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DETERMINISTIC SQEMA AND APPLICATION FOR PRE-CONTACT LOGIC

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SQEMA is a set of rules for finding first-order correspondents of modal formulas, and can be used for proving axiomatic completeness. SQEMA succeeds for the Sahlqvist and Inductive formulas.

A deterministic, terminating, but sometimes failing algorithm based on SQEMA for a modal language with nominals, reversed modalities and the universal modality - ML(T, U) - is presented. Deterministic SQEMA finds first-order correspondents, and it can be used to prove di-persistence. It succeeds for the Sahlqvist and Inductive formulas.

The axiomatic system for ML(T, U) is shown and its strong completeness is proven. It is shown that adding di-persistent formulas as axioms preserves strong completeness.

Deterministic SQEMA is extended for the language of pre-contact logics using a modified translation into ML(T, U). Deterministic SQEMA succeeds for the Sahlqvist class of pre-contact formulas.

Keywords: SQEMA, correspondence, Sahlqvist, modal, logic

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1. INTRODUCTION

The problem of the existence of first-order correspondent formulas for modal formulas was proposed by van Benthem. This problem is not computable, as shown by Chagrova in her PhD thesis in 1989, see [4]. However, there have been solutions for some modal formulas. The most famous class of formulas for which there is a first-order correspondent is the Sahlqvist class, shown in [19], where one can use the Sahlqvist-van Benthem algorithm as described in [23] and [3] to obtain first-order correspondents.

There are other algorithms for finding first-order correspondents, for example in [11] Gabbay and Ohlbach introduced the SCAN algorithm, and in [20], Szalas introduced DLS. SCAN is based on a resolution procedure applied on a Skolemized translation of the modal formula into the second-order logic, while DLS works on the same translation, but is based on a transformation procedure using a lemma by Ackermann. Both algorithms use a procedure of unskolemization, which is not always successful.

In [6, 7, 8, 10, 9] another algorithm, called SQEMA, for computing first-order correspondents in modal logic is introduced. It is based on a modal version of the Ackermann Lemma. SQEMA works directly on the modal formulas without translating them into the second-order logic and without using Skolemization. SQEMA succeeds not only on all Sahlqvist formulas, but also on the extended class of *inductive formulas* introduced in [5, 16]. There are examples of modal formulas on which SQEMA succeeds, while both SCAN and DLS fail, e.g.: $(\Box(\Box p \leftrightarrow q) \to p)$.

As proved in [6, 7, 8] SQEMA only succeeds on d-persistent (for languages without nominals) or di-persistent (for reversive languages with nominals) — and hence, by [3, 5, 15, 16], canonical formulas, i.e., whenever successful, it not only computes a local first-order correspondent of the input modal formula, but also proves its canonicity and therefore the canonical completeness of the modal logic axiomatized with that formula. This extends to any set of modal formulas on which SQEMA succeeds. Thus, SQEMA can also be used as an automated prover of canonical model completeness of modal logics.

An implementation of SQEMA in Java was given in [13]. Some additional simplifications were added to the implementation thanks to a suggestion by Renate Schmidt, which helps the implementation to succeed on formulas such as $((\Box\Diamond p \to \Diamond\Box p) \vee (\Box p \to \Diamond p))$.

The universal modality and nominals were introduced in [17].

In [14], SQEMA was augmented to $\mathrm{ML}(\Box, [U])$, the basic modal language extended by adding the universal modality. In [7], SQEMA for a reversive language with nominals is discussed, promising an extension with $[U]$. In [10], SQEMA with downwards monotonicity for Ackermann's rule is presented. In [22, 9], an extension of SQEMA for a reversive language with $[U]$ and nominals is introduced, with the output being in the first-order μ-calculus.

In this paper, we define a deterministic and terminating strategy for using the SQEMA rules for the language with universal modality, countably infinitely many couples of converse modalities, and nominals, $\mathrm{ML}(T, U)$. We show that Deterministic SQEMA always succeeds on Sahlqvist and inductive formulas. We show, like in [7], that Deterministic SQEMA succeeds only on di-persistent formulas. We show the axiomatic system for $\mathrm{ML}(T, U)$ and its strong completeness, following closely [17, 18, 12, 3]. Like in [15, 16, 5, 21], we show strong completeness of di-persistent formulas. Therefore, Deterministic SQEMA can be used to prove strong axiomatic completeness of a formula. We extend Deterministic SQEMA to the language of pre-contact logics, using a modified form of the translation from [1] as to obtain Sahlqvist formulas from Sahlqvist formulas of the pre-contact language, as defined in [2], so that Deterministic SQEMA succeeds on them. Completeness of all pre-contact formulas is shown in [1].

2. PRELIMINARIES

We use $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{m}$, $\boldsymbol{n}$ for natural numbers. If a and b are words, we write $a \nearrow b$ iff a occurs in b. If a is a word and b is a sequence or a set of words, $a \nearrow b$ means that a occurs in some of the words of b. The negation of $a \nearrow b$ is denoted by $a \not\nearrow b$.

Definition 1. (Formulas of $\mathrm{ML}(T, U)$) Formulas of $\mathrm{ML}(T, U)$ are:

$$\boldsymbol{\phi} ::= \bot|\top|p_{\boldsymbol{i}}|c_{\boldsymbol{i}}|\neg\boldsymbol{\phi}|(\boldsymbol{\phi} \vee \boldsymbol{\phi})|(\boldsymbol{\phi} \wedge \boldsymbol{\phi})|\Diamond_{\boldsymbol{i}}\boldsymbol{\phi}|\Diamond_{\boldsymbol{i}}^{-1}\boldsymbol{\phi}|\Box_{\boldsymbol{i}}\boldsymbol{\phi}|\Box_{\boldsymbol{i}}^{-1}\boldsymbol{\phi}$$

where $c_0, c_1, \ldots$ are *nominals*, $p_0, p_1, \ldots$ are *propositional variables*, and there are at most countably many pairs of mutually converse *boxes* and *diamonds*. We denote "any box" by $\Box$, its converse by $\Box^{-1}$, "any diamond" by $\Diamond$, its converse by $\Diamond^{-1}$. $\langle U\rangle$ means $\Diamond_0$, $[U]$ means $\Box_0$. $\mathrm{PROP}(\boldsymbol{\phi})$ is the set of propositional variables, occurring in $\boldsymbol{\phi}$. $\mathrm{NOM}(\boldsymbol{\phi})$ is the set of nominals, occurring in $\boldsymbol{\phi}$. $\boldsymbol{\phi}$ is *pure* iff $\mathrm{PROP}(\boldsymbol{\phi}) = \emptyset$. $(\boldsymbol{\phi_1} \rightarrow \boldsymbol{\phi_2})$ stands for $(\neg\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$, $\bigwedge(\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n})$ for $\boldsymbol{n} \geq 0$ and different $\boldsymbol{\phi_i}$ stands for $(\boldsymbol{\phi_1} \wedge \ldots (\boldsymbol{\phi_{n-1}} \wedge \boldsymbol{\phi_n}) \ldots)$ if $\boldsymbol{n} > 0$, and $\top$ otherwise. $\bigvee(\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n})$ for $\boldsymbol{n} \geq 0$ and different $\boldsymbol{\phi_i}$ stands for $(\boldsymbol{\phi_1} \vee \ldots (\boldsymbol{\phi_{n-1}} \vee \boldsymbol{\phi_n}) \ldots)$ if $\boldsymbol{n} > 0$, and $\bot$ otherwise. We emphasize a disjunction in a formula by using $\veebar$ instead of $\vee$. We also use $\boldsymbol{\gamma}$ for formulas. We use the standard definitions for a *positive/negative occurrence* of $\boldsymbol{p}$ in $\boldsymbol{\phi}$, for $\boldsymbol{\phi}$ being *positive/negative* in $\boldsymbol{p}$, and for $\boldsymbol{\phi}$ being *positive/negative*.

Definition 2. (New Nominal) We denote by $c_{\boldsymbol{k}} \not\nearrow^{\infty} \boldsymbol{\phi}$ iff $c_{\boldsymbol{k}}$ is the first nominal, such that for all $\boldsymbol{n} \geq \boldsymbol{k}$: $c_{\boldsymbol{n}} \not\nearrow \boldsymbol{\phi}$. We denote by $c_{\boldsymbol{k}} \not\nearrow^{\infty} \Gamma$ for a set of modal formulas Γ iff for all $\boldsymbol{\phi} \in \Gamma$: $c_{\boldsymbol{k}} \not\nearrow^{\infty} \boldsymbol{\phi}$.

Definition 3. (Kripke Frame) A *Kripke frame* for $\mathrm{ML}(T, U)$, or just a *frame*, is a tuple $\langle W, \mathcal{R}\rangle$, where W is a non-empty set of *possible worlds*, also a *universe*, and for all $\boldsymbol{i}$, $\mathcal{R}(\boldsymbol{i}) \subseteq W \times W$ are *accessibility relations*, where $\mathcal{R}(0) = W \times W$. We use $\mathbf{F}$ for frames, $\boldsymbol{w}$, $\boldsymbol{u}$, $\boldsymbol{v}$ for possible worlds, and $\boldsymbol{s}$ for sets of possible worlds. If $\boldsymbol{w} \in \boldsymbol{W}$, we say that $\boldsymbol{w}$ is *in* $\mathbf{F}$.

Definition 4. (Kripke Model) Let $\mathbf{F} = \langle \boldsymbol{W}, \mathcal{R}\rangle$. A *Kripke model* for $\mathrm{ML}(T, U)$, or just a *model*, is a tuple $\langle \mathbf{F}, V, A\rangle$, where $V : \mathrm{PROP} \rightarrow \mathbb{P}(\boldsymbol{W})$ is a *valuation*, and $A : \mathrm{NOM} \rightarrow \boldsymbol{W}$ is an *assignment*. We say that the model thus defined is *based on* $\boldsymbol{F}$. We use $\mathbf{M}$ for models, $\boldsymbol{V}$ for valuations, $\boldsymbol{A}$ for assignments. If $\boldsymbol{w} \in \boldsymbol{W}$, we say that $\boldsymbol{w}$ is *in* $\mathbf{M}$. $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A}\rangle$ is *named* iff $\boldsymbol{A}$ is surjective.

Definition 5. (Modal Truth and Validity). Let $\mathbf{F} = \langle \boldsymbol{W}, \mathcal{R} \rangle$, $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$, and $\boldsymbol{w} \in \boldsymbol{W}$. We say that, by induction on $\boldsymbol{\phi}$, $\boldsymbol{\phi}$ is *true in* $\mathbf{M}$ *at* $\boldsymbol{w}$, denoted by $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff:

- $\mathbf{M}, \boldsymbol{w} \Vdash \top$
- $\mathbf{M}, \boldsymbol{w} \nVdash \bot$
- $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{p}$ iff $\boldsymbol{w} \in \boldsymbol{V}(\boldsymbol{p})$
- $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{c}$ iff $\boldsymbol{w} = \boldsymbol{A}(\boldsymbol{c})$
- $\mathbf{M}, \boldsymbol{w} \Vdash \neg\boldsymbol{\phi_1}$ iff $\mathbf{M}, \boldsymbol{w} \nVdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$ iff $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ or $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$
- $\mathbf{M}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$ iff $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ and $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Diamond_{\boldsymbol{i}}\boldsymbol{\phi_1}$ iff for some $v \in \boldsymbol{W}$: $\boldsymbol{w}\ \mathcal{R}(\boldsymbol{i})\ v$ and $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Diamond_{\boldsymbol{i}}^{-1}\boldsymbol{\phi_1}$ iff for some $v \in \boldsymbol{W}$: $v\ \mathcal{R}(\boldsymbol{i})\ \boldsymbol{w}$ and $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Box_{\boldsymbol{i}}\boldsymbol{\phi_1}$ iff for all $v \in \boldsymbol{W}$: $\boldsymbol{w}\ \mathcal{R}(\boldsymbol{i})\ v$ implies $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$
- $\mathbf{M}, \boldsymbol{w} \Vdash \Box_{\boldsymbol{i}}^{-1}\boldsymbol{\phi_1}$ iff for all $v \in \boldsymbol{W}$: $v\ \mathcal{R}(\boldsymbol{i})\ \boldsymbol{w}$ implies $\mathbf{M}, v \Vdash \boldsymbol{\phi_1}$

We say that $\boldsymbol{\phi}$ is *true in* $\mathbf{M}$ iff for all $\boldsymbol{w} \in \mathbf{M}$: $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is *valid in* $\mathbf{F}$ *at* $\boldsymbol{w}$ (*local validity*), denoted by $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$, iff for every model $\mathbf{M}$ based on $\mathbf{F}$, $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is *valid in* $\mathbf{F}$ (*frame validity*) iff $\boldsymbol{\phi}$ is true in every model based on $\mathbf{F}$ iff for all $\boldsymbol{w}$ in $\mathbf{F}$, $\boldsymbol{\phi}$ is valid in $\mathbf{F}$ at $\boldsymbol{w}$. We say that $\boldsymbol{\phi}$ is *valid*, denoted by $\Vdash \boldsymbol{\phi}$, iff it is valid in all frames. The *extension* of $\boldsymbol{\phi}$ in $\mathbf{M}$, denoted by $[\![\boldsymbol{\phi}]\!]_{\mathbf{M}}$, is the set of all $w \in \boldsymbol{W}$ such that $\mathbf{M}, w \Vdash \boldsymbol{\phi}$. It is clear that, if $\mathbf{M_1} = \langle \mathbf{F}, \boldsymbol{V_1}, \boldsymbol{A_1} \rangle$ and $\mathbf{M_2} = \langle \mathbf{F}, \boldsymbol{V_2}, \boldsymbol{A_2} \rangle$ *agree on* $\mathrm{NOM}(\boldsymbol{\phi}) \cup \mathrm{PROP}(\boldsymbol{\phi})$, meaning that $\boldsymbol{V_1} \restriction \mathrm{PROP}(\boldsymbol{\phi}) = \boldsymbol{V_2} \restriction \mathrm{PROP}(\boldsymbol{\phi})$ and $\boldsymbol{A_1} \restriction \mathrm{NOM}(\boldsymbol{\phi}) = \boldsymbol{A_2} \restriction \mathrm{NOM}(\boldsymbol{\phi})$, then $[\![\boldsymbol{\phi}]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi}]\!]_{\mathbf{M_2}}$. We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *semantically equivalent*, denoted by $\boldsymbol{\phi_1} \equiv \boldsymbol{\phi_2}$, iff for every model $\mathbf{M}$, their extensions in $\mathbf{M}$ are equal. We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *opposite* iff $\boldsymbol{\phi_1} \equiv \neg\boldsymbol{\phi_2}$ We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally frame-equivalent*, denoted by $\boldsymbol{\phi_1} \sim \boldsymbol{\phi_2}$, iff for every frame, $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$: $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ iff $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$. We define a *modified assignment* as follows: $\boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}](\boldsymbol{c}) := \boldsymbol{w}$ and $\boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}](\boldsymbol{c'}) := \boldsymbol{A}(\boldsymbol{c'})$ for all $\boldsymbol{c'}$ distinct from $\boldsymbol{c}$. We define a *modified model* as follows: $\mathbf{M}[\boldsymbol{c} \to \boldsymbol{w}] := \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A}[\boldsymbol{c} \to \boldsymbol{w}] \rangle$.

If for a $\mathbf{F}$ and for a $\boldsymbol{w}$ in $\mathbf{F}$, we have that for every $\mathbf{M}$ over $\mathbf{F}$, there is a $\boldsymbol{w_2}$ in $\mathbf{F}$: $\mathbf{M}[c_{\boldsymbol{k}} \to \boldsymbol{w}], \boldsymbol{w_2} \Vdash \boldsymbol{\phi}$, then $\boldsymbol{\phi}$ is called *relatively* $\boldsymbol{k}$*-true in* $\mathbf{F}$ *at* $\boldsymbol{w}$, $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi}$.

For a given $\boldsymbol{k}$, two formulas $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally frame-equivalent with respect to* $c_{\boldsymbol{k}}$, denoted by $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\phi_2}$, iff for every frame $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$, we have that $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_1}$ iff $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_2}$.

