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WEIGHTED BANACH SPACES OF HOLOMORPHIC FUNCTIONS WITH LOG-CONCAVE WEIGHTS

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Some theorems on convex functions are proved and an application of these theorems in the theory of weighted Banach spaces of holomorphic functions is investigated, too. We prove that $H_v(G)$ and $H_{v_0}(G)$ are exactly the same spaces as $H_w(G)$ and $H_{w_0}(G)$ where w is the smallest log-concave majorant of v. This investigation is based on the theory of convex functions and some specific properties of the weighted banach spaces of holomorphic functions under consideration.

Keywords: Associated weights, holomorphic function, weighted banach space, convex function

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1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and

$$G = \{z = x + iy \mid x \in (-\infty, \infty),\ y \in (0, \infty)\} \subset \mathbb{C}$$

be the upper half plane of $\mathbb{C}$. Throughout, $v : G \to (0, \infty)$ will be a function such that $v(z) = v(x + iy) = v(iy)$ for every $z = x + iy \in G$, and

$$\inf_{y \in [\frac{1}{c}, c]} v(i\,y) > 0 \quad \text{for every} \quad c > 1. \tag{1.1}$$

We define

$$\varphi_v(y) = -\ln v(iy), \quad y \in (0, \infty)\,,$$

and property (1.1) is reformulated as the following property of $\varphi_v(y)$:

$$\sup_{y\in[\frac{1}{c},c]} \varphi_v(y) < \infty \quad \text{for every } c > 1. \tag{1.1$'$}$$

The weighted Banach spaces of holomorphic functions $H_v(G)$ and $H_{v_0}(G)$ are defined as follows

- $f \in H_v(G)$ if f is holomorphic on G and

$$\| f \|_v = \sup_{z\in G} v(z)|f(z)| < \infty\,;$$

- $f \in H_{v_0}(G)$ if $f \in H_v(G)$ and f is such that for every $\varepsilon > 0$ there exists a compact $\mathcal{K}_\varepsilon \subset G$ for which

$$\sup_{z\in G\setminus\mathcal{K}_\varepsilon} v(z)|f(z)| < \varepsilon.$$

Here, we use notations from [1, 2, 3, 4, 5].

In [1], [2] the authors find an isomorphic classification of the spaces $H_v(G)$ and $H_{v_0}(G)$ provided the weight function v satisfies some growth conditions.

In [3], [4] weighted composition operators between weighted spaces of holomorphic functions on the unit disk in the complex plane are studied and the associated weights are used in order to estimate the norm of the weighted composition operators.

The associated weights are studied in [5].

This paper is about weights that have some of the properties of the associated weights. We prove that $H_v(G)$ and $H_{v_0}(G)$ are exactly the same spaces as $H_w(G)$ and $H_{w_0}(G)$, where w is the smallest log-concave majorant of v. Here, the smallest log-concave majorant of v is exactly the associated weight, but in case of other weighted spaces this coincidence might not take place. Our work is based on the theory of convex functions and some specific properties of the weighted banach spaces of holomorphic functions under consideration.

The results of this paper are communicated at the conferences [7] and [8].

2. DEFINITIONS AND NOTATIONS

Let Φ be the set of functions φ satisfying the following conditions:

- $\varphi : (0,\infty) \to \mathbb{R}$ and
- there exists $a \in \mathbb{R}$ such that

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - ax\big) > -\infty.$$

Note that if $\varphi \in \Phi$, then $-\infty < \varphi(x) < \infty$ for every $x \in (0, \infty)$.

We denote by $\widehat{a}_\varphi$ the limit inferior

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x}, \quad \varphi \in \Phi.$$

If $\varphi \in \Phi$, then

- $\widehat{a}_\varphi \in \mathbb{R} \cup \{\infty\}$, $\widehat{a}_\varphi > -\infty$;
- $\widehat{a}_\varphi = \sup\big\{a \mid a \in \mathbb{R}, \inf\limits_{x\in(0,\infty)} (\varphi(x) - ax) > -\infty\big\}$

If $\varphi \in \Phi$ is convex in $(0, \infty)$, then

$$\widehat{a}_\varphi = \lim_{x\to\infty} \frac{\varphi(x)}{x}$$

By Φ_1, Φ_2, Φ_3 we denote the following subsets of Φ:

$$\Phi_1 = \big\{\varphi \,:\, \varphi \in \Phi,\ \widehat{a}_\varphi = \infty\big\}\,;$$
$$\Phi_2 = \big\{\varphi \,:\, \varphi \in \Phi,\ \widehat{a}_\varphi < \infty,\ \liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) = -\infty\big\}\,;$$
$$\Phi_3 = \big\{\varphi \,:\, \varphi \in \Phi,\ \widehat{a}_\varphi < \infty,\ \liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) > -\infty\big\}\,.$$

Note that Φ_1, Φ_2, Φ_3 are mutually disjoint sets and $\Phi_1 \cup \Phi_2 \cup \Phi_3 = \Phi$.
If $\varphi \in \Phi_2 \cup \Phi_3$ is convex on $(0, \infty)$, then

$$\liminf_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x) = \lim_{x\to\infty}(\varphi(x) - \widehat{a}_\varphi x).$$

Note that a function $\varphi \in \Phi$ is not necessarily continuous. In fact, $\varphi \in \Phi$ is not supposed to satisfy any conditions beside those of the definition of Φ, Φ_1, Φ_2, Φ_3. There are a number of simple functions that belong to Φ, Φ_1, Φ_2, Φ_3, for instance,

- $\varphi_1(x) = x^2$ belongs to Φ_1;
- $\varphi_2(x) = x - \sqrt{x}$ belongs to Φ_2;
- $\varphi_3(x) = x^{-1}$ belongs to Φ_3,

and $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$ are all convex on $(0, \infty)$.

For a $\varphi \in \Phi$ let

$$M_\varphi = \Big\{(a, b) \Big|\ a,\, b \in \mathbb{R},\ \inf_{t\in(0,\infty)} (\varphi(t) - at) > b\Big\}.$$

The function $\varphi^{**} : (0, \infty) \to \mathbb{R}$ is defined as

$$\varphi^{**}(x) = \sup_{(a,b)\in M_\varphi} (ax + b).$$

φ^{**} is referred to as the second Young-Fenhel conjugate of φ and it is the largest convex minorant of φ.

3. MAIN RESULTS

Here we state our main results.

Theorem 3.1. *Let φ, $\psi \in \Phi$. If ψ is convex on $(0,\infty)$, then*

$$\inf_{x\in(0,\infty)} \big(\varphi(x)-\psi(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x)-\psi(x)\big)\,.$$

Theorem 3.2. *Let φ, $\psi \in \Phi$. If ψ is convex on $(0,\infty)$ and, in addition, $\lim\limits_{x\to 0^+} \big(\varphi(x)-\psi(x)\big) = \infty$, then*

$$\lim_{x\to 0^+} \big(\varphi^{**}(x)-\psi(x)\big) = \infty\,.$$

Theorem 3.3. *Let $\varphi \in \Phi$ and $\psi \in \Phi \setminus \Phi_3$. If ψ is convex on $(0,\infty)$ and, in addition, $\lim\limits_{x\to \infty} \big(\varphi(x)-\psi(x)\big) = \infty$, then*

$$\lim_{x\to \infty} \big(\varphi^{**}(x)-\psi(x)\big) = \infty\,.$$

The next examples show that the assumption for convexity of ψ in Theorems 3.1–3.3 cannot be omitted.

