1}{\sqrt{2}}\varepsilon^{\alpha\beta\mu}D_{\mu\gamma}, \quad B_{\alpha\beta}{}^{\gamma} = \frac{1}{\sqrt{2}}\varepsilon_{\alpha\beta\mu}D^{\mu\gamma},$$

where

$$D_{\mu\gamma} = \pm T^{8}_{\mu\gamma} + iT^{7}_{\mu\gamma}, \quad D^{\mu\gamma} = D_{\widehat{\mu}\widehat{\gamma}} = \pm T^{8}_{\widehat{\mu}\widehat{\gamma}} - iT^{7}_{\widehat{\mu}\widehat{\gamma}}.$$

Here T^{φ}_{kj} are the components of the configuration tensor (in Gray's notation [11], or the Euler curvature tensor [12]) of the Hermitian submanifold $M^6 \subset \mathbf{O}$; $\alpha, \beta, \gamma, \mu = 1, 2, 3$; $\widehat{\mu} = \mu + 3$; $k, j = 1, \dots, 6$; $\varphi = 7, 8$; $\varepsilon^{\alpha\beta\mu} = \varepsilon^{\alpha\beta\mu}_{123}$, $\varepsilon_{\alpha\beta\mu} = \varepsilon^{123}_{\alpha\beta\mu}$ are the components of the third order Kronecker tensor [13].

From $(3)_1$ we obtain

$$B^{ab}{}_{c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma c} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{3c} = 0 \Leftrightarrow D_{3c} = 0.$$

The similar reasoning can be applied to the above obtained condition $B^{ab}{}_{3} = 0$:

$$B^{ab}{}_{3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab\gamma}D_{\gamma 3} = 0 \Leftrightarrow \frac{1}{\sqrt{2}}\varepsilon^{ab3}D_{33} = 0 \Leftrightarrow D_{33} = 0.$$

So, $D_{3c} = D_{33} = 0$ and hence

$$D_{3\alpha} = 0. \tag{4}$$

From $(3)_5$ we get

$$\sigma^{b}_{3} = \sigma_{3\widehat{b}} = -iB^{3b}{}_{3} = -i\frac{1}{\sqrt{2}}\varepsilon^{3b\gamma}D_{\gamma 3} = 0.$$

We have $\sigma_{ab} = \sigma_{\widehat{a}\widehat{b}} = \sigma_{3b} = \sigma_{3\widehat{b}} = 0$. We shall compute the rest of the components of the second fundamental form using $(3)_2$:

$$\sigma_{\widehat{a}b} = \sigma^{a}_{b} = i\sqrt{2}B^{a3}{}_{b} = i\sqrt{2}\frac{1}{\sqrt{2}}\varepsilon^{a3\gamma}D_{\gamma b} = i\varepsilon^{a3c}D_{cb}.$$

Then

$$\sigma_{\widehat{1}1} = i\varepsilon^{13c}D_{c1} = i\varepsilon^{132}D_{21} = -iD_{21};$$

$$\sigma_{\widehat{1}2} = i\varepsilon^{13c}D_{c2} = i\varepsilon^{132}D_{22} = -iD_{22};$$

$$\sigma_{\widehat{2}1} = i\varepsilon^{23c}D_{c1} = i\varepsilon^{231}D_{11} = iD_{11};$$

$$\sigma_{\widehat{2}2} = i\varepsilon^{23c}D_{c2} = i\varepsilon^{231}D_{12} = iD_{12};$$

$$\sigma_{1\widehat{1}} = \overline{\sigma_{\widehat{1}1}} = iD^{12}; \quad \sigma_{1\widehat{2}} = \overline{\sigma_{\widehat{1}2}} = iD^{22};$$

$$\sigma_{2\widehat{1}} = \overline{\sigma_{\widehat{2}1}} = -iD^{11}; \quad \sigma_{2\widehat{2}} = \overline{\sigma_{\widehat{2}2}} = -iD^{12}.$$

We thus obtain that the matrix of the second fundamental form of the immersion of the cosymplectic hyperspace N into $M^6 \subset \mathbf{O}$ looks as follows:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & iD^{12} & -iD^{11} \\ 0 & 0 & 0 & iD^{22} & -iD^{12} \\ 0 & 0 & \sigma_{33} & 0 & 0 \\ -iD_{12} & -iD_{22} & 0 & 0 & 0 \\ iD_{11} & iD_{22} & 0 & 0 & 0 \end{pmatrix}.$$

3. THE MAIN RESULT

As the hypersurface N is a totally geodesic submanifold of a Hermitian $M^6 \subset \mathbf{O}$ precisely when the matrix σ vanishes, we can conclude that the conditions

$$D_{11} = D_{12} = D_{22} = D^{11} = D^{12} = D^{22} = \sigma_{33} = 0 \tag{5}$$

are a criterion for N to be a totally geodesic submanifold of M^6.

We recall that the almost Hermitian manifold satisfies the g-cosymplectic hypersurfaces axiom if through every point of this manifold passes a totally geodesic cosymplectic hypersurface. That is why for the Hermitian $M^6 \subset \mathbf{O}$, satisfying the g-cosymplectic hypersurfaces axiom, the equalities (5) hold for every point of M^6. But we have proved previously [9, 14] that the matrix D of a six-dimensional Hermitian submanifold of the octave algebra looks as follows:

$$D = \begin{pmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D^{11} & D^{12} & D^{13} \\ 0 & 0 & 0 & D^{21} & D^{22} & D^{23} \\ 0 & 0 & 0 & D^{31} & D^{32} & D^{33} \end{pmatrix}.$$

If M^6 satisfies the g-cosymplectic hypersurfaces axiom, this matrix D vanishes as a consequence of (4) and (5). But the matrix D vanishes at every point of a six-dimensional almost Hermitian submanifold of Cayley algebra precisely when the given submanifold is Kählerian [9, 14–16]. Hence we have proved the following Theorem.

Theorem. *Every six-dimensional Hermitian submanifold of a Cayley algebra, satisfying the g-cosymplectic hypersurfaces axiom, is a Kählerian manifold.*

REFERENCES

1. Goldberg, S. Totally geodesic hypersurfaces of Kaehler manifolds. *Pacif. J. Math.*, **27**, 1968, 275–281.
2. Tashiro, Y. On contact structure of hypersurfaces in complex manifolds. *Tôhoku Math. J.*, **15**, 1963, 62–78.
3. Kirichenko, V. F., L. V. Stepanova. Geometry of hypersurfaces of quasi-Kählerian manifolds. *Uspehi Mat. Nauk*, No 2, 1995, 213–214.
4. Stepanova, L. V. Contact geometry of hypersurfaces of quasi-Kählerian manifolds. MSPU "V. I. Lenin", Moscow, 1995.
5. Gray, A. Vector cross products on manifolds. *Trans Amer. Math. Soc.*, **141**, 1969, 465–504.
6. Kirichenko, V. F. On nearly-Kählerian structures induced by means of 3-vector cross products on six-dimensional submanifolds of Cayley algebra. *Vestnik MGU*, No 3, 1973, 70–75.
7. Blair, D. E. The theory of quasi-Sasakian structures. *J. Diff. Geom.*, **1**, 1967, 331–345.
8. Stepanova, L. V. Quasi-Sasakian structure on hypersurfaces of Hermitian manifolds. *Sci. Works of MSPU "V. I. Lenin"*, 1995, 187–191.
9. Banaru, M. B. Hermitian geometry of six-dimensional submanifolds of Cayley algebra. MSPU "V. I. Lenin", Moscow, 1993.
10. Kirichenko, V. F. Methods of the generalized Hermitian geometry in the theory of almost contact metric manifolds. In: *Problems of Geometry*, **18**, 1986, 25–71.
11. Gray, A. Some examples of almost Hermitian manifolds. *Ill. J. Math.*, **10**, 1966, 353–366.
12. Cartan, E. Riemannian geometry in an orthogonal frame. MGU, Moscow, 1960.
13. Lichnerowicz, A. Théory globale des connexions et des groupes d'holonomie. Cremonese, Roma, 1955.
14 Banaru, M. B. On almost Hermitian structures induced by means of 3-vector cross products on six-dimensional submanifolds of the octave algebra. Polyanalytical Functions, Smolensk, 1997, 113–117.
15. Kirichenko, V. F. Classification of Kählerian structures induced by means of 3-vector cross products on six-dimensional submanifolds of Cayley algebra. *Izvestia Vuzov, Kazan*, **8**, 1980, 32–38.
16. Banaru, M. B. On Gray-Hervella classes of almost Hermitian manifolds on six-dimensional submanifolds of Cayley algebra. *Sci. Works of MSPU "V. I. Lenin"*, 1994, 36–38.

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HARMONIC MAPS OF COMPACT KÄHLER MANIFOLDS TO EXCEPTIONAL LOCAL SYMMETRIC SPACES OF HODGE TYPE AND HOLOMORPHIC LIFTINGS TO COMPLEX HOMOGENEOUS FIBRATIONS

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Let M be a compact Kähler manifold and G/K be a non-Hermitian Riemannian symmetric space of Hodge type. Certain harmonic maps $f : M \to \Gamma \setminus G/K$ will be proved to admit holomorphic liftings $F_P : M \to \Gamma \setminus G/G \cap P$ to complex homogeneous fibrations, where P are parabolic subgroups of $G^{\mathbb{C}}$. The work studies whether the images $F_P(M) = \Gamma_h \setminus G_h/K_h$ are local equivariantly embedded Hermitian symmetric subspaces of $\Gamma \setminus G/G \cap P$. For each of the cases examples of harmonic maps f which do not admit holomorphic liftings are supplied.

Keywords: harmonic and holomorphic maps, exceptional Riemannian symmetric spaces of Hodge type, complex homogeneous fibrations, abelian subspaces, Levi-Civita connections, equivariant Hermitian symmetric subspaces

Mathematics Subject Classification 2000: 58E20, 53C35, 22E30, 32H02.

1. STATEMENT OF THE RESULTS

Let M be a compact Kähler manifold and $\Gamma \setminus G/K$ be a local Riemannian symmetric space of noncompact type. The results of Eells and Sampson from [7] imply that whenever $\Gamma \setminus G/K$ is compact, any continuous map $c : M \to \Gamma \setminus G/K$ is homotopic to a harmonic map $f : M \to \Gamma \setminus G/K$. Corlette has proved in [6] that a continuous map $c : M \to \Gamma \setminus G/K$ has a unique harmonic representative $f : M \to \Gamma \setminus G/K$ in its homotopy class if and only if the image $c_* \pi_1(M)$ of the induced representation $c_* : \pi_1(M) \to \Gamma$ has reductive real Zariski closure in G.

The present article studies the harmonic maps $f : M \to \Gamma \setminus G/K$ for which there exist parabolic subgroups $P \subset G^{\mathbb{C}}$, complex homogeneous fibrations $\Pi_P : G/G \cap P \to G/K$ and holomorphic liftings $F_P : M \to \Gamma \setminus G/G \cap P$, such that $f = \Pi_P F_P$. For the compact discrete quotients $\Gamma \setminus G/K$ of the irreducible classical Hermitian symmetric spaces G/K of noncompact type and $\dim_{\mathbb{C}} G/K \geq 3$, Siu has established in [14] that the harmonic maps $f : M \to \Gamma \setminus G/K$ of maximum constant $rank_x^{\mathbb{C}} df := \dim_{\mathbb{C}} df_x^{\mathbb{C}}(T_x^{1,0}M) = \dim_{\mathbb{C}} G/K$, $\forall x \in M$, are either holomorphic or anti-holomorphic. Following Helgason's classification [8] of the irreducible Riemannian symmetric spaces G/K of noncompact type, we recall the other known results for the harmonic maps $f : M \to \Gamma \setminus G/K$ of compact Kähler manifolds M. Carlson and Toledo show in [4] that the nonconstant non-holomorphic (non-anti-holomorphic) harmonic $f : M \to \Gamma \setminus SU(n,1)/S(U_n \times U_1)$ either map to a closed geodesic $f(M)$ or factor through a holomorphic map to a Riemann surface. For a harmonic map $f : M \to \Gamma \setminus SL(2n, \mathbb{R})/SO(2n)$ with $n \geq 3$ they establish in [5] that $rank_x^{\mathbb{C}} df \leq \dfrac{n(n+1)}{2}$ for $x \in M$, and the equality is realized only by the holomorphic maps of maximum constant rank onto an equivariantly embedded discrete quotient of $Sp(n,\mathbb{R})/U_n$. In [9] is proved that the harmonic maps $f : M \to \Gamma \setminus SL(2n+1, \mathbb{R})/SO(2n+1)$, $n \geq 4$, are of $rank_x^{\mathbb{C}} df \leq \dfrac{n(n+1)}{2} + 1$ at $x \in M$. The equality is attained by the holomorphic f with $f(M) = \Gamma_h \setminus G_h/K_h$ for a Hermitian symmetric (not necessarily equivariant) subspace $G_h/K_h \subset SL(2n+1, \mathbb{R})/SO(2n+1)$ or by a non-holomorphic f with $f(M) = \Gamma_0 \setminus (Sp(n,\mathbb{R})/U_n \times T^1)$, where $Sp(n,\mathbb{R})/U_n \subset SL(2n+1,\mathbb{R})/SO(2n+1)$ is an equivariant subspace and $T^1 \subset SL(2n+1,\mathbb{R})$ is a noncompact 1-dimensional torus, centralizing $Sp(n,\mathbb{R})$. Carlson and Toledo show in [5] that the harmonic maps $f : M \to \Gamma \setminus SU^*(2n)/Sp(n)$ with $n \geq 3$ have $rank_x^{\mathbb{C}} df \leq \dfrac{n(n-1)}{2}$ at $x \in M$, and the equality is attained by the holomorphic f onto a discrete quotient of an equivariantly embedded $SO^*(2n)/U_n$. For the harmonic maps $f : M \to \Gamma \setminus SO_0(n,1)/SO(n)$, Carlson and Toledo obtain in [4] that either $f(M)$ is a closed geodesic or f factors through a holomorphic map to a Riemann surface. In [5] they establish that the harmonic maps $f : M \to \Gamma \setminus SO_0(2m,2n)/SO(2m) \times SO(2n)$ with $\min(m,n) \geq 3$, $m+n > 6$, have $rank_x^{\mathbb{C}} df \leq mn$, $x \in M$, and the equality is attained by the holomorphic f onto discrete quotients of equivariantly embedded $SU(m,n)/S(U_m \times U_n)$. For the harmonic maps $f_1 : M \to \Gamma \setminus SO_0(2m+1,2n)/SO(2m+1) \times SO(2n)$ or $f_2 : M \to \Gamma \setminus SO_0(2m+1,2n+1)/SO(2m+1) \times SO(2n+1)$ with $\min(m,n) \geq 5$, the work [9] shows that $rank_x^{\mathbb{C}} df_1 \leq mn+1$, $rank_x^{\mathbb{C}} df_2 \leq mn+2$ and the equalities are attained by the holomorphic f_i onto discrete quotients of (not necessarily equivariant) Hermitian symmetric subspaces. In the case of $f : M \to \Gamma \setminus Sp(n,1)/Sp(n) \times Sp(1)$ with $n \geq 3$, Carlson and Toledo prove in [4] that either $f(M)$ is a closed geodesic or f factors through a holomorphic map to a Riemann surface or f has a holomorphic lifting $F : M \to \Gamma \setminus Sp(n,1)/Sp(n) \times U_1$. In [5] Carlson and Toledo establish that the harmonic $f : M \to \Gamma \setminus Sp(m,n)/Sp(m) \times Sp(n)$ with $\min(m,n) \geq 2$ have

$rank_x^{\mathbb{C}} df \leq mn$ and the equality is attained by the holomorphic f onto discrete quotients of equivariantly embedded $SU(m,n)/S(U_m \times U_n)$. Carlson and Hernandez show in [3] that a harmonic $f : M \to \Gamma \setminus F_{4(-20)}/SO(9) = \Gamma \setminus \mathbf{FII}$ either maps to a closed geodesic or factors through a holomorphic map to a Riemann surface, or factors through a holomorphic map $F : M \to \Gamma_0 \setminus SU(2,1)/S(U_2 \times U_1) = S$ to a discrete quotient S of a 2-ball, followed by a geodesic immersion $S \to \Gamma \setminus \mathbf{FII}$.

Let G be a noncompact simple real Lie group and P be a parabolic subgroup of its complexification $G^{\mathbb{C}}$. A necessary condition for the existence of a fibration $G/G \cap P \to G/K$ is the inclusion of $G \cap P$ in K. First of all, that requires the presence of a Cartan subalgebra $\mathbf{h} \subset \mathbf{g} := LieG$, contained in $\mathbf{k} := LieK$. The noncompact semisimple Lie groups G, whose Lie algebras admit common Cartan subalgebras with the Lie algebras of the maximal compact subgroups K of G, are said to be of Hodge type. According to Simpson [13] or Burstall and Rawnsley [2], a noncompact semisimple Lie group G is of Hodge type exactly when the Cartan involution of G is an inner automorphism. The isometry groups G of the irreducible Hermitian symmetric spaces G/K of noncompact type are groups of Hodge type. According to Simpson [13], the remaining noncompact simple Lie groups of Hodge type are

$$SO(m,2n), \quad Sp(m,n), \quad E_{6(2)}, \quad E_{7(7)}, \quad E_{7(-5)},$$
$$E_{8(8)}, \quad E_{8(-24)}, \quad F_{4(4)}, \quad F_{4(-20)}, \quad G_{2(2)}.$$

Let G_c be the compact real form of G. For a simple Lie group G of Hodge type and a parabolic subgroup $P \subset G^{\mathbb{C}}$ the inclusion $G \cap P \subset K$ is equivalent to $G \cap P = G_c \cap P$ and happens exactly when $G_c \cap P$ is a subgroup of K.

Let us recall that $G^{\mathbb{C}}/P = G_c/G_c \cap P$ is a projective algebraic manifold when $P \subset G^{\mathbb{C}}$ is a parabolic subgroup. For G of Hodge type we claim that the orbit $G/G \cap P$ is an open subset of $G_c/G_c \cap P$. If $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ is the Cartan decomposition of $\mathbf{g}$, then the tangent space $T_{\check{o}}^{\mathbb{R}} G/K$ at the origin $\check{o} \in G/K$ can be identified with $\mathbf{p}$. The exponential map $Exp_{\check{o}}^{G/K} : \mathbf{p} \to G/K$ at $\check{o} \in G/K$ is a global diffeomorphism, due to the nonpositiveness of the sectional curvatures of G/K. Let $Exp_{\check{o}}^{G_c/K} : T_{\check{o}}^{\mathbb{R}} G_c/K = i\mathbf{p} \to G_c/K$ be the locally defined exponential map of the compact dual G_c/K at $\check{o} \in G_c/K$ and $\mu_i : \mathbf{p} \to i\mathbf{p}$ be the multiplication by the imaginary unit i. Then $Exp_{\check{o}}^{G_c/K} \mu_i \left(Exp_{\check{o}}^{G/K} \right)^{-1} : G/K \to G_c/K$ is a local diffeomorphism. Since $G/G \cap P$ and $G_c/G_c \cap P$ have coinciding fibers $K/G \cap P = K/G_c \cap P$, the homogeneous space $G/G \cap P$ is immersed in $G_c/G_c \cap P$. In particular, $G/G \cap P$ is a complex (even Kähler) manifold.

Recall also that for a group G of Hodge type and a parabolic subgroup $P \subset G^{\mathbb{C}}$ with $G \cap P \subset K$ the reductive Lie group $G \cap P$ is a centralizer of a torus $T \subset K$ in G. Conversely, any centralizer $Z \subset G$ of a torus $T \subset K$ determines uniquely the parabolic subgroup P, whose semisimple part is the complexification of the semisimple part of Z.

Let $\mathbf{h}$ be a common Cartan subalgebra of $\mathbf{k}, \mathbf{g}$ and

$$\mathbf{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C} X_\sigma + \sum_{\sigma \in \Delta^+} \mathbb{C} X_{-\sigma},$$

$$\mathbf{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta_c^+} \mathbb{C} X_\sigma + \sum_{\sigma \in \Delta_c^+} \mathbb{C} X_{-\sigma}, \qquad \Delta_c^+ \subset \Delta^+,$$

be the corresponding root decompositions of the complexified Lie algebras. A parabolic subalgebra of $\mathbf{g}^{\mathbb{C}} = LieG^{\mathbb{C}} = \mathbf{g} \otimes_{\mathbb{R}} \mathbb{C}$ is of the form

$$LieP = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C} X_{-\sigma} + \sum_{\sigma \in \Delta^+(P)} \mathbb{C} X_\sigma$$

for an appropriate subset $\Delta^+(P) \subset \Delta_c^+$. The minimal parabolic subgroup $B \subset G^{\mathbb{C}}$ with $LieB = \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} + \sum_{\sigma \in \Delta^+} \mathbb{C} X_{-\sigma}$ is called a Borel subgroup. The corresponding $G/G \cap B \to G/K$ is referred to as a maximal complex homogeneous fibration. The Borel subgroup $B \subset G^{\mathbb{C}}$ intersects the real form G in the common maximal torus $T = G \cap B$ of K, G with $LieT = \mathbf{h}$ and centralizes itself. The maximal complex homogeneous fibration G/T contains an equivariant Hermitian symmetric subspace G_h/K_h if and only if G_h/K_h is a polydisc. Any parabolic subgroup $T \subset P \subset G^{\mathbb{C}}$ contains the Borel subgroup $B \supset T$. That determines a fibration $G/G \cap B \to G/G \cap P$ with a holomorphic projection. The existence of a holomorphic lifting $F_B : M \to \Gamma \backslash G/T$ of a harmonic map $f : M \to \Gamma \backslash G/K$ implies the existence of holomorphic liftings $F_P : M \to \Gamma \backslash G/G \cap P$ for all parabolic subgroups $P \supset T$. The complex homogeneous fibrations $G/G \cap P$, associated with centralizers $G \cap P$ of 1-dimensional tori $T^1 \subset K$, are called minimal.

Let $J(G_c/K) \to G_c/K$ be the bundle of the Hermitian almost complex structures on G_c/K. Burstall and Rawnsley show in [2] that for any parabolic subgroup $P \subset G^{\mathbb{C}}$ the quotient $G^{\mathbb{C}}/P = G_c/G_c \cap P$ is a holomorphically embedded subspace of $J(G_c/K)$. Therefore the open subset $G/G \cap P$ of $G_c/G_c \cap P$ is also a holomorphically embedded subspace of the twistor fibration $J(G_c/K) \to G_c/K$. Consequently, any holomorphic lifting $F : M \to \Gamma \backslash G/G \cap P$ of a harmonic map $f : M \to \Gamma \backslash G/K$ can be regarded as a local holomorphic map to the twistor fibration.

