ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 102

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 102

ON VECTOR–PARAMETER FORM OF THE SU(2) → SO(3, ℝ) MAP

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By making use of the *Cayley* maps for the isomorphic Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ we have found the vector parameter form of the well-known *Wigner* group homomorphism W : SU(2) → SO(3,ℝ) and its sections. Based on it and pulling back the group multiplication in SO(3,ℝ) through the *Cayley* map $\mathfrak{su}(2) \to$ SU(2) to the covering space, we present the derivation of the explicit formulas for compound rotations. It is shown that both sections are compatible with the group multiplications in SO(3,ℝ) up to a sign and this allows uniform operations with half-turns in the three-dimensional space. The vector parametrization of SU(2) is compared with that of SO(3,ℝ) generated by the *Gibbs* vectors in order to discuss their advantages and disadvantages.

Keywords: Lie groups and algebras, Cayley map, vector-parametrization of rotations
2000 Math. Subject Classification: 15A16, 15A23, 22E60, 22E70

1. INTRODUCTION

Parameterizations are used to describe Lie groups in an easier way. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$. A vector parametrization of G is a map $\mathfrak{g} \to G$, which is diffeomorphic onto its image. Before studying vector parametrizations, let us compare them with the exponential map $\exp : \mathfrak{g} \to G$. It is locally bijective and need not to be such globally. For example in the case of $G = \mathrm{GL}_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ for arbitrary integers $k_1, \ldots, k_n$ the diagonal matrix $\mathrm{diag}(2\pi \mathrm{i} k_1, \ldots, 2\pi \mathrm{i} k_n)$ is transformed into the unit matrix $\mathcal{I}_n$. If G is connected and compact as it is in cases under consideration the exponential map is surjective, see [3]. Besides, the group multiplication $\mu : G \times G \to G$ admits a local pull-back on the Lie algebra level via the commutative diagram (see Fig. 1).

$$\begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \xrightarrow{(1.1)} & \mathfrak{g} \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ G\times G & \xrightarrow{\ \mu\ } & G \end{array}$$

Figure 1: Local pullback of the multiplication law μ for the Lie group G in the corresponding Lie algebra $\mathfrak{g}$.

This pull-back is given by the *Baker–Campbell–Hausdorff* formula in commutator-free form

$$BCH(X,Y) = X + Y + \sum_{n=2}^{\infty} \sum_{|\omega|=n} g_\omega \omega\,, \tag{1.1}$$

where the inner sum is over all the "words" $\omega = \omega_1 \dots \omega_n$ of length n in the alphabet $\{X, Y\}$. Here, g_ω are the Goldberg's rational coefficients [9, 15]. In general, it is difficult to compute (1.1) and there is an ongoing research in this area (see [1, 4, 17]). However, the first few terms of (1.1) in commutator form are given by the formula

$$\begin{aligned} BCH(X,Y) =& X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[Y,X]] - [Y,[X,Y]]) \\ &- \frac{1}{24}[Y,[X,[X,Y]]] + \cdots \end{aligned} \tag{1.2}$$

The image of the parametrization need not be the whole group G. For SO(3,$\mathbb{R}$), the image of the *Cayley* map consists of all rotations with angles $\theta \neq \pm\pi$, i.e., the matrices $\mathcal{R} \in$ SO(3,$\mathbb{R}$) with no eigenvalues of -1.
In Section 2 of the paper we derive a vector parametrization of SU(2) and make use of it for expressing the composition law in this group. We show that the *Cayley* map $\mathfrak{su}(2) \to$ SU(2) is bijective onto its image. Section 3 provides an explicit formula for the double cover map SU(2) $\to$ SO(3,$\mathbb{R}$) in terms of the vector parameters of the source and the target manifold.

2. VECTOR PARAMETRIZATION OF SU(2) AND THE PULL-BACK OF THE COMPOSITION LAW

2.1. THE CASE OF SO(3,$\mathbb{R}$)

The Lie algebra $\mathfrak{so}(3)$ consists of the real anti-symmetric 3×3 matrices. The *Cayley* map of $\mathfrak{so}(3) \to$ SO(3,$\mathbb{R}$) gives the so called *Gibbs* vector parametrization of SO(3,$\mathbb{R}$). The matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

form a basis of $\mathfrak{so}(3)$ over the filed of the real numbers. For arbitrary $i, j, k \in \{1, 2, 3\}$ let $\varepsilon_{ijk} = 1$ if i, j, k is an even permutation of $1, 2, 3$, $\varepsilon_{ijk} = -1$ for an odd permutation of $1, 2, 3$ and $\varepsilon_{ijk} = 0$ otherwise. The following relations hold:

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad i, j, k \in \{1, 2, 3\}. \tag{2.2}$$

Any $\mathcal{C} \in \mathfrak{so}(3)$ has a unique representation

$$\boldsymbol{c} \mapsto \mathcal{C} = \mathbf{c} \cdot \mathbf{J} = c_1 J_1 + c_2 J_2 + c_3 J_3 = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix},$$

where

$$\boldsymbol{c} = (c_1, c_2, c_3), \qquad \boldsymbol{c}^2 := c_1^2 + c_2^2 + c_3^2 = \boldsymbol{c}.\boldsymbol{c} = |\boldsymbol{c}|^2 = c^2. \tag{2.3}$$

Hereafter we shall use $\boldsymbol{c}$ and c to denote respectively the vector $\boldsymbol{c}$ and its norm c. This convention applies to other vectors as well.

The *Hamilton–Cayley* theorem for $\mathcal{C}$ reads as $\mathcal{C}^3 = -c^2\mathcal{C}$. That is why the exponential map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3,\mathbb{R})$ is given explicitly by the formula

$$\exp(\mathcal{C}) = \mathcal{I} + \frac{\sin c}{c}\mathcal{C} + \frac{1 - \cos c}{c^2}\mathcal{C}^2. \tag{2.4}$$

In order to compare, let us recall that the *Cayley* map for $\mathfrak{so}(3)$ associates with $\mathbf{c} \cdot \mathbf{J} \in \mathfrak{so}(3)$ the matrix

$$\mathcal{R}(\boldsymbol{c}) = \mathrm{Cay}_{\mathfrak{so}(3)}(\boldsymbol{c}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = (\mathcal{I} - \mathcal{C})^{-1}(\mathcal{I} + \mathcal{C}). \tag{2.5}$$

One checks immediately that

$$(\mathcal{I} - \mathcal{C})^{-1} = \mathcal{I} + \frac{1}{1 + c^2}\mathcal{C} + \frac{1}{1 + c^2}\mathcal{C}^2 \tag{2.6}$$

and (2.5) can be expressed in the form

$$\mathrm{Cay}_{\mathfrak{so}(3)}(\boldsymbol{c}) = \mathcal{I} + \frac{2}{1 + c^2}\mathcal{C} + \frac{2}{1 + c^2}\mathcal{C}^2 \tag{2.7}$$

for all $\boldsymbol{c} \in \mathbb{R}^3$. Is is well known that in SO(3,$\mathbb{R}$), the half-turns are described by symmetric rotation matrices. Note that $\mathrm{Cay}_{\mathfrak{so}(3)}$ is bijective onto its image (see [14])

$$\Im\mathrm{Cay}_{\mathfrak{so}(3)} = \{\mathcal{R} \in \mathrm{SO}(3,\mathbb{R});\, \mathcal{R} \neq \mathcal{R}^t\} = \mathrm{SO}(3,\mathbb{R}) \backslash \mathrm{S}(3,\mathbb{R}), \tag{2.8}$$

where $\mathrm{S}(3,\mathbb{R})$ is the set of all symmetric 3×3 matrices with real entries. The image $\mathcal{R}(\boldsymbol{c})$ of $\mathbf{c}$ by $\mathrm{Cay}_{\mathfrak{so}(3)}$ is

$$\mathbf{c} \to \mathcal{R}(\boldsymbol{c}) = \frac{2}{1 + c^2}\begin{pmatrix} 1 + c_1^2 & c_1c_2 - c_3 & c_1c_3 + c_2 \\ c_1c_2 + c_3 & 1 + c_2^2 & c_2c_3 - c_1 \\ c_1c_3 - c_2 & c_2c_3 + c_1 & 1 + c_3^2 \end{pmatrix} - \mathcal{I}. \tag{2.9}$$