Definition 6. (Uniform Substitution) We denote by $\boldsymbol{\phi_1}[\boldsymbol{p}/\boldsymbol{\phi'}]$ the word obtained from $\boldsymbol{\phi_1}$, where each occurrence of $\boldsymbol{p}$ (if any) has been replaced with $\boldsymbol{\phi'}$. According to Definition 1, the word thus constructed is also a formula, $\boldsymbol{\phi_2}$. We call the rule for obtaining $\boldsymbol{\phi_2}$ from $\boldsymbol{\phi_1}$ *uniform substitution* of $\boldsymbol{p}$ by $\boldsymbol{\phi'}$ in $\boldsymbol{\phi_1}$.

Proposition 7. (Properties of the Uniform Substitution)

1. Let $\mathbf{M_1}$ and $\mathbf{M_2}$ be based on $\mathbf{F}$ and be such that they agree on the nominals and variables, occurring in $\boldsymbol{\phi}$, except for $\boldsymbol{p}$. If $[\![\boldsymbol{\phi'}]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi''}]\!]_{\mathbf{M_2}}$, then $[\![\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]]\!]_{\mathbf{M_1}} = [\![\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi''}]]\!]_{\mathbf{M_2}}$.

2. If $\mathbf{F} \Vdash \boldsymbol{\phi}$, then $\mathbf{F} \Vdash \boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]$.

3. If $\boldsymbol{\phi'} \equiv \boldsymbol{\phi''}$, then $\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}] = \boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi''}]$.

Proof. Follows directly from the definitions. □

If the elements of PROP($\boldsymbol{\phi}$) are, in left-to-right order of initial occurrence in $\boldsymbol{\phi}$: $\boldsymbol{p_1}, \dots, \boldsymbol{p_n}$ with $\boldsymbol{n} \geq 0$, and the elements of NOM($\boldsymbol{\phi}$) are, in left-to-right order of initial occurrence in $\boldsymbol{\phi}$: $\boldsymbol{c_1}, \dots, \boldsymbol{c_m}$ with $\boldsymbol{m} \geq 0$, then $[\![\boldsymbol{\phi}]\!]$ is an operator from $\boldsymbol{n}$,$\boldsymbol{m}$-tuples of $\boldsymbol{n}$ sets of states and $\boldsymbol{m}$ states to a set of states, defined thus: if $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$ is a model, $\boldsymbol{V}(\boldsymbol{p_1}) = s_1, \dots, \boldsymbol{V}(\boldsymbol{p_n}) = s_n$, $\boldsymbol{A}(\boldsymbol{c_1}) = w_1, \dots, \boldsymbol{A}(\boldsymbol{c_m}) = w_m$, then $[\![\boldsymbol{\phi}]\!](s_1, \dots, s_n, w_1, \dots, w_m)$ is $[\![\boldsymbol{\phi}]\!]_{\mathbf{M}}$.

Definition 8. (General Discrete Frame) Let $\mathbf{F} = \langle \boldsymbol{W}, \mathcal{R} \rangle$. We say that $\langle \mathbf{F}, \mathbb{W} \rangle$ is a *general discrete frame for* ML(T, U), or just a *general discrete frame*, iff $\mathbb{W} \subseteq \mathbb{P}(\boldsymbol{W})$ is non-empty and the following conditions hold:
- for every $\boldsymbol{w} \in \boldsymbol{W}$, $\{\boldsymbol{w}\} \in \mathbb{W}$
- $\mathbb{W}$ is closed under $[\![\neg p_0]\!]$
- $\mathbb{W}$ is closed under $[\![(p_0 \vee p_1)]\!]$
- $\mathbb{W}$ is closed under $[\![\Diamond p_0]\!]$ for all diamonds $\Diamond$.

$\mathbb{W}$ is the set of *admissible valuations.* It is clear that $\boldsymbol{W}, \emptyset \in \mathbb{W}$. It is clear that $\mathbf{F}$ with universe $\boldsymbol{W}$ is also the *full general discrete frame* $\langle \mathbf{F}, \mathbb{P}(\boldsymbol{W}) \rangle$. We use $\mathfrak{g}$ for general discrete frames, $\mathbb{W}$ for sets of admissible valuations. If $\mathfrak{g} = \langle \mathbf{F}, \mathbb{W} \rangle$, then we denote by $\mathfrak{g}_\#$ the *underlying frame* of $\mathfrak{g}$, $\mathbf{F}$. $\boldsymbol{w}$ is in $\mathfrak{g}$ iff it is in $\mathfrak{g}_\#$. $\mathbf{M} = \langle \mathfrak{g}_\#, \boldsymbol{V}, \boldsymbol{A} \rangle$ is a *model over* $\mathfrak{g}$ iff for each propositional variable p, $\boldsymbol{V}(p) \in \mathbb{W}$. An induction on $\boldsymbol{\phi}$ shows that if $\mathbf{M}$ is a model over $\mathfrak{g}$, then $[\![\boldsymbol{\phi}]\!]_{\mathbf{M}} \in \mathbb{W}$ for any formula. $\boldsymbol{\phi}$ is *valid* in $\mathfrak{g}$, denoted with $\mathfrak{g} \Vdash \boldsymbol{\phi}$, iff it is true in all models over $\mathfrak{g}$.

$\boldsymbol{\phi}$ is *di-persistent* iff for every $\mathfrak{g}$, $\mathfrak{g} \Vdash \boldsymbol{\phi}$ iff $\mathfrak{g}_\# \Vdash \boldsymbol{\phi}$. As we show later, di-persistence is a sufficient condition for strong axiomatic completeness of a formula.

For every named $\mathbf{M}$ over $\mathbf{F}$, there is a $\mathfrak{g} = \langle \mathbf{F}, \{[\![\phi]\!]_{\mathbf{M}} \mid \phi \in \mathrm{ML}(T, U)\} \rangle$. This helps to prove strong axiomatic completeness of di-persistent formulas.

Definition 9. (Local Equivalence for General Discrete Frames) We say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally di-equivalent*, denoted by $\boldsymbol{\phi_1} \approx \boldsymbol{\phi_2}$, iff for every $\mathfrak{g}$ and every $\boldsymbol{w}$ in $\mathfrak{g}$: $\mathfrak{g}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ iff $\mathfrak{g}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$.

If for a $\mathfrak{g}$ and for a $\boldsymbol{w}$ in $\mathfrak{g}$, we have that for every $\mathbf{M}$ over $\mathfrak{g}$, there is a $\boldsymbol{w_2}$ in $\mathfrak{g}$: $\mathbf{M}[c_{\boldsymbol{k}} \to \boldsymbol{w}], \boldsymbol{w_2} \Vdash \boldsymbol{\phi}$, then $\boldsymbol{\phi}$ is called *relatively* $\boldsymbol{k}$*-true in* $\mathfrak{g}$ *at* $\boldsymbol{w}$, $\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi}$.

For a given $\boldsymbol{k}$, we say that $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are *locally di-equivalent with respect to* $c_{\boldsymbol{k}}$, denoted by $\boldsymbol{\phi_1} \approx_{\boldsymbol{k}} \boldsymbol{\phi_2}$, iff for every general discrete frame $\mathfrak{g}$ and every $\boldsymbol{w}$ in $\mathfrak{g}$, we have that $\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_1}$ iff $\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi_2}$.

Because every frame is also a full general discrete frame, if $\boldsymbol{\phi_1} \approx_{\boldsymbol{k}} \boldsymbol{\phi_2}$, then $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\phi_2}$, and if $\boldsymbol{\phi_1} \approx \boldsymbol{\phi_2}$, then $\boldsymbol{\phi_1} \sim \boldsymbol{\phi_2}$.

Proposition 10. (Sufficient Condition for Di-Persistence) Let $\boldsymbol{\phi}$ be a modal formula, let $c_{\boldsymbol{k}}$ be such that $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, let $\boldsymbol{\phi'}$ be such that $\boldsymbol{\phi'}$ is a pure formula and $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi}) \approx_{\boldsymbol{k}} \boldsymbol{\phi'}$. Then $\boldsymbol{\phi}$ is di-persistent.

Proof. Let $\mathfrak{g}$ be a general discrete frame and let $\boldsymbol{w}$ be in $\mathfrak{g}$. Then:

$\mathfrak{g}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff (because $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$)

$\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} (c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$ iff (because of di-equivalence)
$\mathfrak{g}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi'}$ iff (because $\boldsymbol{\phi'}$ is pure)
$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi'}$ iff (because of di-equivalence)
$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} (c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$ iff (because $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$)
$\mathfrak{g}_{\#}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$. □
A direct corollary to the above is that every pure formula is di-persistent.

3. FIRST-ORDER CORRESPONDENCE PROBLEM

We define a first-order language with equality and binary predicate symbols. The language is called FOL. We use $\boldsymbol{\psi}$ for *FOL formulas.*

Definition 11. (First-Order formulas) FOL formulas are:

$$\boldsymbol{\psi} ::= \bot | \top | (\boldsymbol{x'} = \boldsymbol{x''}) | (\boldsymbol{x'}\ r_{\boldsymbol{i}}\ \boldsymbol{x''}) | \neg\boldsymbol{\psi} | (\boldsymbol{\psi} \vee \boldsymbol{\psi}) | (\boldsymbol{\psi} \wedge \boldsymbol{\psi}) | \exists \boldsymbol{x}\boldsymbol{\psi} | \forall \boldsymbol{x}\boldsymbol{\psi},$$

where $x_0, x_1, \ldots$ are *individual variables*, $r_1, r_2, \ldots$ are *binary relational predicate symbols*, $=$ is *equality*, $\exists$ and $\forall$ are *quantifiers*. An occurrence of $\boldsymbol{x}$ in $\boldsymbol{\psi}$ is *bound* iff it occurs in an occurrence of $\exists \boldsymbol{x}\boldsymbol{\psi_1}$ or of $\forall \boldsymbol{x}\boldsymbol{\psi_1}$ in $\boldsymbol{\psi}$. Any occurrence of $\boldsymbol{x}$ in $\boldsymbol{\psi}$ that is not bound is *free.* We say that $\boldsymbol{x}$ is a *free variable of* $\boldsymbol{\psi}$ iff $\boldsymbol{\psi}$ contains a free occurrence of $\boldsymbol{x}$. We say that $\boldsymbol{\psi}$ is *closed*, or that $\boldsymbol{\psi}$ is a *sentence* iff it has no free variables. We denote by FREE($\boldsymbol{\psi}$) the set of all free variables of $\boldsymbol{\psi}$. If the elements of FREE($\boldsymbol{\psi}$) are, in left-to-right order of initial occurrence in $\boldsymbol{\psi}$, $\boldsymbol{x_1}, \ldots, \boldsymbol{x_k}$ for some $\boldsymbol{k} \geq 0$, then we denote $\boldsymbol{\psi}$ by $\boldsymbol{\psi}(\boldsymbol{x_1}, \ldots, \boldsymbol{x_n})$, where $\boldsymbol{n} > 0$ and $\boldsymbol{n} \geq \boldsymbol{k}$.

Definition 12. (Semantics of FOL formulas) Let $\mathbf{F}$ be a Kripke frame and let $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$ be a Kripke model over $\mathbf{F}$. We extend $\boldsymbol{A}$ to all individual variables as follows: $\boldsymbol{A}(x_{\boldsymbol{i}}) := \boldsymbol{A}(c_{\boldsymbol{i}})$. We use the usual semantics of $\mathbf{M} \vDash \boldsymbol{\psi}$. We say that $\boldsymbol{\psi}(\boldsymbol{x})$ is *true in* $\mathbf{F}$ *at* $\boldsymbol{w}$, $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$ iff for some model $\mathbf{M}$ over $\mathbf{F}$: $\mathbf{M}[c_{\boldsymbol{i}} \to \boldsymbol{w}] \vDash \boldsymbol{\psi}(x_{\boldsymbol{i}})$. We say that $\boldsymbol{\psi}$ is *valid* in $\mathbf{F}$ (*Kripke frame validity*) iff $\boldsymbol{\psi}$ is true in every model based on $\mathbf{F}$. Thus, $\boldsymbol{\psi}(\boldsymbol{x})$ is valid in $\mathbf{F}$ iff $\boldsymbol{\phi}(\boldsymbol{x})$ is true in $\mathbf{F}$ at every state in $\mathbf{F}$.

Definition 13. (First-order Correspondence) We say that a modal formula $\boldsymbol{\phi}$ and FOL formula $\boldsymbol{\psi}(\boldsymbol{x})$ are *locally correspondent*, denoted $\boldsymbol{\phi} \sim \boldsymbol{\psi}(\boldsymbol{x})$, iff for every frame $\mathbf{F}$ and every state $\boldsymbol{w}$ in $\mathbf{F}$: $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$. We say that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are *globally correspondent* iff for every frame $\mathbf{F}$: $\mathbf{F} \Vdash \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}$. It is clear that, if $\boldsymbol{\phi}$ and $\boldsymbol{\psi}(\boldsymbol{x})$ are locally correspondent, then they are globally correspondent.

For a given $\boldsymbol{k}$, we say that $\boldsymbol{\phi}$ and $\boldsymbol{\psi}(x_{\boldsymbol{k}})$ are *locally correspondent with respect to* $c_{\boldsymbol{k}}$, denoted by $\boldsymbol{\phi} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, iff for every frame $\mathbf{F}$ and every $\boldsymbol{w}$ in $\mathbf{F}$, we have that: $\mathbf{F}, \boldsymbol{w} \Vdash_{\boldsymbol{k}} \boldsymbol{\phi}$ iff $\mathbf{F} \vDash \boldsymbol{\psi}[\boldsymbol{w}]$

An easy argument shows that if $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, and $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi}) \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, then $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Also, if $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\phi_2}$, and if $\boldsymbol{\phi_2} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$, then $\boldsymbol{\phi_1} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$.

Combined with the properties of local frame equivalence, we now have a sufficient condition for the existence of a local first-order correspondent:

Proposition 14. (Sufficient Condition for First-order Correspondence) Let $\boldsymbol{\phi}$ be a modal formula, let $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, let $\boldsymbol{\phi'}$ be such that $\boldsymbol{\phi'}$ is a pure formula, $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi}) \sim_{\boldsymbol{k}} \boldsymbol{\phi'}$, and $\boldsymbol{\phi'} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Then $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$. □

An immediate corollary of propositions 14 and 10 is the following:

Proposition 15. (Sufficient Condition for Di-Persistence and First-order Correspondence) Let $\boldsymbol{\phi}$ be a modal formula, let $c_{\boldsymbol{k}}$ be such that $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$, let $\boldsymbol{\phi_1}, \ldots, \boldsymbol{\phi_n}$ be a sequence, such that $\boldsymbol{\phi_1}$ is $(c_{\boldsymbol{k}} \wedge \boldsymbol{\phi})$, $\boldsymbol{\phi_i} \approx_{\boldsymbol{k}} \boldsymbol{\phi_j}$ for any $1 \leq i \leq j \leq n$, and $\boldsymbol{\phi_n}$ is a pure formula, such that $\boldsymbol{\phi_n} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$. Then it follows that:

1. $\boldsymbol{\phi} \sim \boldsymbol{\psi}(x_{\boldsymbol{k}})$.
2. $\boldsymbol{\phi}$ is di-persistent. □

As we show later, it is enough to have $c_{\boldsymbol{k}}$ and $\boldsymbol{\phi_n}$ to find a $\boldsymbol{\psi}(x_{\boldsymbol{k}})$, and also $c_{\boldsymbol{k}}$ is uniquely defined for $\boldsymbol{\phi}$. Therefore, if such a sequence for $\boldsymbol{\phi}$ exists, we call $\boldsymbol{\psi}(x_{\boldsymbol{k}})$ a *solution for* $\boldsymbol{\phi}$.

Therefore, a good approach for both finding first-order correspondents and for proving that a formula is di-persistent is to have rules for elimination of propositional variables that replace formulas with formulas that are locally di-equivalent with respect to a given nominal.

For reducing the size of the problem, we need a lemma for conjunctions.

