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ON A GENERALIZATION OF CRITERIA *A* AND *D* FOR CONGRUENCE OF TRIANGLES

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The conditions determining that two triangles are congruent play a basic role in planimetry. By comparing not congruent triangles with respect to given sets of corresponding elements it is important to discover if they have any common geometric properties characterizing them. The present paper is devoted to an answer of this question. We give a generalization of criteria A and D for congruence of triangles and apply it to prove some selected geometric problems.

Keywords: Congruence of triangles, comparison of triangles

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1. INTRODUCTION

There are six essential elements of every triangle - three angles and three sides. The method of constructing a triangle varies according to the facts which are known about its sides and angles.

It is important to know what is the minimum knowledge about the sides and angles which is necessary to construct a particular triangle.

Clearly all triangles constructed in the same way with the same data must be identically equal, i. e. they must be of exactly the same size and shape and their areas must be the same.

Triangles which are equal in all respects are called *congruent triangles*.

The four sets of minimal conditions for two triangles to be congruent are set out in the following geometric criteria.

Criterion A. *Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other.*

Criterion B. *Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and a side of the other.*

Criterion C. *Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other.*

Criterion D. *Two triangles are congruent if two sides and the angle opposite the greater side of one triangle are respectively equal to two sides and the angle opposite the greater side of the other.*

We notice that in *criteria* A and D the sets of corresponding equal elements are two sides and an angle. The given angle may be any one of the three angles of the triangle. The problem "*Construct a triangle with two of its sides* a *and* b, $a < b$, *and angle* α *opposite the smaller side*" has not a unique solution. There are two triangles each of which satisfies the given conditions.

In the present paper we compare not congruent triangles with respect to given sets of corresponding elements and answer the question what are the geometric properties characterizing such couples of triangles.

2. THEORETICAL BASIS OF THE PROPOSED METHOD FOR COMPARING TRIANGLES

Throughout, for the elements of two triangles $\triangle ABC$ and $\triangle A_1B_1C_1$ we shall use the notations $AB = c$, $BC = a$, $CA = b$; $A_1B_1 = c_1$, $B_1C_1 = a_1$, $C_1A_1 = b_1$. Moreover, θ and θ_1 will stand for two corresponding angles of $\triangle ABC$ and $\triangle A_1B_1C_1$, respectively.

Suppose that in $\triangle ABC$ and $\triangle A_1B_1C_1$ the relations $a = a_1$, $b = b_1$ and $\theta = \theta_1$ hold. We consider four possible cases.

- The angle θ is included between the sides a and b, i.e., $\theta = \sphericalangle ACB$ and $\theta_1 = \sphericalangle A_1C_1B_1$. The triangles are congruent by *Criterion A*.
- Let $a = b$, i.e., $\triangle ABC$ and $\triangle A_1B_1C_1$ are isosceles. Since $\theta = \theta_1$, the triangles are congruent as a consequence of *Criterion A*.
- Let $a > b$ and the angle θ be opposite the greater side a. In this case the triangles are congruent in view of *Criterion D*.
- Let $a > b$ and the angle θ is opposite the smaller side b. In this case the triangles are either congruent or not.

- If the triangles are congruent, then the angles opposite the greater sides are necessarily equal. It could happen that the sum of the equal angles opposite the greater sides equals 180^0, then obviously the triangles are right-angled.
- If the triangles are not congruent, then we show that the sum of the angles opposite the greater sides is always equal to 180^0.

Lemma 2.1. *Let* $\triangle ABC$ *and* $\triangle ABD$ *be not congruent triangles, and let* $AC = AD$. *If* $\sphericalangle ABC = \sphericalangle ABD$, *then* $\sphericalangle ACB + \sphericalangle ADB = 180^0$.

Proof. Since $\triangle ABC$ and $\triangle ABD$ are not congruent, then $AC < AB$ (and hence $AD < AB$). Let us denote $\sphericalangle ACB = \alpha$ and $\sphericalangle ADB = \beta$.