The results of the present article are summarized in the following

Theorem 1. (i) *There are two minimal complex homogeneous fibrations $G_{2(2)}/G_{2(2)} \cap P_i \to G_{2(2)}/SO(4)$, $i = 1, 2$, with fibers $\mathbf{CP}^1$ and a maximal complex homogeneous fibration $G_{2(2)}/T^2 \to G_{2(2)}/SO(4)$ with fiber $\mathbf{CP}^1 \times \mathbf{CP}^1$. A harmonic map $f : M \to \Gamma \backslash G_{2(2)}/SO(4)$ with $df^{\mathbb{C}}(T_x^{1,0}M)$, $\forall x \in M$, consisting of nilpotents and of maximum constant $rank_x^{\mathbb{C}} df = 3$, admits a holomorphic lifting to either of the complex homogeneous fibrations. Neither of the corresponding holomorphic images is an equivariantly embedded local Hermitian symmetric subspace.*

(ii) *Any harmonic map $f : M \to \Gamma \backslash F_{4(4)}/Sp(3) \times SU(2)$ with $df^{\mathbb{C}}(T_x^{1,0}M)$, $\forall x \in M$, consisting of nilpotents and maximum constant $rank_x^{\mathbb{C}} df = 7$ admits holomorphic liftings $F_P : M \to \Gamma \backslash F_{4(4)}/F_{4(4)} \cap P$ to all complex homogeneous fibrations. The images of these F_P are not equivariant local Hermitian symmetric subspaces.*

(iii) *A harmonic map $f : M \to \Gamma \backslash E_{6(2)}/SU(6) \times SU(2)$ of maximum constant $rank_x^{\mathbb{C}} df = 10$ with $ad\mathbf{h}$-invariant $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J)) g_x^{-1}$,* *labeled by an* $E_{6(2)}$*-admissible index set* J*, has a holomorphic lifting* $F_B : M \to \Gamma \backslash E_{6(2)}/T^6$ *to a maximal complex homogeneous fibration. There are sufficient conditions for nonexistence of equivariant Hermitian symmetric subspaces* $G_h/K_h \subset G/G \cap P$ *with* $F_P(M) = \Gamma_h \backslash G_h/K_h$.

(iv) *If the harmonic map* $f : M \to \Gamma \backslash E_{7(7)}/SU(8)$ *has* $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J,K))g_x^{-1}$ *for an* $E_{7(7)}$*-admissible set of indices* J, K*, then there is a holomorphic lifting* $F_B : M \to \Gamma \backslash E_{7(7)}/T^7$ *to a maximal complex homogeneous fibration. There are sufficient conditions for nonexistence of equivariant Hermitian symmetric subspaces* $G_h/K_h \subset E_{7(7)}/E_{7(7)} \cap P$ *with* $F_P(M) = \Gamma_h \backslash G_h/K_h$.

(v) *For any harmonic* $f : M \to \Gamma \backslash E_{7(-5)}/SO(12) \times SU(2)$ *with maximal* $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K))g_x^{-1}$ *there is a holomorphic lifting* $F_P : M \to \Gamma \backslash E_{7(-5)}/SO(12) \times T^1$ *to a minimal complex homogeneous fibration. For an* $E_{7(-5)}$*-admissible set of indices* I, K*, there exists a holomorphic lifting* $F_B : M \to \Gamma \backslash E_{7(-5)}/T^7$ *to a maximal complex homogeneous fibration. There is a list of sufficient conditions for nonexistence of equivariantly embedded Hermitian symmetric subspaces* $G_h/K_h \subset E_{7(-5)}/E_{7(-5)} \cap P$ *with* $F_P(M) = \Gamma_h \backslash G_h/K_h$.

(vi) *If a harmonic map* $f : M \to \Gamma \backslash E_{8(8)}/SO(16)$ *has* $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K))g_x^{-1}$ *for an* $E_{8(8)}$*-semi-admissible index set* J, K *of second kind, then there is a holomorphic lifting* $F_P : M \to \Gamma \backslash E_{8(8)}/U_8 \times T^1$ *to a minimal complex homogeneous fibration. Whenever* $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i)g_x^{-1}$, $i = 1,2$*, for a commutative root system* $C_1(I,J,K)$ *or* $C_2(J,K)$ *with* $E_{8(8)}$*-admissible index sets of first or second kind, there is a holomorphic lifting* $F_B : M \to \Gamma \backslash E_{8(8)}/T^8$ *to a maximal complex homogeneous fibration. There is a set of sufficient conditions for nonexistence of equivariant Hermitian symmetric* $G_h/K_h \subset E_{8(8)}/E_{8(8)} \cap P$ *with* $F_P(M) = \Gamma_h \backslash G_h/K_h$.

(vii) *Any harmonic map* $f : M \to \Gamma \backslash E_{8(-24)}/E_7 \times SU(2)$ *with maximal* $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J))g_x^{-1}$, $i = 1,2$*, admits a holomorphic lifting* $F_P : M \to \Gamma \backslash E_{8(-24)}/E_7 \times T^1$ *to a minimal complex homogeneous fibration. If, moreover,* I_1, I_2, J *is an* $E_{8(-24)}$*-admissible set of indices of i-th kind, then there exists a holomorphic lifting* $F_B : M \to \Gamma \backslash E_{8(-24)}/T^8$ *to a maximal complex homogeneous fibration. Under certain conditions on* I_1, I_2, J *there is no equivariant Hermitian symmetric image* $F_P(M) = \Gamma_h \backslash G_h/K_h$.

The notions of admissible index sets and the sufficient conditions for nonexistence of equivariant locally Hermitian symmetric images will be clearified separately for each exceptional Riemannian symmetric space under consideration.

Here is an interpretation of a part of the already mentioned results on harmonic maps as existence of holomorphic liftings, whenever they exist. Since the Hermitian symmetric G/K of noncompact type are complex homogeneous spaces, Siu's result [14] can be viewed as an existence of a holomorphic lifting to the fibration with a trivial fiber. Similarly, Carlson and Toledo's article [4] specifies that a harmonic map $f : M \to \Gamma \backslash SU(n,1)/S(U_n \times U_1)$, whose image is not a closed geodesic and which does not factor through a holomorphic map to a Riemann surface, admits a holomorphic lifting to the complex homogeneous fibration with a trivial fiber. For a harmonic map $f : M \to \Gamma \backslash SO_0(2m, 2n)/SO(2m) \times SO(2n)$ with $\min(m, n) \geq 3$, $m + n > 6$, $rank_x^{\mathbb{C}} df = mn$ for all $x \in M$, Carlson and Toledo's results from [5] can be interpreted as an existence of a holomorphic lifting $F_P : M \to \Gamma \backslash SO_0(2m, 2n)/U_m \times U_n \times T^2$ to a complex homogeneous fibration. The results of [9] imply that a harmonic map $f : M \to \Gamma \backslash SO_0(2m+1, 2n)/SO(2m+1) \times SO(2n)$ with $\min(m, n) \geq 5$ and a constant $rank_x^{\mathbb{C}} df = mn + 1$ admits a holomorphic lifting $F_P : M \to \Gamma \backslash SO_0(2m+1, 2n)/U_m \times U_{n-1} \times T^3$. Concerning the harmonic maps $f : M \to \Gamma \backslash Sp(n, 1)/Sp(n) \times Sp(1)$ with $n \geq 3$, which do not map to a closed geodesic and do not factor through holomorphic maps to Riemann surfaces, Carlson and Toledo prove in [4] the existence of a holomorphic lifting $F_P : M \to \Gamma \backslash Sp(n, 1)/Sp(n) \times U_1$ to a complex homogeneous fibration. Carlson and Toledo's results from [5] reveal that a harmonic map $f : M \to \Gamma \backslash Sp(m, n)/Sp(m) \times Sp(n)$ with $\min(m, n) \geq 2$ and constant $rank_x^{\mathbb{C}} df = mn$ admits a holomorphic lifting $F_P : M \to \Gamma \backslash Sp(m, n)/U_m \times U_n \times T^2$ to a complex homogeneous fibration. In [3] Carlson and Hernandez establish that the harmonic maps $f : M \to \Gamma \backslash F_{4(-20)}/SO(9)$, whose image is not a closed geodesic and which do not factor through holomorphic maps to Riemann surfaces, admit holomorphic liftings $F_P : M \to \Gamma \backslash F_{4(-20)}/S(U_2 \times U_3)$ to complex homogeneous fibrations.

2. BASIC TECHNIQUES OF THE ARGUMENT

The proof of Theorem 1 is based on Sampson's result [11] for the harmonic maps $f : M \to \Gamma \backslash G/K$ of compact Kähler manifolds M into local Riemannian symmetric spaces $\Gamma \backslash G/K$ of noncompact type. It asserts that such f are pluriharmonic and $df^{\mathbb{C}}(T_x^{1,0} M)$ are abelian subspaces of $T_{f(x)}^{\mathbb{C}} \Gamma \backslash G/K$ for all $x \in M$.

In order to formulate precisely, let us recall few basics of the structure theory of semisimple Lie algebras. Assume that $\mathbf{g} := LieG$ for a noncompact simple Lie group G of Hodge type and fix a common Cartan subalgebra $\mathbf{h}$ of $\mathbf{k}(G) := LieK$ and $\mathbf{g}$, where K is a maximal compact subgroup of G. There is a Killing orthogonal Cartan decomposition $\mathbf{g} := \mathbf{k}(G) \oplus \mathbf{p}(G)$. Its complexification $\mathbf{g}^{\mathbb{C}} = \mathbf{k}^{\mathbb{C}}(G) \oplus \mathbf{p}^{\mathbb{C}}(G)$ is invariant under the adjoint action of $\mathbf{h}^{\mathbb{C}} := \mathbf{h} \otimes_{\mathbb{R}} \mathbb{C}$. More precisely, $\mathbf{k}^{\mathbb{C}}(G) := \mathbf{k}(G) \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{h}^{\mathbb{C}} + \sum_{\sigma \in \Delta_c(G)} \mathbb{C} X_\sigma$ and $\mathbf{p}^{\mathbb{C}}(G) := \mathbf{p}(G) \otimes_{\mathbb{R}} \mathbb{C} = \sum_{\sigma \in \Delta_{nc}(G)} \mathbb{C} X_\sigma$ for an appropriate decomposition $\Delta(G) = \Delta_c(G) \cup \Delta_{nc}(G)$ into a disjoint union of compact and noncompact roots. An arbitrary ordering on $\Delta(G)$, compatible with the Lie bracket of the corresponding root vectors, introduces splittings into disjoint unions $\Delta_c(G) = \Delta_c^+(G) \cup \Delta_c^-(G)$, $\Delta_{nc}(G) = \Delta_{nc}^+(G) \cup \Delta_{nc}^-(G)$, whereas $\Delta(G) = \Delta^+(G) \cup \Delta^-(G)$ with $\Delta^+(G) = \Delta_c^+(G) \cap \Delta_{nc}^+(G)$, $\Delta^-(G) = \Delta_c^-(G) \cup \Delta_{nc}^-(G)$. The pairs of positive and negative root vectors are complex conjugate to each other, $\overline{X_\sigma} = X_{-\sigma}$. Observe also that the root system $\Delta(G)$ and its decomposition $\Delta(G) = \Delta^+(G) \cup \Delta^-(G)$ depend only on the complexification $G^{\mathbb{C}}$ but not on the real form G. We take $\Delta(G_{2(2)}) = \Delta(G_2^{\mathbb{C}})$ from Sato and Kimura's paper [12] and borrow the other $\Delta(G) = \Delta(G^{\mathbb{C}})$ from the Table of Bourbaki's book [1]. The notation $G_{(n)}$ stands for the real form of $G^{\mathbb{C}}$ with $\dim_{\mathbb{R}} \mathbf{p}(G_{(n)}) - \dim_{\mathbb{R}} \mathbf{k}(G_{(n)}) = n$

(cf. [8]). In order to avoid the explicit matrix realization of the root vectors X_σ, $\sigma \in \Delta(G)$, and the calculation of their Lie brackets, let us introduce structure constants $N_{\sigma,\tau}$, such that $[X_\sigma, X_\tau] = N_{\sigma,\tau} X_{\sigma+\tau}$ whenever $\sigma + \tau \in \Delta(G)$.

At the origin $\check{o} \in G/K$, the complexified tangent space $T^{\mathbb{C}}_{\check{o}} G/K := T^{\mathbb{R}}_{\check{o}} G/K \otimes_{\mathbb{R}} \mathbb{C} = \mathbf{p}^{\mathbb{C}}(G)$ and the holomorphic tangent space of a Hermitian symmetric G_h/K_h is $T^{1,0}_{\check{o}} G_h/K_h = \mathbf{p}^+_h$. At an arbitrary point $gK \in G/K$ the tangent spaces $T^{\mathbb{R}}_{gK} G/K = g\mathbf{p}(G)g^{-1}$ and $T^{\mathbb{C}}_{gK} G/K = g\mathbf{p}^{\mathbb{C}}(G)g^{-1}$.

Carlson and Toledo have established in [5] that for any abelian subspace $\mathbf{a} \subset \mathbf{p}^{\mathbb{C}}$, which consists entirely of nilpotent elements, there exists a Cartan subalgebra $\mathbf{h} \subset \mathbf{g}$ with respect to which $\mathbf{a} \subset \mathbf{p}^+$. The construction of $\mathbf{h}$ reveals that whenever G is of Hodge type, this Cartan subalgebra is contained in $\mathbf{k}(G)$. Whenever $df^{\mathbb{C}}(T^{1,0}_{x_0} M)$ is an abelian subspace of $\mathbf{p}^+$, $f(x_0) = \check{o}$, the complexified differential of f is represented by $\theta + \overline{\theta}$ for an appropriate $\theta \in \Omega^{1,0}_M(\mathbf{p}^+)$. Let ∇ be the flat Levi-Civita connection of the locally trivial bundle $f^* T^{\mathbb{R}}(\Gamma \setminus G/K)$. It decomposes into a sum $\nabla = D + \theta + \overline{\theta}$, where D is a $\mathbf{k}(G)$-valued connection. Further decomposition into $(1,0)$- and $(0,1)$-types provides $D = D' + D''$ with $\overline{D'} = D''$. The pluriharmonic equation for f reads as

$$D''\theta = 0. \tag{1}$$

On the other hand, the $(2,0)$-component with values in $\mathbf{p}^{\mathbb{C}}$ of the flatness equation $\nabla^2 = 0$ provides

$$D'\theta = 0. \tag{2}$$

For some specific $df^{\mathbb{C}}(T^{1,0}_{x_0} M)$ or, equivalently, θ, the equations (1) and (2) reduce D to a $Lie(G \cap P)$-valued connection for an appropriate parabolic subgroup $P \subset G^{\mathbb{C}}$. That implies the existence of a lifting $F_P : M \to \Gamma \setminus G/G \cap P$ of $f : M \to \Gamma \setminus G/K$. If $f(x) = \Gamma g_x K$ and

$$dF^{\mathbb{C}}_P(T^{1,0}_x M) = df^{\mathbb{C}}(T^{1,0}_x M) \subset g_x \mathbf{p}^+ g_x^{-1} \subset g_x(\mathbf{p}^+ \oplus \sum_{\sigma \in \Delta^+_c(G) - \Delta^+(P)} \mathbb{C} X_\sigma) g_x^{-1}$$

$$= T^{1,0}_{\Gamma g_x (G \cap P)} \Gamma \setminus G/G \cap P$$

for all $x \in M$, then the lifting F_P is holomorphic. That is why, it is natural to assume that $df^{\mathbb{C}}(T^{1,0}_x M)$ consist entirely of nilpotent elements for all $x \in M$, in order to look for holomorphic liftings of $f : M \to \Gamma \setminus G/K$.

For the proof of the main Theorem 1, we have to characterize the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ of maximum $\dim_{\mathbb{C}} \mathbf{a} = 3$ and the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+(F_{4(4)})$ of maximum $\dim_{\mathbb{C}} \mathbf{a} = 7$. To this end, we apply Malcev's method of the leading root vectors for studying abelian subspaces of nilpotents in semisimple Lie algebras (cf. [10]). More precisely, Gauss-Jordan elimination on a basis $Y_1, \ldots, Y_k$

of an abelian subspace $\mathbf{a} \subset \mathbf{p}^+$ allows to represent

$$\begin{array}{llllll} Y_1 = & X_{\sigma_1} & & & + \sum_{\tau \neq \sigma_1, \dots, \sigma_k} & y_1^\tau X_\tau, \\ Y_2 = & & X_{\sigma_2} & & + \sum_{\tau \neq \sigma_1, \dots, \sigma_k} & y_2^\tau X_\tau, \\ \dots & \dots & \dots & \dots & & \dots \\ Y_k = & & & X_{\sigma_k} & + \sum_{\tau \neq \sigma_1, \dots, \sigma_k} & y_k^\tau X_\tau \end{array}$$

by positive noncompact root vectors with $\sigma_1 < \sigma_2 < \ldots < \sigma_k$, $\sigma_i < \tau$, for all $X_\tau \in SuppY_i$ and $y_i^\tau \in \mathbb{C}$. According to the compatibility of the Lie bracket with the ordering, the equality $0 = [Y_i, Y_j] = [X_{\sigma_i}, X_{\sigma_j}] + [X_{\sigma_i}, \sum y_j^\tau X_\tau] + [\sum y_i^\tau X_\tau, X_{\sigma_j}] + [\sum y_i^\tau X_\tau, \sum y_j^\tau X_\tau]$ implies the vanishing of the minimal term $[X_{\sigma_i}, X_{\sigma_j}] = 0$. Thus, the root system $C = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is commutative, i.e.,

$$\forall \sigma_i, \sigma_j \in C \Rightarrow \sigma_i + \sigma_j \not\in \Delta(G).$$

The commutative root systems $C \subset \Delta_{nc}^+(G)$ are studied up to the Weyl group action. Accordingly, the abelian subspaces $\mathbf{a} \subset \mathbf{p}^+$ are described modulo the adjoint action of $K^{\mathbb{C}}$.

Let us assume that there exists an equivariant Hermitian symmetric subspace $G_h/K_h \subset G/G \cap P$ with $T_{\check{o}}^{1,0} G_h/K_h = \mathbf{a}$ for some parabolic subgroup $P \subset G^{\mathbb{C}}$. Then the Lie bracket of $\mathbf{g}_h := LieG_h$ is the restriction of the Lie bracket of $\mathbf{g} := LieG$. The same holds for the corresponding complexifications. If $\mathbf{a} = \mathbf{p}_h^+$, then $[\mathbf{a}, \bar{\mathbf{a}}] \subseteq \mathbf{k}_h^{\mathbb{C}}$ and $[\mathbf{a}, [\mathbf{a}, \bar{\mathbf{a}}]] \subseteq \mathbf{a}$. When $\mathbf{a} = Span_{\mathbb{C}}(X_\sigma | \sigma \in C)$, the presence of $\sigma_1, \sigma_2, \sigma_3 \in C$ with $\sigma_2 - \sigma_3 \in \Delta_c(G)$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(G) - C$ rejects the existence of an equivariant Hermitian symmetric $G_h/K_h \subset G/G \cap P$ with $T_{\check{o}}^{1,0} G_h/K_h = \mathbf{a}$.

For each of the noncompact exceptional simple Lie groups $G \neq E_{6(-14)}, E_{7(-25)}, F_{4(-20)}$ of Hodge type are constructed examples of harmonic maps $f : M \to \Gamma \setminus G/K$, which do not admit holomorphic liftings. Let $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma, X_{-\sigma}, X_\tau | \sigma \in S_1, \tau \in S_2) g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x K$, where $S_1 \neq \varnothing$ and the disjoint union $S = S_1 \cup S_2 \subset \Delta_{nc}^+(G)$ is strongly commutative, i.e.,

$$\forall \sigma, \tau \in S \Rightarrow \sigma + \tau \not\in \Delta(G) \text{ and } \sigma - \tau \not\in \Delta(G).$$

Then a lifting $F_P : M \to \Gamma \setminus G/G \cap P$ to a complex homogeneous fibration is not holomorphic, according to $dF_P^{\mathbb{C}}(T_x^{1,0}M) = df^{\mathbb{C}}(T_x^{1,0}M) \not\subset T_{\Gamma g_x(G\cap P)}^{1,0} \Gamma \setminus G/G \cap P$. For specific examples of strongly commutative $S \subset \Delta_{nc}^+(G)$, we refer to the next sections.

3. $\mathbf{G} = G_{2(2)}/SO(4)$

The complexified Lie algebra $\mathbf{g}_2^{\mathbb{C}}$ admits a representation by (7×7)-matrices and can be identified with the derivations of the Cayley numbers (cf. [12]). We use the system of the positive roots $\Delta^+(G_2^{\mathbb{C}}) = \{e_1, e_2, e_1+e_2, e_1-e_2, e_1+2e_2, 2e_1+e_2\}$, borrowed from [12]. The Lie algebra $\mathbf{g}_{2(2)}$ of Hodge type admits a 2-dimensional

Cartan subalgebra $\mathbf{h} \subset so(4)$. The complexified isotropy subalgebra $so(4,\mathbb{C}) = \mathbf{h}^{\mathbb{C}} + \sum_{i=1}^{2} \mathbb{C}X_{\sigma_i} + \sum_{i=1}^{2} \mathbb{C}X_{-\sigma_i}$, where the compact roots σ_1, σ_2 have one and the same length and $\sigma_1 + \sigma_2 \notin \Delta(G_2^{\mathbb{C}})$. Bearing in mind that $\sigma, \tau \in \Delta^+_{nc}(G_{2(2)}), \sigma + \tau \in \Delta^+(G_{2(2)}) \Rightarrow \sigma + \tau \in \Delta^+_c(G_{2(2)})$, one specifies that $\Delta^+_c(G_{2(2)}) = \{e_1 - e_2, e_1 + e_2\}$, whereas $\Delta^+_{nc}(G_{2(2)}) = \{e_1, e_2, e_1 + 2e_2, 2e_1 + e_2\}$. This choice is also subject to $\sigma \in \Delta^+_c(G_{2(2)}), \tau \in \Delta^+_{nc}(G_{2(2)}), \sigma + \tau \in \Delta^+(G_{2(2)}) \Rightarrow \sigma + \tau \in \Delta^+_{nc}(G_{2(2)})$.

The only restriction to which a commutative root system $C \subset \Delta^+_{nc}(G_{2(2)})$ obeys is not to contain simultaneously e_1 and e_2. Up to the action of the Weyl group of $SO(4,\mathbb{C})$, which is generated by the permutation of e_1 with e_2 and their simultaneous sign changes, one can assume that the maximal commutative root system $C = \{e_1, e_1 + 2e_2, 2e_1 + e_2\}$.

Lemma 2. *The 3-dimensional abelian subspaces $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ are $SO(4,\mathbb{C})$-conjugate to*

$$\mathbf{a}_4 = Span_{\mathbb{C}}(X_{e_1}, \quad X_{e_1+2e_2}, \quad X_{2e_1+e_2}).$$

Proof. An abelian $\mathbf{a} \subset \mathbf{p}^+(G_{2(2)})$ with a leading root system $C = \{e_1, e_1 + 2e_2, 2e_1 + e_2\}$ has generators

$$Y_1' = X_{e_1} + a_1 X_{e_2}, \quad Y_2' = X_{e_1+2e_2} + a_2 X_{e_2}, \quad Y_3' = X_{2e_1+e_2} + a_3 X_{e_2}.$$

After the action of

$$AdExp\left(\frac{-a_1}{N_{-e_1+e_2,e_1}} X_{-e_1+e_2}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} ad^k \left(\frac{-a_1}{N_{-e_1+e_2,e_1}} X_{-e_1+e_2}\right),$$

followed by an elimination of $X_{e_1+2e_2}$ from the image of Y_3', one gets $Y_1'' = X_{e_1}$, $Y_2'' = X_{e_1+2e_2} + a_2 X_{e_2}$, $Y_3'' = X_{2e_1+e_2} + a_3 X_{e_2}$. The commutations $[Y_1'', Y_2''] = 0$ and $[Y_1'', Y_3''] = 0$ reveal the vanishing of a_2 and a_3, Q.E.D.

