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ALMOST CONTACT B-METRIC STRUCTURES AND THE BIANCHI CLASSIFICATION OF THE THREE-DIMENSIONAL LIE ALGEBRAS

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The object of investigation are the almost contact manifolds with B-metric in the lowest dimension three, constructed on Lie algebras. It is considered a relation between the classes in the Bianchi classification of three-dimensional real Lie algebras and the classes of a classification of the considered manifolds. There are studied some geometrical characteristics in some special classes.

Keywords: Almost contact structure, B-metric, Lie group, Lie algebra, indefinite metric

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1. INTRODUCTION

The differential geometry of the manifolds equipped with an almost contact structure is well studied (see, e.g. [3]). The almost contact manifolds with B-metric are introduced and classified in [6]. These manifolds are the odd-dimensional counterpart of the almost complex manifolds with Norden metric [5, 7].

An object of special interest is the case of the lowest dimension of the considered manifolds. We investigate the almost contact B-metric manifolds in dimension three and get explicit results. Some curvature identities of the three-dimensional manifolds of this type are studied in [11, 12].

Almost contact manifolds with B-metric can be constructed on Lie algebras. It is known that all three-dimensional real Lie algebras are classified in [1, 2]. The main goal of this paper is to find a relation between the classes in the Bianchi classification and the classification of almost contact B-metric manifolds given in [6]. Moreover, the present work gives some geometrical characteristics of the considered manifolds in certain special classes.

The paper is organized as follows. In Section 2 we recall some preliminary facts about the almost contact B-metric manifolds. In Section 3 we equip each Bianchi-type Lie algebra with an almost contact B-metric structure. In Section 4 we give the relation between the Bianchi classification and the classification given in [6]. Section 5 is devoted to the curvature properties of some of the considered manifolds.

2. PRELIMINARIES

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, where M is a $(2n+1)$-dimensional differentiable manifold, (φ,ξ,η) is an almost contact structure consisting of an endomorphism φ of the tangent bundle, a Reeb vector field ξ and its dual contact 1-form η. Moreover, M is equipped with a pseudo-Riemannian metric g, called a *B-metric*, such that the following algebraic relations are satisfied [6]:

$$\varphi\xi = 0, \qquad \varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \qquad \eta\circ\varphi = 0, \qquad \eta(\xi) = 1,$$
$$g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y),$$

where Id is the identity. In the latter equalities and further, x, y, z, w will stand for arbitrary elements of the algebra of the smooth vector fields on M or vectors in the tangent space T_pM of M at an arbitrary point p in M.

The associated B-metric $\tilde{g}$ of g is determined by $\tilde{g}(x,y) = g(x,\varphi y) + \eta(x)\eta(y)$. The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is also an almost contact B-metric manifold. The signature of both metrics g and $\tilde{g}$ is necessarily $(n+1,n)$. We denote the Levi-Civita connection of g and $\tilde{g}$ by ∇ and $\tilde{\nabla}$, respectively.

A classification of almost contact B-metric manifolds, consisting of eleven basic classes $\mathcal{F}_1$, $\mathcal{F}_2$, …, $\mathcal{F}_{11}$, is given in [6]. This classification is made with respect to the tensor F of type (0,3) defined by

$$F(x,y,z) = g\big((\nabla_x\varphi)\,y, z\big) \tag{2.1}$$

and having the following properties:

$$F(x,y,z) = F(x,z,y) = F(x,\varphi y,\varphi z) + \eta(y)F(x,\xi,z) + \eta(z)F(x,y,\xi).$$

The special class determined by the condition $F(x,y,z) = 0$ is denoted by $\mathcal{F}_0$. This class is the intersection of all the basic classes. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi = \nabla\xi = \nabla\eta = \nabla g = \nabla\tilde{g} = 0$. Therefore $\mathcal{F}_0$ is the class of the *cosymplectic manifolds with B-metric.*

According to [10], the *square norm of* $\nabla\varphi$ is defined by:

$$\|\nabla\varphi\|^2 = g^{ij}g^{ks}g\big((\nabla_{e_i}\varphi)\, e_k, (\nabla_{e_j}\varphi)\, e_s\big). \tag{2.2}$$

It is clear that $\|\nabla\varphi\|^2 = 0$ is valid if $(M, \varphi, \xi, \eta, g)$ is a cosymplectic manifold with B-metric, but the inverse implication is not always true. An almost contact B-metric manifold having a zero square norm of $\nabla\varphi$ is called an *isotropic-cosymplectic B-metric manifold.*

If $\{e_i; \xi\}$ $(i = 1, 2, \ldots, 2n)$ is a basis of T_pM and $\left(g^{ij}\right)$ is the inverse matrix of (g_{ij}), then the 1-forms θ, θ^*, ω, called *Lee forms*, are associated with F and defined by:

$$\theta(z) = g^{ij}F(e_i, e_j, z), \quad \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z).$$

Let now consider the case of the lowest dimension of the almost contact B-metric manifold M, i.e. $\dim M = 3$.