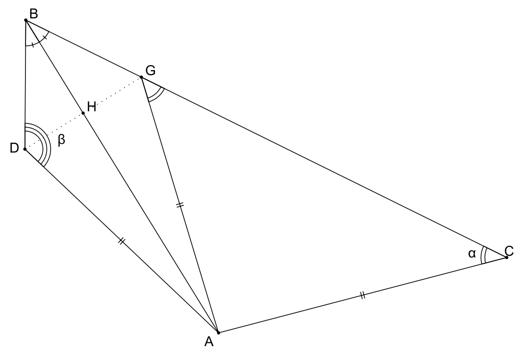

Fig. 1.

There are two possible locations of the points C and D with respect to the straight line AB.

(i) *The points* C *and* D *lie on opposite sides of* AB.

The symmetry with respect to the straight line AB transforms $\triangle ABD$ into a congruent $\triangle ABG$ which lies on the same side of AB as $\triangle ABC$ (see Fig. 1). Since $\triangle ABC \ncong \triangle ABD$, then $\triangle ABC \ncong \triangle ABG$. The condition $\sphericalangle ABC = \sphericalangle ABD$ implies that the straight line AB is the bisector of $\sphericalangle DBC$. From the symmetry with respect to AB it follows that $G \in BC$ and $BG \neq BC$. Let, e.g., G/BC (the case C/BG is analogous). Clearly, if the assumptions of Lemma 2.1 are fulfilled for $\triangle ABC$ and $\triangle ABD$, then they are also valid for $\triangle ABC$ and $\triangle ABG$ and vice versa.

Let us consider $\triangle ABC$ and $\triangle ABG$. The side AB and $\sphericalangle ABC$ are common for both triangles. In view of the symmetry with respect to AB and $AC = AD$, we get $AD = AG = AC$. Hence, $\triangle ACG$ is isosceles and $\sphericalangle ACG = \alpha = \sphericalangle AGC$. The angles $\sphericalangle AGC$ and $\sphericalangle AGB = \sphericalangle ADB = \beta$ are adjacent and hence $\sphericalangle AGC + \sphericalangle AGB = \sphericalangle ACB + \sphericalangle ADB = \alpha + \beta = 180^0$.

Remark 2.2. The quadrilateral $ACBD$ can be inscribed in a circle.

(*ii*) *The points* C *and* D *lie on one and the same side of* AB.

This case was already considered in (i), with $D \equiv G$. □

Remark 2.3. In the case when $\triangle ABC$ and $\triangle A_1B_1C_1$ are not congruent, the relations $AB = A_1B_1$, $AC = A_1C_1$ and $\sphericalangle ABC = \sphericalangle A_1B_1C_1$ are fulfilled and the triangles have no common side, we can choose a suitable congruence and transform $\triangle A_1B_1C_1$ into a congruent $\triangle ABD$ so that $\triangle ABC$ and $\triangle ABD$ satisfy the assumptions of Lemma 2.1.

Based on the above arguments we formulate a theorem, which is a generalization of *criteria* A and D for congruence of triangles (see also [6], p. 12).

Theorem 2.4. *Assume that* $\triangle ABC$ *and* $\triangle A_1B_1C_1$ *have two pairs of equal sides,* $a = a_1$, $b = b_1$, *and equal corresponding angles,* $\theta = \theta_1$. *Then* $\triangle ABC$ *and* $\triangle A_1B_1C_1$ *are either congruent, or not congruent, in which case the sum of the other two angles, not included between the given sides, is equal to* 180^0.

Lemma 2.1 and Theorem 2.4 can be used as alternative methods of comparing different triangles.

3. APPLICATION OF THEOREM 2.4 TO TWO GEOMETRIC PROBLEMS

The solutions of next selected problems are based on Theorem 2.4.

Problem 3.1 ([4, Problems 4.20 and 4.23]; [5]). *Let the middle points of the sides* BC, CA *and* AB *of* $\triangle ABC$ *be* F, D, *and* E, *respectively. If the center* G *of the circumscribed circle* k *of* $\triangle FDE$ *lies on the bisector of* $\sphericalangle ACB$, *prove that* $\triangle ABC$ *is either isosceles* ($CA{=}CB$), *or not isosceles, in which case* $\sphericalangle ACB{=}60^0$.