Lemma 16. (Conjunction Lemma)

1. Let $\boldsymbol{\phi_1} \sim \boldsymbol{\psi_1}(x_{\boldsymbol{k}})$ and $\boldsymbol{\phi_2} \sim \boldsymbol{\psi_2}(x_{\boldsymbol{k}})$. Then $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \sim (\boldsymbol{\psi_1}(x_{\boldsymbol{k}}) \wedge \boldsymbol{\psi_2}(x_{\boldsymbol{k}}))$.
2. If $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are di-persistent, then so is $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$.

Proof. For 1, let $\boldsymbol{w}$ be a world in $\mathbf{F}$. Then, by the hypothesis, $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ iff $\mathbf{F} \Vdash \boldsymbol{\psi_1}[\boldsymbol{w}]$ and $\mathbf{F}, \boldsymbol{w} \vDash \boldsymbol{\phi_2}$ iff $\mathbf{F} \vDash \boldsymbol{\psi_2}[\boldsymbol{w}]$. Let $\mathbf{F}, \boldsymbol{w} \Vdash (\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$. Then, $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_1}$ and $\mathbf{F}, \boldsymbol{w} \Vdash \boldsymbol{\phi_2}$. Therefore, $\mathbf{F} \vDash \boldsymbol{\psi_1}[\boldsymbol{w}]$ and $\mathbf{F} \vDash \boldsymbol{\psi_2}[\boldsymbol{w}]$, so $\mathbf{F} \vDash (\boldsymbol{\psi_1}(x_{\boldsymbol{k}}) \wedge \boldsymbol{\psi_2}(x_{\boldsymbol{k}}))[\boldsymbol{w}]$. The converse direction is analogous.

For 2, it follows directly from the definition of di-persistence. □

Therefore, to find a solution for $\bigwedge(\gamma_1, \ldots, \gamma_n)$, it is enough to find solutions for each of $\gamma_1, \ldots, \gamma_n$ with respect to the same $c_{\boldsymbol{k}}$, such that $c_{\boldsymbol{k}} \not\nearrow \bigwedge(\gamma_1, \ldots, \gamma_n)$, and to take the conjunction of the solutions, then this becomes a solution for the whole formula.

4. DETERMINISTIC SQEMA

A formula $\boldsymbol{\phi}$ is in *negation normal form* iff $\neg$ occurs only in front of atomic formulas.

We follow [6, 10]. First, we give a simplified informal definition of the algorithm. Let $\boldsymbol{\phi}$ be the input modal formula. The goal is to obtain a nominal $c_{\boldsymbol{k}}$, and a pure formula $\boldsymbol{\phi'}$, such that $c_{\boldsymbol{k}} \not\nearrow \boldsymbol{\phi}$ and $\boldsymbol{\phi} \approx_{\boldsymbol{k}} \boldsymbol{\phi'}$. Then it is very easy, as we show below, to obtain a local first-order correspondent for $\boldsymbol{\phi}$.

First, we negate $\boldsymbol{\phi}$ and rewrite it in negation normal form, obtaining $\boldsymbol{\gamma}$. We start eliminating variables by a process similar to Gaussian elimination. Thus, we solve a *system of equations* (actually a conjunction of disjunctions), starting with a system with the single equation $(\neg c_{\boldsymbol{k}} \vee \boldsymbol{\gamma})$. We eliminate each variable separately, so let $\boldsymbol{p}$ be the current variable to eliminate. The elimination is carried out by applying the following rules:
Ackermann rule:

$$\begin{cases} \bigwedge((\boldsymbol{\alpha}_1 \vee \boldsymbol{p}), \ldots, (\boldsymbol{\alpha}_{n_a} \vee \boldsymbol{p})) \wedge \\ \bigwedge(\boldsymbol{\beta}_1(\neg \boldsymbol{p}), \ldots, \boldsymbol{\beta}_{n_b}(\neg \boldsymbol{p})) \wedge \\ \bigwedge(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}) \end{cases} \Rightarrow \begin{cases} \bigwedge(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b})[\boldsymbol{p}/\neg \bigwedge(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a})] \wedge \\ \bigwedge(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}) \end{cases}$$

where $\boldsymbol{p} \not\nearrow \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}\}$ and $\bigwedge(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b})$ is negative in $\boldsymbol{p}$.
$\Box$-rule: $(\boldsymbol{\phi_1} \vee \Box \boldsymbol{\phi_2}) \Rightarrow (\Box^{-1} \boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$
$\Diamond$-rule: $(\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{\phi}) \Rightarrow (\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{c''}) \wedge (\neg \boldsymbol{c''} \vee \boldsymbol{\phi})$, where $\boldsymbol{c''}$ is a new nominal.

Now we are ready to formalize the algorithm.

Proposition 17. (SQEMA rules)

1. *Equivalence rule.*
   If $\boldsymbol{\phi_1} \equiv \boldsymbol{\phi_2}$, then $\boldsymbol{\phi_1} \approx_{\boldsymbol{k}} \boldsymbol{\phi_2}$. As per Proposition 7, we can also replace (occurrences of) subformulas with semantically equivalent ones.
2. *Polarity reversing rule.*
   $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\neg \boldsymbol{p}]$.
3. *Positive elimination rule.*
   Let $\boldsymbol{\phi}$ be positive in $\boldsymbol{p}$. Then $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\top]$.
4. *Negative elimination rule.*
   Let $\boldsymbol{\phi}$ be negative in $\boldsymbol{p}$. Then $\neg \boldsymbol{\phi} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi}[\boldsymbol{p}/\bot]$.
5. $\Box$*-rule.*
   $\neg(\boldsymbol{\phi'} \wedge (\boldsymbol{\phi_1} \vee \Box \boldsymbol{\phi_2})) \approx_{\boldsymbol{k}} \neg(\boldsymbol{\phi'} \wedge (\Box^{-1} \boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}))$.
6. $\Diamond$*-rule.*
   Let $\boldsymbol{c''}$ be such that $\boldsymbol{c''} \not\nearrow \{c_{\boldsymbol{k}}, \boldsymbol{c'}, \boldsymbol{\phi'}, \boldsymbol{\phi}\}$. Then:
   $\neg(\boldsymbol{\phi'} \wedge (\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{\phi})) \approx_{\boldsymbol{k}} \neg(\boldsymbol{\phi'} \wedge ((\neg \boldsymbol{c'} \vee \Diamond \boldsymbol{c''}) \wedge (\neg \boldsymbol{c''} \vee \boldsymbol{\phi})))$.
7. *The Ackermann rule.* Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}$ be formulas which contain no occurrences of $\boldsymbol{p}$, let $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}$ be formulas which are either *negative* or *downwards monotone* in $\boldsymbol{p}$. Let:
   $\boldsymbol{\gamma'} := \neg \bigwedge((\boldsymbol{\alpha}_1 \vee \boldsymbol{p}), \ldots, (\boldsymbol{\alpha}_{n_a} \vee \boldsymbol{p}), \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$
   $\boldsymbol{\gamma''} := \neg \bigwedge(\bigwedge(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b})[\boldsymbol{p}/\neg \bigwedge(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a})], \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$
   Then: $\boldsymbol{\gamma'} \approx_{\boldsymbol{k}} \boldsymbol{\gamma''}$.

Proof. For the equivalence rule, the result follows immediately.

The rest of the rules are in the form $\neg \boldsymbol{\phi'} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi''}$ for some formulas $\boldsymbol{\phi'}$ and $\boldsymbol{\phi''}$. Let $\mathfrak{g}$ be a general discrete frame, and let $\boldsymbol{w}$ be a world in $\mathfrak{g}$. To prove that $\neg \boldsymbol{\phi'} \approx_{\boldsymbol{k}} \neg \boldsymbol{\phi''}$, it is enough to prove that for every model $\mathbf{M}$ over $\mathfrak{g}$, such that

$[\![c_{\boldsymbol{k}}]\!]_{\mathbf{M}} = \{\boldsymbol{w}\}$ and $\mathbf{M} \Vdash \boldsymbol{\phi'}$, there is a model $\mathbf{M'}$ over $\mathfrak{g}$, such that $[\![c_{\boldsymbol{k}}]\!]_{\mathbf{M'}} = \{\boldsymbol{w}\}$ and $\mathbf{M'} \Vdash \boldsymbol{\phi''}$, and vice versa.

Polarity reversing rule: Because negations of admissible valuations are admissible, we set $\mathbf{M'}$ to be equal to $\mathbf{M}$, except $[\![\boldsymbol{p}]\!]_{\mathbf{M'}}$ is set to be the complement of $[\![\boldsymbol{p}]\!]_{\mathbf{M}}$. The implication follows by Definition 5. The converse follows analogously.

Positive elimination rule: Let $\boldsymbol{W}$ be the universe of $\mathbf{M}$. By induction on $\boldsymbol{\phi}$, we get that $[\![\boldsymbol{\phi}]\!]_{\mathbf{M}} \subseteq [\![\boldsymbol{\phi}[\boldsymbol{p}/\top]]\!]_{\mathbf{M}}$.

First, let $[\![c_{\boldsymbol{k}}]\!]_{\mathbf{M}} = \{\boldsymbol{w}\}$ and $\mathbf{M} \Vdash \boldsymbol{\phi}$. We set $\mathbf{M'}$ to be equal to $\mathbf{M}$, except $[\![\boldsymbol{p}]\!]_{\mathbf{M'}}$ is set to be $\boldsymbol{W}$, which is admissible. We have that $\boldsymbol{W} = [\![\boldsymbol{\phi}]\!]_{\mathbf{M}} \subseteq [\![\boldsymbol{\phi}[\boldsymbol{p}/\top]]\!]_{\mathbf{M}} = [\![\boldsymbol{\phi}[\boldsymbol{p}/\top]]\!]_{\mathbf{M'}}$, by Proposition 7. Therefore, $[\![\boldsymbol{\phi}[\boldsymbol{p}/\top]]\!]_{\mathbf{M'}} = \boldsymbol{W}$.

Now, let $[\![c_{\boldsymbol{k}}]\!]_{\mathbf{M}} = \{\boldsymbol{w}\}$ and $\mathbf{M} \Vdash \boldsymbol{\phi}[\boldsymbol{p}/\top]$. We construct $\mathbf{M'}$ in the same way, and it is straightforward to prove that $[\![\boldsymbol{\phi}]\!]_{\mathbf{M'}} = \boldsymbol{W}$.

Negative elimination rule: Follows from the polarity reversing rule and the positive elimination rule.

$\Box$-rule: Let $R_{\Box}$ be the (converse) relation of $\mathbf{M}$, which corresponds to $\Box$.

First, let $\mathbf{M} \Vdash (\boldsymbol{\phi'} \wedge (\boldsymbol{\phi_1} \vee \Box\boldsymbol{\phi_2}))$, suppose that $\mathbf{M} \nVdash (\boldsymbol{\phi'} \wedge (\Box^{-1}\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}))$. Then, there is a $\boldsymbol{w_1} \in \boldsymbol{W}$: $\mathbf{M}, \boldsymbol{w_1} \nVdash (\Box^{-1}\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})$. Then, $\mathbf{M}, \boldsymbol{w_1} \nVdash \Box^{-1}\boldsymbol{\phi_1}$ and $\mathbf{M}, \boldsymbol{w_1} \nVdash \boldsymbol{\phi_2}$. Therefore, there is a $\boldsymbol{w_2} \in \boldsymbol{W}$: $\mathbf{M}, \boldsymbol{w_2} \nVdash \boldsymbol{\phi_1}$ and $\boldsymbol{w_2}\ R_{\Box}\ \boldsymbol{w_1}$. However, $\mathbf{M} \Vdash (\boldsymbol{\phi_1} \vee \Box\boldsymbol{\phi_2})$, therefore $\mathbf{M}, \boldsymbol{w_2} \Vdash \Box\boldsymbol{\phi_2}$, so $\mathbf{M}, \boldsymbol{w_1} \Vdash \boldsymbol{\phi_2}$, contradiction.

Now, let $\mathbf{M} \Vdash (\boldsymbol{\phi'} \wedge (\Box^{-1}\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}))$, suppose that $\mathbf{M} \nVdash (\boldsymbol{\phi'} \wedge (\boldsymbol{\phi_1} \vee \Box\boldsymbol{\phi_2}))$. Analogously to the above, we derive a contradiction.

$\Diamond$-rule: Let $\mathbf{M} = \langle \mathfrak{g}_{\#}, \boldsymbol{V}, \boldsymbol{A} \rangle$. Let $R_{\Diamond}$ be the relation or converse relation of $\mathbf{M}$, corresponding to $\Diamond$.

First, let $\mathbf{M} \Vdash (\boldsymbol{\phi'} \wedge (\neg \boldsymbol{c'} \vee \Diamond\boldsymbol{\phi}))$, and let $\boldsymbol{w_1} := \boldsymbol{A}(\boldsymbol{c'})$. Then, $\mathbf{M}, \boldsymbol{w_1} \Vdash \Diamond\boldsymbol{\phi}$. So, there is a $\boldsymbol{w_2} \in \boldsymbol{W}$: $\boldsymbol{w_1}\ R_{\Diamond}\ \boldsymbol{w_2}$ and $\mathbf{M}, \boldsymbol{w_2} \Vdash \boldsymbol{\phi}$. We set $\mathbf{M'} := \mathbf{M}[\boldsymbol{c''} \to \boldsymbol{w_2}]$. By Proposition 7, and by the hypothesis on $\boldsymbol{c''}$, the condition holds.

Now, let $\mathbf{M} \Vdash (\boldsymbol{\phi'} \wedge ((\neg \boldsymbol{c'} \vee \Diamond\boldsymbol{c''}) \wedge (\neg \boldsymbol{c''} \vee \boldsymbol{\phi})))$. Then, $\mathbf{M} \Vdash (\boldsymbol{\phi'} \wedge (\neg \boldsymbol{c'} \vee \Diamond\boldsymbol{\phi}))$.