We introduce an almost contact structure (φ, ξ, η) on M defined by

$$\begin{gathered} \varphi e_1 = e_2, \quad \varphi e_2 = -e_1, \quad \varphi e_3 = 0, \quad \xi = e_3, \\ \eta(e_1) = \eta(e_2) = 0, \quad \eta(e_3) = 1 \end{gathered} \tag{2.3}$$

and a B-metric g such that

$$g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_i, e_j) = 0, \;\; i \neq j \in \{1, 2, 3\}. \tag{2.4}$$

Let us denote the components $F_{ijk} = F(e_i, e_j, e_k)$ of F with respect to a φ*-basis* $\{e_1, e_2, e_3\}$ of T_pM.

According to [8], the components of the Lee forms are

$$\begin{array}{lll} \theta_1 = F_{111} - F_{221}, & \theta_2 = F_{112} - F_{211}, & \theta_3 = F_{113} - F_{223}, \\ \theta^*_1 = F_{112} + F_{211}, & \theta^*_2 = F_{111} + F_{221}, & \theta^*_3 = F_{123} + F_{213}, \\ \omega_1 = F_{331}, & \omega_2 = F_{332}, & \omega_3 = 0. \end{array}$$

Then, if F_s $(s = 1, 2, \ldots, 11)$ are the components of F in the corresponding basic classes $\mathcal{F}_s$ and $x = x^ie_i$, $y = y^je_j$, $z = z^ke_k$ for arbitrary vectors in T_pM, we have [8]:

$$\begin{aligned} &F_1(x, y, z) = \left(x^1\theta_1 - x^2\theta_2\right)\left(y^1z^1 + y^2z^2\right), \\ &\qquad \theta_1 = F_{111} = F_{122}, \qquad \theta_2 = -F_{211} = -F_{222}; \\ &F_2(x, y, z) = F_3(x, y, z) = 0; \\ &F_4(x, y, z) = \frac{1}{2}\theta_3\Big\{x^1\left(y^3z^1 + y^1z^3\right) - x^2\left(y^3z^2 + y^2z^3\right)\Big\}, \\ &\qquad \frac{1}{2}\theta_3 = F_{131} = F_{113} = -F_{232} = -F_{223}; \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&F_5(x,y,z) = \frac{1}{2}\theta_3^*\left\{x^1\left(y^3z^2+y^2z^3\right)+x^2\left(y^3z^1+y^1z^3\right)\right\},\\
&\qquad \frac{1}{2}\theta_3^* = F_{132} = F_{123} = F_{231} = F_{213};\\
&F_6(x,y,z) = F_7(x,y,z) = 0;\\
&F_8(x,y,z) = \lambda\left\{x^1\left(y^3z^1+y^1z^3\right)+x^2\left(y^3z^2+y^2z^3\right)\right\},\\
&\qquad \lambda = F_{131} = F_{113} = F_{232} = F_{223};\\
&F_9(x,y,z) = \mu\left\{x^1\left(y^3z^2+y^2z^3\right)-x^2\left(y^3z^1+y^1z^3\right)\right\},\\
&\qquad \mu = F_{132} = F_{123} = -F_{231} = -F_{213};\\
&F_{10}(x,y,z) = \nu x^3\left(y^1z^1+y^2z^2\right), \qquad \nu = F_{311} = F_{322};\\
&F_{11}(x,y,z) = x^3\left\{\left(y^1z^3+y^3z^1\right)\omega_1+\left(y^2z^3+y^3z^2\right)\omega_2\right\},\\
&\qquad \omega_1 = F_{313} = F_{331}, \qquad \omega_2 = F_{323} = F_{332}.
\end{aligned}
\tag{2.6}
$$