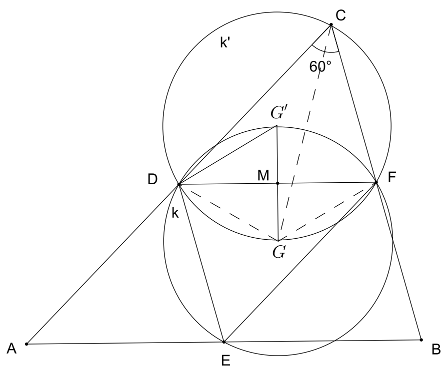

Fig. 2.

Proof. Let the center G of the circumscribed circle k of $\triangle FDE$ lie on the bisector of $\sphericalangle ACB$ (Fig. 2). Since $\triangle CGD$ and $\triangle CGF$ have a common side

CG, equal corresponding angles $\sphericalangle DCG = \sphericalangle FCG$ and equal corresponding sides $DG = FG$ (as radii of k), the assumptions of Theorem 2.4 are satisfied.

(i) If $\triangle CGD$ and $\triangle CGF$ are congruent, then $CD = CF$ and hence $CA = CB$, i.e., $\triangle ABC$ is isosceles.

Remark 3.2. There are two possibilities for $\sphericalangle ACB$: either $\sphericalangle ACB = 60^0$, in which case $\triangle ABC$ is equilateral, or $\sphericalangle ACB \neq 60^0$, and then $\triangle ABC$ is isosceles.

(ii) If $\triangle CGD$ and $\triangle CGF$ are not congruent, then in view of Lemma 2.1 $\sphericalangle CDG + \sphericalangle CFG = 180^0$ and the quadrilateral $CDGF$ can be inscribed in a circle k' (Fig. 2).

It is easily seen that $\triangle EFD \cong \triangle CDF$ and their circumscribed circles k and k' have equal radii. The circles k and k' are symmetrically located with respect to their common chord FD. Since the center G of k lies on k', then the center G' of k' lies on k. Hence, $\triangle DGG' \cong \triangle FGG'$, both triangles are equilateral, $\sphericalangle DGF = 120^0$ and $\sphericalangle ACB = 60^0$. □

Problem 3.3 ([3, Problem 8]; [4, Problem 4.12]). *Let in $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, be the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively. Let also $AA_1 \cap BB_1 = J$. If $JA_1 = JB_1$, prove that $\triangle ABC$ is either isosceles ($CA = CB$), or not isosceles, in which case $\sphericalangle ACB = 60^0$.*

Proof. Let $\sphericalangle BAC = 2\alpha$, $\sphericalangle ABC = 2\beta$, $\sphericalangle ACB = 2\gamma$. Since J is the cut point of the angle bisectors AA_1 and BB_1 of $\triangle ABC$, then the straight line CJ is the bisector of $\sphericalangle ACB$ and $\alpha + \beta + \gamma = 90^0$ (Fig. 3).

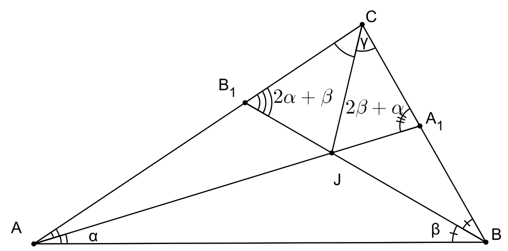

Fig. 3.

Since $\sphericalangle CB_1J$ is an exterior angle of $\triangle ABB_1$, then $\sphericalangle CB_1J = 2\alpha + \beta$. Since $\sphericalangle CA_1J$ is an exterior angle of $\triangle ABA_1$, then $\sphericalangle CA_1J = 2\beta + \alpha$.

Let us compare $\triangle CA_1J$ and $\triangle CB_1J$. They have a common side CJ, corresponding equal sides $JA_1 = JB_1$ and angles $\sphericalangle A_1CJ = \sphericalangle B_1CJ$. We observe that $\triangle CA_1J$ and $\triangle CB_1J$ satisfy the assumptions of Theorem 2.4.