The Ackermann rule: It is easy to show that if $\boldsymbol{\beta}$ is negative in $\boldsymbol{p}$, then it is downwards monotone in $\boldsymbol{p}$. Let $\boldsymbol{\alpha}$ be $\bigwedge(\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_{n_a}})$, $\boldsymbol{\beta}$ be $\bigwedge(\boldsymbol{\beta_1}, \ldots, \boldsymbol{\beta_{n_b}})$, and $\boldsymbol{\beta}$ be downwards monotone in $\boldsymbol{p}$. First, let $\mathbf{M} \Vdash \neg\boldsymbol{\gamma'}$, so $\mathbf{M} \Vdash (\boldsymbol{\alpha} \vee \boldsymbol{p})$ and $\mathbf{M} \Vdash \boldsymbol{\beta}$. Then, $[\![\neg\boldsymbol{\alpha}]\!]_{\mathbf{M}} \subseteq [\![\boldsymbol{p}]\!]_{\mathbf{M}}$, therefore $\boldsymbol{W} = [\![\boldsymbol{\beta}]\!]_{\mathbf{M}} \subseteq [\![\boldsymbol{\beta}[\boldsymbol{p}/\neg\boldsymbol{\alpha}]]\!]_{\mathbf{M}}$, so $\mathbf{M} \Vdash \neg\boldsymbol{\gamma''}$. Now, let $\mathbf{M} \Vdash \neg\boldsymbol{\gamma''}$, and let $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$. Let $\boldsymbol{V'}(\boldsymbol{p}) := [\![\neg\boldsymbol{\alpha}]\!]_{\mathbf{M}}$, and let $\boldsymbol{V'}(\boldsymbol{p'}) := \boldsymbol{V}(\boldsymbol{p'})$ for other variables $\boldsymbol{p'}$. Let $\mathbf{M'} := \langle \mathbf{F}, \boldsymbol{V'}, \boldsymbol{A} \rangle$. Then, $\mathbf{M'} \Vdash \neg\boldsymbol{\gamma'}$. □

Definition 18. (Standard Translation) In the function definition below, $st(\boldsymbol{n}, \boldsymbol{x}, \boldsymbol{\phi})$ stands for *special standard translation for pure formulas*, or simply *standard translation.* For $st(\boldsymbol{n}, \boldsymbol{x}, \boldsymbol{\phi})$, we assume that $\boldsymbol{\phi}$ is pure, that $\boldsymbol{x}$ is $x_{\boldsymbol{i}}$, such that $c_{\boldsymbol{i}}$ does not occur in $\boldsymbol{\phi}$, and that $\boldsymbol{n}$ is such that $c_{\boldsymbol{n}} \not\nearrow^{\infty} \{c_{\boldsymbol{i}}, \boldsymbol{\phi}\}$.
$st(\boldsymbol{n}, \boldsymbol{x}, \bot) := \bot$
$st(\boldsymbol{n}, \boldsymbol{x}, \top) := \top$
$st(\boldsymbol{n}, \boldsymbol{x}, c_{\boldsymbol{i}}) := (\boldsymbol{x} = x_{\boldsymbol{i}})$
$st(\boldsymbol{n}, \boldsymbol{x}, \neg\boldsymbol{\phi}) := \neg st(\boldsymbol{n}, \boldsymbol{x}, \boldsymbol{\phi})$

$st(\boldsymbol{n}, x_{\boldsymbol{i}}, (\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2})) := (st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi_1}) \vee st(\boldsymbol{n'}, x_{\boldsymbol{i}}, \boldsymbol{\phi_2}))$, where $\boldsymbol{n'}$ is the least number such that $\boldsymbol{n'} \geq \boldsymbol{n}$, $\boldsymbol{n'} > \boldsymbol{i}$ and for all $x_{\boldsymbol{j}}$, occurring in $st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi_1})$, $\boldsymbol{n'} > \boldsymbol{j}$.
$st(\boldsymbol{n}, x_{\boldsymbol{i}}, (\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})) := (st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi_1}) \wedge st(\boldsymbol{n'}, x_{\boldsymbol{i}}, \boldsymbol{\phi_2}))$, where $\boldsymbol{n'}$ is the least number such that $\boldsymbol{n'} \geq \boldsymbol{n}$, $\boldsymbol{n'} > \boldsymbol{i}$ and for all $x_{\boldsymbol{j}}$, occurring in $st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi_1})$, $\boldsymbol{n'} > \boldsymbol{j}$.
$st(\boldsymbol{n}, \boldsymbol{x}, \langle U \rangle \boldsymbol{\phi}) := \exists x_{\boldsymbol{n}} st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi})$
$st(\boldsymbol{n}, \boldsymbol{x}, \Diamond_0{}^{-1} \boldsymbol{\phi}) := \exists x_{\boldsymbol{n}} st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi})$
$st(\boldsymbol{n}, \boldsymbol{x}, [U] \boldsymbol{\phi}) := \forall x_{\boldsymbol{n}} st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi})$
$st(\boldsymbol{n}, \boldsymbol{x}, \Box_0{}^{-1} \boldsymbol{\phi}) := \forall x_{\boldsymbol{n}} st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi})$
$st(\boldsymbol{n}, \boldsymbol{x}, \Diamond_{\boldsymbol{i}} \boldsymbol{\phi}) := \exists x_{\boldsymbol{n}}((\boldsymbol{x}\ r_{\boldsymbol{i}}\ x_{\boldsymbol{n}}) \wedge st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi}))$
$st(\boldsymbol{n}, \boldsymbol{x}, \Diamond_{\boldsymbol{i}}{}^{-1} \boldsymbol{\phi}) := \exists x_{\boldsymbol{n}}((x_{\boldsymbol{n}}\ r_{\boldsymbol{i}}\ \boldsymbol{x}) \wedge st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi}))$
$st(\boldsymbol{n}, \boldsymbol{x}, \Box_{\boldsymbol{i}} \boldsymbol{\phi}) := \forall x_{\boldsymbol{n}}(\neg(\boldsymbol{x}\ r_{\boldsymbol{i}}\ x_{\boldsymbol{n}}) \vee st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi}))$
$st(\boldsymbol{n}, \boldsymbol{x}, \Box_{\boldsymbol{i}}{}^{-1} \boldsymbol{\phi}) := \forall x_{\boldsymbol{n}}(\neg(x_{\boldsymbol{n}}\ r_{\boldsymbol{i}}\ \boldsymbol{x}) \vee st(\boldsymbol{n}+1, x_{\boldsymbol{n}}, \boldsymbol{\phi}))$
It is immediate that *st* defines a unique function if the conditions for it hold. It is also clear that the result of *st* can be effectively obtained.

An easy, but somewhat tedious, induction on pure formulas $\boldsymbol{\phi}$ shows that, under the above assumptions for $\boldsymbol{n}$ and $x_{\boldsymbol{i}}$, for any model $\mathbf{M}$ and any world $\boldsymbol{w}$ in $\mathbf{M}$, it is the case that $\mathbf{M}, \boldsymbol{w} \Vdash \boldsymbol{\phi}$ iff $\mathbf{M}[c_{\boldsymbol{i}} \to \boldsymbol{w}] \vDash st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi})$. We call this the *main property of st*.

Lemma 19. (Standard Translation Lemma) Let $c_{\boldsymbol{k}}$ be a nominal, and let $\boldsymbol{\phi}$ be a pure formula. Then for $\boldsymbol{\phi}$ there can be effectively obtained a first-order formula $\boldsymbol{\psi}(x_{\boldsymbol{k}})$, such that $\boldsymbol{\phi} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$.

Proof. Let $\boldsymbol{i}$ be such that $c_{\boldsymbol{i}} \not\nearrow \boldsymbol{\phi}$. Consider $\boldsymbol{\psi}$: $\forall x_{j_1} \ldots \forall x_{j_m} \exists x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi})$, where $c_{\boldsymbol{n}} \not\nearrow^{\infty} \{c_{\boldsymbol{i}}, \boldsymbol{\phi}\}$, and $[j_1, \ldots, j_m]$ are such that $[c_{j_1}, \ldots, c_{j_m}]$ is the list of members of $\mathrm{NOM}(\boldsymbol{\phi}) \setminus \{c_{\boldsymbol{k}}\}$ in left-to-right order of initial occurrence in $\boldsymbol{\phi}$. Note that $\boldsymbol{\psi}$ can be denoted by $\boldsymbol{\psi}(x_{\boldsymbol{k}})$, because the only free variable, if any, is $x_{\boldsymbol{k}}$. We show that $\boldsymbol{\phi} \sim_{\boldsymbol{k}} \boldsymbol{\psi}(x_{\boldsymbol{k}})$. For convenience, denote $\neg\boldsymbol{\phi}$ by $\boldsymbol{\phi'}$.

For given $\mathbf{F}$ and $\boldsymbol{w}$ in $\mathbf{F}$, let $\mathbf{M} = \langle \mathbf{F}, \boldsymbol{V}, \boldsymbol{A} \rangle$ be a model over $\mathbf{F}$ such that $\mathbf{M}, \boldsymbol{w} \Vdash c_{\boldsymbol{k}}$ and $\mathbf{M} \Vdash \boldsymbol{\phi'}$. By the main property of *st*, for every $\boldsymbol{w_1}$ in $\mathbf{F}$: $\mathbf{M}[c_{\boldsymbol{i}} \to \boldsymbol{w_1}] \vDash st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$ iff $\mathbf{M}, \boldsymbol{w_1} \Vdash \boldsymbol{\phi'}$. Therefore, $\mathbf{M} \vDash \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$. Because $\boldsymbol{A}$ assigns every nominal, $\mathbf{M} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$. Because $\boldsymbol{A}(c_{\boldsymbol{k}}) = \boldsymbol{w}$, and because $x_{\boldsymbol{k}}$ is the only free variable in $\exists x_{j_1} \ldots \exists x_{j_m} \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$, if any, we have that $\mathbf{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})[\boldsymbol{w}]$.

Now, for given $\mathbf{F}$ and $\boldsymbol{w}$ in $\mathbf{F}$, let $\mathbf{F} \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})[\boldsymbol{w}]$. Let $\mathbf{M}$ be any model over $\mathbf{F}$, then $\mathbf{M}[c_{\boldsymbol{k}} \to \boldsymbol{w}] \vDash \exists x_{j_1} \ldots \exists x_{j_m} \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$. We define the model $\mathbf{M'}$ over $\mathbf{F}$, such that $\mathbf{M'}, \boldsymbol{w} \Vdash c_{\boldsymbol{k}}$ and $\mathbf{M'} \Vdash \boldsymbol{\phi'}$. Because there are states $v_{j_1}, \ldots, v_{j_m}$ in $\mathbf{F}$, such that $\mathbf{M}[c_{\boldsymbol{k}} \to \boldsymbol{w}][c_{j_1} \to v_{j_1}] \ldots [c_{j_m} \to v_{j_m}] \vDash \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$, we set $\mathbf{M'}$ to the above modification of $\mathbf{M}$. We show that $\mathbf{M'} \vDash \forall x_{\boldsymbol{i}} st(\boldsymbol{n}, x_{\boldsymbol{i}}, \boldsymbol{\phi'})$ iff $\mathbf{M'} \Vdash \boldsymbol{\phi'}$. But this follows by the main property of *st* and Definition 12. □

Now, we need a deterministic and terminating strategy for applying the SQEMA rules. *Equations* are formulas of the kind $(\boldsymbol{c'} \to \boldsymbol{\Diamond c''})$ or of the kind $(\boldsymbol{\phi'} \veebar \boldsymbol{\phi''})$, such that $\boldsymbol{\phi'}$ and $\boldsymbol{\phi''}$ are in negation normal form. A *system* is a formula of the kind $\neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n})$ for some $\boldsymbol{n} \geq 0$, where $\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n}$ are equations. We use $\boldsymbol{\sigma}$ for systems of equations and $\boldsymbol{\chi}$ for equations. $\boldsymbol{\sigma}$ is *solved for* $\boldsymbol{p}$ iff there are no occurrences of $\boldsymbol{p}$ in $\boldsymbol{\sigma}$. $\boldsymbol{\sigma}$ is *solved* iff it is pure. The algorithm first splits the input formula, by the conjunction lemma, into several systems of equations, trying to solve each of them in sequence. Below, we say that $\boldsymbol{c}$ is a *new nominal*, if $\boldsymbol{c}$ is such that: if $\boldsymbol{\gamma_1}, \dots, \boldsymbol{\gamma_n}$ are all formulas that have occurred as input or during the execution of any branch of the algorithm so far, it is the case that $\boldsymbol{c} \not\nearrow^{\infty} \{\boldsymbol{\gamma_1}, \dots, \boldsymbol{\gamma_n}\}$ (see Definition 2).

If $\boldsymbol{\sigma}$ is $\neg \bigwedge(\boldsymbol{\chi_1}, \dots, \boldsymbol{\chi_m})$, we denote by $\boldsymbol{\sigma}[\boldsymbol{\chi_j} // \boldsymbol{\chi'_1}, \dots, \boldsymbol{\chi'_m}]$: $\neg \bigwedge(\boldsymbol{\chi_1}, \dots, \boldsymbol{\chi_{j-1}}, \boldsymbol{\chi'_1}, \dots, \boldsymbol{\chi'_m}, \boldsymbol{\chi_{j+1}}, \dots, \boldsymbol{\chi_n})$. We denote by $\boldsymbol{\sigma}[\boldsymbol{p} // \neg \boldsymbol{p}]$ the system of equations, produced from $\boldsymbol{\sigma}$, where, simultaneously, every occurrence of $\boldsymbol{p}$ has been replaced with $\neg \boldsymbol{p}$ and every occurrence of $\neg \boldsymbol{p}$ has been replaced with $\boldsymbol{p}$.

We now describe a deterministic version of the SQEMA algorithm from [6].

The algorithm Deterministic SQEMA

INPUT: $\phi \in \mathrm{ML}(T, U)$

OUTPUT: $\langle success, fol(\phi) \rangle$ or $\langle failure \rangle$

STEP 1: Rewrite ϕ in negation normal form. Then, distribute all boxes, which are not in the scope of a diamond, and all disjunctions, over conjunctions as much as possible, using the semantic equivalences:

Rule 1.1: $\Box(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \equiv (\Box\boldsymbol{\phi_1} \wedge \Box\boldsymbol{\phi_2})$

Rule 1.2: $((\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2}) \vee \boldsymbol{\phi_3}) \equiv ((\boldsymbol{\phi_1} \vee \boldsymbol{\phi_3}) \wedge (\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3}))$

Rule 1.3: $(\boldsymbol{\phi_1} \vee (\boldsymbol{\phi_2} \wedge \boldsymbol{\phi_3})) \equiv ((\boldsymbol{\phi_1} \vee \boldsymbol{\phi_2}) \wedge (\boldsymbol{\phi_1} \vee \boldsymbol{\phi_3}))$

Thus, obtain $\phi \equiv \bigwedge(\phi_1, \dots, \phi_n)$ where no further applications of rules 1.1, 1.2 or 1.3 are possible on any $\phi_{\boldsymbol{i}}$. Now reserve the nominal c_k, such that $c_k \not\nearrow^{\infty} \phi$ (see Definition 2), and use it throughout the steps. Proceed with STEP 2, applied separately on each of the subformulas $\phi_{\boldsymbol{i}}$, and if it succeeds for all $\phi_{\boldsymbol{i}}$, proceed to STEP 5. Otherwise, if anyone of the branches for a single $\boldsymbol{i}$ fails, then return $\langle failure \rangle$ as output and stop.

STEP 2: Let $\phi_{\boldsymbol{i}}$ be one of the conjuncts from STEP 1. Let ϕ' be the *normalized* form, of $\neg\phi_{\boldsymbol{i}}$, which we define below, but for now it suffices to know that it means that ϕ' is in negation normal form, and any variable, that occurs only positively or negatively in $\neg\phi_{\boldsymbol{i}}$ has been replaced, by the positive or negative elimination rules, with $\top$, or $\bot$, respectively. Now, construct the equation $(\neg c_k \underline{\vee} \phi')$, where c_k is the nominal from STEP 1. By the sufficient condition and the equivalence rule, try solving σ: $\neg \bigwedge((\neg c_k \vee \phi'))$ by proceeding to STEP 3.

STEP 3: Let the current system be σ. For every permutation of $\mathrm{PROP}(\sigma)$, try it as the *variable elimination order*, trying to eliminate each variable in that order by proceeding to STEP 4 with a new, empty *backtracking stack*. If a permutation succeeds, and thus, all propositional variables have been eliminated from the current system, proceed to STEP 5. If all elimination orders fail, report failure for the current system and go back to executing STEP 2.

STEP 4: Take the propositional variable p that has to be eliminated and the system σ_0 as input. Save a backtracking context $\langle p, \sigma_0 \rangle$, to the stack for the application of the polarity reversing rule, but only if the input hasn't come out of the stack. Deterministically apply the SQEMA rules in order to try eliminating all occurrences of p, converting σ_0 to σ_1. Use the deterministic strategy for SQEMA rules application which is shown below. If p has been eliminated, report success and return the normalized form of σ_1 (defined below) to STEP 3 to try eliminating the remaining variables. If this fails, check if the backtracking stack is empty. If it is empty, report failure to eliminate p and resume executing STEP 3 to try other permutations. Otherwise, backtrack to the context $\langle p', \sigma_0' \rangle$ from the top of the stack, which may apply to a previous variable, then execute STEP 4 with p' and $\sigma_0'[p'//\neg p']$, skipping the saving of backtracking context.

STEP 5: If this step is reached by all branches of the execution, then all propositional variables have been eliminated from all systems resulting from the input formula. Let all pure systems be $\sigma_1, \ldots, \sigma_{\boldsymbol{n}}$. For each pure system $\sigma_{\boldsymbol{i}}$, let $\mathrm{NOM}(\sigma_{\boldsymbol{i}}) \backslash \{c_k\} = \{c_{\boldsymbol{j_1^i}}, \ldots, c_{\boldsymbol{j_{l_i}^i}}\}$, and let $c_{\boldsymbol{m_i}} \overset{\infty}{\not\nearrow} \{c_k, \sigma_{\boldsymbol{i}}\}$ (see Definition 2). Using the standard translation lemma, let $fol_{\boldsymbol{i}}(\phi)$ be: $\forall x_{\boldsymbol{j_1^i}} \ldots \forall x_{\boldsymbol{j_{l_i}^i}} \exists x_{\boldsymbol{m_i}} st(\boldsymbol{m_i}+1, x_{\boldsymbol{m_i}}, \sigma_{\boldsymbol{i}})$. Let $fol(\phi)$ be $\bigwedge(fol_1(\phi), \ldots, fol_{\boldsymbol{n}}(\phi))$, by the conjunction lemma, Lemma 19. Return the result $\langle \mathit{success},\ fol(\phi) \rangle$.