Obviously, the class of three-dimensional almost contact B-metric manifolds is

$$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_8 \oplus \mathcal{F}_9 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}.$$

Let $R = [\nabla,\nabla] - \nabla_{[\,,\,]}$ be the curvature (1,3)-tensor of ∇. The corresponding curvature $(0,4)$-tensor is denoted by the same letter: $R(x,y,z,w) = g(R(x,y)z,w)$. The following properties are valid:

$$
\begin{aligned}
&R(x,y,z,w) = -R(y,x,z,w) = -R(x,y,w,z),\\
&R(x,y,z,w) + R(y,z,x,w) + R(z,x,y,w) = 0.
\end{aligned}
$$

It is known from [11] that every 3-dimensional cosymplectic B-metric manifold is flat, i.e. $R = 0$.

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities are defined respectively by

$$
\begin{aligned}
&\rho(y,z) = g^{ij}R(e_i,y,z,e_j), \qquad && \tau = g^{ij}\rho(e_i,e_j),\\
&\rho^*(y,z) = g^{ij}R(e_i,y,z,\varphi e_j), \qquad && \tau^* = g^{ij}\rho^*(e_i,e_j),
\end{aligned}
$$

where $\{e_1,e_2,\dots,e_{2n+1}\}$ is an arbitrary basis of T_pM.

Let α be a non-degenerate 2-plane (section) in T_pM. It is known that the special 2-planes with respect to (φ,ξ,η,g) are: a *totally real section* if α is orthogonal to its φ-image $\varphi\alpha$, a *φ-holomorphic section* if α coincides with $\varphi\alpha$ and a *ξ-section* if ξ lies on α.

The sectional curvature $k(\alpha;p)(R)$ of α with an arbitrary basis $\{x,y\}$ at p is

$$k(\alpha;p)(R) = \frac{R(x,y,y,x)}{g(x,x)g(y,y) - g(x,y)^2}.$$

According to [9], a manifold M whose Ricci tensor satisfies

$$\rho = \lambda g + \mu\tilde{g} + \nu\eta\otimes\eta$$

is said to be an *η-complex-Einstein manifold.*

3. EQUIPPING OF EACH BIANCHI-TYPE LIE ALGEBRA WITH ALMOST CONTACT B-METRIC STRUCTURE

It is known that L. Bianchi has categorized all three-dimensional real (and complex) Lie algebras. He proved that every three-dimensional Lie algebra is isomorphic to one, and only one, Lie algebra of his list (cf. [1, 2]). These isomorphism classes form the so-called Bianchi classification and are noted by Bia(I), Bia(II), Bia(IV), Bia(V), Bia(VI$_h$) ($h \leq 0$), Bia(VII$_h$) ($h \geq 0$), Bia(VIII) and Bia(IX). The class Bia(III) coincides with Bia(VI$_{-1}$). The following theorem introduces the Bianchi classification.

Theorem A. ([1, 2]) *Let $\mathfrak{l}$ be a real three-dimensional Lie algebra. Then $\mathfrak{l}$ is isomorphic to exactly one of the following Lie algebras* $(\mathbb{R}^3, [\cdot,\cdot])$, *where the Lie bracket is given on the canonical basis* $\{e_1, e_2, e_3\}$ *as follows:*

Bia(I) :	$[e_1, e_2] = o,$	$[e_2, e_3] = o,$	$[e_3, e_1] = o;$
Bia(II) :	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = o;$
Bia(IV) :	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - e_2,$	$[e_3, e_1] = e_1;$
Bia(V) :	$[e_1, e_2] = o,$	$[e_2, e_3] = e_2,$	$[e_3, e_1] = e_1;$
Bia(VI$_h$) ($h \leq 0$) :	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 - e_2;$
Bia(VII$_h$) ($h \geq 0$) :	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 + e_2;$
Bia(VIII) :	$[e_1, e_2] = -e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2;$
Bia(IX) :	$[e_1, e_2] = e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2.$

Here, o is the zero vector in $\mathfrak{l}$.

The geometrization conjecture, associated with W. Thurston, states that every closed manifold of dimension three could be decomposed in a canonical way into pieces, connected to one of the eight types of Thurston's geometric structures ([13]): Euclidean geometry E^3, Spherical geometry S^3, Hyperbolic geometry H^3, the geometry of $S^2 \times \mathbb{R}$, the geometry of $H^2 \times \mathbb{R}$, the geometry of the universal cover $\widetilde{SL}(2,\mathbb{R})$ of the special linear group $SL(2,\mathbb{R})$, the Nil geometry, the $Solv$ geometry.