Let us describe the parabolic subgroups $P \subset G_2^{\mathbb{C}}$. According to $card\Delta^+_c = 2$, there are a Borel subgroup $B \subset G_2^{\mathbb{C}}$ with $LieB = \mathbf{h} + \sum_{\sigma \in \Delta^+(G_2^{\mathbb{C}})} \mathbb{C}X_{-\sigma}$ and two maximal parabolic subgroups $P_1, P_2 \subset G_2^{\mathbb{C}}$ with $LieP_1 = LieB + \mathbb{C}X_{e_1-e_2}$ and $LieP_2 = LieB + \mathbb{C}X_{e_1+e_2}$. Clearly, $G_{2(2)} \cap B = T^2 = Exp_1^{G_{2(2)}}(\mathbb{R}H_1 + \mathbb{R}H_2)$ centralizes itself, $G_{2(2)} \cap P_1 \simeq SU(2) \times T^1_+$ centralizes the 1-dimensional torus $T^1_+ = Exp_1^{G_{2(2)}}(\mathbb{R}(H_1 + H_2))$ and $G_{2(2)} \cap P_2 \simeq SU(2) \times T^1_-$ centralizes the 1-dimensional torus $T^1_- = Exp_1^{G_{2(2)}}(\mathbb{R}(H_1 - H_2))$. Bearing in mind that $SO(4) = SU(2) \times SU(2)$, one observes that the complex homogeneous fibrations $G_{2(2)}/G_{2(2)} \cap P_i \to G_{2(2)}/SO(4)$, $i = 1, 2$, have fibers $SU(2)/S^1 = SU(2)/S(U_1 \times U_1) = \mathbf{CP}^1$ and $G_{2(2)}/T^2 \to G_{2(2)}/SO(4)$ has a fiber $\mathbf{CP}^1 \times \mathbf{CP}^1$. There are also fibrations $G_{2(2)}/T^2 \to G_{2(2)}/G_{2(2)} \cap P_i$ with fibers $\mathbf{CP}^1$ and holomorphic projections.

Lemma 3. *Let $f : M \to \Gamma \setminus G_{2(2)}/SO(4)$ be a harmonic map of a compact Kähler manifold M with maximum constant $rank^{\mathbb{C}}_x df = 3$ and $df^{\mathbb{C}}(T^{1,0}_x M)$, $\forall x \in M$, consisting entirely of nilpotent elements. Then there exists a holomorphic lifting*

$F_B : M \to \Gamma \setminus G_{2(2)}/T^2$ to a maximal complex homogeneous fibration. Neither of $f(M)$, $F_B(M)$, $F_{P_1}(M)$ or $F_{P_2}(M)$ is a local equivariantly embedded Hermitian symmetric subspace.

Proof. The $(0,1)$-part of the $so(4)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{2} \overline{\xi_i} \otimes H_i + \overline{p} \otimes X_{e_1-e_2} + \overline{r} \otimes X_{-e_1+e_2} + \overline{\zeta} \otimes X_{e_1+e_2} + \overline{z} \otimes X_{-e_1-e_2}$$

for some $\xi_i, \eta_{ij}, \zeta, z \in \Omega^{1,0}_M$. The holomorphic differential of f is represented by the 1-form

$$\theta = dx^1 \otimes X_{e_1} + dx^2 \otimes X_{e_1+2e_2} + dx^3 \otimes X_{2e_1+e_2}.$$

Wedging the forms and computing the Lie bracket of the root vectors and Cartan generators, one obtains $D''\theta = (\overline{\xi_1} \wedge dx^1 + N_{-e_1-e_2,2e_1+e_2}\overline{z} \wedge dx^3) \otimes X_{e_1} + (N_{-e_1+e_2,e_1}\overline{r} \wedge dx^1 + N_{-e_1-e_2,e_1+2e_2}\overline{z} \wedge dx^2) \otimes X_{e_2} + N_{-e_1-e_2,e_1}\overline{z} \wedge dx^1 \otimes X_{-e_2} + [(\overline{\xi_1}+2\overline{\xi_2}) \wedge dx^2 + N_{-e_1+e_2,2e_1+e_2}\overline{r} \wedge dx^3] \otimes X_{e_1+2e_2} + [(2\overline{\xi_1}+\overline{\xi_2}) \wedge dx^3 + N_{e_1-e_2,e_1+2e_2}\overline{p} \wedge dx^2 + N_{e_1+e_2,e_1}\overline{\zeta} \wedge dx^1] \otimes X_{2e_1+e_2} = 0$. Bearing in mind the $\mathbb{C}$-linear independence of the root vectors and the functional independence of dx^1, dx^2, dx^3, one derives that $D'' = \overline{\partial}$. Therefore $D = \partial + \overline{\partial} = d$ takes values in $\mathbf{h} = Lie(G_{2(2)} \cap B)$ and there is a holomorphic lifting $F_B : M \to \Gamma \setminus G_{2(2)}/T^2$.

The only 3-dimensional Hermitian symmetric spaces of noncompact type are $SU(3,1)/S(U_3 \times U_1) \simeq SO^*(6)/U_3$ with 9-dimensional complexified isotropy subalgebra and $SO(3,2)/SO(3) \times SO(2) \simeq Sp(2)/U_2$ with 3-dimensional complex isotropy subalgebra. They both satisfy $\mathbf{k}_h^{\mathbb{C}} = [\mathbf{p}_h^{\mathbb{C}}, \mathbf{p}_h^{\mathbb{C}}]$. For the abelian subspace $\mathbf{a} = Span_{\mathbb{C}}(X_{e_1}, X_{e_1+2e_2}, X_{2e_1+e_2})$ it is straightforward that $[\mathbf{a}+\overline{\mathbf{a}}, \mathbf{a}+\overline{\mathbf{a}}] = so(4,\mathbb{C})$ is of dimension 6. That would contradict an assumption $\mathbf{a} = \mathbf{p}_h^+ = \mathbf{p}^+(G_h)$ for an equivariant Hermitian symmetric subspace G_h/K_h, Q.E.D.

The strongly commutative subsets $S \subset \Delta^+_{nc}(G_{2(2)})$ are commutative. Therefore, one can assume that $S \subset \{e_1, e_1+2e_2, 2e_1+e_2\}$. Bearing in mind that $(2e_1 + e_2) - e_1 = e_1 + e_2$, $(2e_1+e_2)-(e_1+2e_2) = e_1 - e_2$, one determines $S = \{e_1, e_1+2e_2\}$ of maximal cardinality, up to $Weyl(SO(4,\mathbb{C}))$-action. The harmonic maps $f : M \to \Gamma \setminus G_{2(2)}/SO(4)$ with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{-e_1}, X_{e_1+2e_2}, X_{-e_1-2e_2}) g_x^{-1}$,

$$df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{-e_1}, X_{e_1+2e_2}) g_x^{-1}$$

or

$$df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_{e_1}, X_{e_1+2e_2}, X_{-e_1-2e_2}) g_x^{-1}$$

have no holomorphic liftings to complex homogeneous fibrations.

4. $\mathbf{FI} = F_{4(4)}/Sp(3) \times SU(2)$

Let us recall from Bourbaki's Table [1]

$$\Delta^+(F_4^{\mathbb{C}}) = \Big\{ e_i (1 \leq i \leq 4), \quad e_i \pm e_j (1 \leq i < j \leq 4),$$

$$\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \mu e_4)(\varepsilon, \nu, \mu \in \{\pm 1\}) \Big\}.$$

One needs to decompose into a disjoint union $\Delta^+(F_4^{\mathbb{C}}) = \Delta_c^+(F_{4(4)}) \cup \Delta_{nc}^+(F_{4(4)})$, where $\Delta_c^+(F_{4(4)}) = \Delta^+(Sp(3,\mathbb{C})) \cup \Delta^+(SL(2,\mathbb{C}))$. Observe that the positive roots of $Sp(3,\mathbb{C})$ can be expressed by two short simple roots α_1, α_2 and a long simple root α_3. Among the short roots e_i, $\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \mu e_4)$ and the long roots $e_i \pm e_j$ of $F_4^{\mathbb{C}}$, the only possible choices are $\alpha_1 = e_i$, $\alpha_2 = \frac{1}{2}(e_1 - e_i - e_j \mp e_k)$, $\alpha_3 = e_j \pm e_k$. Up to the action of the Weyl group of $F_4^{\mathbb{C}}$, let us specify $\alpha_1 = e_2$, $\alpha_2 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, $\alpha_3 = e_3 + e_4$. As far as $Sp(3)$ and $SU(2)$ are in a direct product in the isotropy subgroup, for any $\sigma \in \Delta^+(Sp(3,\mathbb{C}))$ and the only positive root τ of $SL(2,\mathbb{C})$ the sum $\sigma + \tau$ is not a root of $F_4^{\mathbb{C}}$. That determines $\tau = e_3 - e_4$, so that

$$\Delta_c^+(F_{4(4)}) = \left\{ e_i (1 \leq i \leq 2),\quad e_1 \pm e_2,\quad e_3 \pm e_4,\quad \frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 + \nu e_4)(\varepsilon, \nu \in \{\pm 1\}) \right\},$$

$$\Delta_{nc}^+(F_{4(4)}) = \Big\{ e_i (3 \leq i \leq 4),\quad e_i \pm e_j (1 \leq i \leq 2, 3 \leq j \leq 4),$$
$$\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4)(\varepsilon, \nu \in \{\pm 1\}) \Big\}.$$

A maximal commutative root system $C \subset \Delta_{nc}^+(F_{4(4)})$ decomposes into a disjoint union of the commutative root systems $C_1 := C \cap \{e_i | 3 \leq i \leq 4\}$, $C_2 := C \cap \{\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4) | \varepsilon, \nu \in \{\pm 1\}\}$, $C_3 := C \cap \{e_i \pm e_3 | 1 \leq i \leq 2\}$ and $C_4 := C \cap \{e_i \pm e_4 | 1 \leq i \leq 2\}$. Therefore $C_1 \subseteq \{e_i\}$ for some fixed $i = 3$ or 4, $C_2 \subseteq \{\frac{1}{2}(e_1 + \varepsilon e_2 + \nu e_3 - \nu e_4) | \varepsilon = \pm 1\}$ for some fixed $\nu \in \{\pm 1\}$, and $C_j \subseteq \{e_i \pm e_j\}$ for some fixed $1 \leq i \leq 2$ or $C_j \subseteq \{e_i + \varepsilon e_j | 1 \leq i \leq 2\}$ for some fixed $\varepsilon \in \{\pm 1\}$ whenever $3 \leq j \leq 4$. Preventing the presence of $\rho_i \in C_i$, $\rho_j \in C_j$ with $\rho_i + \rho_j \in C$, $i \neq j$, one obtains the following commutative root systems $C \subset \Delta_{nc}^+(F_{4(4)})$ of maximal $card C = 7$, up to the action of the Weyl group of $Sp(3,\mathbb{C}) \times SL(2,\mathbb{C})$:

$$C' = \left\{ e_3,\quad e_1 + e_3,\quad e_2 + e_3,\quad e_1 \pm e_4,\quad \frac{1}{2}(e_1 \pm e_2 + e_3 - e_4) \right\}$$

and

$$C'' = \left\{ e_3,\quad e_1 + e_3,\quad e_2 + e_3,\quad e_1 - e_4,\quad e_2 - e_4,\quad \frac{1}{2}(e_1 \pm e_2 + e_3 - e_4) \right\}.$$

Lemma 4. *The abelian subalgebras* $\mathbf{a} \subset \mathbf{p}^+(F_{4(4)})$ *of maximal* $\dim_{\mathbb{C}} \mathbf{a} = 7$ *are* $Sp(3,\mathbb{C}) \times SL(2,\mathbb{C})$*-conjugate to* $\mathbf{a}' = Span_{\mathbb{C}}(X_\sigma | \sigma \in C')$ *or* $\mathbf{a}'' = Span_{\mathbb{C}}(X_\sigma | \sigma \in C'')$, *where* $C', C'' \subset \Delta_{nc}^+(F_{4(4)})$ *are the aforementioned commutative root systems of maximal cardinality* 7.

Proof. On $Y_\sigma = X_\sigma + \sum_{\tau\in\Delta^+_{nc}(F_{4(4)})-C} y^\tau_\sigma X_\tau$, for $\sigma \in C$, $C = C'$ or C'', one applies the adjoint action of $Exp\left(-\frac{y^{e_4}_{e_3}}{N_{-e_3+e_4,e_3}} X_{-e_3+e_4}\right)$, in order to annihilate the coefficient of X_{e_4} from Y_{e_3}. After eliminating X_{σ_i} from the expressions of Y_{σ_j} for $\sigma_i, \sigma_j \in C$, $\sigma_i \neq \sigma_j$, one calculates the commutators $[Y_{\sigma_i}, Y_{\sigma_j}] = 0$ for all different $\sigma_i, \sigma_j \in C$ and concludes the vanishing of y^τ_σ except $y^{e_1+e_4}_{e_2-e_4}$ for $C = C''$. If $Y_{e_2-e_4} = X_{e_2-e_4} + y^{e_1+e_4}_{e_2-e_4} X_{e_1+e_4}$ with $y^{e_1+e_4}_{e_2-e_4} \neq 0$, one can reduce the considerations to the leading root system C', Q.E.D.

Lemma 5. *If $f : M \to \Gamma \setminus F_{4(4)}/Sp(3) \times SU(2)$ is a harmonic map with 7-dimensional abelian spaces of nilpotents $df^{\mathbb{C}}(T^{1,0}_x M)$, $\forall x \in M$, then there is a holomorphic lifting $F_B : M \to \Gamma \setminus F_{4(4)}/T^4$ to a maximal complex homogeneous fibration. There are no parabolic subgroup $P \subset F_4^{\mathbb{C}}$ and equivariant Hermitian symmetric subspace $G_h/K_h \subset F_{4(4)}/F_{4(4)} \cap P$ such that $F_P(M) = \Gamma_h \setminus G_h/K_h$.*

Proof. The $(0,1)$-part of the $sp(3) \oplus su(2)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{4} \overline{\xi_i} \otimes H_i + \sum_{\sigma\in\Delta^+_c(F_{4(4)})} \overline{\eta_\sigma} \otimes X_\sigma + \sum_{\sigma\in\Delta^+_c(F_{4(4)})} \overline{\zeta_\sigma} \otimes X_{-\sigma}.$$

The differential of a harmonic map f with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x \mathbf{a}' g_x^{-1}$ or $df^{\mathbb{C}}(T^{1,0}_x M) = g_x \mathbf{a}'' g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x(Sp(3) \times SU(2))$ is represented, respectively, by $\theta_1 = \sum_{\tau\in C'} dx^\tau \otimes X_\tau$ or $\theta_2 = \sum_{\tau\in C''} dx^\tau \otimes X_\tau$. The pluriharmonic equations $D''\theta_i = 0$ imply $D'' = \overline{\partial} + \overline{\eta_{e_1+e_2}} \otimes X_{e_1+e_2}$ in both cases. Then the consequences $D'\theta_i = 0$ of the flatness equation force $D = d$.

Let us assume that there is an equivariant Hermitian symmetric subspace $G_h/K_h \subset F_{4(4)}/F_{4(4)} \cap P$ with $\mathbf{p}^+_h = \mathbf{a}'$ or $\mathbf{a}''$. Then $[X_{e_1+e_3}, X_{-e_3}] = N_{e_1+e_3,-e_3} X_{e_1} \in \mathbf{k}^{\mathbb{C}}_h$, whereas $[X_{-e_1}, X_{e_1-e_4}] = N_{-e_1,e_1-e_4} X_{-e_4} \in \mathbf{p}^{\mathbb{C}}_h$, which is not true in either case, Q.E.D.

After detecting the pairs σ, τ from C' or C'' with $\sigma - \tau \in \Delta_c(F_{4(4)})$, one drops out at least one member of these pairs and obtains the strongly commutative root systems

$$S_1 = \{e_3,\ e_1 \pm e_4\} \quad \text{and} \quad S_2 = \left\{\frac{1}{2}(e_1 + e_2 + e_3 - e_4),\ e_2 - e_3,\ e_1 + e_4\right\}$$

in $\Delta^+_{nc}(F_{4(4)})$, up to $Weyl(Sp(3,\mathbb{C}) \times SL(2,\mathbb{C}))$-action.

5. EII $= E_{6(2)}/SU(6) \times SU(2)$

Let us recall that there is a chain of subgroups $E_6^{\mathbb{C}} \subset E_7^{\mathbb{C}} \subset E_8^{\mathbb{C}}$. In terms of the root decompositions of the corresponding Lie algebras, if $H_1, \ldots H_8$ generate a

Cartan subalgebra $\mathbf{h}_8^{\mathbb{C}}$ of $LieE_8^{\mathbb{C}}$, then

$$\mathbf{h}_7^{\mathbb{C}} = \Big\{ \sum_{i=1}^{8} x_i H_i | x_i \in \mathbb{C}, x_7 + x_8 = 0 \Big\} \subset \mathbf{h}_8^{\mathbb{C}}$$

is a Cartan subalgebra of $LieE_7^{\mathbb{C}}$ and

$$\mathbf{h}_6^{\mathbb{C}} = \Big\{ \sum_{i=1}^{8} x_i H_i | x_i \in \mathbb{C}, x_6 - x_7 = 0, x_7 + x_8 = 0 \Big\} \subset \mathbf{h}_7^{\mathbb{C}}$$

is a Cartan subalgebra of $LieE_6^{\mathbb{C}}$. The positive root system

$$\Delta^+(E_8^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 8),\ e_i + e_j (1 \le i < j \le 8),$$
$$\frac{1}{2}\Big(\sum_{i=1}^{7} \varepsilon_i e_i + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{7} \varepsilon_i = 1 \Big) \Big\}$$

contains

$$\Delta^+(E_7^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 6),\ e_i + e_j (1 \le i < j \le 6),\ -e_7 + e_8,$$
$$\frac{1}{2}\Big(\sum_{i=1}^{6} \varepsilon_i e_i - e_7 + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{6} \varepsilon_i = -1 \Big) \Big\},$$

which, in turn, contains

$$\Delta^+(E_6^{\mathbb{C}}) = \Big\{ -e_i + e_j (1 \le i < j \le 5),\ e_i + e_j (1 \le i < j \le 5),$$
$$\frac{1}{2}\Big(\sum_{i=1}^{5} \varepsilon_i e_i - e_6 - e_7 + e_8 \Big) \Big(\varepsilon_i = \pm 1, \prod_{i=1}^{5} \varepsilon_i = 1 \Big) \Big\}$$

(cf. [1]).

For the study of the Riemannian symmetric spaces **EII**, **EV**, **EVI**, **EVIII** and **EIX**, let us introduce the notations

$$\lambda_{ij} := -e_i + e_j \quad (1 \le i < j \le 8), \qquad \mu_{ij} := e_i + e_j \quad (1 \le i < j \le 8),$$

$$\alpha := \frac{1}{2}\Big(\sum_{i=1}^{8} e_i \Big),$$

$$\beta_{ij} := \frac{1}{2}(-e_i - e_j + e_k + e_l + e_m + e_n + e_p + e_8) \quad (1 \le i < j \le 7),$$

$$\gamma_{ijk} := \frac{1}{2}(e_i + e_j + e_k - e_l - e_m - e_n - e_p + e_8) \quad (1 \le i < j < k \le 7),$$

$$\delta_i := \frac{1}{2}(e_i - e_j - e_k - e_l - e_m - e_n - e_p + e_8) \quad (1 \le i \le 7),$$

where i, j, k, l, m, n, p stand for a permutation of $1, 2, \ldots, 7$. It is convenient to put also $\lambda_{ji} := -\lambda_{ij}$, $\mu_{ji} := \mu_{ij}$, $\beta_{ji} := \beta_{ij}$ for $i < j$ and $\gamma_{ikj} = \gamma_{jik} = \gamma_{jki} = \gamma_{kij} = \gamma_{kji} := \gamma_{ijk}$ for $i < j < k$.

In order to recognize the subset $\Delta_c^+(E_{6(2)}) \subset \Delta^+(E_6^{\mathbb{C}})$, let us observe that $sl(6, \mathbb{C}) = su(6) \otimes_{\mathbb{R}} \mathbb{C}$ has 5 simple roots α_i, such that $\sum_{k=i}^{j} \alpha_k$ is a root for any $1 \leq i \leq j \leq 5$. Looking at the Dynkin diagram of $E_6^{\mathbb{C}}$, one notes that the only unramified path with 5 vertices corresponds to the simple roots δ_1, λ_{12}, λ_{23}, λ_{34}, λ_{45}. Therefore $\Delta^+(SL(6, \mathbb{C})) = \{\lambda_{ij}(1 \leq i < j \leq 5), \delta_i(1 \leq i \leq 5)\}$. Since $SU(6)$ and $SU(2)$ are in a direct product in the isotropy group of **EII**, the only positive root σ of $sl(2, \mathbb{C})$ is such that $\sigma + \tau \notin \Delta^+(E_6^{\mathbb{C}})$ for all $\tau \in \Delta^+(SL(6, \mathbb{C}))$. That specifies $\Delta^+(SL(2, \mathbb{C})) = \{\beta_{67}\}$. Thus,

$$\Delta_c^+(E_{6(2)}) = \{\lambda_{ij}(1 \leq i < j \leq 5), \quad \delta_i(1 \leq i \leq 5), \quad \beta_{67}\}$$

and

$$\Delta_{nc}^+(E_{6(2)}) = \{\mu_{ij}(1 \leq i < j \leq 5), \quad \gamma_{ijk}(1 \leq i < j < k \leq 5)\}.$$

The maximal commutative $C \subset \Delta_{nc}^+(E_{6(2)})$ are of the form $C(J) = \{\mu_{ij}((i,j) \in J), \gamma_{klm}((i,j) \notin J)\}$ for some subsets J of unordered pairs $1 \leq i, j \leq 5$ .In particular, $card C(J) = 10$.

A generic abelian subspace $\mathbf{a} \subset \mathbf{p}^+(E_{6(2)})$ of maximal $\dim_{\mathbb{C}} \mathbf{a} = 10$ is not invariant under the adjoint action of the Cartan subalgebra. Therefore, the pluriharmonic equation $D''\theta = 0$ and the consequence $D'\theta = 0$ of the flatness $\nabla^2 = 0$ do not force a reduction of D to a $Lie(G \cap P)$-valued connection.

Definition 6. The set J of unordered pairs is $E_{6(2)}$*-admissible* if there hold simultaneously the following conditions:

(i) for an arbitrary $i \neq j$ with $(j, k) \in J$ for all $k \notin \{i, j\}$ there exists $(i, k) \in J$;

(ii) for an arbitrary i there exists $(j, k) \in J$ with different i, j, k;

(iii) for an arbitrary i there exists $(j, k) \notin J$ with different i, j, k.