Now, we define: a) the normalization of a formula used in STEP 2 with diamond extraction, b) the normalization of a system of equations used in STEP 4, and c) the deterministic SQEMA rules application strategy.

a) It is clear how we can obtain a formula in negation normal form for a given $\boldsymbol{\gamma}$, such that $\Box_0^{-1}$ and $\Diamond_0^{-1}$ do not occur, because these are semantically equivalent to $[U]$ and $\langle U \rangle$. We use this procedure to reduce the number of subformulas in the output, by applying the equivalence rule for some obvious boolean and modal laws, as well as the following rules for the universal modality:

For $j \in \{1, 2\}$, we use U_j for either $[U]$ or $\langle U \rangle$, we use $\overset{\wedge}{\vee}$ for either $\vee$ or $\wedge$.

Replace	with	Replace	with
$(c_1 \to \langle U \rangle c_2)$	$\top$	$(\langle U \rangle \gamma_1 \vee \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\top$
$U_1 U_2 \gamma$	$U_2 \gamma$	$(\langle U \rangle \gamma \vee \Diamond\gamma)$	$\langle U \rangle \gamma$
$\Box U_1 \gamma$	$(U_1 \gamma \vee \Box\bot)$	$(\langle U \rangle \gamma \vee \gamma)$	$\langle U \rangle \gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(\langle U \rangle \gamma \wedge \Diamond\gamma)$	$\Diamond\gamma$
$[U](U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} [U]\gamma_2)$	$(\langle U \rangle \gamma \wedge \gamma)$	γ
$[U]\neg c$	$\bot$	$([U]\gamma_1 \wedge \gamma_2)$, for $\gamma_2 \equiv \neg\gamma_1$	$\bot$
$\Diamond U_1 \gamma$	$(U_1\gamma \wedge \Diamond\top)$	$([U]\gamma \wedge \Box\gamma)$	$[U]\gamma$
$\langle U \rangle(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} U_2\gamma_2)$	$([U]\gamma \wedge \gamma)$	$[U]\gamma$
$\langle U \rangle(U_1\gamma_1 \overset{\wedge}{\vee} \gamma_2)$	$(U_1\gamma_1 \overset{\wedge}{\vee} \langle U \rangle\gamma_2)$	$([U]\gamma \vee \Box\gamma)$	$\Box\gamma$
$\langle U \rangle c$	$\top$	$([U]\gamma \vee \gamma)$	γ

Then, we define a procedure for constructing a *conjunctive normal form*, using the standard definition of this notion. It is clear how this normal form can be constructed. During this construction, also perform *diamond extraction*, applying the rule $(\Diamond\boldsymbol{\phi'} \vee \Diamond\boldsymbol{\phi''}) \equiv \Diamond(\boldsymbol{\phi'} \vee \boldsymbol{\phi''})$. Attempt to eliminate semantically equivalent or opposite members of any disjunction, by comparing their normal forms. The output must not have subformulas of the kind $(\bot \overset{\Diamond}{\vee} \gamma)$ or $(\gamma \overset{\Diamond}{\vee} \bot)$.

Two improvements can be made: during the elimination, a tableaux method for $\mathrm{ML}(T, U)$ could be used to prove an equivalence, instead of comparing normal forms. Also, in the conjunction construction phase, modal resolution can be performed, as in example 6.14 of [7].

This is the normalization procedure for $\boldsymbol{\gamma}$, which produces the *normal form of* $\boldsymbol{\gamma}$: First, convert $\boldsymbol{\gamma}$ to negation normal form, then convert the result to conjunctive normal form simultaneously performing diamond extraction, by the equivalence rule, then perform box extraction using the semantic equivalence $(\Box\boldsymbol{\phi_1} \wedge \Box\boldsymbol{\phi_2}) \equiv \Box(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$, and finally replace any variables that occur only positively or negatively in $\boldsymbol{\gamma}$ with $\top$, or $\bot$, respectively. Repeat the whole process until no further changes to the formula can be made.

b) Now, we normalize a system of equations $\boldsymbol{\sigma}$. Let $\boldsymbol{\sigma}$ be $\neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n})$. Let $\boldsymbol{\phi'}$ be the normal form of $\bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n})$. If $\boldsymbol{\phi'}$ is of the kind $(\neg \boldsymbol{c} \vee \boldsymbol{\phi''})$, then the *normal form of* $\boldsymbol{\sigma}$ is $\neg \bigwedge((\neg \boldsymbol{c} \,\underline{\vee}\, \boldsymbol{\phi''}))$; otherwise, it is $\neg \bigwedge((\bot \,\underline{\vee}\, \boldsymbol{\phi'}))$.

c) The deterministic strategy for applying the SQEMA rules for a given variable $\boldsymbol{p}$ is to use the *step* function (given below) repeatedly until either a formula without occurrences of $\boldsymbol{p}$ is reached, or *failure* is obtained.

Definition 20. (Deterministic SQEMA Step) We describe a single step of the strategy, which is uniquely defined for $\boldsymbol{\sigma}$ and $\boldsymbol{p}$.

(1) If $\boldsymbol{p} \not\nearrow \boldsymbol{\sigma}$, then the result is $\boldsymbol{\sigma}$.

(2) Else, if $\boldsymbol{\sigma}$ is $\neg \bigwedge((\boldsymbol{\alpha}_1 \,\underline{\vee}\, \boldsymbol{p}), \ldots, (\boldsymbol{\alpha}_{n_a} \,\underline{\vee}\, \boldsymbol{p}), \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$, where $n_a \geq 0$, $n_b \geq 0$, $n_t \geq 0$, $\boldsymbol{p} \not\nearrow \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}\}$, and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}$ are formulas which are *negative* in $\boldsymbol{p}$, then we can apply the Ackermann rule for $\boldsymbol{p}$ and $\boldsymbol{\sigma}$. Let for $1 \leq \boldsymbol{l} \leq n_b$, $\boldsymbol{\beta'_l}$ be obtained from $\boldsymbol{\beta_l}$ by replacing all occurrences of $\neg \boldsymbol{p}$ with $\bigwedge(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a})$. Then the result for $\boldsymbol{\sigma}$ is $\neg \bigwedge(\boldsymbol{\beta}'_1, \ldots, \boldsymbol{\beta}'_{n_b}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$.
This can be improved by allowing $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}$ to be *downwards monotone* in $\boldsymbol{p}$. This can be tested by proving $[U]([U](\boldsymbol{p''} \to \boldsymbol{p'}) \to [U](\boldsymbol{\beta_l}[\boldsymbol{p}/\boldsymbol{p'}] \to \boldsymbol{\beta_l}[\boldsymbol{p}/\boldsymbol{p''}]))$, such that $\boldsymbol{p'} \not\nearrow \boldsymbol{\beta_l}$ and $\boldsymbol{p''} \not\nearrow \boldsymbol{\beta_l}$, using a tableaux method.

(3) If we are not in any of the above two cases, then there is at least one positive occurrence of $\boldsymbol{p}$ in $\boldsymbol{\sigma}$, which is not in an equation of the kind $(\boldsymbol{\alpha} \,\underline{\vee}\, \boldsymbol{p})$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}$. For convenience, let $\boldsymbol{\sigma}$ be $\neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_m})$, let $\boldsymbol{j}$ be the least number, such that $\boldsymbol{p}$ occurs positively in $\boldsymbol{\chi_j}$, $\boldsymbol{\chi_j}$ is not as described, and let let $\boldsymbol{\chi_j}$ be $(\boldsymbol{\phi'} \,\underline{\vee}\, \boldsymbol{\phi_1})$.

(3.1) If $\boldsymbol{\phi_1}$ is $(\boldsymbol{\phi_2} \wedge \boldsymbol{\phi_3})$, then, by the equivalence rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j} // (\boldsymbol{\phi'} \,\underline{\vee}\, \boldsymbol{\phi_1}), (\boldsymbol{\phi'} \,\underline{\vee}\, \boldsymbol{\phi_2})]$.

(3.2) If $\boldsymbol{\phi_1}$ is $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$, then there are three cases. If $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$, then by the equivalence rule the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j} // ((\boldsymbol{\phi'} \vee \boldsymbol{\phi_2}) \,\underline{\vee}\, \boldsymbol{\phi_3})]$. Otherwise, if

$\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$, then, by the equivalence rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//((\boldsymbol{\phi'} \vee \boldsymbol{\phi_3}) \veebar \boldsymbol{\phi_2})]$. Otherwise, the result for $\boldsymbol{\sigma}$ is *failure*.

(3.3) If $\boldsymbol{\phi_1}$ is $\Box\boldsymbol{\phi_2}$, by the box rule, the result for $\boldsymbol{\sigma}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//(\Box^{-1}\boldsymbol{\phi'} \veebar \boldsymbol{\phi_1})]$.

(3.4) If $\boldsymbol{\phi_1}$ is $\Diamond\boldsymbol{\phi_2}$ and $\boldsymbol{\phi'}$ is either $\neg\boldsymbol{c'}$ or $(\bot \vee \neg\boldsymbol{c'})$, then, by the diamond rule, let $\boldsymbol{c''}$ be a new nominal, then the result for $\boldsymbol{\sigma'}$ is $\boldsymbol{\sigma}[\boldsymbol{\chi_j}//(\boldsymbol{c'} \to \Diamond\boldsymbol{c''}), (\neg\boldsymbol{c''} \veebar \boldsymbol{\phi_1})]$.

(3.5) If we are not in any of the above four cases, the result for $\boldsymbol{\sigma}$ is *failure*.

It is immediate that the above describes a uniquely defined effective function over the systems of equations and propositional variables. We denote the function by *step*.

Immediately by the definition of *step*, we have that $\boldsymbol{\sigma} \approx_{\boldsymbol{k}}$ *step*$(\boldsymbol{\sigma}, \boldsymbol{p})$ by the SQEMA rules, Proposition 17.

We prove that the application of *step* can be composed only finitely many times for $\boldsymbol{\sigma}$ and $\boldsymbol{p}$, before reaching either a $\boldsymbol{\sigma'}$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\sigma'}$, or *failure*.

Indeed, if the result is ever obtained by (1), (2), (3.5), or the failing condition of (3.2), it is clear that this is the final application of *step*. Therefore, suppose there is an infinite sequence of results, obtained by (3.1), (3.3), (3.4), or the non-failing conditions of (3.2). Then, there is an infinite sequence $\boldsymbol{\sigma_0}, \boldsymbol{\sigma_1}, \dots$, and let $S_0, S_1, \dots$ be the sum of lengths of right-hand sides of equations in the corresponding $\boldsymbol{\sigma}$-s. It is clear that $S_0 > 0$ and for $i < j$, $S_i > S_j$, which is impossible. Therefore, we can only apply *step* a finite number of times. $\Box$

This concludes our definition of Deterministic SQEMA and the proof for its soundness and termination.

5. SAHLQVIST AND INDUCTIVE FORMULAS

We now examine some famous classes of elementary formulas and we prove that Deterministic SQEMA succeeds for them.

We use ideas from the proofs in [6].

Definition 21. (Sahlqvist formulas) A *boxed atom* is a formula $\boldsymbol{\phi}$, which is either a propositional variable $\boldsymbol{p}$ or $\Box\boldsymbol{\phi'}$, where $\boldsymbol{\phi'}$ is a boxed atom. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, boxed atoms and negative formulas, using $\wedge$, $\vee$ and $\Diamond$. A *Sahlqvist implication* is of the form $(\boldsymbol{\phi'} \to \boldsymbol{\phi''})$, where $\boldsymbol{\phi'}$ is a Sahlqvist antecedent and $\boldsymbol{\phi''}$ is positive. A *Sahlqvist formula* (in the classical definition) is built up from Sahlqvist implications by using boxes and conjunctions, and by applying disjunctions only between formulas which do not share propositional variables. An *extended Sahlqvist formula* is built up from Sahlqvist implications by using boxes, conjunctions, and disjunctions. From now on, we simply say *Sahlqvist formula* instead of *extended Sahlqvist formula*.

A *boxed piece* is a formula $\boldsymbol{\phi}$ which is either $\boldsymbol{p}$, $\Box\boldsymbol{\phi'}$, $\boldsymbol{Neg}$, $(\boldsymbol{pure} \vee \boldsymbol{\phi'})$, $(\boldsymbol{\phi'} \vee \boldsymbol{pure})$ or $(\boldsymbol{\phi'_1} \wedge \boldsymbol{\phi'_2})$, where $\boldsymbol{\phi'}$, $\boldsymbol{\phi'_1}$, and $\boldsymbol{\phi'_2}$ are boxed pieces, $\boldsymbol{Neg}$ is a negative formula,

$\boldsymbol{pure}$ is a pure formula, $\boldsymbol{\phi}$ is in negation normal form, disjunction over conjunction distribution may only apply to negative or pure subformulas of $\boldsymbol{\phi}$.

A *good piece* is a formula $\boldsymbol{\phi}$ which is built up from boxed pieces using $\wedge$ and $\Diamond$ such that $\boldsymbol{\phi}$ is in negation normal form, disjunction over conjunction distribution may only apply to negative or pure subformulas of $\boldsymbol{\phi}$, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas.

We denote by $\boldsymbol{\delta}$ a formula which is either a boxed piece, or of the form $(\neg \boldsymbol{c} \vee \boldsymbol{\phi})$ where $\boldsymbol{\phi}$ is a good piece. We denote by $\boldsymbol{\delta'}$ a formula which is either a $\boldsymbol{\delta}$, or of the kind $((\bot \vee \neg \boldsymbol{c}) \vee \boldsymbol{\phi})$ where $\boldsymbol{\phi}$ is a good piece. We denote by $\boldsymbol{\delta''}$ a formula which is either a $\boldsymbol{\delta}$, or of the kind $(\boldsymbol{\phi} \vee \neg \boldsymbol{c})$ where $\boldsymbol{\phi}$ is a good piece.

Proposition 22. If $\boldsymbol{\sigma}$ is a system of equations, where each equation $\boldsymbol{\chi}$ of $\boldsymbol{\sigma}$ is such that either $\boldsymbol{\chi}$ is a $\boldsymbol{\delta'}$, or $\boldsymbol{\chi}$ is of the form $(\bot \veebar \bigwedge(\boldsymbol{\delta''_1}, \ldots, \boldsymbol{\delta''_n}))$, then

1) Applying *step* gives a system of the same kind, and never *failure*.

2) The result of a system normalization procedure on $\boldsymbol{\sigma}$ is also a system of the same kind.