Seven of the eight Thurston geometries can be associated to a class of the Bianchi classification as it is shown in the following table. The Thurston geometry on $S^2 \times \mathbb{R}$ has no such a realization (see, e.g., [4]).

TABLE 1. Relations between the Bianchi types and the Thurston geometries

Bia(I)	E^3
Bia(II)	Nil
Bia(III)	$H^2 \times \mathbb{R}$
Bia(IV)	
Bia(V)	H^3
Bia(VI$_0$)	$Solv$

Bia(VI$_{h<0}$)	
Bia(VII$_0$)	E^3
Bia(VII$_{h>0}$)	
Bia(VIII)	$\widetilde{SL}(2,\mathbb{R})$
Bia(IX)	S^3

Let us consider each Lie algebra from the Bianchi classification, equipped with an almost contact structure (φ, ξ, η) and a B-metric g as in (2.3) and (2.4).

The presence of the structure (φ, ξ, η, g) gives us a reason to consider the relation between the Bianchi types and the classification of almost contact B-metric manifolds in [6].

We obtain immediately the following

Proposition 3.1. *Some Bianchi types can be equipped with a structure (φ, ξ, η, g) in several ways. In the cases* Bia(I) *and* Bia(IX) *there is only one variant. In the remaining cases, there are three possible subtypes of each type, obtained from each other by a cyclic change of the basic vectors e_1, e_2 and e_3. All subtypes are given in Table 2:*

TABLE 2. Equipping of the Bianchi types Lie algebras with a (φ, ξ, η, g) structure

Bia(I)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = o,$	$[e_3, e_1] = o$
Bia(II)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = o$
(2)	$[e_1, e_2] = o,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2$
(3)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = o$
Bia(III) $\equiv$ Bia(VI_{-1})			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 + e_2,$	$[e_3, e_1] = -e_1 - e_2$
(2)	$[e_1, e_2] = -e_2 - e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 + e_3$
(3)	$[e_1, e_2] = e_1 + e_3,$	$[e_2, e_3] = -e_1 - e_3,$	$[e_3, e_1] = o$
Bia(IV)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - e_2,$	$[e_3, e_1] = e_1$
(2)	$[e_1, e_2] = e_2,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - e_3$
(3)	$[e_1, e_2] = -e_1 + e_3,$	$[e_2, e_3] = e_3,$	$[e_3, e_1] = o$
Bia(V)			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_2,$	$[e_3, e_1] = e_1$
(2)	$[e_1, e_2] = e_2,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_3$
(3)	$[e_1, e_2] = e_1,$	$[e_2, e_3] = e_3,$	$[e_3, e_1] = o$
Bia(VI_h), $h \leq 0$			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 - e_2$
(2)	$[e_1, e_2] = he_2 - e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - he_3$
(3)	$[e_1, e_2] = -he_1 + e_3,$	$[e_2, e_3] = -e_1 + he_3,$	$[e_3, e_1] = o$
Bia(VII_h), $h \geq 0$			
(1)	$[e_1, e_2] = o,$	$[e_2, e_3] = e_1 - he_2,$	$[e_3, e_1] = he_1 + e_2$
(2)	$[e_1, e_2] = he_2 + e_3,$	$[e_2, e_3] = o,$	$[e_3, e_1] = e_2 - he_3$
(3)	$[e_1, e_2] = -he_1 + e_3,$	$[e_2, e_3] = e_1 + he_3,$	$[e_3, e_1] = o$
Bia(VIII)			
(1)	$[e_1, e_2] = -e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2$
(2)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = -e_1,$	$[e_3, e_1] = e_2$
(3)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = -e_2$
Bia(IX)			
(1)	$[e_1, e_2] = e_3,$	$[e_2, e_3] = e_1,$	$[e_3, e_1] = e_2$

4. ALMOST CONTACT B-METRIC MANIFOLDS OF EACH BIANCHI TYPE

Let us consider the Lie group L corresponding to the given Lie algebra $\mathfrak{l}$. Each definition of a Lie algebra for the different subtypes in Proposition 3.1 generates a corresponding almost contact B-metric manifold denoted by $(L, \varphi, \xi, \eta, g)$. In this section we characterize the obtained manifolds with respect to the classification in [6].