Example 3.1. *Theorem 3.1 does not hold with the functions*

$$\varphi(x) = \min\{x,1\}+1, \quad \psi(x) = \frac{x}{x+1}, \quad x\in(0,\infty)\,.$$

Note that $\varphi \in \Phi$, and $\psi \in \Phi$ is not convex on $(0,\infty)$. We have $\varphi^{**}(x) = 1$, $x\in(0,\infty)$, and

$$1 = \inf_{x\in(0,\infty)} \big(\varphi(x)-\psi(x)\big) \neq \inf_{x\in(0,\infty)} \big(\varphi^{**}(x)-\psi(x)\big) = 0\,.$$

Example 3.2. *Theorem 3.2 does not hold with*

$$\varphi(x) = \frac{1}{x^2}+\frac{1}{x}\sin\frac{1}{x}+\frac{2}{x}, \quad \psi(x) = \varphi(x)-\frac{2}{x}, \quad x\in(0,\infty)\,.$$

Note that φ, $\psi \in \Phi$, the function ψ is not convex on $(0,\infty)$ and

$$\infty = \lim_{x\to 0^+} \big(\varphi(x)-\psi(x)\big) > \liminf_{x\to 0^+} \big(\varphi^{**}(x)-\psi(x)\big)$$

This fact is proved in Proposition 4.1.

Example 3.3. *Theorem 3.3 does not hold with*

$$\varphi(x) = x^2+x\sin x+2x, \quad \psi(x) = \varphi(x)-2x, \quad x\in(0,\infty).$$

Note that $\varphi\in\Phi$, $\psi \in \Phi\setminus\Phi_3$, ψ is not convex on $(0,\infty)$ and

$$\infty = \lim_{x\to \infty} \big(\varphi(x)-\psi(x)\big) > \liminf_{x\to \infty} \big(\varphi^{**}(x)-\psi(x)\big)$$

This fact is proved in Proposition 4.2.

Corollary 3.1. *If φ, $\psi \in \Phi$ are such that*

$$\lim_{x\to 0^+} \big(\varphi(x) - \psi(x)\big) = \infty\,,$$

then

$$\lim_{x\to 0^+} \big(\varphi^{**}(x) - \psi^{**}(x)\big) = \infty\,. \tag{3.1}$$

Proof. Since $\psi^{**} \le \psi$, we have

$$\lim_{x\to 0^+} \big(\varphi(x) - \psi^{**}(x)\big) \ge \lim_{x\to 0^+} \big(\varphi(x) - \psi(x)\big) = \infty.$$

Now Theorem 3.2 applied to φ, ψ^{**} proves (3.1). □

Corollary 3.2. *Let $\varphi \in \Phi$ and $\psi \in \Phi \setminus \Phi_3$. If φ and ψ satisfy*

$$\lim_{x\to \infty} \big(\varphi(x) - \psi(x)\big) = \infty\,,$$

then

$$\lim_{x\to \infty} \big(\varphi^{**}(x) - \psi^{**}(x)\big) = \infty\,. \tag{3.2}$$

Proof. Note that

- $\psi^{**} \in \Phi \setminus \Phi_3$ by the Lemma 4.1;
- $\psi^{**} \le \psi$.

Theorem 3.3 applied to φ, ψ^{**} implies (3.2). □

Example 3.4. *Let $\varphi(x) = x^2 + x$ and*

$$\psi(x) = \begin{cases} 3x - 1, & x \in (0, 1] \\ 5 - 3x, & x \in (1, 2] \\ x^2 + x - 7, & x \in (2, \infty)\,. \end{cases}$$

We observe that φ, $\psi \in \Phi$ and

- *φ is convex on $(0, \infty)$, and therefore $\varphi^{**} = \varphi$,*
- *ψ is not convex on $(0, \infty)$ and*

$$\psi^{**}(x) = \begin{cases} -1, & x \in (0, 2] \\ x^2 + x - 7, & x \in (2, \infty)\,. \end{cases}$$

A direct calculation shows that

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - \psi(x)\big) = 0 \ne 1 = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x) - \psi^{**}(x)\big).$$

Thus, there is no analog of Theorem 3.1 involving φ^{**} and ψ as in Corollaries 3.1 and 3.2. $\square$

4. AUXILIARY RESULTS

Proposition 4.1. *Let*

$$\varphi(x) = \frac{1}{x^2} + \frac{1}{x}\sin\frac{1}{x} + \frac{2}{x} \quad \textit{and} \quad \psi(x) = \varphi(x) - \frac{2}{x}, \qquad x \in (0,\infty).$$

Then φ, $\psi \in \Phi$, *the function* ψ *is not convex on* $(0,\infty)$ *and*

$$\infty = \lim_{x\to 0^+}\bigl(\varphi(x) - \psi(x)\bigr) > \liminf_{x\to 0^+}\bigl(\varphi^{**}(x) - \psi(x)\bigr).$$

Proof. The function ψ satisfies

$$\psi(x) \ge \begin{cases} \frac{1}{x^2} - \frac{1}{x}, & x \in (0,1), \\ \frac{1}{x^2}, & x \in [1,\infty), \end{cases}$$

hence, $\psi(x) \ge 0$ for every $x \in (0,\infty)$, and this implies that $\psi \in \Phi$.

Since $\varphi \ge \psi$, we have also $\varphi \in \Phi$.

Note that

$$\lim_{x\to 0^+}\bigl(\varphi(x) - \psi(x)\bigr) = \lim_{x\to 0^+}\frac{2}{x} = \infty.$$

Let

$$x_k = \frac{1}{\frac{3\pi}{2} + 2k\pi}, \quad \widetilde{x}_k = \frac{1}{\frac{5\pi}{2} + 2k\pi}, \qquad k = 0,1,2,\ldots.$$

We observe that $x_k > \widetilde{x}_k > x_{k+1} > 0$, $\lim\limits_{k\to\infty} x_k = 0$, and the harmonic mean of x_k, x_{k+1} is equal to $\widetilde{x}_k$. A direct computation shows that $\psi''(\widetilde{x}_0) < 0$, therefore ψ is not convex on $(0,\infty)$.

Let

$$f(x) = \frac{1}{x^2} + \frac{1}{x}, \qquad x \in (0,\infty).$$

The function f is convex on $(0,\infty)$ and $f(x) \le \varphi(x)$, $x \in (0,\infty)$. So, f is a convex minorant of φ and thus $f \le \varphi^{**}$.