3) On Sahlqvist input formulas, Deterministic SQEMA only works on systems of the above kind. □

Corollary 23. Deterministic SQEMA succeeds on every Sahlqvist formula at the first permutation of its variables, without backtracking. □

Definition 24. (Inductive formulas) Let # be a symbol, which is not in the alphabet of ML(T, U). # is a *box-form* of #. If $B(\#)$ is a box-form of #, then $\Box B(\#)$ is a box-form of # for any $\Box$, and $(\boldsymbol{\phi} \to B(\#))$ is a box-form of # for any *positive* formula $\boldsymbol{\phi}$. Replacing all occurrences of # in $B(\#)$ with $\boldsymbol{p}$, we get $B(\boldsymbol{p})$, a *box-formula* of $\boldsymbol{p}$. The only positive occurrence of $\boldsymbol{p}$ in $B(\boldsymbol{p})$ is the *head of* $B(\boldsymbol{p})$, and any other occurrence of a propositional variable in $B(\boldsymbol{p})$ is *inessential*. For convenience, we also say that $\boldsymbol{p}$ is the *head of* $B(\boldsymbol{p})$ and the variables which have inessential occurrences in $B(\boldsymbol{p})$ are *inessential*. A *monadic regular formula (MRF)* is a modal formula built up from $\top$, $\bot$, positive formulas and negated box-formulas by applying $\wedge$, $\vee$ and $\Box$. The *dependency graph* of a set of box-formulas $\mathfrak{B} = \{B_1(\boldsymbol{p_1}), \ldots, B_n(\boldsymbol{p_n})\}$ is a directed graph $G(\mathfrak{B}) = \langle V, E\rangle$ where $V = \{\boldsymbol{p_1}, \ldots, \boldsymbol{p_n}\}$ is the set of heads in $\mathfrak{B}$ and E is the set of edges, such that $\langle \boldsymbol{p_i}, \boldsymbol{p_j}\rangle \in E$ iff $\boldsymbol{p_i}$ occurs as an inessential variable in a box-formula from $\mathfrak{B}$ with head $\boldsymbol{p_j}$. A directed graph is *acyclic* iff it does not contain directed cycles. The dependency graph of an MRF $\boldsymbol{\phi}$ is the dependency graph of the set of box-formulas which occur in the construction of $\boldsymbol{\phi}$ as an MRF. A *monadic inductive formula (MIF)* is a monadic regular formula with an acyclic dependency graph. We say that a conjunction of MIFs is an *inductive formula*.

We extend the definitions to negation normal forms of the above.

We define an *extended box-formula of* $\boldsymbol{p}$ thusly: $\boldsymbol{p}$ is an $\boldsymbol{EB}(\boldsymbol{p})$, $\Box\boldsymbol{EB}(\boldsymbol{p})$ is an $\boldsymbol{EB}(\boldsymbol{p})$, $(\boldsymbol{EB_1}(\boldsymbol{p}) \wedge \boldsymbol{EB_2}(\boldsymbol{p}))$ is an $\boldsymbol{EB}(\boldsymbol{p})$, if $\boldsymbol{Neg'}$ and $\boldsymbol{Neg''}$ are negative formulas, then each of $(\boldsymbol{Neg'} \vee \boldsymbol{EB}(\boldsymbol{p}))$, $(\boldsymbol{EB}(\boldsymbol{p}) \vee \boldsymbol{Neg'})$, $(\boldsymbol{Neg''} \wedge \boldsymbol{EB}(\boldsymbol{p}))$ and $(\boldsymbol{EB}(\boldsymbol{p}) \wedge \boldsymbol{Neg''})$ is an $\boldsymbol{EB}(\boldsymbol{p})$, and also $\boldsymbol{EB}(\boldsymbol{p})$ is in negation normal form. Here, $\boldsymbol{p}$ is the *head* of the extended box-formula, any occurrences of propositional variables in any of the $\boldsymbol{Neg'}$ formulas is *inessential*. The dependency graph of $\boldsymbol{EB}(\boldsymbol{p})$ is defined analogously to the above, but note that the variables of any $\boldsymbol{Neg''}$ do not count as inessential.

PureBox is a pure formula built up from negated nominals, $\bot$, $\vee$ and $\Box$.

We say that a formula $\boldsymbol{\phi}$ is a ***Good*** formula if it is such that $\boldsymbol{\phi}$ is in negation normal form, and $\boldsymbol{\phi}$ is either $\boldsymbol{EB}(\boldsymbol{p})$, $\boldsymbol{Neg}$, $(\boldsymbol{\phi_1} \wedge \boldsymbol{\phi_2})$, $\Diamond\boldsymbol{\phi'}$ outside the scope of boxes and disjunctions, $\Box\boldsymbol{\phi'}$, $(\boldsymbol{\phi'} \vee \boldsymbol{PureBox})$, or $(\boldsymbol{PureBox} \vee \boldsymbol{\phi'})$, where $\boldsymbol{Neg}$ is a negative formula, $\boldsymbol{\phi'}$, $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ are ***Good*** formulas, and also the following diamond distribution rule — $\Diamond(\gamma_1 \vee \gamma_2) \Rightarrow (\Diamond\gamma_1 \vee \Diamond\gamma_2)$ — may only be applied to diamonds within negative or pure subformulas. The *dependency graph of* ***Good*** is the union of the dependency graphs of the occurring formulas of kind $\boldsymbol{EB}(\boldsymbol{p})$, and we require that all ***Good*** formulas have an acyclic dependency graph.

A *good system* is a system of equations $\boldsymbol{\sigma} = \neg \bigwedge(\boldsymbol{\chi_1}, \ldots, \boldsymbol{\chi_n})$, such that every $\boldsymbol{\chi_i}$ is a *good equation* with an acyclic *dependency graph* $G(\boldsymbol{\chi_i})$ defined below, $G(\boldsymbol{\sigma}) = \bigcup\{G(\chi_1), \ldots, G(\chi_n)\}$, $G(\boldsymbol{\sigma})$ is acyclic, where exactly one of the following holds for each $\boldsymbol{\chi_i}$:

good.1. $\boldsymbol{\chi_i}$ is either $(\boldsymbol{Neg_{i_1}} \veebar \boldsymbol{Neg_{i_2}})$ or $(\boldsymbol{c'_i} \to \Diamond\boldsymbol{c''_i})$, with $G(\boldsymbol{\chi_i}) = \langle\emptyset, \emptyset\rangle$,

good.2.1. $\boldsymbol{\chi_i}$ is not of kind good.1, but is either $(\neg\boldsymbol{c_i} \veebar \boldsymbol{Good_i})$ or $((\bot \vee \neg\boldsymbol{c_i}) \veebar \boldsymbol{Good_i})$, with $G(\boldsymbol{\chi_i}) = G(\boldsymbol{Good_i})$,

good.2.2. $\boldsymbol{\chi_i}$ is not of the kind good.1 or good.2.1., but is $(\boldsymbol{PureBox} \veebar \boldsymbol{Good'_i})$, such that 1. there are no diamonds in $\boldsymbol{Good'_i}$ outside of box-formulas or negative subformulas, 2. $G(\boldsymbol{\chi_i}) = G(\boldsymbol{Good'_i})$,

good.3. $\boldsymbol{\chi_i}$ is not of the above kinds, but $\boldsymbol{\chi_i}$ is $(\boldsymbol{Neg'_i} \veebar \boldsymbol{EB'_i}(\boldsymbol{p_i}))$, such that $\boldsymbol{\chi_i}$ is some $\boldsymbol{EB_i}(\boldsymbol{p_i})$ with an acyclic graph, and $G(\boldsymbol{\chi_i}) = G(\boldsymbol{EB_i}(\boldsymbol{p_i}))$,

good.4. $\boldsymbol{\chi_i}$ is not of the above kinds, but $\boldsymbol{\chi_i}$ is $(\bot \veebar \bigwedge(\delta_1, \ldots, \delta_m))$, where each δ_j is either 1. negative with $G(\delta_j) = \langle\emptyset, \emptyset\rangle$, 2. $(\neg\boldsymbol{c} \vee \boldsymbol{Good})$ or $(\boldsymbol{Good} \vee \neg\boldsymbol{c})$ with $G(\delta_j) = G(\boldsymbol{Good})$, 3. $(\boldsymbol{PureBox} \vee \boldsymbol{Good'})$ or $(\boldsymbol{Good'} \vee \boldsymbol{PureBox})$ with $G(\delta_j) = G(\boldsymbol{Good'})$, such that there are no diamonds in $\boldsymbol{Good'}$ outside of box-formulas or negative formulas, or 4. an $\boldsymbol{EB}$ with $G(\delta_j) = G(\boldsymbol{EB})$ - an acyclic graph. The graph $G(\boldsymbol{\chi_i}) = \bigcup\{G(\delta_1), \ldots, G(\delta_m)\}$ is acyclic.

Claim 25. Every output of *step*, where the input is a good system, is a good system.

Proof. Consider ***result***, which is $step(\boldsymbol{\sigma}, \boldsymbol{p})$, where for $\boldsymbol{\sigma}$ the invariant holds. We show that ***result*** is not *failure* and that the invariant holds for ***result***.

If ***result*** is obtained from (1), then the invariant holds.

If ***result*** is obtained from (2), the Ackermann rule, then ***result*** is $\boldsymbol{\sigma'}$, $\boldsymbol{\sigma}$ is $\neg \bigwedge((\boldsymbol{\alpha}_1 \veebar \boldsymbol{p}), \ldots, (\boldsymbol{\alpha}_{n_a} \veebar \boldsymbol{p}), \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_{n_b}, \boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t})$, such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{n_a}$, $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_{n_t}$. Then, each $(\boldsymbol{\alpha} \veebar \boldsymbol{p})$ is of the form good.2.1, good.2.2, good.3, or good.4, so each $\boldsymbol{\alpha}$ is a $\boldsymbol{Neg}$, and the occurrences of $\neg\boldsymbol{p}$ within every $\boldsymbol{\beta}$ are in occurrences of a $\boldsymbol{Neg}$ within $\boldsymbol{\beta}$.

It remains to prove that $G(\boldsymbol{\sigma'})$ is acyclic. It would follow that the graph of every resulting equation is acyclic and that each of the resulting equations are in some of the good equation forms.

Because of the replacement, for every edge $\langle q_1, q_2\rangle$ of $G(\boldsymbol{\sigma'})$ either $\langle q_1, q_2\rangle$ is an edge of $G(\boldsymbol{\sigma})$ or there are edges $\langle q_1, \boldsymbol{p}\rangle$ and $\langle \boldsymbol{p}, q_2\rangle$ of $G(\boldsymbol{\sigma})$. Then for every cycle in $G(\boldsymbol{\sigma'})$ there is a corresponding cycle in $G(\boldsymbol{\sigma})$. Hence $G(\boldsymbol{\sigma'})$ is acyclic.

Then $\boldsymbol{\sigma'}$ is a good system.

If ***result*** is obtained from (3.1), then ***result*** is $\boldsymbol{\sigma'}$. We have split on $\wedge$ an equation of type good.2.1, good.2.2, good.3, or good.4. Equations of type good.2.1 split into two equations of type good.2.1, or one of type good.1 and one of type good.2.1. Equations of type good.2.2 split into two equations of type good.2.2, or one of type good.1 and one of type good.2.2. Equations of type good.3 split into two equations of the same kind, or one of kind good.1 and one of kind good.3. Equations of kind good.4 split into two equations, each of them of type either good.1, good.2.2, good.3, or good.4. All resulting equations are good equations, because the resulting equations have graphs that are subgraphs of the original ones. Hence $\boldsymbol{\sigma'}$ is a good system.

If ***result*** is obtained from (3.2), then let the changed equation of $\boldsymbol{\sigma}$ be $\boldsymbol{\chi}$, which is $(\boldsymbol{\phi'} \veebar (\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3}))$. We have that $\boldsymbol{\chi}$ is not negative, and because of the invariant for $\boldsymbol{\sigma}$ and the definition of ***Good***, we have that $\boldsymbol{\chi}$ is either of type good.2.1, good.2.2, good.3, or good.4 with $m = 1$.

First, let $\boldsymbol{\chi}$ be of type good.2.1, good.2.2 or good.3. Then $\boldsymbol{\phi'}$ is negative. Because the graph of $\boldsymbol{\chi}$ is acyclic, either $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$, with $\boldsymbol{\phi_2}$ negative or pure and $\boldsymbol{\phi_3}$ a ***Good*** formula, or vice versa. So ***result*** is $\boldsymbol{\sigma'}$, not ***failure***, and the invariant holds for $\boldsymbol{\sigma'}$ because we have converted $\boldsymbol{\chi}$ to an equation of type good 2.1, good 2.2 or good.3 with a graph that is the same.

Now, let $\boldsymbol{\chi}$ be of type good.4 with $m = 1$. Then $\boldsymbol{\phi'}$ is $\bot$. Then, because $\boldsymbol{p}$ occurs positively in $\boldsymbol{\chi}$, there are three cases for $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is $(\neg \boldsymbol{c} \vee \boldsymbol{Good})$ or $(\boldsymbol{PureBox} \vee \boldsymbol{Good'})$, then $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is $(\boldsymbol{Good} \vee \neg \boldsymbol{c})$ or $(\boldsymbol{Good'} \vee \boldsymbol{PureBox})$, then $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$. In these two cases we have converted $\boldsymbol{\chi}$ into an equation of type good.2.1 or good.2.2. If $(\boldsymbol{\phi_2} \vee \boldsymbol{\phi_3})$ is an $\boldsymbol{EB}(\boldsymbol{p'})$ with an acyclic graph, then clearly $\boldsymbol{p}$ is $\boldsymbol{p'}$. Either $\boldsymbol{\phi_2}$ is negative and $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_2}$ or $\boldsymbol{\phi_3}$ is negative and $\boldsymbol{p} \not\nearrow \boldsymbol{\phi_3}$. In this case we have converted $\boldsymbol{\chi}$ into an equation of type good.3.

In either case, ***result*** is $\boldsymbol{\sigma'}$ and the invariant holds for $\boldsymbol{\sigma'}$.

If ***result*** is obtained from (3.3), then ***result*** is $\boldsymbol{\sigma'}$. Suppose for the sake of contradiction that we have changed an equation $\boldsymbol{\chi}$ of kind good.4. Then either $\boldsymbol{\chi}$ is a negative formula, which contradicts the fact that $\boldsymbol{\chi}$ is not of kind good.1, or the right-hand side of $\boldsymbol{\chi}$ is a box, which contradicts the fact that $\boldsymbol{\chi}$ is not of kind good.3. Now, because $\boldsymbol{p}$ occurs positively in the changed equation of σ, there are three cases. First, an equation of type good.3 was changed, then we have converted the equation into another one of type good.3 with a graph that is the same. Second, we have converted an equation of type good.2.2 into another one of the same kind, with a graph that is the same. Third, we have converted an equation of type good.2.1 into an equation of type good.2.2 with a graph that is the same. Therefore, the invariant holds.

If ***result*** is obtained from (3.4) or from (3.5), let the first equation of $\boldsymbol{\sigma}$ where $\boldsymbol{p}$ occurs positively and which is not of kind $(\boldsymbol{\alpha} \veebar \boldsymbol{p})$ such that $\boldsymbol{p} \not\nearrow \boldsymbol{\alpha}$, be $\boldsymbol{\chi}$, which is $(\boldsymbol{\phi'} \veebar \Diamond\boldsymbol{\phi_2})$. Because $\boldsymbol{\chi}$ is not negative, $\boldsymbol{\chi}$ can only be of type good.2.1, and the result can only have been obtained from (3.4). The invariant holds because we have converted $\boldsymbol{\chi}$ into an equation of type good.1 and an equation of type good.2.1. $\Box$

Claim 26. Every output result of the system normalization procedure, where the input is a good system, is a good system.

Proof. It can be verified that the negation normal form procedure, followed by the conjunctive normal form procedure with diamond extraction, followed by the box extraction procedure, output a good system with a graph which is a subgraph of a graph of the original. $\Box$

Claim 27. On inductive input formulas, Deterministic SQEMA only works on good systems, with the starting equation being either of kind good.1 or of kind good.2.1.

Proof. By Claim 25 and Claim 26, it is enough to show that every initial equation is one of the kinds good.1, good.2.1, good.2.2, good.3, or good.4.

It can be verified that every initial equation on inductive formula inputs is of kind good.1 or good.2.1. $\Box$

Corollary 28. Deterministic SQEMA succeeds on every inductive formula at the first permutation of its variables, without backtracking. $\Box$

6. EXAMPLES

Let us consider the formula $(\Box_1 p_0 \to [U]p_0)$. After negation and normalization, the initial equation is $(\neg c_0 \veebar (\langle U\rangle\neg p_0 \wedge \Box_1 p_0))$. The system is split into two equations using the Equivalence Rule: $(\neg c_0 \veebar \langle U\rangle\neg p_0), (\neg c_0 \veebar \Box_1 p_0)$. Then, the Box-Rule is applied: $(\neg c_0 \veebar \langle U\rangle\neg p_0), (\Box_1^{-1}\neg c_0 \veebar p_0)$. After that, the Ackermann Rule is applied: $(\neg c_0 \veebar \langle U\rangle\Box_1^{-1}\neg c_0)$. The final result is: $\langle \mathit{success}, \forall x_1(x_0\ r_1\ x_1)\rangle$.