Lemma 7. *Let $f : M \to \Gamma \backslash E_{6(2)}/SU(6) \times SU(2)$ be a harmonic map of maximum dimension with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J)) g_x^{-1}$ for a commutative root system $C(J)$, labeled by an $E_{6(2)}$-admissible set of indices J. Then f lifts to a holomorphic map $F_B : M \to \Gamma \backslash E_{6(2)}/T^6$ to a maximal complex homogeneous fibration.*

Proof. In general,

$$D'' = \overline{\partial} + \sum_{i=1}^{5} \overline{\xi_i} \otimes H_i + \overline{\xi_6} \otimes (H_6 + H_7 - H_8) + \sum_{1 \leq i \neq j \leq 5} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$

$$+ \sum_{i=1}^{5} \overline{\zeta_i} \otimes X_{\delta_i} + \sum_{i=1}^{5} \overline{z_i} \otimes X_{-\delta_i} + \overline{\rho} \otimes X_{\beta_{67}} + \overline{\tau} \otimes X_{-\beta_{67}}$$

and

$$\theta = \sum_{(i,j)\in J} dx^{ij} \otimes X_{\mu_{ij}} + \sum_{(i,j)\notin J} dx^{klm} \otimes X_{\gamma_{klm}}.$$

Making use of the table of $\sigma \in \Delta_c(E_{6(2)})$, $\tau \in \Delta^+_{nc}(E_{6(2)})$ with $\sigma + \tau \in \Delta_{nc}(E_{6(2)})$, one derives from the pluriharmonic equation $D''\theta = 0$ that $\eta_{ij} = 0$ if $\exists(i,k) \in J$ or $\exists(j,k) \notin J$, $\zeta_i = 0$ if $\exists(j,k) \in J$, $z_i = 0$ if $\exists(j,k) \notin J$ and $r = 0$. For an $E_{6(2)}$-admissible index set J there follows

$$D'' = \overline{\partial} + \sum_{i=1}^{5} \overline{\xi_i} \otimes H_i + \overline{\xi_6} \otimes (H_6 + H_7 - H_8) + \overline{p} \otimes X_{\beta_{67}}.$$

Then the consequence $D'\theta = 0$ of the flatness equation $\nabla^2 = 0$ reveals the vanishing of ρ, Q.E.D.

Lemma 8. *Each of the following conditions is sufficient for the nonexistence of a Hermitian symmetric G_h/K_h, where G_h is a subgroup of $E_{6(2)}$ and $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J))$ is associated with a maximal commutative root system $C(J) \subset \Delta^+_{nc}(E_{6(2)})$:*

(a) *the existence of different i, j, k, l with $(i,j), (i,k), (k,l) \in J$ and $(j,l) \notin J$;*

(b) *the existence of different i, j, k, l with $(i,k) \in J$ and $(i,j), (j,l), (k,l) \notin J$;*

(c) *the existence of different i, j, k, l with $(i,j), (i,k), (j,l) \in J$ and $(k,l) \notin J$.*

Proof. In either case, it suffices to exhibit $\sigma_1, \sigma_2, \sigma_3 \in C(J)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{6(2)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{6(2)}) - C(J)$. Namely,

(a) $\gamma_{ikm} + (\mu_{ij} - \mu_{ik}) = \gamma_{ikm} + \lambda_{kj} = \gamma_{ijm}$;

(b) $\mu_{ik} + (\gamma_{ijm} - \gamma_{ikm}) = \mu_{ik} + \lambda_{kj} = \mu_{ij}$;

(c) $\mu_{ik} + (\gamma_{ijm} - \mu_{ij}) = \mu_{ik} + \delta_m = \gamma_{ikm}$, Q.E.D.

Towards the construction of strongly commutative root systems $S \subset \Delta^+_{nc}(E_{6(2)})$, let us associate them to graphs with 5 vertices. For $\mu_{ij} \in S$ draw a "blue" edge, connecting the i-th and the j-th vertices. When $\gamma_{klm} \in S$, the complementing vertices i and j are connected by a "red" edge. According to $\mu_{ij} - \mu_{ik} = \lambda_{kj} \in \Delta_c(E_{6(2)})$ and $\gamma_{klm} - \gamma_{jlm} = \lambda_{jk} \in \Delta_c(E_{6(2)})$, no edges of one and the same color have a common vertex. Further, $\gamma_{klm} - \mu_{kl} = \delta_m \in \Delta_c(E_{6(2)})$ requires the nonexistence of disjoint "blue" and "red" edges. Putting all together, one obtains $S = \{\gamma_{345}, \ \mu_{23}, \ \gamma_{125}, \ \mu_{45}\}$ up to $Weyl(SL(6,\mathbb{C}))$-action.

6. EV $= E_{7(7)}/SU(8)$

The elements of $\Delta^+_c(E_{7(7)})$ are expressed by simple roots $\alpha_1, \ldots, \alpha_7$, such that $\sum_{k=i}^{j} \alpha_k \in \Delta^+_c(E_{7(7)})$ for all $1 \leq i \leq j \leq 7$. From the Dynkin diagram of $E_7^{\mathbb{C}}$ (cf. [1]) one recognizes $\alpha_1 := \delta_1$ and $\alpha_i := \lambda_{i-1,i}$ for $2 \leq i \leq 6$ with

$$\Delta'_c(E_7^{\mathbb{C}}) = \left\{ \sum_{k=i}^{j} \alpha_k | 1 \leq i \leq j \leq 6 \right\}$$
$$= \{\delta_i (1 \leq i \leq 6), \lambda_{ij} (1 \leq i < j \leq 6)\} \subset \Delta^+(E_7^{\mathbb{C}}).$$

The existence of $\alpha_0 \in \Delta^+(E_7^{\mathbb{C}}) - \Delta_c'(E_7^{\mathbb{C}})$ with $\alpha_0 + \delta_i \in \Delta^+(E_7^{\mathbb{C}})$ for all $1 \leq i \leq 6$ is contradicted by $\mu_{ij} + \delta_i \notin \Delta(E_7^{\mathbb{C}})$, $\lambda_{78} + \delta_i \notin \Delta(E_7^{\mathbb{C}})$, $\beta_{i7} + \delta_j \notin \Delta(E_7^{\mathbb{C}})$ for $i \neq j$ and $\gamma_{ijk} + \delta_i \notin \Delta(E_7^C)$ for different i,j,k. Therefore, there is $\alpha_7 \in \Delta^+(E_7^{\mathbb{C}}) - \Delta_c'(E_7^{\mathbb{C}})$ with $\alpha_7 + \lambda_{i6} \in \Delta^+(E_7^{\mathbb{C}})$ for $1 \leq i \leq 5$ and $\alpha_7 + \delta_6 \in \Delta^+(E_7^{\mathbb{C}})$. For any $1 \leq i < j \leq 5$ there exists $k \in \{1,\dots,5\} - \{i,j\}$ such that $\mu_{ij} + \lambda_{k6} \notin \Delta^+(E_7^{\mathbb{C}})$. Clearly, $\lambda_{78} + \lambda_{i6} \notin \Delta^+(E_7^{\mathbb{C}})$, $\beta_{i7} + \lambda_{i6} \notin \Delta^+(E_7^{\mathbb{C}})$ for $1 \leq i \leq 5$ and $\gamma_{ijk} + \delta_6 \notin \Delta^+(E_7^{\mathbb{C}})$, regardless whether $6 \in \{i,j,k\}$ or $6 \notin \{i,j,k\}$. Finally, $\beta_{67} + \lambda_{i6} = \beta_{i7}$ for $1 \leq i \leq 5$ and $\beta_{67} + \delta_6 = \lambda_{78}$ reveal that $\alpha_7 = \beta_{67}$, whereas

$$\Delta_c^+(E_{7(7)}) = \{\lambda_{ij}(1 \leq i < j \leq 6),\ \ \lambda_{78},\ \ \beta_{i7}(1 \leq i \leq 6),\ \ \delta_i(1 \leq i \leq 6)\}$$

and

$$\Delta_{nc}^+(E_{7(7)}) = \{\mu_{ij}(1 \leq i < j \leq 6),\ \ \gamma_{ijk}(1 \leq i < j < k \leq 6)\}.$$

After listing the pairs $\sigma, \tau \in \Delta_{nc}^+(E_{7(7)})$ with $\sigma + \tau \in \Delta_c^+(E_{7(7)})$, one characterizes the commutative root systems

$$C(J,K) := \{\mu_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\} \subset \Delta_{nc}^+(E_{7(7)})$$

by the conditions $(i,j,k) \in K \Rightarrow (l,m) \notin J$ and $(i,j,k) \in K \Rightarrow (l,m,n) \notin K$ for different i,j,k,l,m,n.

A generic abelian subspace $\mathbf{a} \subset \mathbf{p}^+(E_{7(7)})$ with a leading root system $C(J,K)$ is not invariant under the adjoint action of the Cartan subalgebra $\mathbf{h}_7^{\mathbb{C}}$. The associated $su(8)$-valued connection D is not reduced to a $Lie(E_{7(7)} \cap P)$-valued one. Even when $T_x^{1,0}M$, $x \in M$, map to $ad\mathbf{h}_7^{\mathbb{C}}$-invariant abelian subspaces of $g_x \mathbf{p}^+(E_{7(7)}) g_x^{-1}$, $f(x) = \Gamma g_x SU(8)$, the existence of a holomorphic lifting to a complex homogeneous fibration is not clear.

Definition 9. The set of indices $J \subseteq \{(i,j) | 1 \leq i \neq j \leq 6\}$, $K \subset \{(i,j,k) \mid 1 \leq i,j,k \leq 6\}$, labeling a commutative root system $C(J,K) \subset \Delta_{nc}^+(E_{7(7)})$, is $E_{7(7)}$-*admissible* if there hold simultaneously the conditions:

(i) for an arbitrary $i \neq j$ with $(i,k) \notin J$ for all $k \notin \{i,j\}$ there exists $(i,k,l) \in K$;

(ii) for an arbitrary i with $(j,k) \notin J$ for all j,k different from i there exists $(j,k,l) \in K$ with $l \notin \{i,j,k,\}$;

(iii) for an arbitrary i there exists $(i,j,k) \in K$.

Lemma 10. *Let us suppose that for the harmonic map $f : M \to \Gamma \backslash E_{7(7)}/SU(8)$ there holds $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J,K)) g_x^{-1}$ for some $E_{7(7)}$-admissible index set J, K. Then there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(7)}/T^7$ to a maximal complex homogeneous fibration.*

Proof. In general, the $(0,1)$-component of the $su(8)$-valued connection D is

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \sum_{1 \leq i \neq j \leq 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \overline{\eta_{78}} \otimes X_{\lambda_{78}}$$

$$+ \overline{\eta_{87}} \otimes X_{\lambda_{87}} + \sum_{i=1}^{6} \overline{\zeta_i} \otimes X_{\beta_{i7}} + \sum_{i=1}^{6} \overline{z_i} \otimes X_{-\beta_{i7}} + \sum_{i=1}^{6} \overline{\rho_i} \otimes X_{\delta_i} + \sum_{i=1}^{6} \overline{r_i} \otimes X_{-\delta_i}.$$

The $(1,0)$-component of $df^{\mathbb{C}}$ is represented by

$$\theta = \sum_{(i,j)\in J} dx^{ij} \otimes X_{\mu_{ij}} + \sum_{(i,j,k)\in K} dx^{ijk} \otimes X_{\gamma_{ijk}}.$$

The pluriharmonic equation $D''\theta = 0$ implies that $\eta_{ij} = 0$ if there exist $(i,k) \in J$ or $(i,k,l) \in K$, $\eta_{87} = 0$, $z_i = 0$ if there exist $(j,k) \in J$ or $(j,k,l) \in K$, and $r_i = 0$ if there exists $(i,j,k) \in K$. For $E_{7(7)}$-admissible J, K there follows

$$D' = \partial - \sum_{i=1}^{6} \xi_i \otimes H_i - \xi_7 \otimes (H_7 - H_8) + \eta_{78} \otimes X_{\lambda_{87}} + \sum_{i=1}^{6} \zeta_i \otimes X_{-\beta_{i7}} + \sum_{i=1}^{6} \rho_i \otimes X_{-\delta_i}.$$

The consequence $D'\theta = 0$ of the flatness equation $\nabla^2 = 0$ forces

$$D = d + \sum_{i=1}^{6} (\overline{\xi_i} - \xi_i) \otimes H_i + (\overline{\xi_7} - \xi_7) \otimes (H_7 - H_8),$$

which suffices for the existence of a holomorphic F_B, Q.E.D.

Lemma 11. *Each of the following conditions is sufficient for the nonexistence of an equivariant Hermitian symmetric $G_h/K_h \subset E_{7(7)}/E_{7(7)} \cap P$ with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C(J,K))$:*

(a) *the existence of $(i,k), (j,k), (i,l) \in J$ with $(j,l) \notin J$;*

(b) *the existence of $(i,k,l), (j,k,l), (i,p,q) \in K$ with $(j,p,q) \notin K$, regardless of $\{k,l\} \cap \{p,q\}$;*

(c) *the existence of $(i,j,k), (i,p,q) \in K$ and $(j,k) \in J$ with $(p,q) \notin J$, regardless of $\{j,k\} \cap \{p,q\}$.*

Proof. In either case, we choose $\sigma_1, \sigma_2, \sigma_3 \in C(J,K)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{7(7)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{7(7)}) - C(J,K)$. More precisely,

(a) $\mu_{il} + (\mu_{jk} - \mu_{ik}) = \mu_{il} + \lambda_{ij} = \mu_{jl}$;

(b) $\gamma_{ipq} + (\gamma_{jkl} - \gamma_{ikl}) = \gamma_{ipq} + \lambda_{ij} = \gamma_{jpq}$;

(c) $\gamma_{ipq} + (\mu_{jk} - \gamma_{ijk}) = \gamma_{ipq} - \delta_i = \mu_{pq}$, Q.E.D.

The root system

$$S(J,K) = \{\mu_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\} \subset \Delta_{nc}^+(E_{7(7)})$$

is *strongly commutative* if its index set J, K satisfies the following conditions:

(i) any two pairs from J are disjoint;

(ii) any two triples from K intersect in exactly one index;

(iii) any triple from K intersects any pair from J in exactly one index.

7. EVI $= E_{7(-5)}/SO(12) \times SU(2)$

The complexified isotropy subalgebra

$$so(12,\mathbb{C}) = \mathbf{h}_7^{\mathbb{C}} + \sum_{1\le i\ne j\le 6} \mathbb{C}X_{\lambda_{ij}} + \sum_{1\le i<j\le 6} (\mathbb{C}X_{\mu_{ij}} + \mathbb{C}X_{-\mu_{ij}}).$$

Since $SU(2)$ is in a direct product with $SO(12)$, the only positive root σ of $sl(2,\mathbb{C})$ satisfies $\sigma + \tau \notin \Delta(E_7^{\mathbb{C}})$ for all $\tau \in \Delta(so(12,\mathbb{C}))$. That specifies $\sigma = \lambda_{78}$. Consequently,

$$\Delta_c^+(E_{7(-5)}) = \{\lambda_{ij}(1 \le i < j \le 6), \quad \mu_{ij}(1 \le i < j \le 6), \quad \lambda_{78}\}$$

and

$$\Delta_{nc}^+(E_{7(-5)}) = \{\beta_{i7}(1 \le i \le 6), \quad \gamma_{ijk}(1 \le i < j < k \le 6), \quad \delta_i(1 \le i \le 6)\}.$$

The maximal commutative subsets $C \subset \Delta_{nc}^+(E_{7(-5)})$ are of the form $C(I,K) = \{\beta_{i7}(i \in I), \gamma_{ijk}((i,j,k) \in K), \delta_i(i \notin I)\}$ for $I \subseteq \{1,\dots,6\}$ and a subset K of unordered triples, subject to $(i,j,k) \in K \Rightarrow (l,m,n) \notin K$. In particular, the maximum cardinality of a commutative root system $C \subset \Delta_{nc}^+(E_{7(-5)})$ is 16.

In order to study the holomorphic liftings to a maximal complex homogeneous fibration, let us introduce

Definition 12. An index set I,K of a commutative root system $C(I,K)$ is $E_{7(-5)}$*-admissible* if it satisfies the following conditions:

(i) for any $i \in I$, $j \notin I$ there exists $(i,k,l) \in K$ with different i,j,k,l;

(ii) if $\{1,\dots,6\} - \{i,j\} \subseteq I$, then there exists $(k,l,m) \in K$ with different i,j,k,l,m;

(iii) if $I \subseteq \{i,j\}$, then there exists $k \notin \{i,j\}$ with $(i,j,k) \in K$.

For example, $I = \{1\}$, $K = \{(1,i,j) | 2 \le i < j \le 6\}$ and $I = \{1,2\}$, $K \supset \{(1,2,i) | 3 \le i \le 6\}$ are $E_{7(-5)}$-admissible.

Lemma 13. *Let $f : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times SU(2)$ be a harmonic map with maximum dimensional $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K)) g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x(SO(12) \times SU(2))$. Then there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times T^1$ to the minimal complex homogeneous fibration, whose associated parabolic subgroup P has semisimple part $SO(12,\mathbb{C})$. If, moreover, the index set I,K is $E_{7(-5)}$-admissible, then f admits a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(-5)}/T^7$ to a maximal complex homogeneous fibration.*

Proof. The $(0,1)$-part of the $so(12) \times su(2)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \sum_{1 \le i \ne j \le 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$
$$+ \overline{\eta_{78}} \otimes X_{\lambda_{78}} + \overline{\eta_{87}} \otimes X_{\lambda_{87}} + \sum_{1 \le i < j \le 6} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i < j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}}.$$

Under the assumptions of the lemma, the $(1,0)$-component of $df^{\mathbb{C}}$ is represented by

$$\theta = \sum_{i \in I} dx^i \otimes X_{\beta_{i7}} + \sum_{(i,j,k) \in K} dx^{ijk} \otimes X_{\gamma_{ijk}} + \sum_{i \notin I} dx^i \otimes X_{\delta_i}.$$

The pluriharmonic equation $D''\theta = 0$ reveals that $\eta_{87} = 0$, $\eta_{ij} = 0$ for $i \notin I$ or $j \in I$ or if there is $(i,k,l) \in K$, $\zeta_{ij} = 0$ if $\exists (k,l,m) \in K$ or $\exists k \notin I$ and $z_{ij} = 0$ if $\exists k \in I$ or $\exists (i,j,k) \in K$ for different i,j,k,l,m. Further, $D'\theta = 0$ implies that $\eta_{78} = 0$.

Consequently, $D = \overline{D''} + D''$ takes values in $so(12)$ and there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{7(-5)}/SO(12) \times T^1$.

For $E_{7(-5)}$-admissible I, K, the pluriharmonic equation forces

$$D'' = \overline{\partial} + \sum_{i=1}^{6} \overline{\xi_i} \otimes H_i + \overline{\xi_7} \otimes (H_7 - H_8) + \overline{\eta_{78}} \otimes X_{\lambda_{78}}$$

and $D'\theta = 0$ specifies that D takes values in $\mathbf{h}_7$. In other words, there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{7(-5)}/T^7$, Q.E.D.

Lemma 14. *Each of the following conditions is sufficient for the nonexistence of a Hermitian symmetric space G_h/K_h with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C(I,K))$, equivariantly embedded in some complex homogeneous fibration $E_{7(-5)}/E_{7(-5)} \cap P$:*

(a) *$k \in I$, $l \notin I$, $(i,j,l) \in K$ and $(l,m,n) \notin K$ for different i,j,k,l,m,n;*

(b) *$(i,j,k), (l,m,n) \in K$ for different i,j,k,l,m,n;*

(c) *$k, l \notin I$, $(i,j,l) \in K$ and $(i,j,k) \notin K$ for different i,j,k,l.*

Proof. The aforementioned conditions provide the following $\sigma_1, \sigma_2, \sigma_3 \in C(I,K)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{7(-5)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \in \Delta_{nc}(E_{7(-5)}) - C(I,K)$:

(a) $\beta_{k7} + (\delta_l - \gamma_{ijl}) = \beta_{k7} - \mu_{ij} = \gamma_{lmn}$;

(b) $\gamma_{ijk} + (\gamma_{lmn} - \beta_{k7}) = \gamma_{ijk} - \mu_{ij} = \delta_k$ for $k \in I$ or $\gamma_{lmn} + (\gamma_{ijk} - \delta_k) = \gamma_{lmn} + \mu_{ij} = \beta_{k7}$ for $k \notin I$;

(c) $\delta_k + (\gamma_{ijl} - \delta_l) = \delta_k + \mu_{ij} = \gamma_{ijk}$, Q.E.D.

The strongly commutative root systems $S \subset \Delta_{nc}^+(E_{7(-5)})$ of maximum cardinality are equivalent to $S = \{\beta_{17}, \gamma_{134}, \gamma_{156}, \delta_2\}$ modulo the action of the Weyl group of $SO(12,\mathbb{C}) \times SL(2,\mathbb{C})$.

8. EVIII $= E_{8(8)}/SO(16)$

As far as

$$\Delta^+(so(16,\mathbb{C})) = \Delta_c^+(E_{8(8)}) = \{\lambda_{ij}(1 \le i < j \le 8),\ \ \mu_{ij}(1 \le i < j \le 8)\},$$

there follows

$$\Delta_{nc}^+(E_{8(8)}) = \{\alpha,\ \ \beta_{ij}(1 \le i < j \le 7),\ \ \gamma_{ijk}(1 \le i < j < k \le 7),\ \ \delta_i(1 \le i \le 7)\}.$$

The commutative root systems $C \subset \Delta_{nc}^+(E_{8(8)})$ are of the form

$$C_1(I,J,K) = \{\delta_i(i \in I), \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

or

$$C_2(J,K) = \{\alpha, \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

with $(i,j) \in J \Rightarrow p \notin I$ for $p \in \{i,j\}$; $(i,j,k) \in K \Rightarrow (p,q) \notin J$ for $p,q \in \{i,j,k\}$; $(i,j,k) \in K \Rightarrow (l,m,n) \notin K$ for different i,j,k,l,m,n.

Definition 15. For a commutative root system $C_1(I,J,K)$ one says that I,J,K is an $E_{8(8)}$-admissible index set of first kind whenever there hold the following conditions:

(i) if $i \notin I$, $i \notin SuppK$, then there exists $(j,k) \in J$ for different i,j,k;

(ii) if $(i,j) \notin J$, $I \subseteq \{i,j\}$, then there exists $(k,l,m) \in K$ with different i,j,k,l,m;

(iii) if $(i,j,k) \notin K$ for fixed $i \neq j$ and all $k \notin \{i,j\}$, then there exists $(k,l) \in J$.

Definition 16. If $C_2(J,K)$ is a commutative root system, then J,K is an $E_{8(8)}$-semi-admissible set of indices of second kind whenever for any unordered pair $(i,j) \notin J$ there exists an unordered triple $(k,l,m) \in K$ with different i,j,k,l,m.

Definition 17. For a commutative root system $C_2(J,K)$ the pair J,K is an $E_{8(8)}$-admissible index set of second kind if it is $E_{8(8)}$-semi-admissible and for any $i \neq j$ with $(j,k) \notin J$ for all $k \notin \{i,j\}$ there exists $l \notin \{i,j,k\}$ with $(i,k,l) \in K$.