(i) If $\triangle CA_1J$ and $\triangle CB_1J$ are congruent, then their corresponding elements are equal, in particular,

$$\sphericalangle CB_1J = \sphericalangle CA_1J \;\Leftrightarrow\; 2\alpha + \beta = 2\beta + \alpha \;\Leftrightarrow\; \alpha = \beta.$$

Hence, $\triangle ABC$ is isosceles with $CA = CB$.

Remark 3.4. There are two possibilities for $\sphericalangle ACB$: either $\sphericalangle ACB = 60^0$, and then $\triangle ABC$ is equilateral, or $\sphericalangle ACB \neq 60^0$, in which case $\triangle ABC$ is isosceles.

(ii) If $\triangle CA_1J$ and $\triangle CB_1J$ are not congruent, then by Theorem 2.4,

$$\sphericalangle CB_1J + \sphericalangle CA_1J = 180^0 \Leftrightarrow (2\alpha + \beta) + (2\beta + \alpha) = 180^0 \Leftrightarrow \alpha + \beta = 60^0.$$

Hence, $\sphericalangle ACB = 180^0 - 2(\alpha + \beta) = 60^0$. $\square$

4. GROUPS OF PROBLEMS

In this section we illustrate the composing technology of new problems as an interpretation of specific logical models.

Our aim is the *basic problem* in each of the groups under consideration to be with (exclusive or not exclusive) disjunction as a logical structure in the conclusion and its proof to be based on Lemma 2.1 or Theorem 2.4.

4.1. PROBLEMS OF GROUP I

Suitable logical models for formulation of *equivalent* problems and *generating* problems from a given problem are described in detail in [3, 4]. The basic statements we need in this group of problems are:

$t := \{$ *A square with center O is inscribed in ΔABC so that the vertices of the square lie on the sides of ΔABC and two of them are on the side AB.*$\}$

$p := \{\sphericalangle ACB = 90^0\}$

$q := \{CA = CB\}$

$r := \{\sphericalangle ACO = \sphericalangle BCO\}$

We describe the logical scheme for the composition of Basic problem 4.4, which has not exclusive disjunction as a logical structure in the conclusion:

- First we formulate (and prove) the *generating* problems - Problem 4.1 with a logical structure $t \wedge p \to r$ and Problem 4.3 with a logical structure $t \wedge q \to r$.
- To generate problems with logical structure $(*)$ $t \wedge (p \vee q) \to r$ we use the logical equivalence

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow t \wedge (p \vee q) \to r.$$

- Finally, the formulated *inverse* problem - Basic problem 4.4 - to the problem with structure $(*)$ has the logical structure $t \wedge r \to p \vee q$.

Problem 4.1. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of $\triangle ABC$ and two of them are on the side AB. If $\sphericalangle ACB = 90^0$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (Fig. 4). Since the diagonals of a square are

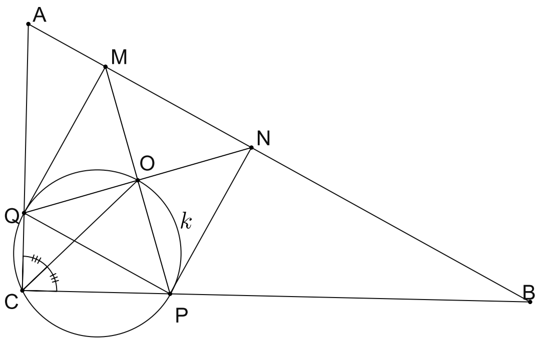

Fig. 4.

equal, intersect at right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\sphericalangle POQ = 90^0$. The quadrilateral $OPCQ$ can be inscribed in a circle k with diameter PQ. To the equal chords OQ and OP of k correspond equal angles, hence $\sphericalangle ACO = \sphericalangle BCO$. □

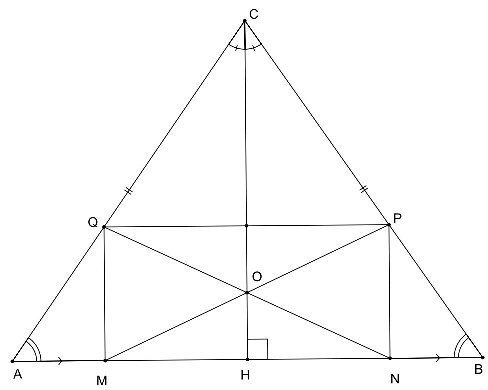

Fig. 5.