Therefore, $f(x_k) \le \varphi^{**}(x_k) \le \varphi(x_k) = f(x_k)$ and this implies that

$$f(x_k) = \varphi^{**}(x_k), \qquad k = 1,2,3,\ldots.$$

Furthermore,

$$\psi(\widetilde{x}_k) = f(\widetilde{x}_k) \le \varphi^{**}(\widetilde{x}_k) \le \frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}}\,\varphi^{**}(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}}\,\varphi^{**}(x_k),$$

because of the convexity of φ^{**}. Thus,

$$0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}} f(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}} f(x_k) - f(\widetilde{x}_k)\,.$$

After some simple calculations we obtain

$$\frac{x_k - \widetilde{x}_k}{x_k - x_{k+1}} f(x_{k+1}) + \frac{\widetilde{x}_k - x_{k+1}}{x_k - x_{k+1}} f(x_k) - f(\widetilde{x}_k) = (3 + \widetilde{x}_k)\pi^2\,.$$

Consequently, $0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le (3 + \widetilde{x}_k)\pi^2$, $k = 1, 2, 3, \ldots$, and

$$\liminf_{x \to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) < \infty. \qquad \square$$

Proposition 4.2. *Let*

$$\varphi(x) = x^2 + x \sin x + 2x \quad \textit{and} \quad \psi(x) = \varphi(x) - 2x, \qquad x \in (0, \infty)\,.$$

Then $\varphi \in \Phi$, $\psi \in \Phi \setminus \Phi_3$, *the function* ψ *is not convex on* $(0, \infty)$ *and*

$$\infty = \lim_{x \to \infty} \big(\varphi(x) - \psi(x)\big) > \liminf_{x \to \infty} \big(\varphi^{**}(x) - \psi(x)\big)\,.$$

Proof. The function ψ satisfies the inequalities

$$\psi(x) \ge \begin{cases} x^2, & x \in (0, \pi)\,, \\ x^2 - x, & x \in [\pi\,, \infty)\,, \end{cases}$$

therefore $\psi(x) \ge 0$, $x \in (0, \infty)$, and thus $\psi \in \Phi$. Moreover,

$$\widehat{a}_\psi = \liminf_{x \to \infty} \frac{\psi(x)}{x} \ge \liminf_{x \to \infty} \frac{x^2 - x}{x} = \infty\,,$$

therefore $\widehat{a}_\psi = \infty$ and thus $\psi \in \Phi_1 \subset \Phi \setminus \Phi_3$.

Now $\varphi \in \Phi$ since $\varphi \ge \psi$ and $\psi \in \Phi$. Moreover,

$$\lim_{x \to \infty} \big(\varphi(x) - \psi(x)\big) = \lim_{x \to \infty} 2x = \infty\,.$$

Let

$$x_k = \frac{3\pi}{2} + 2k\pi, \quad \widetilde{x}_k = \frac{5\pi}{2} + 2k\pi\,, \quad k = 0, 1, 2, \ldots$$

Note that, for $k \in \mathbb{N}$, $0 < x_k < \widetilde{x}_k < x_{k+1}$, $x_k + x_{k+1} = 2\widetilde{x}_k$ and $\lim_{k \to \infty} x_k = \infty$.

A direct computation shows that $\psi''(\widetilde{x}_0) < 0$, hence ψ is not convex on $(0, \infty)$.

Let

$$f(x) = x^2 + x, \quad x \in (0, \infty).$$

The function f is convex on $(0,\infty)$ and $f \le \varphi$ therein. So, f is a convex minorant of φ and thus $f \le \varphi^{**}$. Therefore, $f(x_k) \le \varphi^{**}(x_k) \le \varphi(x_k) = f(x_k)$, and this implies

$$f(x_k) = \varphi^{**}(x_k), \quad k \in \mathbb{N}.$$

Furthermore, by the convexity of φ^{**}, we have

$$\psi(\widetilde{x}_k) = f(\widetilde{x}_k) \le \varphi^{**}(\widetilde{x}_k) \le \frac{x_{k+1} - \widetilde{x}_k}{x_{k+1} - x_k}\, \varphi^{**}(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1} - x_k}\, \varphi^{**}(x_{k+1}).$$

Thus,

$$0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \frac{x_{k+1} - \widetilde{x}_k}{x_{k+1} - x_k}\, f(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1} - x_k}\, f(x_{k+1}) - f(\widetilde{x}_k).$$

After some simple calculations we obtain

$$\frac{x_{k+1} - \widetilde{x}_k}{x_{k+1} - x_k}\, f(x_k) + \frac{\widetilde{x}_k - x_k}{x_{k+1} - x_k}\, f(x_{k+1}) - f(\widetilde{x}_k) = \pi^2, \quad k \in \mathbb{N}.$$

Consequently, $0 \le \varphi^{**}(\widetilde{x}_k) - \psi(\widetilde{x}_k) \le \pi^2$, $k \in \mathbb{N}$, and

$$\liminf_{x\to\infty} \big(\varphi^{**}(x) - \psi(x)\big) < \infty.$$

□

Lemma 4.1. *If $\varphi \in \Phi$, then*

(1) $\liminf\limits_{x\to 0^+} \varphi(x) = \lim\limits_{x\to 0^+} \varphi^{**}(x)$*;*

(2) $\liminf\limits_{x\to\infty} \dfrac{\varphi(x)}{x} = \lim\limits_{x\to\infty} \dfrac{\varphi^{**}(x)}{x}$.

Proof. Let $\varphi \in \Phi$. Then

$$\liminf_{x\to 0^+} \varphi(x) \ge \liminf_{x\to 0^+} \varphi^{**}(x) = \lim_{x\to 0^+} \varphi^{**}(x),$$
$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} \ge \liminf_{x\to\infty} \frac{\varphi^{**}(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x}.$$

Let $a_0,\ b_0 \in \mathbb{R}$ be such that $a_0 x + b_0 \le \varphi(x)$, $x \in (0,\infty)$. Then

$$\liminf_{x\to 0^+} \varphi(x) \ge b_0 > -\infty, \tag{4.1}$$
$$\liminf_{x\to\infty} \frac{\varphi(x)}{x} \ge a_0 > -\infty. \tag{4.2}$$

Let b be such that $\liminf\limits_{x\to 0^+} \varphi(x) > b > -\infty$. We choose $\delta > 0$ so that

$$\inf_{0<x<\delta} \varphi(x) > b.$$

Then

$$\inf_{x>0} \frac{\varphi(x)-b}{x} \geq \min\Big\{\inf_{0<x<\delta} \frac{\varphi(x)-b}{x}\,,\ \inf_{\delta\leq x} \frac{\varphi(x)-b}{x}\Big\}$$
$$\geq \min\Big\{0,\ \inf_{\delta\leq x}\big(a_0+\frac{b_0-b}{x}\big)\Big\} > -\infty\,.$$

Set $a=\min\big\{0,\ \inf_{\delta\leq x}\big(a_0+\frac{b_0-b}{x}\big)\big\}$, then $(a,b)\in M_\varphi$ and consequently $\varphi^{**}(x)\geq ax+b$, $x\in(0,\infty)$. Thus,

$$\lim_{x\to 0^+} \varphi^{**}(x) \geq b$$

and, by our choice of b,

$$\lim_{x\to 0^+} \varphi^{**}(x) \geq \liminf_{x\to 0^+} \varphi(x)\,.$$

Hence, assertion (1) of Lemma 4.1 is proved.