Lemma 18. *Let $f : M \to \Gamma \setminus E_{8(8)}/SO(16)$ be a harmonic map of a compact Kähler manifold M with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K)) g_x^{-1}$ for $x \in M$, $f(x) = \Gamma g_x SO(16)$ and an $E_{8(8)}$-semi-admissible index set J,K of second kind. Then f admits a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(8)}/U_8 \times T^1$ to a minimal complex homogeneous fibration $\Gamma \setminus E_{8(8)}/U_8 \times T^1 \to \Gamma \setminus E_{8(8)}/SO(16)$ with fiber* **DIII**$_c(8)/T^1$*. For a harmonic map $f : M \to \Gamma \setminus E_{8(8)}/SO(16)$ with $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_1(I,J,K)) g_x^{-1}$, where I,J,K is an $E_{8(8)}$-admissible index set of first kind, or $df^{\mathbb{C}}(T^{1,0}_x M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J,K)) g_x^{-1}$, where J,K is an $E_{8(8)}$-admissible index set of second kind, there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$ to the maximal complex homogeneous fibration.*

Proof. The $(0,1)$-component of the $so(16)$-valued connection D is of the form

$$D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \neq j \le 8} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \sum_{1 \le i < j \le 8} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i < j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}}.$$

The $(1,0)$-component of the differential $df^{\mathbb{C}}$, associated with a commutative root system $C_2(J,K)$, is

$$\theta_2 = dx^0 \otimes X_\alpha + \sum_{(k,l) \in J} dx^{kl} \otimes X_{\beta_{kl}} + \sum_{(p,q,r) \in K} dx^{pqr} \otimes X_{\gamma_{pqr}}.$$

The pluriharmonic equation $D''\theta_2 = 0$ implies that $z_{ij} = 0$ for all $1 \le i < j \le 7$, $z_{i8} = 0$ for all $1 \le i \le 7$, $\eta_{ij} = 0$ if $\exists (j,k) \in J$ or $\exists (i,k,l) \in K$, $\zeta_{ij} = 0$ if $(i,j) \in J$ or $\exists (k,l,m) \in K$ for different i,j,k,l,m. For $E_{8(8)}$-semi-admissible J,K of second kind that provides

$$D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \neq j \le 8} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}}$$

and an existence of a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(8)}/U_8 \times T^1$. Here $U_8 \times T^1$ is the centralizer of $Exp_1^{E_{8(8)}}\left(\mathbb{R}\left(\sum_{i=1}^{8} H_i\right)\right)$. Whenever J,K is an $E_{8(8)}$-admissible index set of second kind, there follows $D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i$, which suffices for the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$.

If the harmonic map f is associated with a commutative root system $C_1(I, J, K)$ of first kind, then

$$\theta_1 = \sum_{i \in I} dx^i \otimes X_{\delta_i} + \sum_{(i,j) \in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j,k) \in K} dx^{ijk} \otimes X_{\gamma_{ijk}}.$$

The pluriharmonic equation $D''\theta_1 = 0$ implies that $\eta_{ij} = 0$ if $i \in I$ or there exist $(j,k) \in J$ or $(i,k,l) \in K$, $\zeta_{ij} = 0$ for $(i,j) \in J$ or $\exists(k,l,m) \in K$ or $\exists k \in I$, $z_{ij} = 0$ for $1 \leq i < j \leq 7$ if $\exists(k,l) \in J$ or $\exists(i,j,k) \in K$, $z_{i8} = 0$ if $i \in I$ or $\exists(j,k) \in J$ or $\exists(i,j,k) \in K$. The $E_{8(8)}$-admissibility of the index set I, J, K of first kind suffices for $D'' = \overline{\partial} + \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i$ and the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(8)}/T^8$, Q.E.D.

Lemma 19. *Each of the following conditions implies the nonexistence of an equivariant Hermitian symmetric $G_h/K_h \subset E_{8(8)}/E_{8(8)} \cap P$ with $\mathfrak{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_1(I, J, K))$ or $\mathfrak{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_2(J, K))$:*

(a) *the existence of $(i,k), (j,k) \in J$, $(i,k,l) \in K$ with $(j,k,l) \notin K$;*

(b) *the existence of $i, j \in I$, $(j,k) \in J$ with $(i,k) \notin J$ in the case of $C_1(I, J, K)$;*

(c) *the existence of $(i,j) \in J$, $(i,j,k) \in K$ in the case of $C_2(J, K)$.*

Proof. Below are exhibited $\sigma_1, \sigma_2, \sigma_3 \in C_i$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{8(8)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \notin C_i$:

(a) $\gamma_{ikl} + (\beta_{ik} - \beta_{jk}) = \gamma_{ikl} + \lambda_{ij} = \gamma_{jkl}$;

(b) $\beta_{jk} + (\delta_j - \delta_i) = \beta_{jk} + \lambda_{ij} = \beta_{ik}$;

(c) $\gamma_{ijk} + (\beta_{ij} - \alpha) = \gamma_{ijk} - \mu_{ij} = \delta_k$, Q.E.D.

In order to describe the strongly commutative root systems $S \subset \Delta_{nc}^+(E_{8(8)})$, one lists $\sigma, \tau \in \Delta_{nc}^+(E_{8(8)})$ with $\sigma - \tau \in \Delta_c(E_{8(8)})$ and observes that

$$S = S_1(I, J, K) = \{\delta_i(i \in I), \beta_{ij}((i,j) \in J), \gamma_{ijk}((i,j,k) \in K)\}$$

are subject to the conditions:

(i) $cardI \leq 1$; (ii) the pairs from J are disjoint; (iii) any two triples from K intersect in a single element; (iv) $(i,j) \in J \Rightarrow i,j \notin I$; (v) $(i,j,k) \in K \Rightarrow (i,j), (i,k), (j,k) \notin J$; (vi) $(i,j,k) \in K \Rightarrow (l,m) \notin J$ for different i,j,k,l,m; (vii) $(i,j,k) \in K \Rightarrow i,j,k \notin I$.

The strongly commutative root systems $S_2(J = \varnothing, K) = \{\alpha, \gamma_{ijk}((i,j,k) \in K)\}$ are characterized by the fact that any pair of triples from K intersect in a single index.

9. EIX $= E_{8(-24)}/E_7 \times SU(2)$

The positive roots of E_7 are listed at the beginning of Section 5. The only positive root τ of $sl(2, \mathbb{C})$ is subject to the property $\sigma + \tau \notin \Delta(E_8^{\mathbb{C}})$ for all $\sigma \in \Delta(E_7^{\mathbb{C}})$. Thus, $\tau = \mu_{78}$ and

$$\Delta_c^+(E_{8(-24)}) = \{\lambda_{ij}(1 \le i < j \le 6), \mu_{ij}(1 \le i < j \le 6), \lambda_{78}, \mu_{78},$$
$$\beta_{i7}(1 \le i \le 6), \gamma_{ijk}(1 \le i < j < k \le 6), \delta_i(1 \le i \le 6)\},$$
$$\Delta_{nc}^+(E_{8(-24)}) = \{\lambda_{ij}, \mu_{ij}(1 \le i \le 6, 7 \le j \le 8),$$
$$\alpha, \beta_{ij}(1 \le i < j \le 6), \gamma_{ij7}(1 \le i < j \le 6), \delta_7\}.$$

For a commutative root system $C \subset \Delta_{nc}^+(E_{8(-24)})$ of maximal cardinality, let I_1 be the set of the indices i with $\lambda_{i7} \in C$, I_2 be the set of the indices i with $\lambda_{i8} \in C$, and J be the set of the unordered pairs (i,j) with $\beta_{ij} \in C$. Then either

$$C = C_1(I_1, I_2, J) = \{\alpha, \lambda_{i7}(i \in I_1), \mu_{i8}(i \notin I_1), \lambda_{i8}(i \in I_2),$$
$$\mu_{i7}(i \notin I_2), \beta_{ij}((i,j) \in J), \gamma_{ij7}((i,j) \notin J)\}$$

or

$$C = C_2(I_1, I_2, J) = \{\delta_7, \lambda_{i7}(i \in I_1), \mu_{i8}(i \notin I_1), \lambda_{i8}(i \in I_2),$$
$$\mu_{i7}(i \notin I_2), \beta_{ij}((i,j) \in J), \gamma_{ij7}((i,j) \notin J)\}.$$

Definition 20. A maximal commutative root system $C_1(I_1, I_2, J)$ is labeled by $E_{8(-24)}$-admissible indices of first kind if there hold the following conditions:

(i) for any $i \in I_1 \cap I_2$ and $j \notin I_1 \cup I_2$ there exist $(j,k) \in J$ or $(i,k) \notin J$;

(ii) if $I_1 = \varnothing$, then $I_2 \ne \{1, \ldots, 6\}$;

(iii) if $I_2 = \varnothing$, then $I_1 \ne \{1, \ldots, 6\}$;

(iv) if $(i,j) \notin J$ and $i \notin I_1 \cup I_2$, then there exists $(k,l) \notin J$;

(v) if $i \in I_2$ and $I_1 \subseteq \{i\}$, then there exists $(i,j) \notin J$;

(vi) if $\{i,j,k\} \cap I_1 = \varnothing$, $l, m, n \in I_2$ and $(l,m), (l,n), (m,n) \in J$, then at least one of the pairs $(i,j), (j,k)$ and (i,k) belongs to J;

(vii) if $\{i,j,k\} \cap I_1 = \varnothing$, $\{l,m,n\} \cap I_2 = \varnothing$ and $(i,j), (i,k), (j,k) \in J$, then at least one of the pairs $(l,m), (l,n), (m,n)$ belongs to J;

(viii) if $i \in I_1$ and $I_2 \subseteq \{i\}$, then there exists $(i,j) \notin J$.

Definition 21. A maximal commutative root system $C_2(I_1, I_2, J)$ has an $E_{8(-24)}$-admissible index set I_1, I_2, J of second kind when there hold the following conditions:

(i) if $j \notin I_1 \cup I_2$ and $i \in I_1 \cap I_2$, then there exists $(j,k) \in J$;

(ii) if $I_1 = \varnothing$, then $I_2 \ne \{1, \ldots, 6\}$;

(iii) if $I_2 = \varnothing$, then $I_1 \ne \{1, \ldots, 6\}$;

(iv) if $i \in I_1 \cap I_2$, then there exists $(i,j) \notin J$;

(v) if $i \notin I_2$, then there exists $(i,j) \in J$;

(vi) for any permutation i,j,k,l,m,n of $1, \ldots, 6$ with $\{l,m,n\} \subseteq I_2$ and $(l,m), (l,n), (m,n) \in J$ there exists $(i,j), (j,k)$ or (i,k) from J;

(vii) for $\{i,j,k\} \subseteq I_1$ with $(i,j), (j,k), (i,k) \in J$ there exists $(l,m) \in J$;

(viii) if $i \notin I_1$, then there exists $(i,j) \in J$.

Lemma 22. *If $f : M \to \Gamma \setminus E_{8(-24)}/E_7 \times SU(2)$ is a harmonic map with $df^{\mathbb{C}}(T_x^{1,0}M) = g_x Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J))g_x^{-1}$ for a maximal commutative root system $C_i(I_1, I_2, J) \subset \Delta_{nc}^+(E_{8(-24)})$, then there is a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$ to a minimal complex homogeneous fibration. If $C_i(I_1, I_2, J)$ is* *labeled by an $E_{8(-24)}$-admissible index set of i-th kind, $i = 1$ or 2, then f admits a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$ to a maximal complex homogeneous fibration.*

Proof. The $(0,1)$-component of the $\mathbf{e}_7 \oplus su(2)$-valued connection D has the form

$$
\begin{aligned}
D'' = \bar{\partial} &+ \sum_{i=1}^{8} \overline{\xi_i} \otimes H_i + \sum_{1 \le i \neq j \le 6} \overline{\eta_{ij}} \otimes X_{\lambda_{ij}} + \overline{\eta_{78}} \otimes X_{\lambda_{78}} + \overline{\eta_{87}} \otimes X_{\lambda_{87}} \\
&+ \sum_{1 \le i<j \le 6} \overline{\zeta_{ij}} \otimes X_{\mu_{ij}} + \sum_{1 \le i<j \le 6} \overline{z_{ij}} \otimes X_{-\mu_{ij}} + \overline{\zeta_{78}} \otimes X_{\mu_{78}} + \overline{z_{78}} \otimes X_{-\mu_{78}} \\
&+ \sum_{i=1}^{6} \overline{\sigma_i} \otimes X_{\beta_{i7}} + \sum_{i=1}^{6} \overline{s_i} \otimes X_{-\beta_{i7}} + \sum_{1 \le i<j<k \le 6} \overline{\tau_{ijk}} \otimes X_{\gamma_{ijk}} \\
&+ \sum_{1 \le i<j<k \le 6} \overline{t_{ijk}} \otimes X_{-\gamma_{ijk}} + \sum_{i=1}^{6} \overline{\rho_i} \otimes X_{\delta_i} + \sum_{i=1}^{6} \overline{r_i} \otimes X_{-\delta_i}.
\end{aligned}
$$

The $(1,0)$-component of the differential of f, associated with $C_1(I_1, I_2, J)$, is

$$
\begin{aligned}
\theta_1 = &dx^0 \otimes X_\alpha + \sum_{i \in I_1} dx^i \otimes X_{\lambda_{i7}} + \sum_{i \notin I_1} dy^i \otimes X_{\mu_{i8}} + \sum_{i \in I_2} du^i \otimes X_{\lambda_{i8}} \\
&+ \sum_{i \notin I_2} dv^i \otimes X_{\mu_{i7}} + \sum_{(i,j) \in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j) \notin J} dy^{ij} \otimes X_{\gamma_{ij7}}.
\end{aligned}
$$

The pluriharmonic equation $D''\theta_1 = 0$ implies that $\eta_{ij} = 0$ for $i \notin I_1 \cap I_2$ or $j \in I_1 \cup I_2$ or $\exists (j,k) \in J$ or $\exists (i,k) \notin J$, $\eta_{78} = 0$ if $I_1 \neq \varnothing$ or $I_2 \neq \{1, \dots, 6\}$, $\eta_{87} = 0$ for $I_2 \neq \varnothing$ or $I_1 \neq \{1, \dots, 6\}$, $\zeta_{ij} = 0$ if $(i,j) \in J$ or $\exists (k,l) \notin J$ or $i \in I_1 \cup I_2$, $z_{ij} = 0$ for all $1 \le i < j \le 6$, $z_{78} = 0$, $\sigma_i = 0$ if $i \notin I_2$ or $\exists (i,j) \notin J$ or $\exists j \in I_1$, $s_i = 0$ for all $1 \le i \le 6$, $\tau_{ijk} = 0$ if $\exists (l,m) \notin J$ or $\exists (i,j) \in J$ or $\exists i \in I_1$ or $\exists l \notin I_2$, $t_{ijk} = 0$ if $\exists (i,j) \notin J$ or $\exists (l,m) \in J$ or $\exists i \in I_1$ or $\exists l \in I_2$, $\rho_i = 0$ for all $1 \le i \le 6$ and $r_i = 0$ if $i \notin I_1$ or $\exists j \in I_2$ or $\exists (i,j) \notin J$. Then the consequence $D'\theta_1 = 0$ of the flatness equation $\nabla^2 = 0$ yields $\zeta_{78} \wedge dx^i = 0$ for $i \in I_1$ and $\zeta_{78} \wedge dy^i = 0$ for $i \notin I_1$. If $card I_1 \ge 2$, then $\zeta_{78} \in \langle dx^{i_1} \rangle \cap \langle dx^{i_2} \rangle$ for $i_1, i_2 \in I_1$ forces the vanishing of ζ_{78}. Otherwise, $card(\{1, \dots, 6\} - I_1) \ge 5$ and the containment of ζ_{78} in at least two different differential ideals $\langle dy^{i_1} \rangle$, $\langle dy^{i_2} \rangle$, where $i_1, i_2 \notin I_1$, leads to $\zeta_{78} = 0$. That suffices for the existence of a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$. Observe that $E_7 \times T^1$ is the centralizer of the torus $T^1 = Exp_1^{E_{8(-24)}}(H_7 + H_8)$. If, moreover, the index set I_1, I_2, J is $E_{8(-24)}$-admissible, then $D = d + \sum_{i=1}^{8} (\overline{\xi_i} - \xi_i) \otimes H_i$ justifies the existence of a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$.

For a harmonic map f with an associated commutative root system $C_2(I_1, I_2, J)$ one has

$$\theta_2 = dx^0 \otimes X_{\delta_7} + \sum_{i\in I_1} dx^i \otimes X_{\lambda_{i7}} + \sum_{i\notin I_1} dy^i \otimes X_{\mu_{i8}} + \sum_{i\in I_2} du^i \otimes X_{\lambda_{i8}}$$
$$+ \sum_{i\notin I_2} dv^i \otimes X_{\mu_{i7}} + \sum_{(i,j)\in J} dx^{ij} \otimes X_{\beta_{ij}} + \sum_{(i,j)\notin J} dy^{ij} \otimes X_{\gamma_{ij7}}.$$

The pluriharmonic equation provides $\eta_{ij} = 0$ for $j \in I_1 \cup I_2$ or $i \notin I_1 \cap I_2$ or $\exists(j,k) \in J$, $\eta_{78} = 0$ for $I_1 \neq \varnothing$ or $I_2 \neq \{1,\dots,6\}$, $\eta_{87} = 0$ for $I_2 \neq \varnothing$ or $I_1 \neq \{1,\dots,6\}$, $\zeta_{ij} = 0$ for all $1 \leq i < j \leq 6$, $z_{ij} = 0$ for $i \notin I_1 \cap I_2$ or $(i,j) \notin J$, $z_{78} = 0$, $\sigma_i = 0$ for all $1 \leq i \leq 6$, $s_i = 0$ if $i \in I_2$ or $\exists(i,j) \in J$, $\tau_{ijk} = 0$ if $\exists l \notin I_2$ or $\exists(i,j) \in J$ or $\exists(l,m) \notin J$, $t_{ijk} = 0$ if $\exists i \notin I_1$ or $\exists(i,j) \notin J$ or $\exists(l,m) \in J$, $\rho_i = 0$ if $i \in I_1$ or $\exists(i,j) \in J$ and $r_i = 0$ for all $1 \leq i \leq 6$. The consequence $D'\theta_2 = 0$ of the flatness equation specifies $\zeta_{78} = 0$. Therefore, f admits a holomorphic lifting $F_P : M \to \Gamma \setminus E_{8(-24)}/E_7 \times T^1$. If I_1, I_2, J is an $E_{8(-24)}$-admissible index set of second kind, then D takes values in $\mathbf{h}$ and there is a holomorphic lifting $F_B : M \to \Gamma \setminus E_{8(-24)}/T^8$, Q.E.D.

Lemma 23. *Each of the following conditions on the index set I_1, I_2, J implies the nonexistence of an equivariant Hermitian symmetric subspace $G_h/K_h \subset E_{8(-24)}/E_{8(-24)} \cap P$ with $\mathbf{p}_h^+ = Span_{\mathbb{C}}(X_\sigma | \sigma \in C_i(I_1, I_2, J))$ for $i = 1$ or 2:*
(a) *the existence of $i \in I_1$, $j \in I_2$, $(k,l) \in J$ with $(m,n) \notin J$;*
(b) *the existence of $i \in I_1$, $j \notin I_2$ with $(i,j) \notin J$ in the case of $C_1(I_1, I_2, J)$;*
(c) *the existence of $i \notin I_2$, $j \in I_1$ with $(i,j) \in J$ in the case of $C_2(I_1, I_2, J)$.*

Proof. Here are the appropriate $\sigma_1, \sigma_2, \sigma_3 \in C_i(I_1, I_2, J)$ with $\sigma_2 - \sigma_3 \in \Delta_c(E_{8(-24)})$ and $\sigma_1 + (\sigma_2 - \sigma_3) \notin C_i(I_1, I_2, J)$:
(a) $\lambda_{i7} + (\beta_{kl} - \gamma_{mn7}) = \lambda_{i7} + \mu_{ij} = \mu_{j7}$;
(b) $\alpha + (\lambda_{i7} - \mu_{j7}) = \alpha - \mu_{ij} = \beta_{ij}$;
(c) $\mu_{i7} + (\beta_{ij} - \lambda_{j7}) = \mu_{i7} + \beta_{i7} = \alpha$, Q.E.D.

Applying the very definition, one observes that the strongly commutative root systems $S \subset \Delta_{nc}^+(E_{8(-24)})$ of the form

$$S = \{\lambda_{i7}(i \in I_1), \lambda_{i8}(i \in I_2), \mu_{i8}(i \in I_3), \mu_{i7}(i \in I_4),$$
$$\beta_{ij}((i,j) \in J_1), \gamma_{ij7}((i,j) \in J_2)\}$$

are $S = \{\lambda_{17}, \lambda_{28}, \beta_{13}, \beta_{24}, \gamma_{127}\}$, $S = \{\mu_{18}, \mu_{27}, \beta_{12}, \gamma_{137}, \gamma_{247}\}$, $S = \{\lambda_{17}, \lambda_{28}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{17}, \mu_{28}, \beta_{23}, \gamma_{137}\}$, $S = \{\lambda_{17}, \mu_{18}, \beta_{12}, \gamma_{137}\}$, $S = \{\lambda_{17}, \mu_{18}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{18}, \mu_{17}, \beta_{12}, \gamma_{127}\}$, $S = \{\lambda_{18}, \mu_{17}, \beta_{12}, \gamma_{137}\}$ and $S = \{\mu_{18}, \mu_{27}, \beta_{12}, \gamma_{127}\}$ up to $Weyl(E_7^{\mathbb{C}} \times SL(2,\mathbb{C}))$-action.

REFERENCES

1. Bourbaki, N. Groupes et Algèbres de Lie. Chap. 4, 5 et 6, Eléments de mathématique, fascicule XXXIV, Hermann, 1968.
2. Burstall, F. E., J. H. Rawnsley. Twistor theory for Riemannian Symmetric Spaces. *Lecture Notes in Mathematics*, **1424**, Springer, 1990.

3. Carlson, J., L. Hernandez. Harmonic maps of compact Kähler manifolds to exceptional hyperbolic spaces. *J. Geom. Anal.*, **1**, 1991, 339–357.
4. Carlson, J., D. Toledo. Harmonic mappings of Kähler manifolds to locally symmetric spaces. *Publ. Math. I.H.E.S.*, **69**, 1989, 173–201.
5. Carlson, J., D. Toledo. Rigidity of harmonic maps of maximum rank. *J. Geom. Anal.*, **3**, 1993, 99–140.
6. Corlette, K. Flat G-bundles with canonical metrics. *J. Diff. Geom.*, **28**, 1988, 361–382.
7. Eells, J., J. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. Jour. Math.*, **86**, 1964, 109–160.
8. Helgason, S. Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, Orlando-Florida, 1978.
9. Kasparian, A. Hermitian symmetry of infinitesimally trivial harmonic images of compact Kähler manifolds. In: *Aspects of Complex Analysis, Differential Geometry, Mathematical Physics and Applications*, World Scientific, 1999, 121–134.
10. Malcev, A. Commutative subalgebras of semisimple Lie algebras. *Izv. Akad. Nauk USSR*, **9**, 1945, 125–133 (in Russian).
11. Sampson, J. H. Applications of harmonic maps to Kähler geometry. *Contemp. Math.*, **49**, 1986, 125–133.
12. Sato, M., T. Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. Jour.*, **65**, 1977, 1–155.
13. Simpson, C. Higgs bundles and local systems. *Publ. Math. I.H.E.S.*, **75**, 1992, 5–95.
14. Siu, Y. Complex analyticity of harmonic maps and strong rigidity of compact Kähler manifolds. *Ann. Math.*, **112**, 1980, 73–111.

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A NEW BOUND ON THE ABSORPTION COEFFICIENT OF A TWO-PHASE MEDIUM

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The Doi bound on the effective absorption coefficient of a random two-phase medium is revisited in this brief note. Defects are created in one of the constituents, being absorbed by the other one, which thus act as a perfect sink. Use is made of a variational principle due to Rubinstein and Torquato. The trial fields generalize the ones, originally proposed by Doi, and hence the new bound is more restrictive than the original Doi's one for an arbitrary medium. In the particular case of an array of nonoverlapping array of spherical sinks, the new bound however coincides with Doi's and with the one, derived by Talbot and Willis. In passing, besides the known "particle-particle" bound, a curious new "surface-surface" bound is extracted. Though a bit weaker than the Doi's, this bound relies only upon the two-point "surface" statistics. In the dilute case it reproduces the classical Smoluchowski result.