Using (2.5)–(2.6), we obtain the corresponding components of F in each subtypes (1), (2), (3) in Proposition 3.1 and determine the corresponding class of almost contact B-metric manifolds. The results are given in the following

Theorem 4.1. *The manifold $(L, \varphi, \xi, \eta, g)$, determined by each type of Lie algebra given in Proposition 3.1, belongs to a class from the classification in* [6] *as given in Table 3:*

TABLE 3. Relations between the Bianchi types and the classes in [6]

Bia(I)	
(1)	$\mathcal{F}_0$
Bia(II)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
Bia(III)	
(1)	$\mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(IV)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(V)	
(1)	$\mathcal{F}_9$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_{11}$
Bia(VI$_0$)	
(1)	$\mathcal{F}_{10}$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_8$
(3)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$

Bia(VI$_h$), $h < 0$	
(1)	$\mathcal{F}_5 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
Bia(VII$_0$)	
(1)	$\mathcal{F}_4$
(2)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_4 \oplus \mathcal{F}_8$
Bia(VII$_h$), $h > 0$	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_5$
(2)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10} \oplus \mathcal{F}_{11}$
(3)	$\mathcal{F}_1 \oplus \mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{11}$
Bia(VIII)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$
(2)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
(3)	$\mathcal{F}_8 \oplus \mathcal{F}_{10}$
Bia(IX)	
(1)	$\mathcal{F}_4 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$

Proof. We give our arguments for the case of Bia(II), the other cases are proven in a similar way.

Using Theorem A, Eq. (2.4) and the Koszul equality

$$2g\left(\nabla_{e_i} e_j, e_k\right) = g\left([e_i, e_j], e_k\right) + g\left([e_k, e_i], e_j\right) + g\left([e_k, e_j], e_i\right),$$

we obtain the components of the Levi-Civita connection ∇ of g. Then, by them, (2.1) and (2.3), we get the following non-zero components F_{ijk} and θ_k for the different subtypes:

$$\begin{array}{ll} (1) & F_{113} = F_{131} = -F_{223} = -F_{232} = -\frac{1}{2}, \quad F_{311} = F_{322} = -1, \quad \theta_3 = -1; \\ (2) & F_{113} = F_{131} = -F_{223} = -F_{232} = -\frac{1}{2}, \quad F_{311} = F_{322} = 1, \quad \theta_3 = -1; \\ (3) & F_{113} = F_{131} = F_{223} = F_{232} = \frac{1}{2}, \quad F_{311} = F_{322} = 1. \end{array}$$

Bearing in mind (2.5)–(2.6), we conclude that the corresponding classes of each subtype of Bia(II) are as follows:

$$\begin{array}{ll} (1) & (L, \varphi, \xi, \eta, g) \in \mathcal{F}_4 \oplus \mathcal{F}_{10}; \\ (2) & (L, \varphi, \xi, \eta, g) \in \mathcal{F}_4 \oplus \mathcal{F}_{10}; \\ (3) & (L, \varphi, \xi, \eta, g) \in \mathcal{F}_8 \oplus \mathcal{F}_{10}. \end{array}$$

□

5. CURVATURE PROPERTIES OF THE CONSIDERED MANIFOLDS IN SOME BIANCHI CLASSES

Now we focuss our considerations on the Bianchi classes depending on a real parameter h. They are Bia(VI_h) and Bia(VII_h). Actually, these two classes are families of manifolds whose properties are functions of h. The classes regarding F corresponding to Bia(VI_h), $h < 0$ and Bia(VII_h), $h > 0$, according to Theorem 4.1, can not be restricted for special values of h.

In this section our interest is in the curvature properties of these manifolds in terms of h.

In view of Proposition 3.1, it is reasonable to investigate all three subtypes of the Bianchi classes Bia(VI_h), $h \leq 0$ and Bia(VII_h), $h \geq 0$.