Let α be such that $\liminf_{x\to\infty} \frac{\varphi(x)}{x} > \alpha > -\infty$. We choose $\Delta>0$ such that

$$\inf_{x>\Delta} \frac{\varphi(x)}{x} > \alpha.$$

Then

$$\inf_{x>0} \big(\varphi(x)-\alpha x\big) \geq \min\big\{\inf_{0<x\leq\Delta} (\varphi(x)-\alpha x),\ \inf_{x>\Delta} (\varphi(x)-\alpha x)\big\}$$
$$\geq \min\big\{\inf_{0<x\leq\Delta} (a_0x+b_0-\alpha x),\ 0\big\} > -\infty$$

Let $\beta=\min\big\{\inf_{0<x\leq\Delta} (a_0x+b_0-\alpha x),\ 0\big\}$. Then $(\alpha,\beta)\in M_\varphi$ and consequently, $\varphi^{**}(x)\geq \alpha x+\beta$ for every $x\in(0,\infty)$. Therefore,

$$\lim_{x\to\infty} \frac{\varphi^{**}(x)}{x} \geq \alpha$$

and, by our choice of α,

$$\lim_{x\to\infty} \frac{\varphi^{**}(x)}{x} \geq \liminf_{x\to\infty} \frac{\varphi(x)}{x}$$

Thus, assertion (2) of Lemma 4.1 is proved. □

Lemma 4.2. $\varphi\in\Phi_i \iff \varphi^{**}\in\Phi_i,\ \ i=1,\,2,\,3.$

Proof. The assertion $\varphi\in\Phi_1 \iff \varphi^{**}\in\Phi_1$ is proved as (1) of Lemma 4.1. The proof of Lemma 4.2 will be completed once we prove that

$$\varphi\in\Phi_3 \iff \varphi^{**}\in\Phi_3.$$

Let $\varphi \in \Phi_2 \cup \Phi_3$ and

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x}.$$

If $\varphi^{**} \in \Phi_3$, then $\varphi \geq \varphi^{**}$ implies $\varphi \in \Phi_3$.

Now let us suppose that $\varphi \in \Phi_3$. Let $a_0,\ b_0 \in \mathbb{R}$ be such that $a_0 x + b_0 \leq \varphi(x)$ for every $x \in (0,\infty)$. Let $b \in \mathbb{R}$ be such that $\liminf\limits_{x\to\infty} \big(\varphi(x) - \widehat{a}_\varphi x\big) > b > -\infty$.

Let $\Delta > 0$ satisfy

$$\inf_{x>\Delta} \big(\varphi(x) - \widehat{a}_\varphi x\big) > b.$$

Then,

$$\begin{aligned}\inf_{x>0} \big(\varphi(x) - \widehat{a}_\varphi x\big) &\geq \min\big\{\inf_{0<x\leq\Delta} (\varphi(x) - \widehat{a}_\varphi x),\ \inf_{x>\Delta} (\varphi(x) - \widehat{a}_\varphi x)\big\} \\ &\geq \min\big\{\inf_{0<x\leq\Delta} (a_0 x + b_0 - \widehat{a}_\varphi x),\ b\big\} > -\infty\,.\end{aligned}$$

Let $\widehat{b} = \min\big\{\inf\limits_{0<x\leq\Delta} (a_0 x + b_0 - \widehat{a}_\varphi x),\ b\big\}$. Then $(\widehat{a}_\varphi, \widehat{b}) \in M_\varphi$ and consequently $\varphi^{**}(x) \geq \widehat{a}_\varphi x + \widehat{b}$ for every $x \in (0,\infty)$. Thus,

$$\liminf_{x\to\infty}\big(\varphi^{**}(x) - \widehat{a}_\varphi x\big) \geq \widehat{b} > -\infty\,,$$

and $\varphi^{**} \in \Phi_3$. □

Lemma 4.3. *Let $\varphi \in \Phi$. If a is such that $a < \widehat{a}_\varphi$, then*

$$\inf_{x>0} \big(\varphi(x) - ax\big) > -\infty$$

and

$$\lim_{x\to\infty} \big(\varphi(x) - ax\big) = \infty\,.$$

Proof. Let $a_0,\ b_0 \in \mathbb{R}$ be such that $(a_0, b_0) \in M_\varphi$, let a and a_1 satisfy the inequalities $-\infty < a < a_1 < \widehat{a}_\varphi$, and $\Delta > 0$ be such that

$$\inf_{x>\Delta} \frac{\varphi(x)}{x} > a_1.$$

So, $\varphi(x) - ax > (a_1 - a)x$ for $x > \Delta$ and $\lim\limits_{x\to\infty} \big(\varphi(x) - ax\big) = \infty$. Therefore,

$$\begin{aligned}\inf_{x>0} \big(\varphi(x) - ax\big) &= \min\Big\{\inf_{0<x\leq\Delta} \big(\varphi(x) - ax\big);\ \inf_{x>\Delta} \big(\varphi(x) - ax\big)\Big\} \\ &\geq \min\Big\{\inf_{0<x\leq\Delta} \big(a_0 x + b_0 - ax\big);\ \inf_{x>\Delta} (a_1 - a)x\Big\} > -\infty.\end{aligned}$$

Lemma 4.3 is proved. □

Lemma 4.4. *Let $\psi : (0,\infty) \to \mathbb{R}$ be convex on $(0,\infty)$ and*

$$\widehat{\psi}(x) = \psi(x) - \psi'(x^-)x, \quad x \in (0,\infty),$$

where $\psi'(x^-) = \lim_{t\to x^-} \frac{\psi(t)-\psi(x)}{t-x}$. If $0 < x_1 < x_2$, then

$$\widehat{\psi}(x_1) \geq \widehat{\psi}(x_2).$$

Proof. Let $x_3 = \frac{x_1+x_2}{2}$. Note that

- $2\psi(x_3) \leq \psi(x_1) + \psi(x_2)$,
- $f(u,v) = \frac{\psi(u)-\psi(v)}{u-v}$ is a monotone non-decreasing function of each variable u, $v > 0$, $u \neq v$, and

$$-\infty < \lim_{t\to x^-} \frac{\psi(t)-\psi(x)}{t-x} = \psi'(x^-) \leq \psi'(x^+) = \lim_{v\to x^+} \frac{\psi(v)-\psi(x)}{v-x} < \infty, \quad x > 0.$$