Now, let us take Löb's formula $(\Box_1(\Box_1 p_0 \to p_0) \to \Box_1 p_0)$. The initial equation is $(\neg c_0 \veebar (\Diamond_1\neg p_0 \wedge \Box_1(\Diamond_1\neg p_0 \vee p_0)))$. The Equivalence Rule is applied: $(\neg c_0 \veebar \Diamond_1\neg p_0), (\neg c_0 \veebar \Box_1(\Diamond_1\neg p_0 \vee p_0))$. The Box-Rule is applied: $(\neg c_0 \veebar \Diamond_1\neg p_0), (\Box_1^{-1}\neg c_0 \veebar (\Diamond_1\neg p_0 \vee p_0))$. This is where we have our first failure to eliminate p_0, so backtracking occurs. We backtrack to the initial equation, reversing the polarity of p_0: $(\neg c_0 \veebar (\Diamond_1 p_0 \wedge \Box_1(\Diamond_1 p_0 \vee \neg p_0)))$. The Equivalence Rule is applied: $(\neg c_0 \veebar \Diamond_1 p_0), (\neg c_0 \veebar \Box_1(\Diamond_1 p_0 \vee \neg p_0))$. The Box-Rule is applied: $(\neg c_0 \veebar \Diamond_1 p_0), (\Box_1^{-1}\neg c_0 \veebar (\Diamond_1 p_0 \vee \neg p_0))$. Here we fail again. The backtracking stack is empty, so the result is $\langle \mathit{failure}\rangle$.

7. AXIOMATIZATION OF ML(T,U)

Here, we follow the axiomatic system for nominals and universal modality, described in [17, 18, 12], with some differences in the proofs.

We show an axiomatic system for the valid formulas from the language ML(T,U). For simplicity of the axiomatic system, we use implications and we only use $\Diamond$, $\wedge$, $\vee$, $\neg$ and $\top$ as *defined symbols.* We use p and q for variables. Therefore, our language for this section becomes:

$$\boldsymbol{\phi} ::= \bot|\boldsymbol{p}|\boldsymbol{c}|(\boldsymbol{\phi} \to \boldsymbol{\phi})|\Box_{\boldsymbol{i}}\boldsymbol{\phi}|\Box_{\boldsymbol{i}}^{-1}\boldsymbol{\phi}$$

Definition 29. (Admissible Form) Let $\#$ be a symbol, which is not in the alphabet of ML(T,U). $\#$ is an *admissible form.* If $AF(\#)$ is an admissible form, then so are $\Box AF(\#)$ and $(\boldsymbol{\phi} \to AF(\#))$. The formula, obtained by replacing all occurrences of $\#$ with $\boldsymbol{\phi}$ in $AF(\#)$ is denoted by $AF(\boldsymbol{\phi})$.

We use the same notation for *nominal substitution*, replacing a nominal with another nominal, as the notation for uniform substitution.

Axioms:
The axioms of propositional calculus.
(K) $(\Box(p \to q) \to (\Box p \to \Box q))$ for every box $\Box$
(T for U) $([U]p \to p)$
(B for U) $(p \to [U]\langle U\rangle p)$
(4 for U) $([U]p \to [U][U]p)$
(U) $([U]p \to \Box p)$ for every box $\Box$
(GP) $(p \to \Box_{\boldsymbol{i}}\Diamond_{\boldsymbol{i}}^{-1}p)$ for every number $\boldsymbol{i}$
(HF) $(p \to \Box_{\boldsymbol{i}}^{-1}\Diamond_{\boldsymbol{i}}p)$ for every number $\boldsymbol{i}$
(Nom1) $\langle U\rangle c$
(Nom2) $(\langle U\rangle(c \wedge p) \to [U](c \to p))$

Rules:

$$\text{Modus Ponens (MP): } \frac{\boldsymbol{\phi_1}, (\boldsymbol{\phi_1} \to \boldsymbol{\phi_2})}{\boldsymbol{\phi_2}}, \text{ Gen: } \frac{\boldsymbol{\phi}}{\Box\boldsymbol{\phi}},$$

$$\text{Uniform Substitution: } \frac{\boldsymbol{\phi}}{\boldsymbol{\phi}[\boldsymbol{p}/\boldsymbol{\phi'}]}, \text{ Nominal Substitution: } \frac{\boldsymbol{\phi}}{\boldsymbol{\phi}[\boldsymbol{c'}/\boldsymbol{c''}]},$$

$$\text{Cov*: } \frac{AF(\neg\boldsymbol{c}) \text{ for some } \boldsymbol{c} \not\nearrow AF(\#)}{AF(\bot)}.$$

A *normal modal logic*, or just *logic*, is a set of formulas Λ such that Λ contains all axioms and is closed under applications of the five rules.

$K_{(T,U)}$ is the smallest logic. Let $\boldsymbol{\phi}$ be a formula. We denote the smallest logic, which contains $\boldsymbol{\phi}$ by $K_{(T,U)}+\boldsymbol{\phi}$, and $\boldsymbol{\phi}$ is called *the axiom of* $K_{(T,U)}+\boldsymbol{\phi}$. We denote by $\vdash_\Lambda \boldsymbol{\phi}$ iff $\boldsymbol{\phi} \in \Lambda$. We use the capital greek letters Γ, Δ, Σ for sets of formulas. A

Λ-*theory* Γ is a set of formulas Γ such that $\Lambda \subseteq \Gamma$ and Γ is closed under applications of MP and the infinitary rule Cov:

$$\text{Cov: } \frac{AF(\neg \boldsymbol{c}) \text{ for all } \boldsymbol{c}}{AF(\bot)}.$$

The Λ-*theory of a set of formulas* Γ, $Th_\Lambda(\Gamma)$, is the smallest Λ-theory such that $\Gamma \subseteq Th_\Lambda(\Gamma)$. Despite the infinitary rule, the deduction lemma holds:

Lemma 30. (Deduction Lemma) $(\phi_1 \to \phi_2) \in Th_\Lambda(\Gamma)$ iff $\phi_2 \in Th_\Lambda(\Gamma \cup \{\phi_1\})$.

Proof. The left to right direction is obvious. Let $\phi_2 \in Th_\Lambda(\Gamma \cup \{\phi_1\})$ and let $\Gamma' := \{\phi' \mid (\phi_1 \to \phi') \in Th_\Lambda(\Gamma)\}$. Easily, $\phi_1 \in \Gamma'$ and $\Lambda \subseteq Th_\Lambda(\Gamma) \subseteq \Gamma'$. Also, Γ' is closed under applications of MP. To see that Γ' is closed under applications of Cov, let $AF(\#)$ be an admissible form, and suppose that for each nominal $\boldsymbol{c}$, $AF(\neg \boldsymbol{c}) \in \Gamma'$. Then, by propositional reasoning, for each nominal $\boldsymbol{c}$: $(\phi_1 \to AF(\neg \boldsymbol{c})) \in Th_\Lambda(\Gamma)$. Applying Cov to $(\phi_1 \to AF(\#))$, we get that $(\phi_1 \to AF(\bot)) \in Th_\Lambda(\Gamma)$, therefore $AF(\bot) \in \Gamma'$, so Γ' is closed under applications of Cov. Therefore, $Th_\Lambda(\Gamma \cup \{\phi_1\}) \subseteq \Gamma'$, so by the definition of Γ', $(\phi_1 \to \phi_2) \in Th_\Lambda(\Gamma)$. □

A set Γ is Λ-*consistent* iff $\bot \notin Th_\Lambda(\Gamma)$, and is Λ-*inconsistent*, otherwise. Γ is a *complete* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for every formula $\boldsymbol{\phi}$, it is the case that either $\boldsymbol{\phi} \in \Gamma$ or $\neg\boldsymbol{\phi} \in \Gamma$. Γ is a *maximal* Λ-*theory*, iff Γ is a Λ-consistent Λ-theory, and for any set Σ such that $\Gamma \subsetneq \Sigma$, Σ is Λ-inconsistent.

Corollary 31. A theory is maximal iff it is complete.

Proof. First, let Γ be a complete Λ-theory and let for some set Σ such that $\Gamma \subseteq \Sigma$, $\boldsymbol{\phi} \in \Sigma \setminus \Gamma$. Then $\neg\boldsymbol{\phi} \in \Gamma$, so by propositional reasoning $\bot \in Th_\Lambda(\Sigma)$. Now, let Γ be a maximal Λ-theory and let $\phi \notin \Gamma$. Then, $\bot \in Th_\Lambda(\Gamma \cup \{\phi\})$, so by the deduction lemma, $(\phi \to \bot) \in \Gamma$, therefore $\neg\phi \in \Gamma$. □

Note that the classical Lindenbaum lemma here has the following form:

Lemma 32. (Lindenbaum Lemma) Let Γ be Λ-consistent. Then Γ can be extended to a complete Λ-theory.

Proof. Let $\phi_1, \phi_2, \ldots$ be an enumeration of all formulas of $\mathrm{ML}(T, U)$. We construct by induction an infinite chain of Λ-consistent Λ-theories $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ with the property that for every $i \geq 1$, either $\phi_i \in \Gamma_i$ or $\neg\phi_i \in \Gamma_i$ in the following way. Let Γ_0 be $Th_\Lambda(\Gamma)$. Thus Γ_0 is Λ-consistent. Suppose that Γ_i is defined.

1. If $\Gamma_i \cup \{\phi_i\}$ is Λ-consistent, let $\Gamma_{i+1} = Th_\Lambda(\Gamma_i \cup \{\phi_i\})$.

2. If $\Gamma_i \cup \{\phi_i\}$ is Λ-inconsistent, then $\neg\phi_i \in \Gamma_i$. There are two cases.

2.1. If ϕ_i is not in the form $AF(\bot)$, then let $\Gamma_{i+1} = \Gamma_i$.

2.2. If ϕ is $AF(\bot)$ for some admissible form $AF(\#)$, then we show that there is a nominal $\boldsymbol{c}_i$ such that $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\}$ is Λ-consistent. Suppose for the sake of contradiction that for all $\boldsymbol{c}$: $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c})\}$ is Λ-inconsistent. Then by the deduction lemma, for all $\boldsymbol{c}$: $(\neg AF(\neg \boldsymbol{c}) \to \bot) \in \Gamma_i$, hence for all $\boldsymbol{c}$: $AF(\neg \boldsymbol{c}) \in \Gamma_i$. Since Γ_i is a Λ-theory, by Cov, $AF(\bot) \in \Gamma_i$, so $\phi_i \in \Gamma_i$. Thus Γ_i is Λ-inconsistent, which contradicts the Λ-consistency of Γ_i. We conclude that there is a nominal $\boldsymbol{c}_i$ such that $\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\}$ is Λ-consistent. Let Γ_{i+1} be $Th_\Lambda(\Gamma_i \cup \{\neg AF(\neg \boldsymbol{c}_i)\})$.

According to the construction, Γ_{i+1} is a Λ-consistent extension of Γ_i.

Let $\Gamma^+ := \bigcup_{i=0}^{\infty} \Gamma_i$.

First note that $\bot \notin \Gamma^+$ since for all $i \geq 0$, $\bot \notin \Gamma_i$.

Now we show that Γ^+ is closed under applications of Modus Ponens. Let $\phi_1, (\phi_1 \to \phi_2) \in \Gamma^+$. Then there is a step i such that $\phi_1, (\phi_1 \to \phi_2) \in \Gamma_i$. But Γ_i is closed under applications of MP, so $\phi_2 \in \Gamma_i \subseteq \Gamma^+$.

We now show that Γ^+ is closed under applications of Cov. Let there be an $AF(\#)$ such that for all $\boldsymbol{c}$: $AF(\neg \boldsymbol{c}) \in \Gamma^+$ and suppose for the sake of contradiction that $AF(\bot) \notin \Gamma^+$. There is an index i such that $AF(\bot)$ is ϕ_i, and by case 2.2 of the construction, there is a nominal c' such that $\neg AF(\neg c') \in \Gamma_{i+1} \subseteq \Gamma^+$. By propositional reasoning, $\bot \in \Gamma^+$, contradiction. Therefore, Γ^+ is closed under applications of Cov.

Since every formula is ϕ_i for some i, by the construction either $\phi_i \in \Gamma_{i+1}$ or $\neg\phi_i \in \Gamma_{i+1}$. Thus Γ^+ is a complete Λ-theory. □

We denote by $\Gamma \vdash_\Lambda \boldsymbol{\phi}$ if $\boldsymbol{\phi} \in Th_\Lambda(\Gamma)$. Thus $\emptyset \vdash_\Lambda \boldsymbol{\phi}$ iff $\vdash_\Lambda \boldsymbol{\phi}$. $\mathrm{M}, w \Vdash \Gamma$ iff for all $\boldsymbol{\phi} \in \Gamma$, $\mathrm{M}, w \Vdash \boldsymbol{\phi}$. We say that $\boldsymbol{\phi}$ is a *local semantic consequence* of Γ over the class S of frames, denoted by $\Gamma \Vdash_S \boldsymbol{\phi}$, or, if $\Gamma = \emptyset$, as $S \Vdash \boldsymbol{\phi}$, iff for every frame $\mathrm{F} \in S$, every model M over F and every state w from F, it is the case that if $\mathrm{M}, w \Vdash \Gamma$, then $\mathrm{M}, w \Vdash \boldsymbol{\phi}$. The *class of frames of* Λ, $Fr(\Lambda)$, is the class S of all frames F such that $\mathrm{F} \Vdash \Lambda$. Γ is *satisfiable on S* iff there is an $\mathrm{F} \in S$, an M over F and a w in F such that $\mathrm{M}, w \Vdash \Gamma$.

Our goal is to examine the relationship between $\vdash$ and $\Vdash$.

Soundness: If $\Gamma \vdash_\Lambda \boldsymbol{\phi}$, then $\Gamma \Vdash_{Fr(\Lambda)} \boldsymbol{\phi}$.

Proof. All axioms are valid. Every rule preserves validity on any given frame. The result follows in the usual way. □

The converse, known as *strong completeness*, can be proven for some logics. Here, like in [21], we prove it for $K_{(T,U)}$ and $K_{(T,U)} + \boldsymbol{\phi}$ for di-persistent $\boldsymbol{\phi}$.

Strong Completeness, First Form: If $\Gamma \Vdash_{Fr(\Lambda)} \boldsymbol{\phi}$, then $\Gamma \vdash_\Lambda \boldsymbol{\phi}$.

Strong Completeness, Second Form: If Γ is Λ-consistent, then Γ is satisfiable on $Fr(\Lambda)$.

Proposition 33. The two forms of strong completeness are equivalent.

Proof. See [3]. Note that here we use the deduction lemma.

For every box $\Box$, we denote by $\Box\Gamma$ the set $\{\phi \mid \Box\phi \in \Gamma\}$.

Lemma 34. Let Γ, Σ and Δ be Λ-consistent Λ-theories. Then

1. The set $\Gamma' := \Box\Gamma$ is a Λ-theory and if for some formula ϕ, $\Box\phi \notin \Gamma$, then Γ' is Λ-consistent.
2. $[U]\Gamma$ is Λ-consistent, $[U]\Gamma \subseteq \Gamma$ and $[U]\Gamma \subseteq \Box\Gamma$ for every box.

3. If Γ is complete, then $\Box\phi \notin \Gamma$ iff there is a complete Λ-theory Σ such that $\Box\Gamma \subseteq \Sigma$ and $\phi \notin \Sigma$.
4. If Γ and Σ are complete, then $\Box_{\boldsymbol{i}}\Gamma \subseteq \Sigma$ iff $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Gamma$.
5. If Γ and Σ are complete, then $[U]\Gamma \subseteq \Sigma$ iff $[U]\Sigma \subseteq \Gamma$.
6. If Γ, Σ and Δ are complete, $[U]\Gamma \subseteq \Sigma$ and $[U]\Sigma \subseteq \Delta$, then $[U]\Delta \subseteq \Gamma$.
7. If Γ and Σ are complete and $[U]\Gamma \subseteq \Sigma$, then $[U]\Gamma = [U]\Sigma$.