Keywords: random dispersions, correlation functions, effective properties, variational bounds, absorption problem

Mathematics Subject Classification 2000: 60G60, 60H15, 49K45

Consider a two-phase random medium, consisting of a phase '1', immersed into an unbounded matrix (phase '2'). The medium is assumed statistically homogeneous and isotropic. Let a species (defects) be generated at the rate K within the phase '2' (matrix) occupying the region $\mathcal{K}_2$. It is absorbed by the "sink" phase '1' in the region $\mathcal{K}_1 = \mathbb{R}^3 \backslash \mathcal{K}_2$. In the steady-state limit the concentration of the defects $c(x)$ is governed by the well-known equations

$$\Delta c(x) + K = 0, \quad x \in \mathcal{K}_2, \quad c(x) \Big|_{\partial \mathcal{K}_2} = 0. \tag{1}$$

The creation of defects is *exactly* compensated by their removal from the sinks

$$k^{*2} \langle c(x) \rangle = K(1 - \eta_1). \tag{2}$$

(The brackets $\langle\cdot\rangle$ denote ensemble averaging.) The rate constant k^{*2} is the so-called *effective absorption coefficient* (or *the sink strength*) of the medium. Its evaluation and bounding for special kinds of random constitution and, above all, for random dispersion of spheres, have been the subject of numerous works, starting with the classical studies of Smoluchowski [1], see, e.g., [2–7], the survey [8], and the references therein. (Note that we have added the factor $1-\eta_1$ in (2), due to the fact that in the case under study defects are created *only* within the phase '2', see [9] for a discussion.)

We shall confine the analysis to variational bounding of the sink strength k^{*2}, taking into account the two-point statistical information concerning medium constitution. The basic tool to be employed to this end is the variational principle of Rubinstein and Torquato (R-T) [6]. The principle states that in the class of statistically homogeneous trial fields such that

$$\mathcal{A} = \left\{ u(x) \mid \Delta u(x) + K = 0,\ x \in \mathcal{K}_2 \right\}, \tag{3}$$

the following inequality holds:

$$k^{*2} \geq \frac{K^2(1-\eta_1)}{\langle I_2(x)|\nabla u(x)|^2\rangle}\,. \tag{4}$$

Moreover, the equality sign in (4) is achieved if $u(x) = c(x)$ is the actual field that solves the problem (1).

Since $\left\langle I_2(x)|\nabla u(x)|^2\right\rangle \leq \left\langle|\nabla u(x)|^2\right\rangle$, another bound follows from (4):

$$k^{*2} \geq \frac{K^2(1-\eta_1)}{\langle|\nabla u(x)|^2\rangle}\,, \tag{5}$$

see [6]. Though weaker than (4), the evaluation of the bound (5) is simpler, because it obviously employs a smaller amount of statistical information.

Consider the trial fields

$$u(x) = -\frac{K}{\eta_1}\int G(x-y)\,\left(\lambda I_1(y) - \eta_1 + \frac{\mu\eta_1}{S}\,|\nabla I_1(y)|\right)\,\mathrm{d}y\,, \tag{6}$$

where $G(x) = 1/(4\pi|x|)$ and the (nondimensional) constants λ, μ are adjustable. Since $I_1(x)$ is the characteristic function of $\mathcal{K}_1$, $|\nabla I_1(x)|$ is δ-function, concentrated on the interphase boundary. In turn, the quantity S in (6) is the so-called mean surface, defined as $S = \langle|\nabla I_1(x)|\rangle$.

Since $\Delta G(x) + \delta(x) = 0$, one has $\Delta u(x) = K$ if $x \in \mathcal{K}_2$. This means that the fields $u(x)$ in (6) are indeed admissible, $u(x) \in \mathcal{A}$.

Consider now the quantity of central importance

$$\mathcal{U} = \left\langle|\nabla u(x)|^2\right\rangle/K^2 \tag{7}$$

that enters the estimate (5). For the latter to be finite, and hence to produce a nontrivial lower bound (5), it is necessary that the integrand in (6) have a zero mean value. This implies

$$\lambda + \mu = 1\,, \tag{8}$$

since $\langle I_1(y)\rangle = \eta_1$.

Note that the class of trial fields (6) generalizes the one, proposed by Doi himself [2]:

$$u(x) = K \int G(x-y)\left(I_2(y) + \xi\,|\nabla I_1(x)|\right) \mathrm{d}y\,. \tag{9}$$

The condition that mean value of the integrand in (9) vanishes reads

$$\eta_2 + \xi S = 0\,, \tag{10}$$

so that, as pointed out in [6, 7], there is no place for optimization with respect to ξ, as envisaged originally by Doi [2].

A simple check shows that our fields (6) reproduce the Doi's one (9) for a special choice of λ, namely, for

$$\lambda = \eta_1\,, \quad \mu = 1 - \eta_1\,, \quad \xi = \mu/S\,, \tag{11}$$

cf. Eq. (8).

With Eq. (8) taken into account, the class (6) is recast as

$$\begin{gathered} u(x) = -\frac{K}{\eta_1} \int G(x-y)\left\{\lambda I_1'(y) + \frac{\mu\eta_1}{S}\left(|\nabla I_1(x)| - S\right)\right\} \mathrm{d}y\,, \\ \langle I_1'(y)\rangle = 0\,, \quad \langle|\nabla I_1(x)| - S\rangle = 0\,, \\ I_1'(y) = I_1(y) - \eta_1\,, \end{gathered} \tag{12}$$

so that both random variables in the right-hand side of $(11)_1$ are fluctuations. Then the needed quantity $\mathcal{U}$, cf. (7), becomes

$$\mathcal{U} = \frac{a^2}{\eta_1^2}\left(\lambda^2\theta_1^{\mathrm{pp}} + 2\lambda\mu\eta_1\,\theta_1^{\mathrm{ps}} + \mu^2\eta_1^2\,\theta_1^{\mathrm{ss}}\right) \tag{13}$$

after an appropriate integration by parts. Here

$$\begin{aligned} \theta_1^{\mathrm{pp}} &= \int_0^\infty \rho F^{\mathrm{pp}}(\rho)\,\mathrm{d}\rho\,, \\ \theta_1^{\mathrm{ps}} &= \int_0^\infty \rho F^{\mathrm{ps}}(\rho)\,\mathrm{d}\rho\,, \\ \theta_1^{\mathrm{ss}} &= \int_0^\infty \rho F^{\mathrm{ss}}(\rho)\,\mathrm{d}\rho \end{aligned} \tag{14}$$

are the first moments on the semiaxis $(0,\infty)$ of the "particle-particle," "particle-surface," and "surface-surface" (two-point) correlation functions, respectively, defined here as follows:

$$\begin{aligned} F^{\mathrm{pp}}(\rho) &= \langle I_1'(x) I_1'(0)\rangle\,, \\ F^{\mathrm{ps}}(\rho) &= \frac{1}{S}\,\langle I_1'(x)\,(\nabla|I_1(0)| - S)\rangle\,, \\ F^{\mathrm{ss}}(\rho) &= \frac{1}{S^2}\,\langle(\nabla|I_1(x) - S)\,(\nabla|I_1(0)| - S)\rangle\,. \end{aligned} \tag{15}$$

The multipliers $1/S$ and $1/S^2$ have been added in the definitions (14) in order to make the respective correlations dimensionless. In Eq. (14) $\rho = r/a$, $r = |x|$, where a is a certain characteristic length for the phase '1', for example, the mean size of the sinks. If the sinks are identical spheres — a case to be specially discussed below — then obviously a is to be identified with their radius.

In virtue of (8) and (13) we have

$$\mathcal{U} = \mathcal{U}(\lambda) = \frac{a^2}{\eta_1^2}\left\{\lambda^2\left(\theta_1^{\rm pp} - 2\eta_1\theta_1^{\rm ps} + \eta_1^2\theta_1^{\rm ss}\right) + 2\lambda\eta_1\left(\theta_1^{\rm ps} - \eta_1\theta_1^{\rm ss}\right) + \eta_1^2\theta_1^{\rm ss}\right\}. \tag{16}$$

Optimizing (15) with respect to λ gives the estimate

$$k^{*2}a^2 \geq k_{\rm N}^{*2}a^2\,, \quad k_{\rm N}^{*2}a^2 = (1-\eta_1)\,\frac{\theta_1^{\rm pp} - 2\eta_1\theta_1^{\rm ps} + \eta_1^2\theta_1^{\rm ss}}{\theta_1^{\rm pp}\theta_1^{\rm ss} - \left(\theta_1^{\rm ps}\right)^2}\,. \tag{17}$$

This is our generalization of Doi's bound. The latter shows up if $\lambda = \eta_1$ in $\mathcal{U}$, as given in (16):

$$k^{*2} \geq k_{\rm D}^{*2}\,, \quad k_{\rm D}^{*2}a^2 = \frac{1-\eta_1}{\theta_1^{\rm pp} + 2\eta_2\theta_1^{\rm ps} + \eta_2^2\theta_1^{\rm ss}}\,. \tag{18}$$

It is clear that (17) always improves upon Doi's bound (18), since we have allowed for λ to be adjustable. The bounds (17) and (18) will coincide only if the optimal λ, that minimizes $\mathcal{U}(\lambda)$ in (16), is *exactly* η_1. The latter is the case if the moments (14) for a given random constitution are, by chance, interconnected as follows:

$$\theta_1^{\rm pp} + (1-2\eta_1)\theta_1^{\rm ps} - \eta_1\eta_2\theta_1^{\rm ss} = 0\,. \tag{19}$$

It is to be noted that, besides "Doi's choice" $\lambda = \eta_1$, two more particular values of λ deserve a special attention.

The first choice is $\lambda = 1$. Then $\mathcal{U} = a^2\theta_1^{\rm pp}$ and Eqs. (5) and (7) yield

$$k^{*2} \geq k_{\rm pp}^{*2}\,, \quad k_{\rm pp}^{*2}a^2 = \frac{\eta_1^2(1-\eta_1)}{\theta_1^{\rm pp}}\,. \tag{20}$$

This is a known bound, called by Torquato and Rubinstein [5] "particle-particle." The reason behind this term is clear — the evaluation of $k_{\rm pp}^{*2}$ requires only the statistical information incorporated into the "particle" correlation function $F^{\rm pp}(\rho)$.

The second choice of interest is $\lambda = 0$. Then $\mathcal{U} = a^2\theta_1^{\rm ss}$, see (15), and Eqs. (5) and (7) yield

$$k^{*2} \geq k_{\rm ss}^{*2}\,, \quad k_{\rm ss}^{*2}a^2 = \frac{1-\eta_1}{\theta_1^{\rm ss}}\,. \tag{21}$$

This is a new bound, which is natural to be called "surface-surface." The reason for this term is again clear. The evaluation of $k_{\rm ss}^{*2}$ this time requires only the statistical information incorporated into the "surface" correlation function $F^{\rm pp}(\rho)$.

Consider two classical examples that concern dispersions of identical spherical sinks of radius a with number density n.

In the first example the sinks are forbidden to overlap. Their centers are distributed randomly with the two-point distribution function

$$f_2(r) = n^2 g(r) . \tag{22}$$

The interpretation is that $f_2(r)\,dV_A dV_B$ provides the probability of finding spheres centers in the vicinities dV_A and dV_B of the points A and B, respectively, such that the distance between the latter is r; the function $g(r)$ in (21) is the familiar radial distribution function.

The moments (14) have been evaluated in [10], among other statistical characteristics of dispersions of nonoverlapping spheres, making use of the function $g(r)$. The needed, for our purposes, expressions read

$$\begin{aligned} \theta_1^{\rm pp} &= \eta_1^2\left(\frac{2-9\eta_1}{5\eta_1}+m_1\right) , \\ \theta_1^{\rm ps} &= \eta_1\left(\frac{5-26\eta_1}{15\eta_1}+m_1\right) , \\ \theta_1^{\rm ss} &= \frac{1-5\eta_1}{3\eta_1}+m_1 , \end{aligned} \tag{23}$$

where $\eta_1 = \frac{4}{3}\pi n a^3$ is the volume fraction of the sinks (the phase '1') and

$$m_1 = \int_2^\infty \rho \nu_2(\rho)\,d\rho , \quad \nu_2(\rho) = g(\rho) - 1 , \tag{24}$$

so that $\nu_2(r)$ is the so-called total correlation function. A simple check shows that the condition (19) is satisfied, whatever the total correlation. Hence our bound coincides in this case with Doi's one, yielding

$$k^{*2} \geq k_{\rm D}^{*2} , \quad k_{\rm D}^{*2} a^2 \geq \frac{3\eta_1(1-\eta_1)}{1-5\eta_1-\eta_1^2/5+3\eta_1 m_1} = 3\eta_1 + o(\eta_1). \tag{25}$$

As indicated, the bound (25) is *exact* in the dilute limit $\eta_1 \ll 1$, since it reproduces in this case the well-known Smoluchowski's result [1].

The bound (25) first appeared in Willis' lecture [4]; a more precise derivation, together with some generalizations, is due to Talbot and Willis [5], see also [11] for a discussion and an alternative derivation, based on the Rubinstein-Torquato variational principle (4). The fact that the original Doi's bound for a dispersion of nonoverlapping spheres can be recast in the elegant Talbot-Willis' form (24) was noticed also by Beasley and Torquato [12].

It is noted that the "particle-particle" and "surface-surface" bounds for the dispersion under study have, respectively, the form

$$\begin{aligned} k_{\rm pp}^{*2} a^2 &= \frac{5\eta_1(1-\eta_1)}{2-9\eta_1+5\eta_1 m_1} = \tfrac{5}{2}\eta_1 + o(\eta_1) , \\ k_{\rm ss}^{*2} a^2 &= \frac{3\eta_1(1-\eta_1)}{1-5\eta_1+3\eta_1 m_1} = 3\eta_1 + o(\eta_1) , \end{aligned} \tag{26}$$

see Eqs. (20) – (22).

Clearly, the "surface" bound k_{ss}^{*2} is superior to the "particle" one, which is natural — the absorption phenomenon under study, Eq. (1), is governed by the absorption, taking place on the interphase boundary. Moreover, k_{ss}^{*2} reproduces the correct Smoluchowski's value in the dilute limit, unlike k_{pp}^{*2}. Also, k_{ss}^{*2} is very close to Doi's bound k_{D}^{*2}, the sole difference being the term $-\eta_1^2/5$ in the denominator, cf. Eqs. (25) and (26). However, one cannot claim that k_{ss}^{*2} will be superior to k_{pp}^{*2} for all random constitutions and for all values of η_1. An example, when this is not so, will be supplied below, when discussing the appropriate bounds for the Boolean model of overlapping spheres.

In the simplest "well-stirred" dispersion $g(r) - 1 = h_{2a}(r)$, i.e. the spheres are only forbidden to overlap ($h_{2a}(r)$ is the characteristic function for a sphere of radius $2a$ located at the origin). Then $\nu_2(\rho) = 0$, if $\rho \geq 2$, $m_1 = 0$ and the bounds (24), (25) become

$$k_{\text{pp}}^{*2}a^2 = \frac{5\eta_1(1-\eta_1)}{2-9\eta_1}, \quad k_{\text{ss}}^{*2}a^2 = \frac{3\eta_1(1-\eta_1)}{1-5\eta_1}, \quad k_{\text{D}}^{*2}a^2 = \frac{3\eta_1(1-\eta_1)}{1-5\eta_1-\eta_1^2/5}. \tag{27}$$

Obviously, the "particle" bound k_{pp}^{*2} fails in this case if $\eta_1 \geq 2/9$. Similarly, the "surface" one, k_{ss}^{*2}, fails at $\eta_1 \geq 0.2$, and Doi-Talbot-Willis k_{D}^{*2} — at $\eta_1 \geq \eta_1^0$, $\eta_1^0 \approx 0.1984$ — a fact, explicitly underlined in [4, 5]. This means that the "well-stirred" approximation is unrealistic beyond the value η_1^0 of sphere fraction. This fact, however, is of little interest due to the more recent result of Markov and Willis [13, 14], stating that "well-stirred" approximation is already unrealistic if $\eta_1 \geq 1/8$.

More realistic than the well-stirred one is the Percus-Yevick (PY) approximation for a dispersion of nonoverlapping spheres [15], widely used in the liquid state theory. The Laplace transform of the function ν_2 is analytically known due to Wertheim [16]. An appropriate asymptotic analysis of the Wertheim's formula allows one to obtain, in turn, a number of statistical characteristics of a PY dispersion, see [16, 6]. In particular, it turns out that the parameter (24), needed in the bounds under consideration, is simply

$$m_1^{\text{PY}} = \frac{\eta_1(22-\eta_1)}{5(1+2\eta_1)}. \tag{28}$$

(Note that an equivalent, but much more complicated formula for m_1^{PY} is given by Talbot and Willis [5, Eqs. (8.14) and (8.15)].)

The formula (28), when inserted into (25), gives the Doi-Talbot-Willis bound for a Percus-Yevick dispersion in an extremely simple form:

$$k_{\text{D}}^{*2}a^2 = \frac{3\eta_1(1+2\eta_1)}{(1-\eta_1)^2}. \tag{29}$$

The values of k_{D}^{*2} obviously remain finite for all sphere fractions $\eta_1 \in (0,1)$. This fact makes the application of the PY approximation suspicious for higher volume fractions. The reason is that *any* realistic model of dispersions, in which the spheres are forbidden to overlap, should fail for volume fractions higher than 0.64 — the value corresponding approximately to the close packing of the inclusions.

The second case is the well-known randomly imbedded model of spheres [13, 14], called also Boolean [15]. Here an infinite family of points are placed "fully" randomly throughout the space — more precisely, forming a Poissonian system of number density (intensity) n. Identical spheres of the radius a are centered then at these points, with overlapping permitted. The phase '2' (the "sink-free" part) is then defined as the region, empty of spheres. The "sink" phase '1' comprises either the single spheres or the aggregates, formed by families of overlapping spheres.

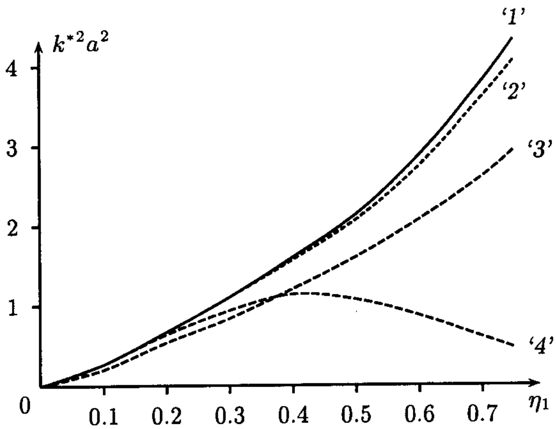

Fig. 1. The various bounds on the dimensionless effective absorption coefficient for the Boolean model. '1' — our new bound $k_{\mathrm{N}}^{*2}a^2$, see (17); '2' — the Doi bound $k_{\mathrm{D}}^{*2}a^2$, see (18); '3' — the "surface-surface" bound $k_{\mathrm{ss}}^{*2}a^2$, see (21); '4' — the "particle-particle" bound $k_{\mathrm{pp}}^{*2}a^2$, see (20)

Here the needed two-point correlations, as evaluated by Doi [2], read

$$\begin{aligned}
\eta_2 &= \exp\left(\tfrac{4}{3}\pi n a^3\right), \quad D(\rho) = 1 + \tfrac{3}{4}\rho - \tfrac{3}{16}\rho^3, \\
F^{\mathrm{pp}}(\rho) &= \left(\eta_2^{D(\rho)} - \eta_2^2\right)(1 - H(\rho - 2)), \\
\frac{1}{S}F^{\mathrm{ps}}(\rho) &= \left(\eta_2 - \frac{(2+\rho)(\eta_2^2 + F^{\mathrm{pp}}(\rho))}{4\eta_2}\right)(1 - H(\rho-2)), \\
\frac{1}{S^2}F^{\mathrm{ss}}(\rho) &= \left\{\left(\frac{1}{6\eta_1\eta_2^2\rho} + \frac{(2+\rho)^2}{16\eta_2^2}\right)\left(\eta_2^2 + F^{\mathrm{pp}}(\rho)\right) - 1\right\}(1 - H(\rho-2)),
\end{aligned} \tag{30}$$

here $\rho = r/a$, $H(r)$ is the Heaviside function.

The moments (14) can be evaluated in this case only numerically. The appropriate bounds on the dimensionless effective absorption coefficient $k^{*2}a^2$ are shown in Fig. 1 as functions on the volume fraction η_1 of the sink constituent. It is seen that our bound (17) does improve upon the Doi one, given in (17), but the improvement is small and shows up only at very high values of the fraction η_1. The behaviour of the "surface-surface" bound $k_{\mathrm{ss}}^{*2}a^2$, see (21), deserves some more attention. For dilute fractions $\eta_1 \ll 1$, it is undoubtedly better than the 'particle-particle" one, see (26), since the Boolean and "nonoverlapping" dispersions share the same effective properties in the dilute limit. However, in the region $\eta_1 \in (0.1, 0.3)$, a bit unexpectedly, $k_{\mathrm{ss}}^{*2}a^2$ falls below $k_{\mathrm{pp}}^{*2}a^2$. Only at $\eta_1 > 0.3$ the "surface-surface" bound becomes superior as compared to the "particle-particle" one. Moreover, it becomes much more sensible when η_1 increases. At the same time the bound $k_{\mathrm{pp}}^{*2}a^2$ deteriorates badly with increasing η_1. The reason is clear: in the Boolean model, when the sink fraction increases, the overlapping becomes more and more frequent, the shape of the aggregates formed by the particles becomes more and more complicated and the specific surface increases considerably as a result.

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REFERENCES

1. Smoluchowski, M. Drei Vorträge über Diffusion, Brownsche Molekularbewegung und Koagulation von Kolloidteichen. *Physik. Zeitschr.*, **17**, 1916, 557–571. Schluß, 585–599.
2. Felderhof, B. U., J. M. Deutch. Concentration dependence of the rate of diffusion-controlled reactions. *J. Chem. Phys.*, **64**, 1976, 4551–4558.
3. Doi, M. A new variational approach to the diffusion and the flow problem in porous media. *J. Phys. Soc. Japan*, **40**, 1976, 567–572.
4. Willis, J. R. Variational principles and bounds for the overall properties of composites. In: *Continuum Models of Discrete Systems*, J. Provan, ed., Univ. of Waterloo Press, Waterloo, Ontario, 1978, 185–215.
5. Talbot, D. R. S., J. R. Willis. The effective sink strength of a random array of voids in irradiated material. *Proc. R. Soc. London A*, **370**, 1980, 351–374.
6. Rubinstein, J., S. Torquato. Diffusion-controlled reactions: Mathematical formulation, variational principles, and rigorous bounds. *J. Chem. Phys.*, **88**, 1988, 6372–6380.
7. Torquato, S., J. Rubinstein. Diffusion-controlled reactions: II. Further bounds on the rate constant. *J. Chem. Phys.*, **90**, 1989, 1644–1647.
8. Calef, D. F., J. M. Deutch. Diffusion-controlled reactions. *Ann. Rev. Phys. Chem.*, **34**, 1983, 493–523.
9. Richards, P. M., S. Torquato. Upper and lower bounds for the rate of diffusion-controlled reactions. *J. Chem. Phys.*, **87**, 1987, 4612–4614.
10. Markov, K. Z. On the two-point correlation functions in random arrays of nonoverlapping spheres. *Annuaire (Godishnik) Univ. Sofia, Fac. Math. Inf., Livre 2, Math. Appl. et Informatique* **91/1997**, 1999, 151–175.