Now, $\widehat{\psi}(x_2) \leq \widehat{\psi}(x_1)$ follows from the inequalities

$$\begin{aligned}
\widehat{\psi}(x_2) &= \psi(x_2) - \psi'(x_2^-)\,x_2 \leq \psi(x_2) - \frac{\psi(x_2)-\psi(x_3)}{x_2-x_3}\,x_2 \\
&= \big(\psi(x_3)-\psi(x_2)\big)\,\frac{2x_2}{x_2-x_1} - \psi(x_2) = \psi(x_3)\,\frac{2x_2}{x_2-x_1} - \psi(x_2)\,\frac{x_2+x_1}{x_2-x_1} \\
&\leq \big(\psi(x_2)+\psi(x_1)\big)\,\frac{x_2}{x_2-x_1} - \psi(x_2)\,\frac{x_2+x_1}{x_2-x_1} \\
&= \psi(x_1)\,\frac{x_2+x_1}{x_2-x_1} - \big(\psi(x_1)+\psi(x_2)\big)\,\frac{x_1}{x_2-x_1} \\
&\leq \psi(x_1)\,\frac{x_2+x_1}{x_2-x_1} - \psi(x_3)\,\frac{2x_1}{x_2-x_1} = \psi(x_1) - \big(\psi(x_3)-\psi(x_1)\big)\,\frac{2x_1}{x_2-x_1} \\
&= \psi(x_1) - \frac{\psi(x_3)-\psi(x_1)}{x_3-x_1}\,x_1 \leq \psi(x_1) - \psi'(x_1^+)\,x_1 \leq \psi(x_1) - \psi'(x_1^-)\,x_1 \\
&= \widehat{\psi}(x_1).
\end{aligned}$$

□

Lemma 4.5. *Let $\psi \in \Phi_2 \cup \Phi_3$. If ψ is convex on $(0,\infty)$ and*

$$\lim_{x\to\infty} \big(\psi(x) - \psi'(x^-)x\big) > -\infty,$$

then $\psi \in \Phi_3$.

Proof. Note that the limit value exists due to Lemma 4.4.

Let $\alpha \in \mathbb{R}$ satisfy

$$\lim_{x\to\infty} \big(\psi(x) - \psi'(x^-)x\big) > \alpha > -\infty.$$

Let $\Delta > 0$ be such that $\inf_{x>\Delta}\big(\psi(x) - \psi'(x^-)x\big) > \alpha$. Then,

$$\psi(x) - \frac{\psi(t) - \psi(x)}{t - x}x > \alpha\,, \qquad \Delta < t < x$$

and therefore

$$\frac{\psi(t) - \alpha}{t} \geq \frac{\psi(x) - \alpha}{x}\,, \qquad \Delta < t < x\,.$$

Consequently,

$$\frac{\psi(x) - \alpha}{x} \geq \lim_{x\to\infty} \frac{\psi(x) - \alpha}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x} = \widehat{a}_\psi\,, \qquad x > \Delta\,.$$

Thus, $\psi(x) - \widehat{a}_\psi x \geq \alpha$ for $x > \Delta$, and

$$\lim_{x\to\infty}\big(\psi(x) - \widehat{a}_\psi x\big) \geq \alpha > -\infty\,,$$

i.e. $\psi \in \Phi_3$. □

As a direct consequence from Lemma 4.5 we obtain

Corollary 4.3. *Let $\psi \in \Phi_2$. If ψ is convex on $(0,\infty)$, then*

$$\lim_{x\to\infty}\big(\psi(x) - \psi'(x^-)x\big) = -\infty\,.$$

5. PROOFS OF THE MAIN RESULTS

Proof of Theorem 3.1. Let $\varphi \in \Phi$, $\psi \in \Phi$ and ψ be convex on $(0,\infty)$. Since $\varphi \geq \varphi^{**}$, we have

$$\inf_{x\in(0,\infty)}\big(\varphi(x) - \psi(x)\big) \geq \inf_{x\in(0,\infty)}\big(\varphi^{**}(x) - \psi(x)\big)\,. \tag{5.1}$$

We consider separately two cases:

Case 1. $\inf_{x\in(0,\infty)}\big(\varphi(x) - \psi(x)\big) = -\infty$. We have

$$\inf_{x\in(0,\infty)}\big(\varphi(x) - \psi(x)\big) = \inf_{x\in(0,\infty)}\big(\varphi^{**}(x) - \psi(x)\big) = -\infty\,.$$

Case 2. $c := \inf_{x\in(0,\infty)}\big(\varphi(x) - \psi(x)\big) > -\infty$. In this case,

$$\varphi(x) \geq \psi(x) + c, \qquad x \in (0,\infty)$$

and $\psi + c$ is a convex minorant of φ. Therefore, $\varphi^{**}(x) \geq \psi(x) + c, \quad x \in (0,\infty)$, i.e. $\inf_{x>0}(\varphi^{**}(x) - \psi(x)) \geq c$ and

$$\inf_{x>0}(\varphi^{**}(x) - \psi(x)) \geq \inf_{x\in(0,\infty)}\big(\varphi(x) - \psi(x)\big)\,.$$

It follows from here and inequality (5.1) that

$$\inf_{x\in(0,\infty)} \big(\varphi(x)-\psi(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x)-\psi(x)\big). \qquad \square$$

Proof of Theorem 3.2. Recall that φ, $\psi \in \Phi$, ψ is convex on $(0,\infty)$ and

$$\lim_{x\to 0^+} \big(\varphi(x)-\psi(x)\big) = \infty\,.$$

Note that $\lim\limits_{x\to 0^+} \psi(x) =: \psi(0^+) \in \mathbb{R}\cup\{\infty\}$ and $\psi(0^+) > -\infty$, because $\psi \in \Phi$ and ψ is convex on $(0,\infty)$. Therefore, $\varphi(0^+) = \infty$ and from Lemma 4.1 we obtain that $\varphi^{**}(0^+) = \infty$.

If $\psi(0^+) < \infty$, then

$$\lim_{x\to 0^+} (\varphi^{**}(x) - \psi(x)) = \varphi^{**}(0^+) - \psi(0^+) = \infty.$$

In order to complete the proof, we have to examine the alternative when ψ satisfies $\psi(0^+) = \infty$. We shall define a new function $\widetilde{\psi}$ that is a convex minorant of φ.

Let a_0, $b_0 \in \mathbb{R}$ be such that $(a_0, b_0) \in M_\varphi$ and $c \in \mathbb{R}$. We choose $\Delta_1 > 0$ such that

$$\inf_{0<x<\Delta_1} \big(\varphi(x)-\psi(x)\big) > c\,.$$

Next, we choose Δ_2 such that $\Delta_1 > \Delta_2 > 0$ and

$$\inf_{0<x<\Delta_2} \big(\psi(x) + c - (a_0 x + b_0)\big) > 0.$$

Now, we choose Δ_3, $\Delta_2 > \Delta_3 > 0$, so that $\psi(x)$ is monotone non-increasing on $(0,\Delta_3)$. For $x \in (0,\Delta_3)$ we have the following inequalities for the convex function ψ:

$$0 \geq \frac{\psi(\Delta_3)-\psi(x)}{\Delta_3} \geq \frac{\psi(\Delta_3)-\psi(x)}{\Delta_3 - x} \geq \limsup_{t\to x^+} \frac{\psi(t)-\psi(x)}{t-x} =: \psi'(x^+)$$

and from $\psi(0^+) = \infty$ it follows that

$$\lim_{x\to 0^+} \psi'(x^+) = -\infty.$$

Further, we choose Δ_4, $\Delta_3 > \Delta_4 > 0$, such that

$$\sup_{0<x<\Delta_4} \psi'(x^+) < a_0.$$

If $x \in (0,\Delta_4)$, then

$$\begin{aligned}
&\limsup_{x\to 0^+}\big(\psi'(x^+)(\Delta_1 - x) + \psi(x) + c\big)\\
&\qquad = \limsup_{x\to 0^+}\big(\psi'(x^+)(\Delta_1/2 - x) + \psi(x) + c + \psi'(x^+)\Delta_1/2\big)\\
&\qquad \leq \limsup_{x\to 0^+}\big(\psi(\Delta_1/2) + c + \psi'(x^+)\Delta_1/2\big) = -\infty\,.
\end{aligned}$$