The rotation $\mathcal{R} = \mathcal{R}(\mathbf{n}, \theta)$ at angle θ about the axis $\mathbf{n}$ is represented by *Gibbs* parameter $\mathbf{c} = \tan\frac{\theta}{2}\,\mathbf{n}$, see [2]. In order to express the group law in SO(3,$\mathbb{R}$) by the means of the *Cayley* map let us denote by $\tilde{\boldsymbol{c}}$ the vector parameter of the product $\mathcal{R}(\tilde{\boldsymbol{c}}) = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\boldsymbol{c})$ of the elements of SO(3,$\mathbb{R}$), corresponding to $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^3$. Then, as pointed out in [7]

$$\mathcal{R}(\tilde{\boldsymbol{c}}) = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\boldsymbol{c}), \qquad \tilde{\boldsymbol{c}} = \tilde{\boldsymbol{c}}(\boldsymbol{a}, \boldsymbol{c}) = \langle \boldsymbol{a}, \boldsymbol{c} \rangle = \frac{\boldsymbol{a} + \boldsymbol{c} + \boldsymbol{a} \times \boldsymbol{c}}{1 - \boldsymbol{a}.\boldsymbol{c}} \cdot \tag{2.10}$$

In the case of $\mathfrak{so}(3)$ it is shown in [6] that the *Baker–Campbell–Hausdorff* formula takes the form

$$BCH(\mathcal{A}, \mathcal{C}) = BCH(\boldsymbol{a} \cdot \mathbf{J}, \boldsymbol{c} \cdot \mathbf{J}) = \alpha\mathcal{A} + \beta\mathcal{C} + \gamma[\mathcal{A}, \mathcal{C}]\,, \tag{2.11}$$

with

$$\alpha = \frac{\sin^{-1}(q)}{q}\frac{m}{\theta}, \qquad \beta = \frac{\sin^{-1}(q)}{q}\frac{n}{\psi}, \qquad \gamma = \frac{\sin^{-1}(q)}{q}\frac{p}{\theta\psi}\,,$$

where $\psi = |\boldsymbol{a}|$, $\theta = |\boldsymbol{c}|$, $\angle(\boldsymbol{a}, \boldsymbol{c}) = \cos^{-1}\left(\frac{\boldsymbol{a}.\boldsymbol{c}}{|\boldsymbol{a}||\boldsymbol{c}|}\right)$ and

$$\begin{aligned} m &= \sin(\theta)\cos^2(\psi/2) - \sin(\psi)\sin^2(\theta/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,, \\ n &= \sin(\psi)\cos^2(\theta/2) - \sin(\theta)\sin^2(\psi/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,, \\ p &= \frac{1}{2}\sin(\theta)\sin(\psi) - 2\sin^2(\theta/2)\sin^2(\psi/2)\cos(\angle(\boldsymbol{a}, \boldsymbol{c}))\,, \\ q &= \sqrt{m^2 + n^2 + 2mn\cos(\angle(\boldsymbol{a}, \boldsymbol{c})) + p^2\sin^2(\angle(\boldsymbol{a}, \boldsymbol{c}))}\,. \end{aligned}$$

Note that equation (2.10) is much simpler and more convenient when compared with (2.11). The vector parameter form of SO(3,$\mathbb{R}$) matrices and the corresponding composition law (2.10) are exploited in the decomposition method of the three dimensional rotations about three almost arbitrary axes, see [2]. In this vector parameter form of SO(3,$\mathbb{R}$), the half-turns, i.e., rotations at angles $\theta = \pm\pi$, can not be described. Henceforth we denote the matrix of the half-turn about the axis $\mathbf{n}$, i.e., $\mathcal{R}(\mathbf{n}, \pi)$, by $\mathcal{O}(\mathbf{n})$. The composition of the two rotations is not well defined also when $1 - \boldsymbol{a}.\boldsymbol{c} = 0$, which is exactly the condition that the compound rotation $\tilde{\boldsymbol{c}}$ is a half-turn.

2.2. DESCRIPTION OF $\mathfrak{su}(2)$

A coordinate free description [11] of $\mathfrak{su}(2)$ can be given. Let i be the imaginary unit and $\sigma_1, \sigma_2, \sigma_3$ be three elements which obey the rules

$$\begin{gathered} \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \\ \sigma_1\sigma_2 = -\sigma_2\sigma_1 = \mathrm{i}\sigma_3, \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = \mathrm{i}\sigma_1, \quad \sigma_3\sigma_1 = -\sigma_1\sigma_3 = \mathrm{i}\sigma_2. \end{gathered} \tag{2.12}$$

If we define the spin vector $\boldsymbol{\sigma}$ as

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3) \tag{2.13}$$

and $\mathbf{n}$ and $\mathbf{m}$ are arbitrary unit vectors in $\mathbb{R}^3$, then the following properties hold:

$$\begin{gathered}(\mathbf{n}\cdot\boldsymbol{\sigma})^2 = 1, \qquad (\mathbf{m}\cdot\boldsymbol{\sigma})(\mathbf{n}\cdot\boldsymbol{\sigma}) = \mathbf{m}.\mathbf{n} + \mathrm{i}(\mathbf{m}\times\mathbf{n})\cdot\boldsymbol{\sigma}\,, \\ \boldsymbol{\sigma}\cdot(\mathbf{n}\cdot\boldsymbol{\sigma}) = \mathbf{n} + \mathrm{i}\mathbf{n}\times\boldsymbol{\sigma}, \qquad (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\boldsymbol{\sigma} = \mathbf{n} - \mathrm{i}\mathbf{n}\times\boldsymbol{\sigma}\,, \\ (\mathbf{m}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}(\mathbf{n}\cdot\boldsymbol{\sigma}) = (\mathbf{m}\cdot\boldsymbol{\sigma})\mathbf{n} + (\mathbf{n}\cdot\boldsymbol{\sigma})\mathbf{m} - \mathrm{i}(\mathbf{m}\times\mathbf{n}) - (\mathbf{m}.\mathbf{n})\cdot\boldsymbol{\sigma}\,.\end{gathered} \tag{2.14}$$

A concrete matrix realization of $\sigma_1, \sigma_2, \sigma_3$ in (2.12) are the Pauli's matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