11. Markov, K. Z. On a two-point correlation function in random dispersions and an application. In: *Continuum Models and Discrete Systems*, eds. E. İnan, K. Z. Markov, World Sci., 1998, 206–215.

12. Beasley, J. D., S. Torquato. New bounds on the permeability of a random array of spheres. *Phys. Fluids A*, **1**, 1989, 199–207.

13. Markov, K. Z. On a statistical parameter in the theory of random dispersions of spheres. In: *Continuum Models of Discrete Systems*, ed. K. Z. Markov, World Sci., 1996, 241–249.

14. Markov, K. Z., J. R. Willis. On the two-point correlation function for dispersions of nonoverlapping spheres. *Mathematical Models and Methods in Applied Sciences*, **8**, 1998, 359–377.

15. Percus J. K., G. J. Yevick. Analysis of classical statistical mechanics by means of collective co-ordinates. *Phys. Rev.*, **110**, 1957, 1–13.

16. Wertheim, M. S. Exact solution of the Percus-Yevick integral equation for hard spheres. *Phys. Rev. Lett.*, **10**, 1963, 321–325.

17. Weissberg, H. L. Effective diffusion coefficients in porous media. *J. Appl. Phys.* **34**, 1963, 2636–2639.

18. Strieder, W., R. Arris. Variational methods applied to problems of diffusion and reaction. Springer-Verlag, Berlin, 1973.

19. Matheron, G. Random sets and integral geometry. Wiley & Sons, New York, 1975.

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FINITE DEFORMATIONS OF TWO DROPS DUE TO ELECTRIC FIELD

IAVOR VARBANOV HRISTOV

The finite deformations of two drops due to electric field are investigated in this article. The radii of the drops and the fluid phases could be different. Reynolds' number is assumed small enough to solve the problem in quasisteady Stokes' approximation. It is also supposed that the initial form of the drops is spherical and the fluids are homogenous, incompressible and Newtonian.

The electric and hydrodynamic problems are separated and the electric one has an influence on the hydrodynamic one through the boundary conditions. The Maxwell's equations are turned to Laplace's equations, and together with Stokes' equations they are solved by semianalytical-seminumerical method. We use boundary-integral type of these equations to solve them by the method of boundary elements. The kinematic condition gives a new form to the particles.

The results obtained indicate that interactions between two and three fluid phases, due to electric field, lead to deformations of the drops. The influence over the deformations of some dimensionless parameters of the problem has been given graphically.

Keywords: electric field, deformation, drop, boundary elements method, fluid phases, interfaces

Mathematics Subject Classification 2000: 76T30

1. INTRODUCTION

Basic ideas for investigating the matter of fluid particles deformations have been presented first by G. I. Taylor (1932). In his next paper (1934) Taylor has found the critical velocity of shear flow, after which a drop set in the flow starts to elongate. In [26] it has been proved that in uniform flow in Stokes approximation an initially spherical particle remains spherical without any deformations.

E. Chervenivanova and Z. Zapryanov obtain small deformations of drop moving with a uniform velocity in spherical container, full with viscous fluid. Although the flow is uniform, there are deformations of the drop, because it is in a container which causes the deformations.

Uijttewaal et al. (1993) solve numerically the three-dimensional problem of drop in linear shear flow moving to a plane wall, using the boundary element method.

The problems of single drop subjected in viscous flow are in the basis for solving problems of compound drops (drop in drop), drop near a plane wall or two separated drops.

The technique "method of reflection", which is used for the first time by Smoluchowski (1911), is in the base of the first systematic investigations of the dynamics of two fluid drops made by Happel & Brenner (1965).

Small deformations of two fluid drops have been presented first in [3]. Deformations of two fluid droplets, drop and bubble, and drop and rigid particle in uniform flow are obtained. A parametric analysis of the small deformations relative to the distance between drops and the ration of viscosities of the different phases is made. "Dimple" formation is one of the basic results of the paper.

The influence of electric field on a water drop has been investigated experimentally in [29] and [12]. The authors have found the critical value of dimensionless parameter (E^*) after which the drop breaks up. Taylor (1964) improves theoretically this value supposing that the drop preserves its spherical form until the break up. In [1, 5, 9, 25] a drop's break up with conical tips is examined. Ramos & Castellanos (1994) present theoretical result for the influence of the coefficients of permittivity and conductivity on the conical tips formation. Torza, Cox & Mason (1971) have found experimentally another model of breaking up a drop, which is divided into two spherical parts connected with a thin "throat". Sherwood (1988), using the method of boundary elements, solves numerically the problem of a single fluid particle deformation under the influence of electric field.

The form which two equal fluid drops achieve in the presence of electric field is given experimentally by O'Konski & Thacker (1953). In the papers [5, 13, 23] the authors show that due to the same electric field but with different parameters of the fluid phases (conductivities, permittivities) there are deformations of the interfaces based on electrostatic charge. In [2, 11, 24] a couple of equal water drops situated in an electric field is investigated experimentally. Sozou (1975), using bipolar coordinate system, presents semianalytical decision for velocities in and out of the drops, presuming keeping the spherical form.

2. FORMULATION OF THE PROBLEM

The problem for defining the finite deformations of two fluid drops due to the electric field is separated into two problems — electrostatic and hydrodynamic. The Navier-Stokes equations and the Maxwell's equations are describing most precisely that problem. In low Reynolds number and quasisteady approximation they turn respectively into Stokes equations for velocities and Laplace's equations for electric potentials, as written below.

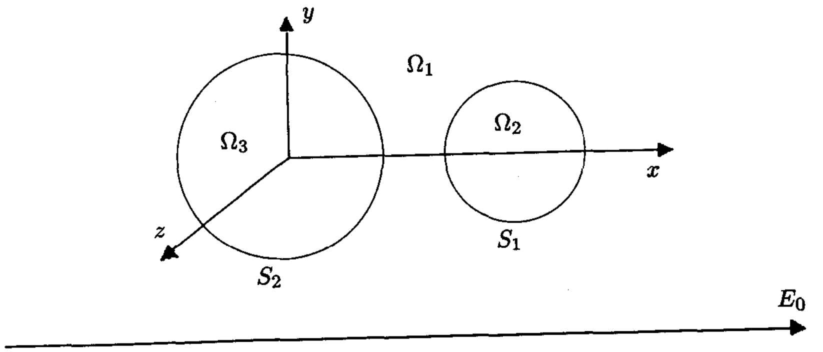

Fig. 1. Scheme of two drops in the presence of electric field

The drops on Fig. 1 are compounded of fluid 2 with viscosity μ_2, conductivity σ_2, permittivity ε_2, and fluid 3 with viscosity μ_3, conductivity σ_3, permittivity ε_3. The electric field that acts on the axis connecting the centres of the drops is with intensity E_0. Under its influence the interfaces of the drops deform. The initial form of fluid drops is spherical with undistorted radius R_1 of the first sphere and undistorted radius R_2 of the second one. With S_1 is marked the interface between phase 1 and phase 2, and with S_2 — the interface between phase 1 and phase 3. The interfacial tensions over S_1 and S_2 are γ_1 and γ_2, respectively. The fluids 1, 2 and 3 are situated in Ω_1, Ω_2 and Ω_3, respectively, while Ω_1 is the infinite area outside the drop (Fig. 1).

At each point the electric potential and the velocity of the flow at each moment is governed by the following equations:

— Laplace's equations: $\Delta\varphi^k = 0 \quad (k = 1, 2, 3);$ (2.1)

— discontinuity equations: $\dfrac{\partial u_i^k}{\partial x_i} = 0 \quad (i, k = 1, 2, 3);$ (2.2)

— Stokes' equations: $\dfrac{\partial \sigma_{ij}^k}{\partial x_j} = 0 \quad (i, j, k = 1, 2, 3)$ (2.3)

where σ_{ij}^k is the stress tensor $\sigma_{ij}^k = -p^k\delta_{ij} + \mu_k\left(\dfrac{\partial u_i^k}{\partial x_j} + \dfrac{\partial u_j^k}{\partial x_i}\right)$.

The index $k = 1$ for $\boldsymbol{x} \in \Omega_1$, $k = 2$ for $\boldsymbol{x} \in \Omega_2$ and $k = 3$ for $\boldsymbol{x} \in \Omega_3$, while p^k is the hydrodynamic pressure of the respective fluid. The electric potential in the three phases satisfies the following boundary conditions:

$$\varphi^1(\boldsymbol{x}_0) \to E_0 x_0^1, \qquad |\boldsymbol{x}_0| \to \infty, \tag{2.1.a}$$

$$\varphi^1(\boldsymbol{x}_0) = \varphi^2(\boldsymbol{x}_0), \qquad \boldsymbol{x}_0 \in S_1, \tag{2.1.b}$$

$$\varphi^1(\boldsymbol{x}_0) = \varphi^3(\boldsymbol{x}_0), \qquad \boldsymbol{x}_0 \in S_2, \tag{2.1.c}$$

$$\sigma_1 \frac{\partial \varphi^1}{\partial n}(x_0) = \sigma_2 \frac{\partial \varphi^2}{\partial n}(x_0), \qquad x_0 \in S_1, \tag{2.1.d}$$

$$\sigma_1 \frac{\partial \varphi^1}{\partial n}(x_0) = \sigma_3 \frac{\partial \varphi^3}{\partial n}(x_0), \qquad x_0 \in S_2, \tag{2.1.e}$$

where E_0 is the intensity of the electric field, x_0^1 is the x-component in Decart coordinate system $Oxyz$ of the vector x_0, and σ_1, σ_2, σ_3 are the electric conductivities of the respective fluids, and $\frac{\partial}{\partial n}$ is the normal derivative to the surface, pointed out of the respective domain.

The flow field is governed by the following boundary conditions:

$$u_i^1(x_0) \to 0, \qquad |x_0| \to \infty, \tag{2.3.a}$$

$$u_i^1(x_0) = u_i^2(x_0), \qquad x_0 \in S_1, \tag{2.3.b}$$

$$\begin{aligned} &\sigma_{ij}^1(x_0)n_j(x_0) - \sigma_{ij}^2(x_0)n_j(x_0) \\ &\quad = \gamma_1 n_i \frac{\partial n_j}{\partial x_j} - \left(\tau_{ij}^1(x_0)n_j(x_0) - \tau_{ij}^2(x_0)n_j(x_0)\right), \qquad x_0 \in S_1, \end{aligned} \tag{2.3.c}$$

$$u_i^1(x_0) = u_i^3(x_0), \qquad x_0 \in S_2, \tag{2.3.d}$$

$$\begin{aligned} &\sigma_{ij}^1(x_0)n_j(x_0) - \sigma_{ij}^3(x_0)n_j(x_0) \\ &\quad = \gamma_2 n_i \frac{\partial n_j}{\partial x_j} - \left(\tau_{ij}^1(x_0)n_j(x_0) - \tau_{ij}^3(x_0)n_j(x_0)\right), \qquad x_0 \in S_2. \end{aligned} \tag{2.3.e}$$

Here n is the single outer normal to the interface S_1 or S_2,

$$\tau_{ij}^k = -\frac{\varepsilon_k}{4\pi}\left(\frac{(\mathbf{E}^k)^2}{2}\delta_{ij} - \mathbf{E}_i^k \mathbf{E}_j^k\right)$$

is the Maxwell's electric stress tensor for the respective phases ($k = 1, 2, 3$), where ε_k is the electric permittivity of the different phases and $\mathbf{E}^k = -\nabla \varphi^k$. Let us assume that S_1 and S_2 are Lyapunov's surfaces. The solution of (2.1) with boundary conditions (2.1.a–e) gives us the electric potentials at each moment and at every point of the three phases. The solution of (2.2), (2.3) with boundary conditions (2.3.a–e) gives us the velocity at each moment and at every point of S_1 and S_2. The deformation of the interfaces is determined at each moment by the normal component of the velocity and the kinematic condition:

$$\frac{dx_s}{dt} = n_i(u_i \cdot n_i) = n_i \cdot u_n. \tag{2.4}$$

Here x_s is a point of the respective surface S_1 or S_2, while u_n is the normal component of the velocity at this point.

Following Greengard & Moura (1994), the integral equations, which determine the potentials of the electric field on the interfaces, are solutions of the system (2.1) with boundary conditions (2.1.a–e) in the single-layer integral form:

$$\varphi(x_0) = \psi(x_0) + \int_S G(x_0, x)\rho(x)\, dS_x, \tag{2.5}$$

where $\varphi(x_0)$ is the total potential of the field at the point x_0:

$$\varphi^1(x_0) = \varphi(x_0)_{|\Omega_1}, \qquad \varphi^2(x_0) = \varphi(x_0)_{|\Omega_2},$$

$$\varphi^3(x_0) = \varphi(x_0)_{|\Omega_3}, \qquad \psi(x_0) = E_0.x_0^1,$$

$\rho(x)$ is an unknown function of distribution, $G(x_0, x)$ is a Green's function for the domain $S = S_1 \cup S_2$, which for our case is $G(x_0, x) = \dfrac{1}{|x - x_0|}$.

By substituting (2.1.d–e) in (2.5) and using the single-layer potential theory we derive:

$$\sigma_k \left[\frac{\partial \psi(x_0)}{\partial n} - \frac{1}{2}\rho(x_0) + \int_S \frac{\partial G}{\partial n}(x_0, x)\rho(x)\, dS_x \right]$$

$$= \sigma_1 \left[\frac{\partial \psi(x_0)}{\partial n} + \frac{1}{2}\rho(x_0) + \int_S \frac{\partial G}{\partial n}(x_0, x)\rho(x)\, dS_x \right], \tag{2.6}$$

$$\rho(x_0) - 2\lambda_k \int_S \frac{\partial G}{\partial n}(x_0, x)\rho(x)\, dS_x = \lambda_k \frac{\partial \psi(x_0)}{\partial n},$$

where $\lambda_k = \dfrac{\sigma_k - \sigma_1}{\sigma_k + \sigma_1}$, $k = 2, 3$.

In order to solve the hydrodynamic problem (2.2), (2.3) with boundary conditions (2.3.a–e), following Power [14], we use Green's integral representation formulae for Stokes equations to get to integral equations, which determine the velocities on the interfaces:

$$\begin{aligned} \frac{1 + (\mu_1/\mu_2)}{2} u_i^1(x_0) = & -\left(1 - \frac{\mu_1}{\mu_2}\right) \int_{S_1} T_{ijk}(x_0, x) u_j^1(x) n_k(x)\, dS_x \\ & -\left(1 - \frac{\mu_1}{\mu_3}\right) \int_{S_2} T_{ijk}(x_0, x) u_j^3(x) n_k(x)\, dS_x \\ & -\frac{1}{\mu_1} \int_{S_1} J_{ij}(x_0, x) \left(\gamma_1 n_j \frac{\partial n_k}{\partial x_k} - \left(\tau_{jk}^1 n_k - \tau_{jk}^2 n_k\right) \right) dS_x \\ & -\frac{1}{\mu_1} \int_{S_2} J_{ij}(x_0, x) \left(\gamma_2 n_j \frac{\partial n_k}{\partial x_k} - \left(\tau_{jk}^1 n_k - \tau_{jk}^3 n_k\right) \right) dS_x \end{aligned} \tag{2.7}$$

for each $x_0 \in S_1$,

$$\begin{aligned} \frac{1 + (\mu_1/\mu_3)}{2} u_i^3(x_0) = & -\left(1 - \frac{\mu_1}{\mu_2}\right) \int_{S_1} T_{ijk}(x_0, x) u_j^1(x) n_k(x)\, dS_x \\ & -\left(1 - \frac{\mu_1}{\mu_3}\right) \int_{S_2} T_{ijk}(x_0, x) u_j^3(x) n_k(x)\, dS_x \end{aligned} \tag{2.8}$$

$$-\frac{1}{\mu_1}\int\limits_{S_2} J_{ij}(x_0,x)\left(\gamma_2 n_j\frac{\partial n_k}{\partial x_k}-\left(\tau^1_{jk}n_k-\tau^3_{jk}n_k\right)\right)dS_x$$

$$-\frac{1}{\mu_1}\int\limits_{S_1} J_{ij}(x_0,x)\left(\gamma_1 n_j\frac{\partial n_k}{\partial x_k}-\left(\tau^1_{jk}n_k-\tau^2_{jk}n_k\right)\right)dS_x$$

for each $x_0 \in S_2$.

The equations (2.5)–(2.8) are dimensionlized to a form that is given in the next part and the following dimensionless parameters are included:

$E_\gamma = \dfrac{\varepsilon_1 E_0^2 R_1}{\gamma_2}$ — dimensionless parameter that indicates the relation between the electric and the capillary forces;

$\mu_{12} = \dfrac{\mu_1}{\mu_2}$, $\mu_{13} = \dfrac{\mu_1}{\mu_3}$ — the relation of the viscosities of the different neighbouring phases;

$\varepsilon_{21} = \dfrac{\varepsilon_2}{\varepsilon_1}$, $\varepsilon_{31} = \dfrac{\varepsilon_3}{\varepsilon_1}$ — the relation of the electric permittivity of the different neighbouring phases;

$\sigma_{12} = \dfrac{\sigma_1}{\sigma_2}$, $\sigma_{13} = \dfrac{\sigma_1}{\sigma_3}$ — the relation of the electric conductivity of the different neighbouring phases;

$\gamma_{12} = \dfrac{\gamma_1}{\gamma_2}$ — the relation of the interface tension coefficient of the two surfaces of the drops;

$R_{12} = \dfrac{R_1}{R_2}$ — the relation between the radii of the two drops.

To accomplish the formulation of the problem, we should say that on each time step we solve first the electrostatic problem, which has an influence on the hydrodynamic one, by Maxwell's electric stress tensor. On its turn, the solving of the hydrodynamic problem gives us the velocities of the fluids in the different phases. Using the kinematic condition for the normal velocity components on the fluid surfaces, we get their deformation. With the new form (changed boundary conditions) we solve once again the electrostatic problem and after that the hydrodynamic one, since the number of time steps determines how many times this procedure will be used. The criteria for ending the procedure are reaching an equilibrium form of the drops or "break up".

3. ALGORITHM FOR DETERMINING THE DEFORMATIONS OF TWO DROPS DUE TO ELECTRIC FIELD

The main steps of the algorithm followed are:

- change of the co-ordinate system from Decart's to cylindrical, in order to transform the boundary integrals to one-dimensional;
- introduction of boundary elements over the boundaries of the domains — arcs of circles;

- introduction of local polar co-ordinate system for each boundary element;
- calculation of the integrals of the single- and double-layer over each boundary element;
- subtraction of the integrals singularities;
- calculation of the velocity on the interfaces;
- determination of the drops form from the kinematic condition.

Due to the axisymmetric flow of the problem, we change the co-ordinate system to a cylindrical one (x, σ, ϕ), in which none of the unknown functions depends on the azimuthal angle ϕ: $\varphi_\phi = \rho_\phi = u_\phi = n_\phi = 0$ (Fig. 2).

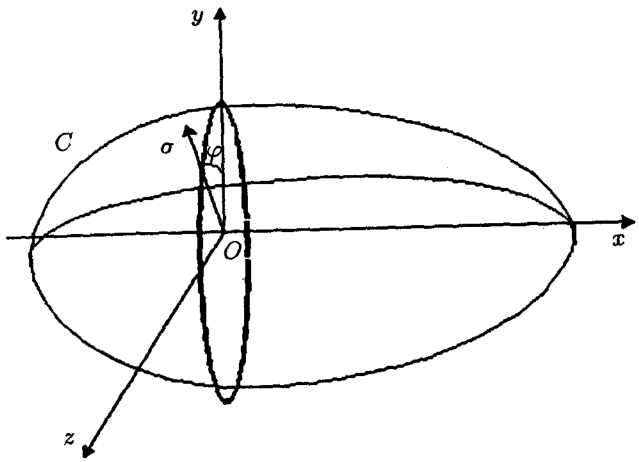

Fig. 2. Scheme of cylindrical co-ordinate system for axisymmetric flow

The normal $\boldsymbol{n}$, the velocity $\boldsymbol{u}$, the electric potential φ and the unknown function of distribution ρ are presented through the new co-ordinates:

$$\boldsymbol{x} = [x, \sigma\cos\phi, \sigma\sin\phi], \quad \boldsymbol{x}_0 = [x_0, \sigma_0\cos\phi_0, \sigma_0\sin\phi_0], \quad r_x = x_0 - x, \quad \boldsymbol{r}_x = \boldsymbol{x}_0 - \boldsymbol{x}$$

$$[n_x, n_y, n_z] = [n_x, n_\sigma\cos\phi, n_\sigma\sin\phi], \quad [u_x, u_y, u_z] = [u_x, u_\sigma\cos\phi, u_\sigma\sin\phi],$$

$$[\varphi_x, \varphi_y, \varphi_z] = [\varphi_x, \varphi_\sigma\cos\phi, \varphi_\sigma\sin\phi], \quad [\rho_x, \rho_y, \rho_z] = [\rho_x, \rho_\sigma\cos\phi, \rho_\sigma\sin\phi].$$

The differential dS is presented through the formula $dS = \sigma\, d\phi dl$, where dl is the elementary length of the curve $C = C_1 \cup C_2$ projection of $S = S_1 \cup S_2$ in the meridian plain Oxy, i.e. C_1 is the projection of S_1, while C_2 is the projection of S_2. Thus, we get to the following equations:

$$\varphi(\boldsymbol{x}_0) = \psi(\boldsymbol{x}_0) + \frac{1}{2\pi}\int_C \rho(l) K^S(\boldsymbol{x}_0, l)\sigma(l)\, dl(\boldsymbol{x}), \tag{3.1}$$

$$\rho(\boldsymbol{x}_0) - \frac{\lambda_k}{\pi}\int_C \rho(l) K^D(\boldsymbol{x}_0, l)\sigma(l)\, dl(\boldsymbol{x}) = \lambda_k \frac{\partial\psi(\boldsymbol{x}_0)}{\partial \boldsymbol{n}}, \tag{3.2}$$

where

$$K^S(\boldsymbol{x}_0, l) = \int_0^{2\pi} \frac{d\phi(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|} = \int_0^{2\pi} \frac{d\phi(\boldsymbol{x})}{|\boldsymbol{r}_x|} = I_{10}(\boldsymbol{x}_0, \sigma, \sigma_0),$$

$$K^D(\boldsymbol{x}_0, l) = \int_0^{2\pi} \frac{(\boldsymbol{x}_0 - \boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})}{|\boldsymbol{x}_0 - \boldsymbol{x}|^3}\, d\phi(\boldsymbol{x}) = \int_0^{2\pi} \frac{\boldsymbol{r}_x \cdot \boldsymbol{n}(\boldsymbol{x})}{|\boldsymbol{r}_x|^3}\, d\phi(\boldsymbol{x})$$

$$= (r_x \boldsymbol{n}_x + \sigma_0 \boldsymbol{n}_\sigma) I_{30}(\boldsymbol{x}_0, \sigma, \sigma_0) - \sigma_0 n_\sigma I_{31}(\boldsymbol{x}_0, \sigma, \sigma_0).$$