5.1. Bia(VI_h), $h \leq 0$.

Let us consider subtype (1) of this Bianchi class as given in Proposition 3.1:

$$[e_1, e_2] = o, \quad [e_2, e_3] = e_1 - he_2, \quad [e_3, e_1] = he_1 - e_2.$$

We calculate the non-zero components of ∇ for Bia(VI_h):

$$\begin{array}{lll} \nabla_{e_1} e_1 = he_3, & \nabla_{e_1} e_3 = -he_1, & \nabla_{e_2} e_2 = -he_3, \\ \nabla_{e_2} e_3 = -he_2, & \nabla_{e_3} e_1 = -e_2, & \nabla_{e_3} e_2 = -e_1. \end{array} \tag{5.1}$$

Using (2.2), (2.3), (2.4) and (5.1), we obtain for the square norm of $\nabla\varphi$

$$\|\nabla\varphi\|^2 = 4(2 - h^2). \tag{5.2}$$

Further, we calculate the basic components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ of the curvature tensor R, $\rho_{jk} = \rho(e_j, e_k)$ of the Ricci tensor ρ, $\rho^*_{jk} = \rho^*(e_j, e_k)$ of the associated Ricci tensor ρ^*, the values of the scalar curvatures τ and τ^* and of the sectional curvatures $k_{ij} = k(e_i, e_j)$. They are as follows:

$$\begin{gathered} R_{1212} = -R_{1313} = R_{2323} = -h^2; \\ \rho_{11} = -\rho_{22} = \rho_{33} = -2h^2, \qquad \rho^*_{12} = \rho^*_{21} = -h^2; \\ \tau = -6h^2, \qquad \tau^* = 0; \\ k_{12} = k_{13} = k_{23} = -h^2. \end{gathered} \tag{5.3}$$

Using (5.3) we obtain the following

Proposition 5.1. *In the case* Bia(VI$_h$)*, subtype (1), the following statements are valid:*

1). *$(L, \varphi, \xi, \eta, g)$ is flat if and only if $h = 0$;*

2). *$(L, \varphi, \xi, \eta, g)$ is an isotropic-cosymplectic B-metric manifold if and only if $h = -\sqrt{2}$;*

3). *The scalar curvature and the sectional curvatures are constant and non-positive;*

4). *$(L, \varphi, \xi, \eta, g)$ is $*$-scalar flat, i.e. $\tau^* = 0$;*

5). *$(L, \varphi, \xi, \eta, g)$ is an Einstein manifold.*

In the same fashion we obtain the analogues of (5.2) and (5.3) and derive the corresponding propositions in the remaining cases. For subtype (2) we have:

$$\begin{gathered} \|\nabla\varphi\|^2 = 2(1 - 5h^2); \\ R_{1212} = -R_{1313} = R_{2323} = -h^2; \\ \rho_{11} = -\rho_{22} = \rho_{33} = -2h^2, \qquad \rho^*_{12} = \rho^*_{21} = -h^2; \\ \tau = -6h^2, \qquad \tau^* = 0; \\ k_{12} = k_{13} = k_{23} = -h^2, \end{gathered}$$

whence we deduce the following

Proposition 5.2. *In the case* Bia*(VI$_h$), subtype (2), all the statements from Proposition 5.1 hold true, with $h = -\sqrt{2}$ replaced by $h = -\frac{\sqrt{5}}{5}$ in statement 2).*

In the case of subtype (3) we obtain:

$$\begin{gathered} \|\nabla\varphi\|^2 = 10(h^2 + 1); \\ R_{1212} = R_{2323} = h^2 + 1, \qquad R_{1313} = 1 - h^2, \qquad R_{1223} = 2h; \\ \rho_{11} = \rho_{33} = 2h^2, \qquad \rho_{13} = \rho_{31} = -2h, \qquad \rho_{22} = -2(h^2 + 1); \\ \rho^*_{12} = \rho^*_{21} = h^2 + 1, \qquad \rho^*_{23} = \rho^*_{32} = -2h; \\ \tau = 2(3h^2 + 1), \qquad \tau^* = 0; \\ k_{12} = k_{23} = h^2 + 1, \qquad k_{13} = h^2 - 1. \end{gathered}$$

The latter equalities imply

Proposition 5.3. *In the case* Bia*(VI_h), subtype (3), the following statements are valid:*

1). The square norm of $\nabla\varphi$ and the scalar curvature are positive;

2). (L,φ,ξ,η,g) is $*$-scalar flat;

3). The sectional curvatures of the φ-holomorphic sections are constant and positive.