The matrices s_1, s_2 and s_3 defined by

$$s_1 = -\frac{\mathrm{i}}{2}\sigma_1, \qquad s_2 = -\frac{\mathrm{i}}{2}\sigma_2, \qquad s_3 = -\frac{\mathrm{i}}{2}\sigma_3 \tag{2.16}$$

form a $\mathbb{R}-$basis of $\mathfrak{su}(2)$. Direct calculation shows that

$$[s_i, s_j] = \epsilon_{ijk}s_k, \qquad i, j, k \in \{1, 2, 3\}. \tag{2.17}$$

Denoting $\mathbf{s} = (s_1, s_2, s_3)$ we express the $\mathfrak{su}(2)$ algebra in the following way:

$$\mathfrak{su}(2) = \left\{\mathbf{c}\cdot\mathbf{s} = \mathrm{c}_1 s_1 + \mathrm{c}_2 s_2 + \mathrm{c}_3 s_3\,;\, \mathbf{c} = (c_1, c_2, c_3) \in \mathbb{R}^3\right\}. \tag{2.18}$$

The corresponding matrix realization of $\mathbf{c}\cdot\mathbf{s}$ is

$$\begin{pmatrix} -\mathrm{i}\dfrac{\mathrm{c}_3}{2} & -\dfrac{\mathrm{c}_2}{2} - \mathrm{i}\dfrac{\mathrm{c}_1}{2} \\ \dfrac{\mathrm{c}_2}{2} - \mathrm{i}\dfrac{\mathrm{c}_1}{2} & \mathrm{i}\dfrac{\mathrm{c}_3}{2} \end{pmatrix}. \tag{2.19}$$

Obviously, the map

$$\mathrm{c}_1 s_1 + \mathrm{c}_2 s_2 + \mathrm{c}_3 s_3 \longrightarrow c_1 J_1 + c_2 J_2 + c_3 J_3 \tag{2.20}$$

is a linear isomorphism between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$.

2.3. CAYLEY MAP FROM $\mathfrak{su}(2)$ TO SU(2)

Till the end of this section $\mathcal{I}$ will stand for the unit matrix with dimension consistent with the context. Let

$$\mathcal{A} = \mathrm{a}_1 s_1 + \mathrm{a}_2 s_2 + \mathrm{a}_3 s_3 = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma} \in \mathfrak{su}(2)\,, \tag{2.21}$$

where

$$\mathbf{a} = (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3), \qquad \mathbf{a}^2 = \mathrm{a}_1^2 + \mathrm{a}_2^2 + \mathrm{a}_3^2 = \mathbf{a}.\mathbf{a} = |\mathbf{a}|^2 = \mathrm{a}^2. \tag{2.22}$$

Let us recall also that (see [8]) the exponential map for $\mathfrak{su}(2)$ is globally defined and surjective. It maps $\mathcal{A} \in \mathfrak{su}(2)$ to

$$\exp(\mathcal{A}) = \cos(\mathrm{a}/2)\mathfrak{I} - \frac{\sin \mathrm{a}/2}{\mathrm{a}/2}\mathcal{A}. \tag{2.23}$$

The *Hamilton–Cayley* theorem implies the identity $\mathcal{A}^2 = -\dfrac{\mathrm{a}^2}{4}\mathfrak{I}$. The image of $\mathcal{A}$ under the *Cayley* map is

$$\mathcal{U}(\mathbf{a}) = \mathrm{Cay}_{\mathfrak{su}(2)}(\mathcal{A}) = (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1}. \tag{2.24}$$

In general, the *Cayley* map $\mathrm{Cay}_{\mathfrak{su}(n)}$ for the Lie algebra $\mathfrak{su}(n)$ of skew-hermitian matrices ($\mathcal{A}^\dagger = \overline{\mathcal{A}}^t = -\mathcal{A}$) with trace zero takes values in $\mathrm{U}(n)$. Indeed, let us take any $\mathcal{A} \in \mathfrak{su}(n)$ and its image $\mathrm{Cay}_{\mathfrak{su}(n)}(\mathcal{A}) = \mathcal{U}$. Taking into account that $(\mathcal{U}^\dagger)^{-1} = (\mathcal{U}^{-1})^\dagger$, we obtain

$$\begin{aligned}
\mathcal{U}\mathcal{U}^\dagger &= (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1}((\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1})^\dagger \\
&= (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1}((\mathfrak{I} - \mathcal{A})^{-1})^\dagger(\mathfrak{I} + \mathcal{A})^\dagger \\
&= (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1}(\mathfrak{I} + \mathcal{A})^{-1}(\mathfrak{I} - \mathcal{A}) \\
&= (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1}(\mathfrak{I} - \mathcal{A})(\mathfrak{I} + \mathcal{A})^{-1} = \mathfrak{I}.
\end{aligned} \tag{2.25}$$

Lemma 1. *For each element $\mathcal{A} \in \mathfrak{su}(2)$ there holds*

$$(\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A}) = (\mathfrak{I} - \mathcal{A})(\mathfrak{I} + \mathcal{A}) = \mathfrak{I} - \mathcal{A}^2 = \big(1 + \frac{\mathrm{a}^2}{4}\big)\mathfrak{I}, \tag{2.26}$$

i.e.,

$$(\mathfrak{I} - \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathfrak{I} + \mathcal{A}), \qquad (\mathfrak{I} + \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathfrak{I} - \mathcal{A}). \tag{2.27}$$

Besides (2.27), from Lemma 1 we also infer

$$\begin{aligned}
\mathcal{U}(\mathbf{a}) &= (\mathfrak{I} + \mathcal{A})(\mathfrak{I} - \mathcal{A})^{-1} = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathfrak{I} + \mathcal{A})^2 \\
&= \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}(\mathfrak{I} + 2\mathcal{A} + \mathcal{A}^2) = \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}\big(\mathfrak{I} + 2\mathcal{A} - \frac{\mathrm{a}^2}{4}\mathfrak{I}\big) \\
&= \big(1 + \frac{\mathrm{a}^2}{4}\big)^{-1}\Big(\big(1 - \frac{\mathrm{a}^2}{4}\big)\mathfrak{I} - \mathrm{i}\mathbf{a} \cdot \boldsymbol{\sigma}\Big).
\end{aligned} \tag{2.28}$$

The matrix form of $\mathcal{U}(\mathbf{a})$ is

$$\mathcal{U}(\mathbf{a}) = \frac{1-\dfrac{\mathrm{a}^2}{4}}{1+\dfrac{\mathrm{a}^2}{4}}\mathcal{I} + \frac{1}{1+\dfrac{\mathrm{a}^2}{4}}\begin{pmatrix} -\mathrm{i}\mathrm{a}_3 & -\mathrm{a}_2-\mathrm{i}\mathrm{a}_1 \\ \mathrm{a}_2-\mathrm{i}\mathrm{a}_1 & \mathrm{i}\mathrm{a}_3 \end{pmatrix}. \tag{2.29}$$