Problem 4.2. *A rectangle with center O is inscribed in $\triangle ABC$ so that the vertices of the rectangle lie on the sides of $\triangle ABC$ and two of them are on the side AB. If $CA = CB$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ rectangle (Fig. 5). Since the diagonals of a rectangle are equal and bisect each other, then $OM = ON = OP = OQ$.

Let $CH \perp AB$, $H \in AB$. Since $\triangle ABC$ is isosceles with $CA = CB$, H is the middle point of AB and CH is the bisector of $\sphericalangle ACB$.

Since $MQ \parallel NP$, $NP \parallel CH$ and $MQ = NP$, it follows that $\triangle AMQ \cong \triangle BNP$ (by *Criterion B*) and $AM = BN$. Hence, H is also the middle point of MN. Since $\triangle MON$ is isosceles, then its median OH is also an altitude, i.e., $OH \perp MN$. This means that $O \in CH$ and $\sphericalangle ACO = \sphericalangle BCO$. □

A special case of Problem 4.2 is Problem 4.3 with a logical structure $t \wedge q \rightarrow r$.

Problem 4.3. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $CA = CB$, prove that $\sphericalangle ACO = \sphericalangle BCO$.*

Now we formulate and prove the *Basic problem* in this group.

Basic problem 4.4. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $\sphericalangle ACO = \sphericalangle BCO$, prove that $CA = CB$ or $\sphericalangle ACB = 90^0$.*

Proof. Let the quadrilateral $MNPQ$, $M \in AB$, $N \in AB$, $P \in BC$, $Q \in AC$, be the inscribed in $\triangle ABC$ square (Fig. 6). Since the diagonals of any square are

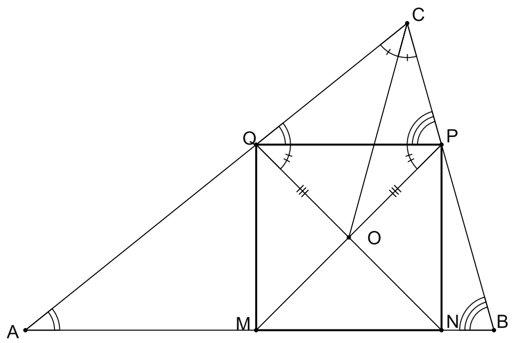

Fig. 6.

equal, intersect at right angles, bisect each other and bisect the opposite angles, then $OP = OQ$ and $\sphericalangle OPQ = \sphericalangle OQP = 45^0$.

We compare $\triangle CQO$ and $\triangle CPO$. They have a common side CO, respectively equal sides $OQ = OP$ and angles $\sphericalangle QCO = \sphericalangle PCO$. We find $\sphericalangle CQO = \sphericalangle CAB + 45^0$ and $\sphericalangle CPO = \sphericalangle CBA + 45^0$ as exterior angles of $\triangle QAN$ and $\triangle PBM$ respectively. Therefore, $\triangle CQO$ and $\triangle CPO$ satisfy the assumptions of Theorem 2.4. We consider separately the two possibilities.

(i) If $\triangle CQO$ and $\triangle CPO$ are congruent, then $\sphericalangle CQO = \sphericalangle CPO$ and hence $\sphericalangle CAB = \sphericalangle CBA$, i.e., $CA = CB$ and $\triangle ABC$ is isosceles.

In this case $\sphericalangle ACB$ is either a right angle and $\triangle ABC$ is isosceles right-angled, or not a right angle and $\triangle ABC$ is only isosceles.

(ii) If $\triangle CQO$ and $\triangle CPO$ are not congruent, then, according to Lemma 2.1, $\sphericalangle CQO + \sphericalangle CPO = 180^0$ and hence $\sphericalangle CAB + \sphericalangle CBA = 90^0$, i.e., $\triangle ABC$ is right-angled with $\sphericalangle ACB = 90^0$. □

Remark 4.5. A logically incorrect version of Basic problem 4.4 is Problem 1.54 in [1].