Finally, we choose Δ_5 so that $\Delta_4 > \Delta_5 > 0$ and

$$\sup_{0<x<\Delta_5} \big(\psi'(x^+)(\Delta_1 - x) + \psi(x) + c\big) < a_0\Delta_1 + b_0\,.$$

Let $x_1 \in (0, \Delta_5)$. We set

$$a_1 := \psi'(x_1^+), \qquad b_1 := -\psi'(x_1^+)x_1 + \psi(x_1) + c\,,$$

hence,

$$\psi'(x_1^+)(x - x_1) + \psi(x_1) + c = a_1x + b_1, \quad x \in (0, \infty).$$

From

$$\begin{aligned} a_1x_1 + b_1 &= \psi(x_1) + c \geq a_0x_1 + b_0, \\ a_1\Delta_1 + b_1 &< a_0\Delta_1 + b_0 \end{aligned}$$

we conclude that there exists $x_2 \in [x_1, \Delta_1]$ such that $a_1x_2 + b_1 = a_0x_2 + b_0$.

We define a function $\widetilde{\psi}$ as follows:

$$\widetilde{\psi}(x) := \begin{cases} \psi(x) + c, & x \in (0, x_1) \\ a_1x + b_1, & x \in [x_1, x_2] \\ a_0x + b_0, & x \in (x_2, \infty). \end{cases}$$

The function $\widetilde{\psi}$ is convex on $(0, \infty)$ because it is continuous, $\psi'(x_1^-) \leq a_1 \leq a_0$ and $\psi + c$ is convex on $(0, x_1)$.

Furthermore,

$$\begin{aligned} &\widetilde{\psi}(x) = \psi(x) + c \leq \varphi(x), \quad x \in (0, x_1)\,, \\ &\widetilde{\psi}(x) = a_1x + b_1 \leq \psi(x) + c \leq \varphi(x), \quad x \in [x_1, x_2]\,, \\ &\widetilde{\psi}(x) = a_0x + b_0 \leq \varphi(x), \quad x \in (x_2, \infty)\,. \end{aligned}$$

Hence, $\widetilde{\psi}$ is a convex minorant of φ, and $\varphi^{**}(x) \geq \widetilde{\psi}(x)$, $x \in (0, \infty)$.

Thus $\varphi^{**}(x) \geq \psi(x) + c$, $x \in (0, x_1)$, and

$$\liminf_{x\to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) \geq c\,,$$

which, according to the choice of c, implies that

$$\lim_{x\to 0^+} \big(\varphi^{**}(x) - \psi(x)\big) = \infty. \qquad \square$$

Proof of Theorem 3.3. Recall that $\varphi \in \Phi$, $\psi \in \Phi \setminus \Phi_3$, ψ is convex on $(0, \infty)$ and $\lim\limits_{x\to\infty} \big(\varphi(x) - \psi(x)\big) = \infty$.

By Lemma 4.1 we have

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \frac{\varphi^{**}(x)}{x},$$
$$\widehat{a}_\psi = \liminf_{x\to\infty} \frac{\psi(x)}{x} = \lim_{x\to\infty} \frac{\psi^{**}(x)}{x}$$

Let $\Delta > 0$ be such that

$$\inf_{x>\Delta} \big(\varphi(x) - \psi(x)\big) > 0,$$

then $\varphi(x) \geq \psi(x)$ for every $x \in (\Delta, \infty)$ and

$$\widehat{a}_\varphi = \liminf_{x\to\infty} \frac{\varphi(x)}{x} \geq \liminf_{x\to\infty} \frac{\psi(x)}{x} = \widehat{a}_\psi.$$

The proof proceeds with separate consideration of several cases:

Case 1: $\widehat{a}_\varphi > \widehat{a}_\psi$.

Let a_1, $a_2 \in \mathbb{R}$ be such that

$$\widehat{a}_\varphi > a_1 > a_2 > \widehat{a}_\psi.$$

We choose $\Delta < \Delta_1$ such that

$$\frac{\varphi^{**}(x)}{x} > a_1 > a_2 > \frac{\psi(x)}{x}, \quad x \in (\Delta_1, \infty).$$

Hence, $\varphi^{**}(x) - \psi(x) > (a_1 - a_2)x$, $x \in (\Delta_1, \infty)$, and

$$\lim_{x\to\infty} \big(\varphi^{**}(x) - \psi(x)\big) = \infty\,.$$

Thus Case 1 is settled.

Case 2: $\widehat{a}_\varphi = \widehat{a}_\psi$. This case is split into three subcases.

Case 2.1: $\varphi \in \Phi_3$. We make the following observations:

- Lemma 3.2 implies that $\varphi^{**} \in \Phi_3$.
- $\psi \in \Phi_2$ and since ψ is convex, we have

$$\liminf_{x\to\infty} \big(\psi(x) - \widehat{a}_\psi x\big) = \lim_{x\to\infty} \big(\psi(x) - \widehat{a}_\psi x\big) = -\infty\,.$$

We claim that

$$\inf_{x>0} \big(\varphi^{**}(x) - \widehat{a}_\varphi x\big) > -\infty. \tag{5.2}$$

Indeed, let us choose the real numbers b, Δ_2, a_0 and b_0 in the following way:

– b is such that

$$\liminf_{x\to\infty}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big)>b\,;$$

– $\Delta_2>0$ is such that

$$\inf_{x>\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big)>b\,;$$

– a_0 and b_0 are such that $(a_0,b_0)\in M_\varphi$ and therefore

$$a_0x+b_0\le\varphi^{**}(x)\,,\qquad x\in(0,\infty).$$

We have

$$\begin{aligned}\inf_{x>0}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big)&=\min\Big\{\inf_{0<x\le\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big);\ \inf_{x>\Delta_2}\big(\varphi^{**}(x)-\widehat{a}_\varphi x\big)\Big\}\\&\ge\min\Big\{\inf_{0<x\le\Delta_2}\big(a_0x+b_0-\widehat{a}_\varphi x\big);\ b\Big\}>-\infty\end{aligned}$$

and claim (5.2) is proved. Now,

$$\begin{aligned}\liminf_{x\to\infty}\big(\varphi^{**}(x)-\psi(x)\big)&=\liminf_{x\to\infty}\Big((\varphi^{**}(x)-\widehat{a}_\varphi x)+(\widehat{a}_\psi x-\psi(x))\Big)\\&\ge\inf_{x>0}(\varphi^{**}(x)-\widehat{a}_\varphi x)+\liminf_{x\to\infty}(\widehat{a}_\psi x-\psi(x))\\&=\inf_{x>0}(\varphi^{**}(x)-\widehat{a}_\varphi x)+\lim_{x\to\infty}(\widehat{a}_\psi x-\psi(x))=\infty\end{aligned}$$

Case 2.1 is settled.