The matrix $\mathcal{U}(\mathbf{a})$ defined in (2.29) is unitary due to (2.25). Direct calculation shows that

$$\det \mathcal{U}(\mathbf{a}) = \det\left(\left(1+\frac{\mathrm{a}^2}{4}\right)^{-1}(\mathcal{I}+\mathcal{A})^2\right) = \left(1+\frac{\mathrm{a}^2}{4}\right)^{-2}(\det(\mathcal{I}+\mathcal{A}))^2 = 1 \tag{2.30}$$

i.e., $\mathcal{U}(\mathbf{a}) \in \mathrm{SU}(2)$. Following *Wigner* [18] we can use the explicit homomorphism map W: $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ given by

$$\begin{gathered}\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha_1+\mathrm{i}\alpha_2 & \beta_1+\mathrm{i}\beta_2 \\ -\beta_1+\mathrm{i}\beta_2 & \alpha_1-\mathrm{i}\alpha_2 \end{pmatrix} \\ \xrightarrow{W} \begin{pmatrix} \alpha_1^2-\alpha_2^2-\beta_1^2+\beta_2^2 & 2(\alpha_1\alpha_2+\beta_1\beta_2) & 2(\alpha_2\beta_2-\alpha_1\beta_1) \\ 2(\beta_1\beta_2-\alpha_1\alpha_2) & \alpha_1^2-\alpha_2^2+\beta_1^2-\beta_2^2 & 2(\alpha_2\beta_1+\alpha_1\beta_2) \\ 2(\alpha_1\beta_1+\alpha_2\beta_2) & 2(\alpha_2\beta_1-\alpha_1\beta_2) & \alpha_1^2+\alpha_2^2-\beta_1^2-\beta_2^2 \end{pmatrix}.\end{gathered} \tag{2.31}$$

The comparison of (2.29) and (2.31) yields

$$\alpha = \alpha_1+\mathrm{i}\alpha_2 = \frac{1-\dfrac{\mathrm{a}^2}{4}}{1+\dfrac{\mathrm{a}^2}{4}} + \mathrm{i}\frac{-\mathrm{a}_3}{1+\dfrac{\mathrm{a}^2}{4}}, \quad \beta = \beta_1+\mathrm{i}\beta_2 = \frac{-\mathrm{a}_2}{1+\dfrac{\mathrm{a}^2}{4}} + \mathrm{i}\frac{-\mathrm{a}_1}{1+\dfrac{\mathrm{a}^2}{4}}. \tag{2.32}$$

In the case of the SU(2) group manifold, which is diffeomorphic to the sphere S^3, there is a homotopy obstruction for the existence of a global diffeomorphism $\mathbb{R}^3 \simeq \mathfrak{su}(2) \to \mathrm{SU}(2) \simeq S^3$, so that no vector parametrization $\mathfrak{su}(2) \to \mathrm{SU}(2)$ exists onto the entire group SU(2). Actually, the *Cayley* map provides a vector parametrization

$$\mathrm{Cay}_{\mathfrak{su}(2)}\colon\ \mathfrak{su}(2) \to \mathrm{SU}(2)\backslash\{-\mathcal{I}\}, \tag{2.33}$$

whose inverse is

$$\begin{gathered}\mathrm{Cay}^{-1}_{\mathfrak{su}(2)}\begin{pmatrix} \alpha_1+\mathrm{i}\alpha_2 & \beta_1+\mathrm{i}\beta_2 \\ -\beta_1+\mathrm{i}\beta_2 & \alpha_1-\mathrm{i}\alpha_2 \end{pmatrix} = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma}, \\ \mathbf{a} = (\mathrm{a}_1,\mathrm{a}_2,\mathrm{a}_3) = -\frac{2}{1+\alpha_1}(\beta_2,\beta_1,\alpha_2).\end{gathered} \tag{2.34}$$

By means of (2.31) and (2.32) one calculates straightforwardly that the image $\mathcal{R}_{\mathcal{U}}(\mathbf{a})$ of $\mathcal{U}(\mathbf{a})$ under the *Wigner* map W is

$$\frac{8}{(4+\mathrm{a}^2)^2}\begin{pmatrix} \frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_2^2-4\mathrm{a}_3^2 & 4\mathrm{a}_1\mathrm{a}_2-\mathrm{a}_3(4-\mathrm{a}^2) & 4\mathrm{a}_1\mathrm{a}_3+\mathrm{a}_2(4-\mathrm{a}^2) \\ 4\mathrm{a}_1\mathrm{a}_2+\mathrm{a}_3(4-\mathrm{a}^2) & \frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_1^2-4\mathrm{a}_3^2 & 4\mathrm{a}_2\mathrm{a}_3-\mathrm{a}_1(4-\mathrm{a}^2) \\ 4\mathrm{a}_1\mathrm{a}_3-\mathrm{a}_2(4-\mathrm{a}^2) & 4\mathrm{a}_2\mathrm{a}_3+\mathrm{a}_1(4-\mathrm{a}^2) & \frac{(4+\mathrm{a}^2)^2}{4}-4\mathrm{a}_1^2-4\mathrm{a}_2^2 \end{pmatrix} - \mathcal{I}. \tag{2.35}$$

Let $\mathcal{A} = -\frac{\mathrm{i}}{2}\mathbf{a}\cdot\boldsymbol{\sigma}$, $\mathcal{C} = -\frac{\mathrm{i}}{2}\mathbf{c}\cdot\boldsymbol{\sigma} \in \mathfrak{su}(2)$. The term of third degree in $BCH(\mathcal{A},\mathcal{C})$ (cf. (1.2)) is $\frac{1}{2}[\mathcal{A},\mathcal{C}] = -\frac{\mathrm{i}}{2}(\mathbf{a}\times\mathbf{c})\cdot\boldsymbol{\sigma}$, and that one of degree four is

$$\frac{1}{12}([\mathcal{A},[\mathcal{C},\mathcal{A}]] - [\mathcal{C},[\mathcal{A},\mathcal{C}]]) = -\frac{\mathrm{i}}{2}\tilde{\mathbf{c}}_4\cdot\boldsymbol{\sigma}, \qquad \tilde{\mathbf{c}}_4 = (u_4, v_4, w_4)\,, \tag{2.36}$$

with

$$\begin{aligned} u_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{a}_2\mathrm{c}_1 + \mathrm{a}_1\mathrm{c}_1\mathrm{c}_2 + \mathrm{a}_2\mathrm{a}_3\mathrm{c}_3 + \mathrm{a}_3\mathrm{c}_2\mathrm{c}_3 - \mathrm{a}_1^2\mathrm{c}_2 - \mathrm{a}_3^2\mathrm{c}_2 - \mathrm{a}_2\mathrm{c}_1^2 - \mathrm{a}_2\mathrm{c}_3^2)\,,\\ v_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{a}_2\mathrm{c}_2 + \mathrm{a}_1\mathrm{a}_3\mathrm{c}_3 + \mathrm{a}_2\mathrm{c}_1\mathrm{c}_2 + \mathrm{a}_3\mathrm{c}_1\mathrm{c}_3 - \mathrm{a}_2^2\mathrm{c}_1 - \mathrm{a}_3^2\mathrm{c}_1 - \mathrm{a}_1\mathrm{c}_2^2 - \mathrm{a}_1\mathrm{c}_3^2)\,,\\ w_4 &= \frac{1}{12}(\mathrm{a}_1\mathrm{c}_1\mathrm{c}_3 + \mathrm{a}_1\mathrm{a}_3\mathrm{c}_1 + \mathrm{a}_2\mathrm{c}_2\mathrm{c}_3 + \mathrm{a}_2\mathrm{a}_3\mathrm{c}_2 - \mathrm{a}_1^2\mathrm{c}_3 - \mathrm{a}_2^2\mathrm{c}_3 - \mathrm{a}_3\mathrm{c}_1^2 - \mathrm{a}_3\mathrm{c}_2^2). \end{aligned} \tag{2.37}$$