We reformulate Problem 4.4 by keeping the condition of homogeneity of the conclusion.

Problem 4.6. *A square with center O is inscribed in $\triangle ABC$ so that the vertices of the square lie on the sides of the triangle and two of them are on the side AB. If $\sphericalangle ACO = \sphericalangle BCO$, then $\triangle ABC$ is either isosceles with $CA = CB$ or not isosceles but right-angled with $\sphericalangle ACB = 90^0$.*

4.2. PROBLEMS OF GROUP II

By formulating appropriate statements and giving suitable logical models we get two *generating* problems that are needed for the construction of Basic problem 4.9. The basic statements we use are:

$t :=$ {*In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively.*}

$p := \{\sphericalangle ACB = 60^0\}$

$q := \{\sphericalangle CAB = 120^0\}$

$r := \{\sphericalangle BB_1A_1 = 30^0\}$

Since the sum of the angles of any triangle is equal to 180^0, statements p and q are mutually exclusive. Hence, if p is true, so is $\neg q$ and vice versa.

We describe the logical scheme for the composition of Basic problem 4.9, which has exclusive disjunction as a logical structure in the conclusion:

- First we formulate (and prove) two *generating* problems - Problem 4.7 with a logical structure $t \wedge p \to r$ and Problem 4.8 with a logical structure $t \wedge q \to r$.

- Since statements p and q are mutually exclusive, the equivalences $p \wedge \neg q \Leftrightarrow p$ and $\neg p \wedge q \Leftrightarrow q$ are true. As a consequence of these facts problems with logical structures $t \wedge p \rightarrow r$ and $t \wedge (p \wedge \neg q) \rightarrow r$ are equivalent. So are the problems with logical structures $t \wedge q \rightarrow r$ and $t \wedge (q \wedge \neg p) \rightarrow r$.

  To generate problems with a logical structure $(**)\quad t \wedge (p \veebar q) \rightarrow r$ we use the logical equivalence

$$(t \wedge (p \wedge \neg q) \rightarrow r) \wedge (t \wedge (\neg p \wedge q) \rightarrow r) \quad \Leftrightarrow \quad t \wedge (p \veebar q) \rightarrow r.$$

- Finally, the formulated *inverse* problem - the Basic problem 4.9 - to the problem with structure $(**)$ has the logical structure $t \wedge r \rightarrow p \veebar q$.

Problem 4.7. *In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$, respectively. If $\sphericalangle ACB = 60^0$, prove that $\sphericalangle BB_1A_1 = 30^0$.*

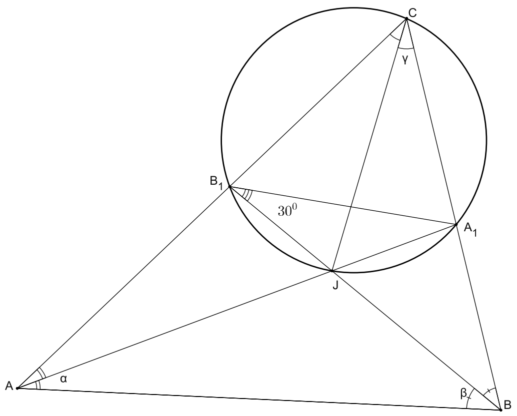

Fig. 7.

Proof. Let $\sphericalangle BAA_1 = \sphericalangle CAA_1 = \alpha$, $\sphericalangle ABB_1 = \sphericalangle CBB_1 = \beta$, $J = AA_1 \cap BB_1$. Since J is the intersection point of the angle bisectors of $\triangle ABC$, we have that $\sphericalangle JCA = \sphericalangle JCB = \gamma = 30^0$ (Fig. 7).