5.2. $\mathrm{Bia}(\mathrm{VII}_h),\ h\geq 0.$

Here we focus on the three subtypes of $\mathrm{Bia}(\mathrm{VII}_h)$. Firstly, let us consider the subtype (1). As in the previous subsection, we find:

$$\begin{gathered}
\|\nabla\varphi\|^2 = 4(1-h^2);\\
R_{1212}=-(h^2+1),\qquad R_{1313}=-R_{2323}=h^2-1,\qquad R_{1323}=-2h;\\
\rho_{11}=-\rho_{22}=-2h^2,\qquad \rho_{12}=\rho_{21}=2h,\qquad \rho_{33}=2(1-h^2);\\
\rho^*_{12}=\rho^*_{21}=-(h^2+1),\qquad \rho^*_{33}=4h;\\
\tau=2(1-3h^2),\qquad \tau^*=4h;\\
k_{12}=-(h^2+1),\qquad k_{13}=k_{23}=1-h^2.
\end{gathered}$$

Applying these results we obtain

Proposition 5.4. *In the case* $\mathrm{Bia}(\mathrm{VII}_h)$*, subtype (1), the following statements are valid:*

1). (L,φ,ξ,η,g) is an isotropic-cosymplectic B-metric manifold if and only if $h=1$;

2). (L,φ,ξ,η,g) is scalar flat if and only if $h=\frac{\sqrt{3}}{3}$;

3). (L,φ,ξ,η,g) is $*$-scalar flat if and only if $h=0$;

4). The sectional curvatures of the φ-holomorphic sections are constant and negative;

5). The sectional curvatures of the ξ-sections are constant;

6). (L,φ,ξ,η,g) is an η-complex-Einstein manifold.

Analogously, we get the corresponding results for subtype (2):

$$\begin{gathered}
\|\nabla\varphi\|^2 = -10(h^2-1);\\
R_{1212}=-R_{1313}=-(h^2-1),\qquad R_{2323}=-(h^2+1),\qquad R_{1213}=2h;\\
\rho_{11}=-2(h^2-1),\qquad \rho_{22}=-\rho_{33}=2h^2,\qquad \rho_{23}=\rho_{32}=-2h;\\
\rho^*_{12}=\rho^*_{21}=-(h^2-1),\qquad \rho^*_{13}=\rho^*_{31}=2h;\\
\tau=-2(3h^2-1),\qquad \tau^*=0;\\
k_{12}=k_{13}=-(h^2-1),\qquad k_{23}=-(h^2+1).
\end{gathered}$$

The latter equalities imply the following

Proposition 5.5. *In the case* Bia*(VII_h), subtype (2), the following statements are valid:*

1). $(L, \varphi, \xi, \eta, g)$ *is an isotropic-cosymplectic B-metric manifold if and only if* $h = 1$*;*

2). $(L, \varphi, \xi, \eta, g)$ *is scalar flat if and only if* $h = \frac{\sqrt{3}}{3}$*;*

3). $(L, \varphi, \xi, \eta, g)$ *is* $*$*-scalar flat;*

4). $(L, \varphi, \xi, \eta, g)$ *is horizontal flat, i.e.* $R|_H = 0$ *for* $H = \ker(\eta)$*, if and only if* $h = 1$*;*

5). ρ^* *and* $\tilde{g}$ *are proportional on* H *as* $\rho^*|_H = (h^2 - 1)\tilde{g}|_H$*;*

6). $(L, \varphi, \xi, \eta, g)$ *is horizontal* $*$*-Ricci flat, i.e.* $\rho^*|_H = 0$*, if and only if* $h = 1$*.*

Finally, for the case of the subtype (3) we have:

$$
\begin{gathered}
\|\nabla\varphi\|^2 = 2(5h^2 + 1); \\
R_{1212} = -R_{1313} = R_{2323} = h^2; \\
\rho_{11} = -\rho_{22} = \rho_{33} = 2h^2; \\
\rho^*_{12} = \rho^*_{21} = h^2; \\
\tau = 6h^2, \qquad \tau^* = 0; \\
k_{12} = k_{13} = k_{23} = h^2,
\end{gathered}
$$

whence we deduce our last proposition:

Proposition 5.6. *In the case* Bia*(VII_h), subtype (3), the following statements are valid:*

1). $(L, \varphi, \xi, \eta, g)$ *is flat if and only if* $h = 0$*;*

2). *The square norm of* $\nabla\varphi$ *is positive;*

3). $(L, \varphi, \xi, \eta, g)$ *is* $*$*-scalar flat;*

4). *The scalar curvature and the sectional curvatures are constant and non-negative;*

5). $(L, \varphi, \xi, \eta, g)$ *is an Einstein manifold.*

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