Note that the coefficients of the term of degree four are homogeneous polynomials of $\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3$ of degree three. It is interesting to compare the composition rule (2.10) of SO(3,$\mathbb{R}$), expressed through the *Gibbs* vector parameter with the following formula

$$\mathcal{A} + \mathcal{C} + \frac{1}{2}[\mathcal{A},\mathcal{C}] = -\frac{\mathrm{i}}{2}\Big(\mathbf{a} + \mathbf{c} + \frac{\mathbf{a}\times\mathbf{c}}{2}\Big)\cdot\boldsymbol{\sigma}. \tag{2.38}$$

2.4. COMPOSITION LAW IN SU(2)

Proposition 1. *Let* $\mathcal{U}_1(\mathbf{c}), \mathcal{U}_2(\mathbf{a}) \in \mathrm{SU}(2)$ *are the images of* $\mathcal{A}_1 = \mathbf{c}\cdot\mathbf{s}$ *and* $\mathcal{A}_2 = \mathbf{a}\cdot\mathbf{s}$ *under the map* (2.24) *of the vectors* $\mathbf{a}, \mathbf{c} \in \mathbb{R}^3$. *Let*

$$\mathcal{U}_3(\langle\mathbf{a},\mathbf{c}\rangle_{\mathrm{SU}(2)}) = \mathcal{U}_2(\mathbf{a}).\mathcal{U}_1(\mathbf{c}) \tag{2.39}$$

denote the composition of $\mathcal{U}_2(\mathbf{a})$ *and* $\mathcal{U}_1(\mathbf{c})$ *in* SU(2). *The corresponding vector-parameter* $\widetilde{\mathbf{a}} \in \mathbb{R}^3$, *for which* $\mathrm{Cay}_{\mathfrak{su}(2)}(\mathcal{A}_3) = \mathcal{U}_3$, $\mathcal{A}_3 = \widetilde{\mathbf{a}}\cdot\mathbf{s}$ *is*

$$\widetilde{\mathbf{a}} = \frac{\left(1 - \frac{\mathrm{c}^2}{4}\right)\mathbf{a} + \left(1 - \frac{\mathrm{a}^2}{4}\right)\mathbf{c} + 4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{1 - 2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2} + \frac{\mathrm{a}^2}{4}\frac{\mathrm{c}^2}{4}}. \tag{2.40}$$

The vector $\widetilde{\mathbf{a}}$ *equals to* $\mathbf{0}$ *if only if* $\mathbf{c} = -\mathbf{a}$ *or* $\mathbf{c} = 2\tan\frac{\theta}{4}\,\mathbf{n}$ *and* $\mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\,\mathbf{n}$, *where* $\mathbf{n}\in\mathbb{R}^3, \mathbf{n}^2 = 1$ *and* $\theta \in [0, 2\pi)$. *In both cases,* $\mathbf{c} = -\mathbf{a}$ *and* $\mathbf{c} = 2\tan\frac{\theta}{4}\,\mathbf{n}$, $\mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\mathbf{n}$, *these vectors represent inverse rotations.*

Proof. From (2.28) we obtain that

$$\begin{aligned}\mathcal{U}_3 &= \big(1+\frac{a^2}{4}\big)^{-1}\big((1-\frac{a^2}{4})\mathcal{I}-i\mathbf{a}\cdot\boldsymbol{\sigma}\big)\big(1+\frac{c^2}{4}\big)^{-1}\big((1-\frac{c^2}{4})\mathcal{I}-i\mathbf{c}\cdot\boldsymbol{\sigma}\big)\\ &\overset{(2.14)}{=} \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})\mathcal{I}-i(1-\frac{a^2}{4})\mathbf{c}\cdot\boldsymbol{\sigma}-i(1-\frac{c^2}{4})\mathbf{a}\cdot\boldsymbol{\sigma}-\mathbf{a}.\mathbf{c}\,\mathcal{I}-i(\mathbf{a}\times\mathbf{c})\cdot\boldsymbol{\sigma}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})} \qquad (2.41)\\ &= \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})-\mathbf{a}.\mathbf{c}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}\mathcal{I}-i\frac{(1-\frac{a^2}{4})\mathbf{c}+(1-\frac{c^2}{4})\mathbf{a}+4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}\cdot\boldsymbol{\sigma}.\end{aligned}$$

The general formulas (2.29) and (2.41) will be compatible if we have simultaneously

$$\begin{aligned}\frac{1-\frac{\tilde{a}^2}{4}}{1+\frac{\tilde{a}^2}{4}} &= \frac{(1-\frac{a^2}{4})(1-\frac{c^2}{4})-\mathbf{a}.\mathbf{c}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})},\\ \frac{\tilde{\mathbf{a}}}{1+\frac{\tilde{a}^2}{4}} &= \frac{(1-\frac{a^2}{4})\mathbf{c}+(1-\frac{c^2}{4})\mathbf{a}+4\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}}{(1+\frac{a^2}{4})(1+\frac{c^2}{4})}.\end{aligned} \qquad (2.42)$$

From (2.42) we get

$$\frac{\tilde{a}^2}{4} = \frac{\frac{a^2}{4}+\frac{c^2}{4}+2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}}{1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}}, \qquad 1+\frac{\tilde{a}^2}{4} = \frac{1+\frac{a^2}{4}+\frac{c^2}{4}+\frac{a^2}{4}\frac{c^2}{4}}{1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}}. \qquad (2.43)$$

Taking into account that

$$1+\frac{a^2}{4}+\frac{c^2}{4}+\frac{a^2}{4}\frac{c^2}{4} = \big(1+\frac{a^2}{4}\big)\big(1+\frac{c^2}{4}\big)$$

and multiplying the numerator and denominator of the second fraction in (2.41) by $1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4}$ (when this expression is non-zero), we get the result in the second case in (2.40), i.e., the composition law in vector-parameter form for SU(2).

To rigorously see when the composition is not well defined, we investigate the case in which the denominator equals zero. According to the identity

$$1-2\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{c^2}{4} = \big(1-\frac{\mathbf{a}}{2}\cdot\frac{\mathbf{c}}{2}\big)^2+\big(\frac{\mathbf{a}}{2}\times\frac{\mathbf{c}}{2}\big)^2, \qquad (2.44)$$

the denominator of (2.40) vanishes if only if $\mathbf{a} = 2\tan\frac{\theta_2}{4}\mathbf{n}$, $\mathbf{c} = 2\tan\frac{\theta_1}{4}\mathbf{n}$ and

$1 = \tan\dfrac{\theta_2}{4}\tan\dfrac{\theta_1}{4}$. This implies $\cos\dfrac{\theta_1+\theta_2}{4} = 0,\ \theta_1+\theta_2 = 2\pi$ and allows to express

$$\mathbf{c} = 2\tan\frac{\theta}{4}\mathbf{n}, \qquad \mathbf{a} = 2\tan\frac{2\pi-\theta}{4}\mathbf{n}. \tag{2.45}$$

Substituting the results from (2.45) in (2.42) gives $\widetilde{\mathbf{a}}\,(\mathbf{a},\mathbf{c}) = 0$, which corresponds to the identity element $\mathcal{I}$. If $\mathbf{c} \equiv -\mathbf{a}$, then $\widetilde{\mathbf{a}} = \mathbf{0}$. ☐