From $\alpha + \beta + \gamma = 90^0$ we find that $\sphericalangle AJB = 120^0$. Hence, the quadrilateral CA_1JB_1 can be inscribed in a circle. Then $\sphericalangle JA_1B_1 = \sphericalangle JCB_1 = 30^0$ and $\sphericalangle JB_1A_1 = \sphericalangle JCA_1 = 30^0$ as angles corresponding to the same segment of this circle. □

Problem 4.8. *In $\triangle ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$ respectively. If $\sphericalangle BAC = 120^0$, prove that $\sphericalangle BB_1A_1 = 30^0$.*

Proof. Let $J = AA_1 \cap BB_1$, $E = A_1B_1 \cap CJ$, $C_1 = CJ \cap AB$. Since $\sphericalangle BAC = 120^0$, its adjacent angles have a measure of 60^0. It is easily seen that the

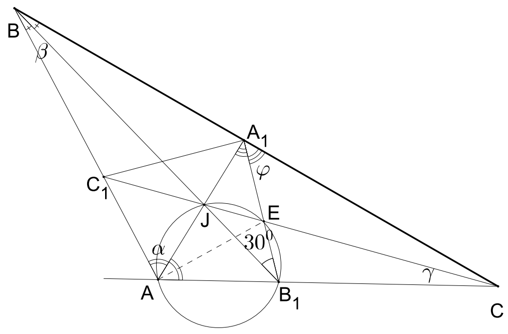

Fig. 8.

point B_1 is equidistant from the straight lines BA, BC, AA_1 and that the straight line A_1B_1 is the bisector of $\sphericalangle CA_1A$ (Fig. 8). The proof that the straight line A_1C_1 is the bisector of $\sphericalangle BA_1A$ is analogous. It follows that $\sphericalangle B_1A_1C_1$ is a right angle (the bisectors of any two adjacent angles are perpendicular to each other) (see also [2], p. 194, Problem 156).

As a consequence we get that E is the intersection point of the angle bisectors CJ and A_1B_1 of $\triangle\, AA_1C$ and hence $\sphericalangle JAE = \sphericalangle EAB_1 = 30^0$.

Let $\varphi = \sphericalangle CA_1B_1 = \sphericalangle B_1A_1A$ and $\gamma = \sphericalangle C_1CA = \sphericalangle C_1CB$. Then $\sphericalangle A_1B_1C = 60^0 + \varphi$ as an exterior angle of $\triangle\, A_1B_1A$, the sum of the angles of $\triangle\, AA_1C$ is $60^0 + 2\varphi + 2\gamma = 180^0$, i. e. $\varphi + \gamma = 60^0$ and hence $\sphericalangle JEB_1 = 120^0$.

Thus, the quadrilateral $AJEB_1$ can be inscribed in a circle. We conclude that $\sphericalangle JAE = \sphericalangle JB_1E = 30^0$ as angles in the same segment of this circle. Hence, $\sphericalangle BB_1A_1 = 30^0$. □

Now we formulate and prove the *Basic problem* in this group.

Basic problem 4.9. *In $\triangle\, ABC$ the straight lines AA_1, $A_1 \in BC$, and BB_1, $B_1 \in AC$, are the bisectors of $\sphericalangle CAB$ and $\sphericalangle CBA$ respectively. If $\sphericalangle BB_1A_1 = 30^0$, prove that either $\sphericalangle ACB = 60^0$ or $\sphericalangle BAC = 120^0$.*

Proof. Let us denote $\sphericalangle BAA_1 = \sphericalangle CAA_1 = \alpha$, $\sphericalangle ABB_1 = \sphericalangle CBB_1 = \beta$, $AA_1 \cap BB_1 = J$. Since J is the intersection point of the angle bisectors of $\triangle\, ABC$, then the straight line CJ is the bisector of $\sphericalangle ACB$. Denoting $\gamma = \sphericalangle JCA = \sphericalangle JCB$ we get $\alpha + \beta + \gamma = 90^0$ (Fig. 9). Let the point A' be orthogonally symmetric to the point A_1 with respect to the axis BB_1. It follows that $A' \neq A$. (If $A' \equiv A$ then $\triangle\, ABC$ does not exist.) The straight line BB_1 is the bisector of $\sphericalangle ABC$ and consequently $A' \in AB$ and $B_1A_1 = B_1A'$. On the other hand, $\sphericalangle BB_1A_1 = 30^0$ and hence $\triangle\, A_1B_1A'$ is equilateral.