Case 2.2: $\varphi\in\Phi_2$ and *Case 2.3:* $\varphi\in\Phi_1$.

Let $c\in\mathbb{R}$, $\Delta>0$ satisfy $\inf_{x>\Delta}(\varphi(x)-\psi(x))>c$, and a_0, b_0 be such that $(a_0,b_0)\in M_\varphi$. In the present cases, the assumptions imply that $\varphi\in\Phi_2$ ($\varphi\in\Phi_1$) and $\widehat{a}_\varphi>a_0$. So, $\widehat{a}_\psi=\widehat{a}_\varphi>a_0$, $\psi\in\Phi_2$ ($\psi\in\Phi_1$), and by Lemma 4.3,

$$\lim_{x\to\infty}(\psi(x)-a_0x)=\infty.$$

Let $\Delta_1>\Delta$ be such that

$$\inf_{x>\Delta_1}\big(\psi(x)+c-(a_0x+b_0)\big)>0.$$

Since $\psi\in\Phi_2$ ($\psi\in\Phi_1$) is a convex function, we have

$$\psi'(x^-)\le\psi'(x^+)<\widehat{a}_\psi,\qquad x>0.$$

For a fixed x' and $\infty>x>x'>0$ we have

$$\begin{aligned}&\frac{\psi(x')-\psi(x)}{x'-x}\le\lim_{t\to x^-}\frac{\psi(t)-\psi(x)}{t-x}=\psi'(x^-)<\widehat{a}_\psi\\\Longrightarrow\ &\lim_{x\to\infty}\frac{\frac{\psi(x)}{x}-\frac{\psi(x')}{x}}{1-\frac{x'}{x}}=\lim_{x\to\infty}\frac{\psi(x')-\psi(x)}{x'-x}\le\lim_{x\to\infty}\psi'(x^-)\le\widehat{a}_\psi\,,\end{aligned}$$

therefore

$$\lim_{x\to\infty} \psi'(x^-) = \widehat{a}_\psi\,. \tag{5.3}$$

Let $\Delta_2 > \Delta_1$ be such that

$$\inf_{x>\Delta_2} \psi'(x^-) > a_0.$$

We claim that there exists $\Delta_3 > \Delta_2$ such that

$$\psi'(x^-)(\Delta - x) + \psi(x) + c < a_0\Delta + b_0, \qquad x > \Delta_3. \tag{5.4}$$

The arguments for the proof of this claim in *Case 2.2* and *Case 2.3* are different.

In *Case 2.2* we have $\widehat{a}_\psi < \infty$, and Corollary 4.3 applied to ψ imply

$$\lim_{x\to\infty} (\psi(x) - \psi'(x^-)x) = -\infty\,.$$

Therefore by (5.3) we obtain

$$\lim_{x\to\infty} \big(\psi'(x^-)(\Delta - x) + \psi(x) + c\big) = -\infty$$

On the other hand, in *Case 2.3* we have $\lim\limits_{x\to\infty} \psi'(x^-) = \widehat{a}_\psi = \infty$ and

$$\begin{aligned}\psi'(x^-)(\Delta - x) + \psi(x) + c &\le \psi'(x^-)(2\Delta - x) + \psi(x) + c - \psi'(x^-)\Delta\\ &\le \psi(2\Delta) + c - \psi'(x^-)\Delta\,,\end{aligned}$$

since $\psi'(x^-)(t - x) + \psi(x) \le \psi(t)$ for x, $t > 0$. Hence,

$$\lim_{x\to\infty} (\psi'(x^-)(\Delta - x) + \psi(x) + c) = -\infty\,.$$

Thus (5.4) is proved and let $\Delta_3 > \Delta_2$ be such that (5.4) is fulfilled. For $x_1 > \Delta_3$ we set

$$a_1 = \psi'(x_1^-), \quad b_1 = -\psi'(x_1^-)x_1 + \psi(x_1) + c.$$

Note that $a_1 > a_0$. Then

$$\begin{aligned}&a_1x + b_1 \le \psi(x) + c, \quad \forall x \in (0,\infty)\,,\\ &a_1x_1 + b_1 = \psi(x_1) + c \ge a_0x_1 + b_0\,,\\ &a_1\Delta + b_1 < a_0\Delta + b_0\,.\end{aligned}$$

We choose $x_2 \in (\Delta, x_1]$ so that

$$a_1x_2 + b_1 = a_0x_2 + b_0,$$

and define a function $\widetilde{\psi} : (0,\infty) \to \mathbb{R}$ as follows:

$$\widetilde{\psi}(x) = \begin{cases} a_0x + b_0, & x \in (0, x_2]\,,\\ a_1x + b_1, & x \in (x_2, x_1]\,,\\ \psi(x) + c, & x \in (x_1, \infty)\,. \end{cases}$$

Notice that $\widetilde{\psi}$ is convex on $(0,\infty)$, because it is continuous, $a_0 \le a_1 \le \psi'(x_1^-)$ and $\psi + c$ is convex on (x_1,∞). Moreover, $\widetilde{\psi}(x) \le \varphi(x)$ for every $x \in (0,\infty)$. Therefore, $\widetilde{\psi}(x) \le \varphi^{**}(x)$, $x \in (0,\infty)$.

Thus for $x > x_1$ we have $\psi(x) + c \le \varphi^{**}(x)$ and

$$\liminf_{x\to\infty}\big(\varphi^{**}(x) - \psi(x)\big) \ge c\,.$$

It follows from our choice of c that

$$\lim_{x\to\infty}\big(\varphi^{**}(x) - \psi(x)\big) = \infty. \qquad \square$$

6. APPLICATION

In this section we apply Theorems 3.1, 3.2 and 3.3 to the theory of spaces $H_v(G)$ and $H_{v_0}(G)$.

We make use of the following notation:

$$Mf(y) = \sup_{x\in(-\infty,\infty)} |f(x+iy)|,$$
$$\psi_f(y) = \ln Mf(y), \qquad \forall y > 0, f \in \Lambda(p)\,,$$

where f is a holomorphic function defined on the upper half plane G.

Note that

$$-\ln \| f \|_v = \inf_{y>0}\big(\varphi_v(y) - \psi_f(y)\big)$$

Here we reformulate our results from [6].

Theorem A. [6, Th. 1.2] *If φ satisfies condition (1.1′), then*

$$H_v(G) \ne \{0\} \iff \varphi \in \Phi\,,$$

where $v = e^{-\varphi}$.