In the particular case when one and the same rotation ($\mathbf{a} \equiv \mathbf{c}$) is applied twice the resulting vector is

$$\widetilde{\mathbf{a}} = \frac{2\big(1-\dfrac{\mathrm{a}^2}{4}\big)\mathbf{a}}{\big(1-\dfrac{\mathrm{a}^2}{4}\big)^2} = \frac{4\dfrac{\mathbf{a}}{2}}{1-\dfrac{\mathrm{a}^2}{4}}\cdot$$

It is important to investigate when the composition $\widetilde{\mathbf{a}}$ is such that $|\,\widetilde{\mathbf{a}}\,| \le 4$. Using (2.43) we obtain

$$\frac{\widetilde{\mathrm{a}}^{\,2}}{4} = \frac{\dfrac{\mathrm{a}^2}{4}+\dfrac{\mathrm{c}^2}{4}+2\dfrac{\mathbf{a}}{2}\cdot\dfrac{\mathbf{c}}{2}}{1-2\dfrac{\mathbf{a}}{2}\cdot\dfrac{\mathbf{c}}{2}+\dfrac{\mathrm{a}^2}{4}\dfrac{\mathrm{c}^2}{4}} \le 1 \tag{2.46}$$

and this is equivalent to the inequality

$$\mathbf{a}.\mathbf{c} \le \big(1-\frac{\mathrm{a}^2}{4}\big)\big(1-\frac{\mathrm{c}^2}{4}\big). \tag{2.47}$$

Similar conditions for $|\,\widetilde{\mathbf{a}}\,| < 4, |\,\widetilde{\mathbf{a}}\,| = 4$ and $|\,\widetilde{\mathbf{a}}\,| > 4$ cases follow immediately.

3. THE COVERING MAP SU(2) → SO(3,ℝ) AND ITS SECTIONS IN VECTOR-PARAMETER FORM

Proposition 2. *Let* $\mathbf{a}$ *be the vector-parameter of a generic* SU(2) *element (i.e., it is not associated with some half-turn,* $\mathbf{a}^2 = 4$*). Then the Gibbs vector* $\mathbf{c}$*, which represents this rotation in* SO(3,ℝ)*, is given by*

$$\boldsymbol{c}(\mathbf{a}) = \frac{\mathbf{a}}{1-\dfrac{\mathrm{a}^2}{4}}\cdot \tag{3.1}$$

On the other hand, if $\boldsymbol{c}$ *is the Gibbs vector, representing a rotation from* SO(3,ℝ)*, then the preimages of this rotation in* SU(2) *correspond to the vector parameters*

$$\mathbf{a}_+(\boldsymbol{c}) = \frac{2(\sqrt{1+c^2}-1)}{c^2}\boldsymbol{c}, \qquad \mathbf{a}_-(\boldsymbol{c}) = -\frac{2(\sqrt{1+c^2}+1)}{c^2}\boldsymbol{c}. \tag{3.2}$$

Moreover, they are connected by the formulas

$$\mathbf{a}_+ = -\frac{4}{\mathrm{a}_-^2}\mathbf{a}_-, \qquad \mathbf{a}_- = -\frac{4}{\mathrm{a}_+^2}\mathbf{a}_+, \qquad \mathrm{a}_-^2\,\mathrm{a}_+^2 = 16. \tag{3.3}$$

Proof. We have to find a *Gibbs* parameter $\boldsymbol{c}$ such that

$$\mathcal{R}(\boldsymbol{c}) = \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2 & c_1c_2-c_3 & c_1c_3+c_2 \\ c_1c_2+c_3 & 1+c_2^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3+c_1 & 1+c_3^2 \end{pmatrix} - \mathcal{I} = \mathcal{R}_{\mathcal{U}}(\mathbf{a}) \tag{3.4}$$

and where $\mathcal{R}_{\mathcal{U}}(\mathbf{a})$ is given by (2.35). Equating the corresponding matrix elements,

$$\begin{aligned} \mathcal{R}(\boldsymbol{c})_{32} - \mathcal{R}(\boldsymbol{c})_{23} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{32} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{23} \\ \mathcal{R}(\boldsymbol{c})_{13} - \mathcal{R}(\boldsymbol{c})_{31} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{13} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{31} \\ \mathcal{R}(\boldsymbol{c})_{21} - \mathcal{R}(\boldsymbol{c})_{12} &= \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{21} - \mathcal{R}_{\mathcal{U}}(\mathbf{a})_{12} \\ \operatorname{tr}\mathcal{R}(\boldsymbol{c}) &= \operatorname{tr}\mathcal{R}_{\mathcal{U}}(\mathbf{a}) \end{aligned} \tag{3.5}$$

we end up with the following equalities

$$\begin{aligned} \frac{2}{1+c^2}\,c_1 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathrm{a}_1\,, \qquad & \frac{2}{1+c^2}\,c_2 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathrm{a}_2\,, \\ \frac{2}{1+c^2}\,c_3 &= \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\mathrm{a}_3\,, \qquad & \frac{2(3+c^2)}{1+c^2} &= \frac{8(-8\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}+6. \end{aligned} \tag{3.6}$$

From (3.6) we have

$$\frac{2}{1+c^2}\,\boldsymbol{c} = \frac{8(4-\mathrm{a}^2)}{(4+\mathrm{a}^2)^2}\,\mathbf{a} \tag{3.7}$$

and separating $1+c^2$ in (3.6) we obtain

$$\frac{2}{1+c^2} = 2\frac{(4+\mathrm{a}^2)^2-16\mathrm{a}^2}{(4+\mathrm{a}^2)^2} = 2\frac{(4-\mathrm{a}^2)^2}{(4+\mathrm{a}^2)^2}.$$

Substituting this expression in (3.7), we obtain (3.1), which is the first statement in the proposition. To invert (3.1), we firstly calculate c^2 and get

$$c^2 = \frac{\mathrm{a}^2}{(1-\frac{\mathrm{a}^2}{4})^2}.$$

If $\mathrm{a}^2 \neq 4$ (i.e., $\mathbf{a}$ does not represent a half-turn), this equality is equivalent to the following quadratic equation for a^2:

$$(\mathrm{a}^2)^2c^2 - 8(2+c^2)\mathrm{a}^2 + 16c^2 = 0. \tag{3.8}$$

The solutions of (3.8) are

$$\mathrm{a}_{\pm}^2 = \frac{4(2+c^2)\mp 8\sqrt{1+c^2}}{c^2}$$

and hence

$$\frac{\mathrm{a}_\pm^2}{4} = \frac{2 + c^2 \mp 2\sqrt{1+c^2}}{c^2} = 1 + \frac{2 \mp 2\sqrt{1+c^2}}{c^2}, \quad 1 - \frac{\mathrm{a}_\pm^2}{4} = -\frac{2(1 \mp \sqrt{1+c^2})}{c^2} \cdot \tag{3.9}$$

Substituting this result in (3.1) we obtain (3.2). It follows from (3.2) that

$$\begin{aligned} \mathbf{a}_+ &= \frac{2\,(\sqrt{1+c^2}-1)}{c^2}\,\boldsymbol{c} = -\frac{\sqrt{1+c^2}-1}{\sqrt{1+c^2}+1}\,\mathbf{a}_- \\ &= -\frac{2+c^2-2\sqrt{1+c^2}}{c^2}\,\mathbf{a}_- = -\frac{\mathrm{a}_+^2}{4}\,\mathbf{a}_-\,, \end{aligned} \tag{3.10}$$

therefore $\mathbf{a}_- = -\dfrac{4}{\mathrm{a}_+^2}\,\mathbf{a}_+$. From $\mathrm{a}_-^2 = \dfrac{16}{\mathrm{a}_+^4}\mathrm{a}_+^2$, $\mathrm{a}_-^2\mathrm{a}_+^2 = 16$ we find $\mathbf{a}_+ = -\dfrac{4}{\mathrm{a}_-^2}\,\mathbf{a}_-$, which completes the proof of Proposition 2. □

The relations obtained above are depicted in Fig. 2. Notice that $\mathbf{a}_\pm$ and $\boldsymbol{c}$ actually act between the algebras and also that the *Cayley* map is not surjective onto the given groups, see equations (2.8) and (2.33).