We find $\sphericalangle AA'B_1 = 30^0 + \beta$ (as an exterior angle of $\triangle\, A'BB_1$), $\sphericalangle AA'A_1 = 90^0 + \beta$ (as an exterior angle of $\triangle\, A'BE$), $\sphericalangle AB_1A' = 60^0 + \gamma - \alpha$ and $\sphericalangle AB_1A_1 = 120^0 + \gamma - \alpha$.

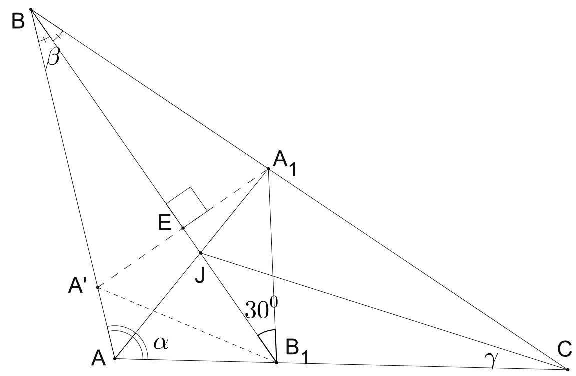

Fig. 9.

Let us compare $\triangle AA_1B_1$ and $\triangle AA_1A'$. They have a common side AA_1, equal corresponding sides $A_1B_1 = A_1A'$ and angles $\sphericalangle B_1AA_1 = \sphericalangle A'AA_1 = \alpha$. Hence Theorem 2.4 is applicable to $\triangle AA_1B_1$ and $\triangle AA_1A'$. We have two possibilities:

(i) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are congruent. Then $\sphericalangle AB_1A_1 = \sphericalangle AA'A_1$, i. e. $120^0 + \gamma - \alpha = 90^0 + \beta$. Hence, $2\gamma = \sphericalangle ACB = 60^0$.

(ii) $\triangle AA_1B_1$ and $\triangle AA_1A'$ are not congruent. By Theorem 2.4 it follows that $\sphericalangle AB_1A_1 + \sphericalangle AA'A_1 = 180^0$, i. e. $(120^0 + \gamma - \alpha) + (90^0 + \beta) = 180^0$. Hence, $2\alpha = \sphericalangle BAC = 120^0$. □

Remark 4.10. An alternative version of Problem 4.9 is Problem 6 in [6].

To formulate a special type equivalent problem (see also [4]) to this Basic problem we need

Proposition 4.11. *If the statements p and q are mutually exclusive, then the following equivalences are true:*

$$(\neg(p \veebar q)) \;\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q) \;\Leftrightarrow\; \neg p \wedge \neg q.$$

Proof. We have

$$(\neg(p \veebar q)) \;\Leftrightarrow\; \neg((p \wedge \neg q) \vee (\neg p \wedge q))$$

$$\Leftrightarrow\; (p \vee \neg q) \wedge (\neg p \vee q) \;\Leftrightarrow\; p \wedge (\neg p \vee q) \vee \neg q \wedge (\neg p \vee q)$$

$$\Leftrightarrow\; (p \wedge \neg p) \vee (p \wedge q) \vee (\neg q \wedge \neg p) \vee (q \wedge \neg q) \;\Leftrightarrow\; \neg p \wedge \neg q.$$

□

By Proposition 4.11, problems with logical structures $t \wedge (\neg(p \veebar q)) \;\rightarrow\; \neg r$ and $t \wedge (\neg p \wedge \neg q) \;\rightarrow\; \neg r$ are equivalent.

The following problem is equivalent to Basic problem 4.9.

Problem 4.12. *In* $\triangle ABC$ *the straight lines* AA_1, $A_1 \in BC$, *and* BB_1, $B_1 \in AC$, *are the bisectors of* $\sphericalangle CAB$ *and* $\sphericalangle CBA$, *respectively. If* $\sphericalangle ACB \neq 60^0$ *and* $\sphericalangle CAB \neq 120^0$, *prove that* $\sphericalangle BB_1A_1 \neq 30^0$.

Proof. Assuming that the opposite statement is true, i.e., $\sphericalangle BB_1A_1 = 30^0$, we would get a contradiction to Basic problem 4.9. □

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