$I_{mn}(x_0, \sigma, \sigma_0)$ are functions defined by:

$$I_{mn} = \int_0^{2\pi} \frac{\cos^n \phi}{(r_x^2 + \sigma^2 + \sigma_0^2 - 2\sigma\sigma_0 \cos\phi)^{m/2}}\, d\phi$$

$$= \frac{4k'^m}{(4\sigma\sigma_0)^{m/2}} \int_0^{\pi/2} \frac{\left(2\cos^2\omega - 1\right)^n}{(1 - k'^2 \cos^2\omega)^{m/2}}\, d\omega,$$

where $k'^2 = \dfrac{4\sigma\sigma_0}{r_x^2 + (\sigma + \sigma_0)^2}$.

$$\frac{1}{2}(\mu_{12} + 1)\boldsymbol{u}_\alpha^1(\boldsymbol{x}_0) + (1 - \mu_{12}) \int_{C_1} q_{\alpha\beta\gamma}(\boldsymbol{x}_0, \boldsymbol{x}) \boldsymbol{u}_\beta^1(\boldsymbol{x}) \boldsymbol{n}_\gamma(\boldsymbol{x})\, dl(\boldsymbol{x}) \tag{3.3}$$

$$+ (1 - \mu_{13}) \int_{C_2} q_{\alpha\beta\gamma}(\boldsymbol{x}_0, \boldsymbol{x}) \boldsymbol{u}_\beta^3(\boldsymbol{x}) \boldsymbol{n}_\gamma(\boldsymbol{x})\, dl(\boldsymbol{x})$$

$$= -\int_{C_1} M_{\alpha\beta}(\boldsymbol{x}_0, \boldsymbol{x}) \left(\gamma_{12} \boldsymbol{n}_\beta \nabla \cdot \boldsymbol{n} - E_\gamma \left(\tau_{\alpha\beta}^1 \boldsymbol{n}_\beta - \tau_{\alpha\beta}^2 \boldsymbol{n}_\beta \varepsilon_{21}\right)\right) dl(\boldsymbol{x})$$

$$- \int_{C_2} M_{\alpha\beta}(\boldsymbol{x}_0, \boldsymbol{x}) \left(\boldsymbol{n}_\beta \nabla \cdot \boldsymbol{n} - E_\gamma \left(\tau_{\alpha\beta}^1 \boldsymbol{n}_\beta - \tau_{\alpha\beta}^3 \boldsymbol{n}_\beta \varepsilon_{31}\right)\right) dl(\boldsymbol{x}),$$

$$\frac{1}{2}(1 + \mu_{13})\boldsymbol{u}_\alpha^3(\boldsymbol{x}_0) + (1 - \mu_{12}) \int_{C_1} q_{\alpha\beta\gamma}(\boldsymbol{x}_0, \boldsymbol{x}) \boldsymbol{u}_\beta^1(\boldsymbol{x}) \boldsymbol{n}_\gamma(\boldsymbol{x})\, dl(\boldsymbol{x}) \tag{3.4}$$

$$+ (1 - \mu_{13}) \int_{C_2} q_{\alpha\beta\gamma}(\boldsymbol{x}_0, \boldsymbol{x}) \boldsymbol{u}_\beta^3(\boldsymbol{x}) \boldsymbol{n}_\gamma(\boldsymbol{x})\, dl(\boldsymbol{x})$$

$$= -\int_{C_1} M_{\alpha\beta}(\boldsymbol{x}_0, \boldsymbol{x}) \left(\gamma_{12} \boldsymbol{n}_\beta \nabla \cdot \boldsymbol{n} - E_\gamma \left(\tau_{\alpha\beta}^1 \boldsymbol{n}_\beta - \tau_{\alpha\beta}^2 \boldsymbol{n}_\beta \varepsilon_{21}\right)\right) dl(\boldsymbol{x})$$

$$- \int_{C_2} M_{\alpha\beta}(\boldsymbol{x}_0, \boldsymbol{x}) \left(\boldsymbol{n}_\beta \nabla \cdot \boldsymbol{n} - E_\gamma \left(\tau_{\alpha\beta}^1 \boldsymbol{n}_\beta - \tau_{\alpha\beta}^3 \boldsymbol{n}_\beta \varepsilon_{31}\right)\right) dl(\boldsymbol{x}).$$

The indices α, β, γ are for x or σ, and we denote by them the axial and radial components, respectively.

The matrices M and q are defined as in [30]:

$$M(x_0,x) = \begin{bmatrix} M_{xx} & M_{x\sigma} \\ M_{\sigma x} & M_{\sigma\sigma} \end{bmatrix} = \sigma \int_0^{2\pi} \begin{bmatrix} J_{xx} & J_{xy}\cos\phi + J_{xz}\sin\phi \\ J_{xy} & J_{yy}\sin\phi + J_{yz}\cos\phi \end{bmatrix} d\phi,$$

$$\begin{bmatrix} q_{xxx} & q_{xx\sigma} \\ q_{x\sigma x} & q_{x\sigma\sigma} \end{bmatrix}(x_0,x)$$

$$= \sigma \int_0^{2\pi} \begin{bmatrix} T_{xxx} & T_{xxy}\cos\phi + T_{xxz}\sin\phi \\ T_{xxy}T_{xzz} + T_{xxz}\sin\phi & T_{xyy}\cos^2\phi + T_{xzz}\sin^2\phi + 2T_{xyz}\sin\phi\cos\phi \end{bmatrix} d\phi,$$

$$\begin{bmatrix} q_{\sigma xx} & q_{\sigma x\sigma} \\ q_{\sigma\sigma x} & q_{\sigma\sigma\sigma} \end{bmatrix}(x_0,x)$$

$$= \sigma \int_0^{2\pi} \begin{bmatrix} T_{xxy} & T_{xyy}\cos\phi + T_{xyz}\sin\phi \\ T_{xyy}T_{xzz} + T_{xyz}\sin\phi & T_{yyy}\cos^2\phi + T_{yzz}\sin^2\phi + 2T_{yyz}\sin\phi\cos\phi \end{bmatrix} d\phi.$$

Formulated by the integrals $I_{mn}(x_0,\sigma,\sigma_0)$, we have

$$M = \sigma \begin{bmatrix} I_{10} + r_x^2 I_{30} & -r_x(\sigma I_{30} - \sigma_0 I_{31}) \\ -r_x(\sigma I_{31} - \sigma_0 I_{30}) & I_{11} + \left(\sigma^2 + \sigma_0^2\right) I_{31} - \sigma\sigma_0(I_{30} + I_{32}) \end{bmatrix},$$

$$\begin{bmatrix} q_{xxx} \\ q_{xx\sigma} = q_{x\sigma x} \\ q_{x\sigma\sigma} \end{bmatrix} = 6\sigma r_x \begin{bmatrix} -r_x^2 I_{50} \\ r_x(\sigma I_{50} - \sigma_0 I_{51}) \\ -\left(\sigma^2 I_{50} + \sigma_0^2 I_{52} - 2\sigma\sigma_0 I_{51}\right) \end{bmatrix},$$

$$\begin{bmatrix} q_{\sigma xx} \\ q_{\sigma x\sigma} = q_{\sigma\sigma x} \\ q_{\sigma\sigma\sigma} \end{bmatrix} = 6\sigma \begin{bmatrix} r_x^2(\sigma I_{51} - \sigma_0 I_{50}) \\ -r_x\left[\left(\sigma^2 + \sigma_0\right)^2 I_{51} - \sigma\sigma_0(I_{50} + I_{52})\right] \\ \sigma^3 I_{51} - \sigma^2\sigma_0(I_{50} + 2I_{52}) + \sigma\sigma_0^2(I_{53} + 2I_{51}) - \sigma_0^3 I_{52} \end{bmatrix}.$$

The integrals I_{mn} could be expressed by elliptic integrals of first and second kind — $F(k')$ and $E(k')$, which are calculated numerically:

$$F(k') = \int_0^{\pi/2} \frac{d\omega}{\left(1 - k'^2\cos^2\omega\right)^{1/2}}, \qquad E(k') = \int_0^{\pi/2} \left(1 - k'^2\cos^2\omega\right)^{1/2} d\omega.$$

The integrals of single and double layer integrals in (3.1)–(3.4) have singularities, and in the singularity point these integrals are calculated using the formulae derived by Pozrikidis [15].

The system (3.1)–(3.4) is solved using the method of boundary elements, and the algebraic system following the electric problem (3.1), (3.2) is solved by Gauss' elimination, while the one following the hydrodynamic part (3.3), (3.4) is solved by the iterative method. For the calculations a project in Code Warrior C has been conducted, as the main results have been obtained through Power Mac 200/6400 in the Laboratory of the Department of Mechanics of Continua at the Faculty of Mathematics and Informatics at the Sofia University "St. Kl. Ohridski".

For the determination of the drops form on each time-step, we use the kinematic condition of the following type:

$$x_s^{\text{new}} = x_s + n_i(u_i \cdot n_i)dt, \qquad \text{where } dt \text{ is a preliminary set time-step.}$$

We assume that the form reaches the equilibrium when the normal component of the velocity becomes less than the preliminary set minimum at every point of the interfaces. Another criterion for the end of the procedure is the normal component to become bigger than the initially set number; then we consider the drop's break-up.

4. RESULTS

The algorithm for obtaining the finite deformations of two drops due to electric field has been tested for a single drop in the presence of an electric field and it has shown a good agreement with the results of Sherwood [18].

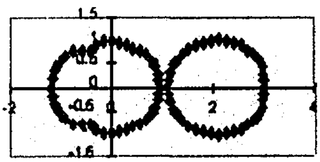

Fig. 3

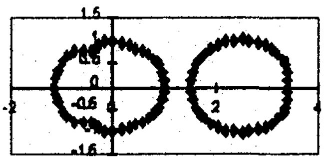

Fig. 4

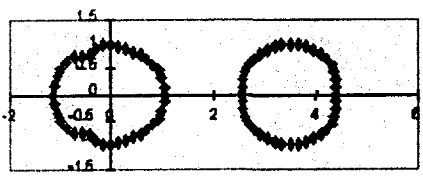

Fig. 5

The deformation of fluid interfaces when they are with equal radii does not depend essentially on the distance between the drops in low intensities $E_\gamma = 0.4$ of the electric field. The initial distance between the centres of the drops on Fig. 3 is 2.1 R1, on Fig. 4 is 2.5 R1, on Fig. 5 is 3.5 R1 with $\mu_{12} = 0.50$, $\mu_{13} = 1.50$, $E_\gamma = 0.4$, $\varepsilon_{21} = 2.0$, $R_{12} = 1.0$, $\varepsilon_{31} = 3.0$, $\sigma_{12} = 10.0$, $\sigma_{13} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$.

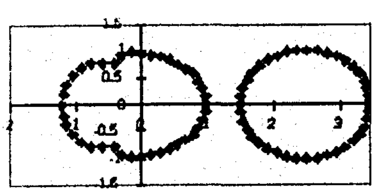

Fig. 6

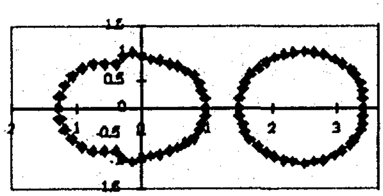

Fig. 7

When we increase the intensity of the electric field, the form reached by the first drop is elongating, till the second one retains almost spherical. On Fig. 6 $E_\gamma = 1.0$, on Fig. 7 $E_\gamma = 2.0$, on Fig. 8 $E_\gamma = 5.0$ with $\mu_{12} = 0.50$, $\varepsilon_{21} = 2.0$, $\mu_{13} = 1.5$, $R_{12} = 1.0$, $\varepsilon_{31} = 3.0$, $\sigma_{12} = 10.0$, $\sigma_{13} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$.

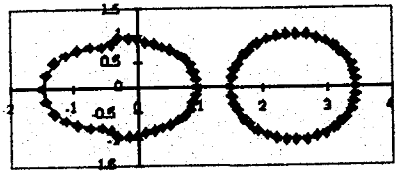

Fig. 8

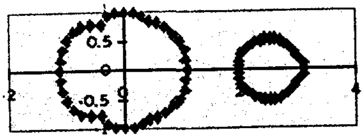

Fig. 9

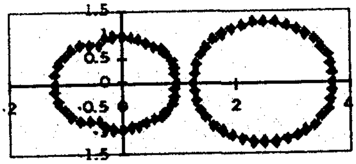

Fig. 10

The ratio between the radii of the two drops causes different pictures of deformation as shown on Fig. 9–11. On Fig. 9 $R_{12} = 0.5$, on Fig. 10 $R_{12} = 1.25$, on Fig. 11 $R_{12} = 1.49$ with $\mu_{12} = 0.5$, $E_\gamma = 0.5$, $\mu_{13} = 2.0$, $\varepsilon_{21} = 2.0$, $\varepsilon_{31} = 3.0$, $\sigma_{12} = 10.0$, $\sigma_{13} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$. The change of the ratio causes increase of the influence of the initially bigger drop to the smaller one.

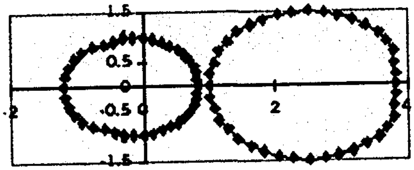

Fig. 11

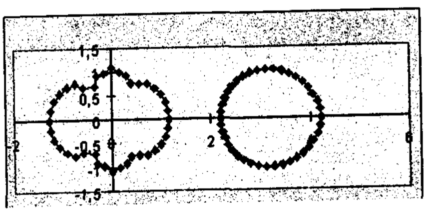

Fig. 12

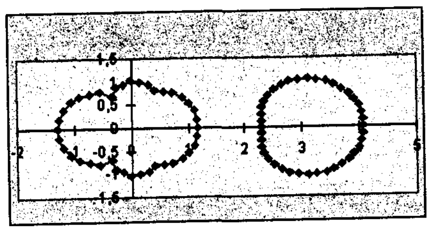

Fig. 13

On Fig. 12–14 the ratio between the viscosities is changed from $\mu_{13} = 1.0$ on Fig. 12, $\mu_{13} = 2.0$ on Fig. 13 and $\mu_{13} = 4.0$ on Fig. 14 with $R_{12} = 1.0$, $E_\gamma = 1.8$, $\mu_{12} = 2.0$, $\varepsilon_{21} = 2.0$, $\varepsilon_{31} = 1.5$, $\sigma_{12} = 20.0$, $\sigma_{13} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$. That shows that a change of viscosity between two phases causes a change of deformations on the both drops.

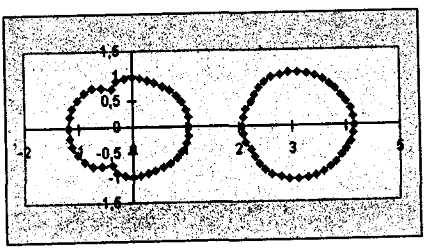

Fig. 14

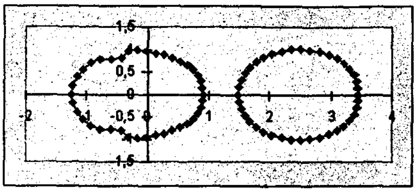

Fig. 15

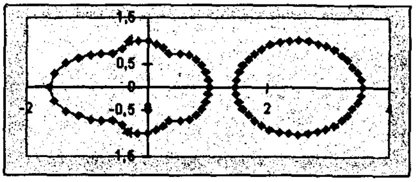

Fig. 16

On Fig. 15 and 16 the permittivity ratio ε_{31} changes from 2.0 to 10.0 on Fig. 16 with $R_{12} = 1.0$, $E_\gamma = 1.8$, $\mu_{13} = 1.5$, $\mu_{12} = 0.5$, $\varepsilon_{21} = 5.0$, $\sigma_{12} = 20.0$, $\sigma_{13} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$. That indicates that the increase of permittivity ratio between the phases 1 and 2 tends the drop to elongate in the direction opposite to the neighbour drop, but the configuration of Fig. 16 ($\varepsilon_{31} = 10.0$) is not stable and after some time steps a "break-up" appears.

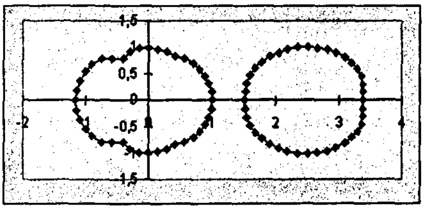

Fig. 17

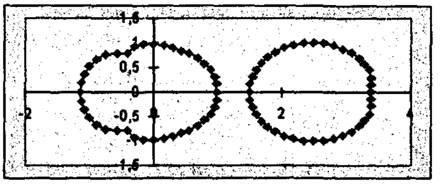

Fig. 18

On Fig. 17 the ratio of conductivities is $\sigma_{13} = 5.0$, and on Fig. 18 it is $\sigma_{13} = 10.0$ with $R_{12} = 1.0$, $E_\gamma = 0.4$, $\mu_{13} = 1.5$, $\mu_{12} = 0.5$, $\varepsilon_{21} = 2.0$, $\varepsilon_{31} = 3.0$, $\sigma_{12} = 15.0$, $dt = 0.01$, $\gamma_{12} = 1.0$, so when the drops are with equal radii and the intensity of the electric field is not strong, the change of conductivity ratio has no significant influence on the deformations.

The problem for the finite deformations of two drops due to electric field has ten dimensionless parameters, each of them having an influence on the process of deformations somehow, so further results will be presented in next papers.

REFERENCES

1. Allan, R. S., S. G. Mason. Particle behavior in shear and electric fields: I. Deformation and burst of fluid drops. *Proc. Roy. Soc. Lond.*, **A267**, 1962, 45.
2. Brazier-Smith, P. R. Stability and shape of isolated and pairs of water drops in an electric field. *Phys. Fluids*, **14**, 1971, 1.
3. Chervenivanova, E., Z. Zapryanov. On the deformation of two droplets in a quasisteady Stokes flow. *Int. J. Multiphase Flow*, **11**, 5, 1985, 721–738.
4. Chervenivanova, E., Z. Zapryanov. On the deformation of a fluid particle moving radially inside a spherical container. *Physico Chem. Hydrodyn*, **8**, 1987, 293.

5. Garton, C. G., Z. Krasucki. Bubble in insulating liquids: stability in an electric field. *Proc. Roy. Soc. Lond.*, **A 280**, 1964, 211.
6. Gradshteyn, I. S., I. M. Ryshik. Tables of integrals, series, and products. Academic Press, 1980.
7. Greengard, L., M. Moura. On the numerical evaluation of electrostatic fields in composite materia. *Acta Numerica*, 1994, 379–410.
8. Happel, J., H. Brener. Low Reynolds number Hydrodynamics. 1-ed, Noordhoff International Publishing: Leyden, 1965.
9. Kozhenkov, V. I., N. A. Fuks. Electrodynamic atomization of liquids. *Russ. Chem. Rev.*, **45**, 1976, 1179.
10. Ladyzhenskaya, O. A. The mathematical theory of viscous incompressible flow. 2nd edn, Gordon & Breach, 1969.
11. Latham, J., I. W. Roxburgh. Disintegration of pairs of water drops in an electric field. *Proc. Roy. Soc.*, **A 295**, 1966, 84.
12. Mackay, W. A. Some investigations on the deformation and breaking of water drops in strong electric field. *Proc. R. Soc. Lond.*, **133**, 1931, 565.
13. O'Konski, C. T., H. C. Thacker. The distortion of aerosol droplets by an electric field. *J. Phys. Chem.*, **57**, 1953, 955.
14. Power, H. Low Reynolds number of compound drops in shear flow. *Math. methods in the applied sciences*, **16**, 1993, 61.
15. Pozrikidis, C. The instability of a moving viscous drop. *J. Fluid Mech.*, **210**, 1990, 1.
16. Pozrikidis, C. Boundary integral and singularity methods for linearized viscous flow. Cambridge Univ. Press, Cambridge - New York, 1992.
17. Ramos, A., A. Castellanos. Conical points in liquid-liquid interfaces subjected to electric fields. *Physics Lett.*, **A 184**, *1994*, 268–272.
18. Sherwood, J. D. Breakup of fluid droplets in electric and magnetic fields. *J. Fluid Mech.*, **188**, 1988, 133–146.
19. Smoluchowski, M. *Bull. Int. Acad. Polonaise Sci.*, Lett. **1A**, 1911, 28.
20. Sozou, C. Electrohydrodynamics of a pair of liquid drops. *J. Fluid Mech.*, **67**, 1975, 339–349.
21. Taylor, G. I. The viscosity of a fluid containing small drops of another fluid. *Proc. Roy. Soc.*, **A 138**, 1932, 41–48.
22. Taylor, G. I. The formation of emulsions in definable fields flow. *Proc. Roy. Soc.*, **A 146**, 1934, 501–523.
23. Taylor, G. I. Disintegration of water drops in an electric field. *Proc. R. Soc. Lond.*, **A 280**, 1964, 383.
24. Taylor, G. I. The circulation produced in a drop by an electric field. *Proc. R. Soc. Lond.*, **A 291**, 1966, 159.
25. Taylor, G. I., A. D. McEwan. The stability of a horizontal fluid interface in a vertical electric field. *J. Fluid Mech.*, **22**, 1965, 1.
26. Taylor, T. D., A. Acrivos. On the deformation and drag of a falling viscous drop at low Reynolds number. *J. Fluid Mech.*, **18**, 1964, 466.
27. Torza, S., R. G. Cox, S. G. Mason. Electrohydrodynamic deformation and bursts of liquid drops. *Phil. Tr. R. Soc. Lond.*, **A 269**, 1971, 295.
28. Uijteewaal, W. S. J., E. J. Nijhof, R. M. Heethaar. *Phys. Fluids*, **A 5**, 819.

29. Wilson, C. T. R., G. I. Taylor. The bursting of soap bubbles in a uniform electric field. *Proc. Camb. Phil. Soc.*, **22**, 1925, 728.

30. Dobreva, P. Master Thesis. Fac. of Math. and Inf., Sofia University "St. Kl. Ohridski", 1998.

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ERRATA

In the article SCATTERING OF ACOUSTOELECTRIC WAVES ON A CYLINDRICAL INHOMOGENEITY IN THE TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIUM by Valery Levin and Thomas Michelitsch, vol. 93, p. 153, instead of (5.20) it has to be

$$C_i(n)$$

$$= \frac{i\pi a^2}{2}\sqrt{\frac{\beta_\perp^3}{2\pi}}e^{-\frac{i\pi}{4}}\left\{\frac{\rho_1}{\rho_0} - \frac{1}{C'_{44}}\left[C_{44}^A + 2\left(\frac{e_{15}^0}{\eta_{11}^0}\right)e_{15}^A - \left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2\eta_{11}^A\right]^2\cos\phi\right\}m_i$$

and instead of (5.21) it has to be

$$Q_{T\perp}(\omega)$$

$$= \frac{\pi^2}{8}(\beta_\perp a)^3 a\left\{\frac{1}{(\rho_0 v_{T\perp}^2)^2}\left[C_{44}^A + 2\left(\frac{e_{15}^0}{\eta_{11}^0}\right)e_{15}^A - \left(\frac{e_{15}^0}{\eta_{11}^0}\right)^2\eta_{11}^A\right]^2 + 2\left(\frac{\rho_1}{\rho_0}\right)^2\right\},$$

$$v_{T\perp}^2 = C'_{44}/\rho_0.$$

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For books: Obreshkoff, N. Higher algebra. Nauka i Izkustvo, 2nd edition, Sofia, 1963 (in Bulgarian).

For journal articles: Frisch, H. L. Statistics of random media. *Trans. Soc. Rheology*, **9**, 1965, 293–312.

For articles in edited volumes or proceedings: Friedman, H. Axiomatic recursive function theory. In: *Logic Colloquim 95*, eds. R. Gandy and F. Yates, North-Holland, 1971, 188–195.