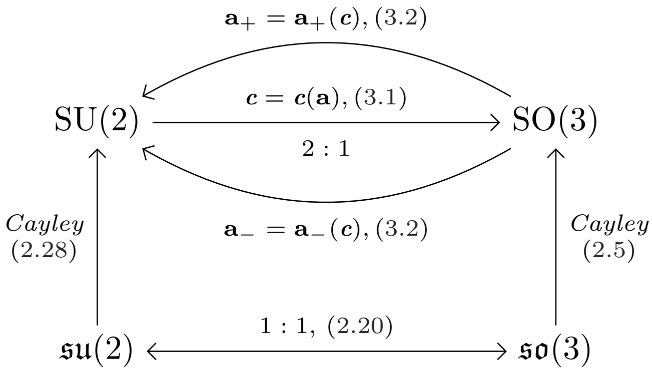

Figure 2: Informal depiction of the relations between the Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ and the Lie groups SU(2) and SO(3,ℝ).

Viewing a_+ and a_- as functions of c (see Fig. 3) one concludes that

$$\mathrm{a}_+(c) \le 2 \le \mathrm{a}_-(c), \qquad \lim_{c\to\infty} \mathrm{a}_+(c) = \lim_{c\to\infty} \mathrm{a}_-(c) = 2. \tag{3.11}$$

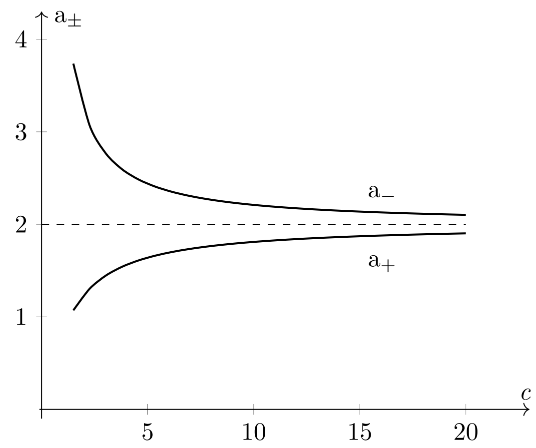

Figure 3: Graphs of a_- and a_+ as functions of c.

In order to obtain the SU(2) elements $\mathcal{U}_\pm(\boldsymbol{c})$ corresponding to the SO(3,$\mathbb{R}$) rotation with vector-parameter $\boldsymbol{c}$, we substitute $\mathbf{a}_\pm(\boldsymbol{c})$ from (3.2) in $\mathcal{U}(\mathbf{a})$ from (2.29) and get

$$\mathcal{U}_\pm(\boldsymbol{c}) = \pm\frac{1}{\sqrt{1+\boldsymbol{c}^2}}\begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix}. \tag{3.12}$$

Let $\boldsymbol{c} = \tan\frac{\theta}{2}\,\mathbf{n}$ represent a SO(3,$\mathbb{R}$) rotation at angle θ about the axis $\mathbf{n}$. The corresponding SU(2) vectors $\mathbf{a}_+(\boldsymbol{c})$ and $\mathbf{a}_-(\boldsymbol{c})$ are

$$\mathbf{a}_+(\boldsymbol{c}) = 2\tan\frac{\theta}{4}\,\mathbf{n}, \qquad \mathbf{a}_-(\boldsymbol{c}) = -2\tan\frac{2\pi-\theta}{4}\,\mathbf{n}. \tag{3.13}$$

The matrix corresponding to $\mathbf{a}_+$ is the familiar axis-angle representation of rotations in SU(2), i.e.,

$$\mathcal{U}(\mathbf{a}_+) = \mathcal{U}(\mathbf{n},\theta) = \cos\frac{\theta}{2}\mathcal{I} + \sin\frac{\theta}{2}\begin{pmatrix} -\mathrm{i}n_3 & -n_2-\mathrm{i}n_1 \\ n_2-\mathrm{i}n_1 & \mathrm{i}n_3 \end{pmatrix}. \tag{3.14}$$

In SU(2) the half-turns about the axis $\mathbf{n}$ are represented by the matrices

$$\mathcal{U}(\pm\mathbf{n},\pi) = \pm\begin{pmatrix} -\mathrm{i}n_3 & -n_2-\mathrm{i}n_1 \\ n_2-\mathrm{i}n_1 & \mathrm{i}n_3 \end{pmatrix}. \tag{3.15}$$

In the derived vector-parameter form the half-turns are represented by the vectors $\pm 2\mathbf{n}$, which are well defined and are of length 2. This is an advantage, because a half-turns $\mathcal{O}(\mathbf{n})$ in the *Gibbs* vector parameter form of SO(3,$\mathbb{R}$) rotations are represented by *vectors* with infinitely large norm and direction $\pm\mathbf{n}$. Such *vectors* will be referred further on as "rays" and will be denoted by $[\mathbf{n}]$ (for more discussion, see e.g. [2] and [12]). Let $\mathcal{R} = \mathcal{O}(\mathbf{n})$ be a half-turn about the axis $\mathbf{n}$, represented by $\pm\mathbf{n}$ in SU(2). Applying the limit $a \to 2$ in (3.1), we can informally write

$\lim\limits_{\mathbf{a}\to\pm 2\mathbf{n}} \boldsymbol{c}(\mathbf{a}) = [\mathbf{n}]$. Roughly speaking, the *Gibbs* parameter, associated with $\mathcal{O}(\mathbf{n})$ is $\boldsymbol{c} = \lim_{\theta\to\pi} \tan\dfrac{\theta}{2}\,\mathbf{n} = [\mathbf{n}]$. Actually, we have

$$\lim_{\theta\to\pi} \mathcal{U}_\pm(\tan\frac{\theta}{2}\mathbf{n}) \overset{(3.12)}{=} \pm \lim_{c^2\to\infty} \frac{1}{\sqrt{1+\boldsymbol{c}^2}} \begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix} \overset{(3.15)}{=} \mathcal{U}(\pm\mathbf{n},\pi). \quad (3.16)$$

We observe that if $\boldsymbol{c} = \tan\dfrac{\theta}{2}\mathbf{n}$ represents an infinitesimal SO(3,$\mathbb{R}$) rotation $\mathcal{R}(\mathbf{n},\theta)$, then as SU(2) element it is represented by two vectors, one with infinitesimal norm $\mathbf{a}_+$ and the other one $\mathbf{a}_-$ with infinite norm, i.e.,

$$\lim_{c\to 0} \mathrm{a}_+^2(\boldsymbol{c}) = 0, \qquad \lim_{c\to 0} \mathrm{a}_-^2(\boldsymbol{c}) = \infty. \quad (3.17)$$

When storing infinitesimal rotations in applications, loss of information may occur because of the operations performed with very small numbers. Equation (3.17) offers an alternative way (by usage of $\mathbf{a}_-$) for computer storage of infinitesimal rotations. This is so because in many of the commercial software systems there are packages for dealing with *large* numbers.