Theorem B. [6, Th. 1.3] *If φ satisfies condition (1.1′), then*

$$H_{v_0}(G) \ne \{0\} \iff \left|\begin{array}{l}\varphi \in \Phi, \\ \varphi(0^+) = \infty\,,\end{array}\right.$$

where $v = e^{-\varphi}$.

Theorem C [6, Th. 1.4] *If φ satisfies condition (1.1′) and $H_{v_0}(G) \ne \{0\}$, then*

$$\psi_f \in \Phi \setminus \Phi_3 \ \textit{ for every } \ f \in H_{v_0}(G) \setminus \{0\},$$

where $v = e^{-\varphi}$.

Note that ψ_f is convex on $(0,\infty)$ and $\psi_f \in \Phi$, $\forall f \in H_v(G) \setminus \{0\}$.

In this section we prove two new theorems.

Theorem 6.1. *If φ satisfies condition (1.1′) and $\varphi \in \Phi$, then*

$$\big(H_v(G), \|\cdot\|_v\big) \equiv \big(H_w(G), \|\cdot\|_w\big)\,,$$

where $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$.

Proof. Let $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$. The following implications hold:

- $\varphi > \varphi^{**} \implies \varphi^{**}$ satisfies condition (1.1′);
- $\varphi \in \Phi \implies \varphi^{**} \in \Phi$, because $M_{\varphi^{**}} = M_\varphi \neq \emptyset$.

Thus, $H_v(G) \neq \{0\}$ and $H_w(G) \neq \{0\}$, by Theorem A. Moreover, $H_v(G) \supset H_w(G)$, because $\| f \|_v \leq \| f \|_w < \infty$, $\forall f \in H_w(G)$.

Note that for every $f \in H_v(G) \neq \{0\}$ the function $\psi_f = \ln Mf$ is convex on $(0,\infty)$ and $\psi_f \in \Phi$. Therefore, by Theorem 3.1,

$$\inf_{x\in(0,\infty)} \big(\varphi(x) - \psi_f(x)\big) = \inf_{x\in(0,\infty)} \big(\varphi^{**}(x) - \psi_f(x)\big)$$

Thus $f \in H_w(G) \neq \{0\}$ and $\| f \|_v = \| f \|_w$. □

Theorem 6.2. *If φ satisfies condition (1.1′), $\varphi \in \Phi$ and $\varphi(0^+) = \infty$, then*

$$\big(H_{v_0}(G), \|\cdot\|_v\big) \equiv \big(H_{w_0}(G), \|\cdot\|_w\big)\,,$$

where $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$.

Proof. Let $v = e^{-\varphi}$ and $w = e^{-\varphi^{**}}$. The following implications hold:

- $\varphi > \varphi^{**} \implies \varphi^{**}$ satisfies condition (1.1′);
- $\varphi \in \Phi \implies \varphi^{**} \in \Phi$, since $M_{\varphi^{**}} = M_\varphi \neq \emptyset$;
- $\varphi^{**}(0^+) = \varphi(0^+) = \infty$, by Lemma 4.1 (1).

By Theorem B, $H_{v_0}(G) \neq \{0\}$ and $H_{w_0}(G) \neq \{0\}$. Moreover, $H_{v_0}(G) \supset H_{w_0}(G)$, because of

$$0 \leq v(iy)|f(x+iy)| \leq w(iy)|f(x+iy)|$$

for every $f \in H_{w_0}(G)$ and $x \in (-\infty,\infty)$, $y \in (0;\infty)$.

By Theorem 6.1, $\| f \|_v = \| f \|_w$ for every $f \in H_{v_0}(G) \neq \{0\}$.

We have to prove that $f \in H_{w_0}(G) \neq \{0\}$ for every $f \in H_{v_0}(G) \neq \{0\}$. Let $f \in H_{v_0}(G) \neq \{0\}$. In view of the definition of $H_{v_0}(G)$,

$$\lim_{\mathcal{K}\uparrow G} \sup_{z\in G\setminus\mathcal{K}} v(z)|f(z)| = 0\,,$$

where $\mathcal{K} \subset G$ and $\mathcal{K}$ is compact. So,

$$\lim_{y\to 0^+} v(iy)Mf(y) = 0, \quad \lim_{y\to\infty} v(iy)Mf(y) = 0\,,$$

and after reformulation,

$$\lim_{y\to 0^+} \big(\varphi(y) - \psi_f(y)\big) = \infty, \qquad \lim_{y\to\infty} \big(\varphi(y) - \psi_f(y)\big) = \infty\,.$$

By Theorem C, $\psi_f \in \Phi \setminus \Phi_3$. By Theorem 3.2 and Theorem 3.3 we have

$$\lim_{y\to 0^+} \big(\varphi^{**}(y) - \psi_f(y)\big) = \infty, \qquad \lim_{y\to\infty} \big(\varphi^{**}(y) - \psi_f(y)\big) = \infty\,,$$

i.e.

$$\lim_{y\to 0^+} w(iy)Mf(y) = 0, \quad \lim_{y\to\infty} w(iy)Mf(y) = 0\,.$$

For an arbitrary $\varepsilon > 0$ we choose $c > 1$ such that

$$\sup_{y<\frac{1}{c}} w(iy)Mf(y) < \varepsilon, \quad \sup_{y>c} w(iy)Mf(y) < \varepsilon\,.$$

The quantity

$$m = \frac{\sup\limits_{\frac{1}{c}\le y\le c} w(iy)}{\inf\limits_{\frac{1}{c}\le y\le c} v(iy)}$$

satisfies $m < \infty$, since $\varphi^{**} \in \Phi$ and therefore $\inf_{\frac{1}{c}\le x\le c} \varphi^{**}(x) > -\infty$ for every $c > 1$.

In view of the definition of $H_{v_0}(G)$ there exist $x_1 > 0$, $c_1 > c$ and a compact

$$\mathcal{K}_1 = \{x + iy \,|\, -x_1 \le x \le x_1, \frac{1}{c_1} \le y \le c_1\}$$

satisfying

$$\sup_{x+iy\in G\setminus\mathcal{K}_1} v(iy)|f(x+iy)| \le \frac{\varepsilon}{m}\,.$$

Let $\mathcal{K} = \{x + iy \,|\, -x_1 \le x \le x_1, \frac{1}{c} \le y \le c\}$, then

$$\begin{aligned}
&\sup_{x+iy\in G\setminus\mathcal{K}} w(iy)|f(x+iy)| \\
&\qquad = \max\Big\{ \sup_{y<\frac{1}{c}} w(iy)Mf(y), \sup_{\substack{|x|>x_1,\\ \frac{1}{c}\le y\le c}} w(iy)Mf(y), \sup_{y>c} w(iy)Mf(y) \Big\} \\
&\qquad \le \max\Big\{ \varepsilon, \sup_{\substack{|x|>x_1,\\ \frac{1}{c}\le y\le c}} v(iy)m|f(x+iy)|, \varepsilon \Big\} \le \varepsilon
\end{aligned}$$

and therefore $f \in H_{w_0}(G)$. □

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