3.1. Compatibility of the composition laws in SU(2) and SO(3,$\mathbb{R}$)

Recall that a map $\varphi : G_1 \to G_2$ of the groups G_1, G_2 is a group homomorphism if it is compatible with the group operations in G_1 and G_2 by the rule $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G_1$. For an arbitrary subset $S_1 \subset G_1$, which is not necessarily a subgroup of G_1, we say that a map $\psi : S_1 \to G_2$ is compatible with the group operations in G_1 and G_2 if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in S_1$.

Proposition 3. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{c}$ *are some non-zero Gibbs parameters of two* SO(3,$\mathbb{R}$) *rotations and such that* $\boldsymbol{a}.\boldsymbol{c} \neq 1$. *Let*

$$\mathcal{U}_1(\boldsymbol{c}) = \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} 1-\mathrm{i}c_3 & -c_2-\mathrm{i}c_1 \\ c_2-\mathrm{i}c_1 & 1+\mathrm{i}c_3 \end{pmatrix}, \ \mathcal{U}_2(\boldsymbol{a}) = \frac{1}{\sqrt{1+a^2}} \begin{pmatrix} 1-\mathrm{i}a_3 & -a_2-\mathrm{i}a_1 \\ a_2-\mathrm{i}a_1 & 1+\mathrm{i}a_3 \end{pmatrix}$$

be the respective images of $\boldsymbol{a}$, $\boldsymbol{c}$ *under the "+" sections of the maps* (**??**) *and* (3.12). *Then the equality*

$$\mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c}) = \mathcal{U}(\tilde{\boldsymbol{c}}) \quad (3.18)$$

holds up to a sign, i.e., the "+" correspondences are compatible up to a sign with the group operations in SO(3,$\mathbb{R}$) *and* SU(2).

Proof. Let $\mathcal{U}_3 = \mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c})$. We will prove that

$$\mathcal{U}_3 = \frac{\pm 1}{\sqrt{1+\tilde{c}^2}} \begin{pmatrix} 1-\mathrm{i}\tilde{c}_3 & -\tilde{c}_2-\mathrm{i}\tilde{c}_1 \\ \tilde{c}_2-\mathrm{i}\tilde{c}_1 & 1+\mathrm{i}\tilde{c}_3 \end{pmatrix}. \quad (3.19)$$

Direct multiplication shows that

$$\mathcal{U}_3 = \frac{1 - \boldsymbol{a}.\boldsymbol{c}}{\sqrt{1+a^2}\sqrt{1+c^2}} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \tag{3.20}$$

where

$$\begin{aligned} \alpha &= 1 - \mathrm{i}\frac{a_3 + c_3 + a_1 c_2 - a_2 c_1}{1 - \boldsymbol{a}.\boldsymbol{c}} = 1 - \mathrm{i}\tilde{c}_3 \\ \beta &= -\frac{a_2 + c_2 + a_3 c_1 - a_1 c_3}{1 - \boldsymbol{a}.\boldsymbol{c}} - \mathrm{i}\frac{a_1 + c_1 + a_2 c_3 - a_3 c_2}{1 - \boldsymbol{a}.\boldsymbol{c}} = -\tilde{c}_2 - \mathrm{i}\tilde{c}_1. \end{aligned} \tag{3.21}$$

For $\tilde{\boldsymbol{c}}$ we have that

$$\tilde{c}^2 = \frac{a^2 + c^2 + (\boldsymbol{a} \times \boldsymbol{c})^2 + 2\boldsymbol{a}.\boldsymbol{c}}{(1 - \boldsymbol{a}.\boldsymbol{c})^2} = \frac{(1+c^2)(1+a^2)}{(1 - \boldsymbol{a}.\boldsymbol{c})^2} - 1. \tag{3.22}$$

Thus

$$\frac{1}{\sqrt{1+\tilde{c}^2}} = \frac{|1 - \boldsymbol{a}.\boldsymbol{c}|}{\sqrt{1+a^2}\sqrt{1+c^2}}. \tag{3.23}$$

Now from (3.19), (3.20) and (3.23) we get that $\mathcal{U}_2(\boldsymbol{a})\mathcal{U}_1(\boldsymbol{c}) = \mathcal{U}(\tilde{\boldsymbol{c}})$ up to a sign. The case $\boldsymbol{a}.\boldsymbol{c} = 1$ in Proposition 3, as well as the cases where half-turns are involved in the composition will be treated elsewhere. □

Note that Proposition 3 holds also for the negative signs of the above sections. If $\boldsymbol{c}_1, \boldsymbol{c}_2$ are represent two SO(3,$\mathbb{R}$) rotations and the vectors $\mathbf{a}_1, \mathbf{a}_2$ are defined by the section $\mathbf{a}_+$ in (3.2) then the SO(3,$\mathbb{R}$) vector parameter corresponding to $\langle \mathbf{a}_2, \mathbf{a}_1 \rangle_{\mathrm{SU}(2)}$ is exactly $\langle \boldsymbol{c}_2, \boldsymbol{c}_1 \rangle_{\mathrm{SO}(3,\mathbb{R})}$, i.e., we have the commutative diagram below. Therefore, the pull-back of the composition in SO(3,$\mathbb{R}$) to the covering group SU(2) allows to bypass the singularities in the vector-parameter description of the base manifold.

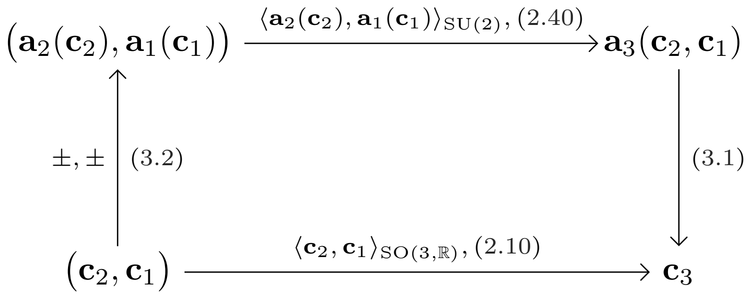

Figure 4: Composition of the three-dimensional rotations through a pull-back to the covering group SU(2).

4. CONCLUDING REMARKS

Despite of the attractive simplicity of the composition law for SO(3,$\mathbb{R}$) rotations, neither the half-turns nor the composition of rotations whose *Gibbs* vector-parameters have a scalar product equal to one are directly manageable. The derived vector-parametrization of SU(2) has the advantage to represent all rotations including the half-turns. Table 1 presents the numbers of operations needed for the composition of two rotations.

Table 1: The numbers of operations necessary to perform when composing two rotations in various representations.

Representations		Multiplications	Additions	Memory needed for the result
SO(3,$\mathbb{R}$)	matrix	27	18	9
	vector-parameter	12	12	3
SU(2)	matrix	16	16	4
	vector-parameter	28	18	3

ACKNOWLEDGEMENTS. The authors would like to thank to the anonymous referee for the helpful suggestions and for recommending various improvements in exposition. The first named author was partially supported by the Sofia University Research Fund through Grant no. 144/2015.

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Received on February 1, 2015
In revised form on April 27, 2015

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