Proof. We only show 1. The proofs for the rest are standard, and follow easily from the axioms, 1., the deduction lemma and the Lindenbaum lemma.

Let $(\phi_1 \to \phi_2), \phi_1 \in \Gamma'$, therefore $\Box(\phi_1 \to \phi_2), \Box\phi_1 \in \Gamma$. Because of (K), $\vdash_\Lambda (\Box(\phi_1 \to \phi_2) \to (\Box\phi_1 \to \Box\phi_2))$, therefore, by MP, $\Box\phi_2 \in \Gamma$, so $\phi_2 \in \Gamma'$.

Now, let for all $\boldsymbol{c}$, $AF(\neg\boldsymbol{c}) \in \Gamma'$. Then for all $\boldsymbol{c}$, $\Box AF(\neg\boldsymbol{c}) \in \Gamma$, so by Cov, $\Box AF(\bot) \in \Gamma$ and hence $AF(\bot) \in \Gamma'$.

Finally, if $\Box\phi \notin \Gamma$, then $\phi \notin \Gamma'$ and hence Γ' is Λ-consistent. □

For given Λ and a complete Λ-theory Γ, let F be $\langle W, \mathcal{R}\rangle$, where W is the set of all complete Λ-theories Σ, such that $[U]\Gamma \subseteq \Sigma$, $\mathcal{R}(0) = W \times W$ and for $i > 0$, $\langle \Sigma_1, \Sigma_2\rangle \in \mathcal{R}(i)$ iff $\Box_i\Sigma_1 \subseteq \Sigma_2$. Then F is called the Λ-*canonical frame for* Γ.

Proposition 35. Let Λ be a logic, Γ be a complete Λ-theory. If $\mathrm{F} = \langle W, \mathcal{R}\rangle$ is the Λ-canonical frame for Γ, then

1. for every $\Sigma \in W$ at least one $\boldsymbol{c} \in \Sigma$.
2. for every $\boldsymbol{c}$ there is exactly one $\Sigma \in W$ such that $\boldsymbol{c} \in \Sigma$.

Proof. 1. Let Σ be a complete Λ-theory. Suppose that for all $\boldsymbol{c}$, $\boldsymbol{c} \notin \Sigma$. Then, by the completeness of Σ, for all $\boldsymbol{c}$, $\neg\boldsymbol{c} \in \Sigma$. Therefore, by Cov, $\bot \in \Sigma$, contradiction.

2. First, we show that for every $\boldsymbol{c}$, there is a $\Sigma \in W$ such that $\boldsymbol{c} \in \Sigma$. Suppose this is not the case, so there is a $\boldsymbol{c}$ such that for all $\Sigma \in W$, $\boldsymbol{c} \notin \Sigma$. Then $\neg\boldsymbol{c} \in \Sigma$, therefore $[U]\neg\boldsymbol{c} \in \Gamma$, which contradicts axiom (Nom1). Second, let for some $\boldsymbol{c}$ there be $\Sigma_1, \Sigma_2 \in W$, such that $\boldsymbol{c} \in \Sigma_1 \cap \Sigma_2$. Let $\phi \in \Sigma_1$. Then, $(\boldsymbol{c} \wedge \phi) \in \Sigma_1$. Suppose $\phi \notin \Sigma_2$, then $(\boldsymbol{c} \to \neg\phi) \in \Sigma_2$. Now, there are two cases. First, if $[U](\boldsymbol{c} \to \neg\phi) \in \Sigma_2$, then because of the definition of W, $(\boldsymbol{c} \to \neg\phi) \in \Sigma_1$, contradiction. Second, if $[U](\boldsymbol{c} \to \neg\phi) \notin \Sigma_2$, then $\neg[U](\boldsymbol{c} \to \neg\phi) \in \Sigma_2$, so $\langle U\rangle(\boldsymbol{c} \wedge \phi) \in \Sigma_2$, then because of (Nom2), $[U](\boldsymbol{c} \to \phi) \in \Sigma_2$, but $[U]\Sigma_2 \subseteq \Sigma_2$, so $\phi \in \Sigma_2$, contradiction. So, we have that $\Sigma_1 \subseteq \Sigma_2$. The converse inclusion is proven similarly, so $\Sigma_1 = \Sigma_2$. □

It easily follows that all axioms of $K_{(T,U)}$ are valid in any Λ-canonical frame.

We are now ready to define the Λ-*canonical model* for a given complete Λ-theory Γ. Let $\mathrm{F} = \langle W, \mathcal{R}\rangle$ be the Λ-canonical frame for Γ, then we define $\mathrm{M} := \langle \mathrm{F}, V, A\rangle$, where $V(p) := \{\Sigma \in W \mid p \in \Sigma\}$, and $A(c) := \Sigma$, where Σ is the only element of W, such that $c \in \Sigma$. The definition of A is correct by Proposition 35. It follows that M is a *named* model.

Lemma 36. (Truth Lemma) Let $\mathrm{M} = \langle\langle W, \mathcal{R}\rangle, V, A\rangle$ be the Λ-canonical model for some complete Λ-theory Γ. Then for any formula ϕ and any Σ in M, $\phi \in \Sigma$ iff $\mathrm{M}, \Sigma \Vdash \phi$.

Proof. Induction on ϕ. For atomic ϕ and for $\bot$, the result follows by the definition of the canonical model. For $(\phi_1 \to \phi_2)$, the result follows by the induction hypothesis and propositional reasoning. For $\Box_{\boldsymbol{i}}^{-1}\phi$: first, let $\Box_{\boldsymbol{i}}^{-1}\phi \in \Sigma$. Let $W' := \{\Sigma' \in W \mid \Box_{\boldsymbol{i}}\Sigma' \subseteq \Sigma\}$. Because for any Λ-complete theory Σ', $[U]\Sigma' \subseteq \Box_{\boldsymbol{i}}\Sigma'$, we have that for all Σ', such that $[U]\Sigma' \subseteq \Box_{\boldsymbol{i}}\Sigma' \subseteq \Sigma$, it is the case that $[U]\Sigma' = [U]\Sigma = [U]\Gamma$, therefore $\Sigma' \in W$. Then, for all these Σ', $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Sigma'$, so $\phi \in \Sigma'$. Therefore, by the induction hypothesis, for all these Σ': $\mathrm{M}, \Sigma' \Vdash \phi$, so, by the definition of $\mathcal{R}(\boldsymbol{i})$ and Definition 5, $\mathrm{M}, \Sigma \Vdash \Box_{\boldsymbol{i}}^{-1}\phi$. Now, let $\mathrm{M}, \Sigma \Vdash \Box_{\boldsymbol{i}}^{-1}\phi$. Then, using the same definition of W', we have that for all such Σ', we can use the induction hypothesis and find that $\phi \in \Sigma'$. Because W' contains exactly all Σ', such that $\Box_{\boldsymbol{i}}^{-1}\Sigma \subseteq \Sigma'$, then it follows that $\Box_{\boldsymbol{i}}^{-1}\phi \in \Sigma$. For $\Box_{\boldsymbol{i}}\phi$, the result follows by Lemma 34. □

Theorem 1 *(1) $K_{(T,U)}$ is strongly complete. (2) $K_{(T,U)} + \phi$ is strongly complete for any di-persistent modal formula ϕ.*

Proof. We use the second form of strong completeness. Like [15, 16, 5, 21]:

We show (1) and (2) together. Let Λ be either $K_{(T,U)}$ or $K_{(T,U)} + \phi$. Let Γ be a Λ-consistent set. By the Lindenbaum lemma, there is a complete Λ-theory Γ^+ extending Γ. We construct the canonical model M for Γ^+ and let its universe be W. By the truth lemma, Γ^+ is satisfiable in M at Γ^+, therefore Γ also is. The frame of M, F, also validates all axioms of $K_{(T,U)}$, which proves (1). For (2), it remains to prove that F validates ϕ. Because M is a named model, we construct $\mathfrak{g} = \langle \mathrm{F}, \mathbb{W} \rangle$, where $\mathbb{W} = \{[\![\phi']\!]_{\mathrm{M}} \mid \phi' \in \mathrm{ML}(T,U)\}$. Because ϕ is di-persistent, it is enough to show that $\mathfrak{g} \Vdash \phi$. Clearly, M is a model over $\mathfrak{g}$ and $\mathrm{M} \Vdash \phi$, so $[\![\phi]\!]_{\mathrm{M}} = W$. If $\mathrm{PROP}(\phi) \cup \mathrm{NOM}(\phi) = \emptyset$, then we are done. Otherwise, let all propositional variables occurring in ϕ be, in left-to-right order of initial occurrence, $p_1, \ldots, p_n$, and let the nominals of ϕ be, in left-to-right order of initial occurrence, $c_1, \ldots, c_m$. Then, clearly, for any model M′ over $\mathfrak{g}$, $[\![\phi]\!]_{\mathrm{M}'} = [\![\phi]\!](s_1, \ldots, s_n, w_1, \ldots, w_m)$ for some $s_1, \ldots, s_n \in \mathbb{W}$ and $w_1, \ldots, w_m \in W$, which, by the definition of $\mathbb{W}$ as the extensions in M of all possible formulas, and the fact that every w_i contains a nominal, is equal to the following set: $[\![\phi[p_1/\phi_1, \ldots, p_n/\phi_n, c_1/c_1', \ldots, c_m/c_m']]\!]_{\mathrm{M}}$ for some formulas $\phi_1, \ldots, \phi_n$ and some nominals $c_1', \ldots, c_m'$. However, $\Lambda \subseteq \Sigma$ for any complete Λ-theory Σ, and Λ is closed under applications of uniform substitution and nominal substitution. Therefore, for all $\Sigma \in W$: $\phi[p_1/\phi_1, \ldots, p_n/\phi_n, c_1/c_1', \ldots, c_m/c_m'] \in \Sigma$. So, by the truth lemma, the result follows. □

Corollary 37. For all formulas $\boldsymbol{\phi}$, for which Deterministic SQEMA succeeds, $K_{(T,U)} + \boldsymbol{\phi}$ is strongly complete.

Proof. All formulas, for which Deterministic SQEMA succeeds, are di-persistent, so the result follows by the above theorem. □

The language of pre-contact logics (PCL) is a first-order language with equality (=) and without quantifiers. It is intended to be a propositional language for point-free theories of space, as outlined in [1].

Boolean terms of PCL are: $\tau ::= p|0|1|-\tau|(\tau \cup \tau)|(\tau \cap \tau)$ where p is a *variable*, 0 and 1 are *boolean constants*. *Atomic formulas* are: $\alpha ::= \bot|\top|(\tau = \tau)|(\tau \leq \tau)|C(\tau, \tau)$ where part-of ($\leq$) and contact (C) are binary predicates. Pre-Contact formulas are: $\psi ::= \alpha|\neg\psi|(\psi \vee \psi)|(\psi \wedge \psi)$. We may use $\rightarrow$ and $\leftrightarrow$ as defined symbols with their usual meaning.

The usual definitions of Kripke frames and Kripke models are used.

If $\mathrm{M} = \langle \mathrm{F}, V \rangle$ is a model, where $\mathrm{F} = \langle W, \mathcal{R} \rangle$, then the valuation V can be extended to all boolean terms in the following way:

$V(0) = \emptyset$, $V(1) = W$

$V(-\tau_1) = W \setminus V(\tau_1)$

$V((\tau_1 \cup \tau_2)) = V(\tau_1) \cup V(\tau_2)$

$V((\tau_1 \cap \tau_2)) = V(\tau_1) \cap V(\tau_2)$

The definition of truth of atomic formulas in a Kripke model M is as follows:

$\mathrm{M} \vDash (\tau_1 = \tau_2)$ iff $V(\tau_1) = V(\tau_2)$

$\mathrm{M} \vDash (\tau_1 \leq \tau_2)$ iff $V(\tau_1) \subseteq V(\tau_2)$

$\mathrm{M} \vDash C(\tau_1, \tau_2)$ iff $\exists x \exists y (x \in V(\tau_1) \wedge y \in V(\tau_2) \wedge x\, \mathcal{R}(1)\, y)$

Truth of pre-contact formulas in M is defined in the standard way.

We say that ψ is *valid* in a frame F, $\mathrm{F} \vDash \psi$, iff ψ is true in all models over F.

It is shown in [1] that pre-contact formulas can be represented as formulas of $\mathrm{ML}(T, U)$. More precisely, there is a translation $\mathtt{t} : \mathrm{PCL} \rightarrow \mathrm{ML}(T, U)$ with the property that for every PCL formula ψ and every Kripke model M, $\mathrm{M} \vDash \psi$ iff $\mathrm{M} \Vdash \mathtt{t}(\psi)$. For describing the translation, we use the defined symbol $\leftrightarrow$ in the language $\mathrm{ML}(T, U)$ with its usual meaning.

This translation $\mathtt{t}$ maps variables to propositional variables. Function symbols map to the corresponding boolean connectives. $\mathtt{t}(0) = \bot \in \mathrm{ML}(T, U)$, $\mathtt{t}(1) = \top \in \mathrm{ML}(T, U)$. Let τ_1, τ_2 be terms. The predicate symbols translate as follows:

$\mathtt{t}((\tau_1 = \tau_2)) = [U](\mathtt{t}(\tau_1) \leftrightarrow \mathtt{t}(\tau_2))$

$\mathtt{t}((\tau_1 \leq \tau_2)) = [U](\mathtt{t}(\tau_1) \rightarrow \mathtt{t}(\tau_2))$

$\mathtt{t}(C(\tau_1, \tau_2)) = \langle U \rangle (\mathtt{t}(\tau_1) \wedge \Diamond_1 \mathtt{t}(\tau_2))$

The boolean connectives translate to themselves.

Now, we discuss Sahlqvist PCL formulas, as defined in [2].

A *positive* term is built up from variables, -0 and 1, using only $\cup$ and $\cap$.

A *negation-free* formula is built up from $\neg(\tau_1 = 0)$ and $C(\tau_1, \tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *positive* formula is built up from $\neg(\tau_1 = 0)$, $(-\tau_1 = 0)$, $(\tau_1 = 1)$, $C(\tau_1, \tau_2)$, and $\neg C(-\tau_1, -\tau_2)$, where τ_1 and τ_2 are positive terms, using only $\top$, $\vee$, and $\wedge$.

A *Sahlqvist* formula ψ is an implication $(\psi_1 \rightarrow \psi_2)$, where ψ_1 is negation-free, and ψ_2 is positive.

To translate Sahlqvist formulas, as defined in [2], into Sahlqvist formulas in $\mathrm{ML}(T, U)$, we define a *modified translation* $\mathtt{t}'$ as follows:

$\mathtt{t}'(p) := p \in \mathrm{ML}(T, U)$

$\mathtt{t}'(0) := \bot \in \mathrm{ML}(T, U)$

$\mathtt{t}'(1) := \top \in \mathrm{ML}(T, U)$

$\mathtt{t}'(-\tau) := \neg \mathtt{t}'(\tau)$ where τ is any term

$\mathtt{t}'((\tau_1 \cup \tau_2)) := (\mathtt{t}'(\tau_1) \vee \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((\tau_1 \cap \tau_2)) := (\mathtt{t}'(\tau_1) \wedge \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'((-\tau = 0)) := [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau = 1)) := [U]\mathtt{t}'(\tau)$, where τ is any term.

$\mathtt{t}'((\tau_1 = \tau_2)) := [U](\mathtt{t}'(\tau_1) \leftrightarrow \mathtt{t}'(\tau_2))$, where $(\tau_1 = \tau_2)$ is not as in the above two cases

$\mathtt{t}'((\tau_1 \leq \tau_2)) := [U](\mathtt{t}'(\tau_1) \rightarrow \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(C(\tau_1, \tau_2)) := \langle U \rangle(\mathtt{t}'(\tau_1) \wedge \Diamond_1 \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg C(-\tau_1, -\tau_2)) := [U](\mathtt{t}'(\tau_1) \vee \Box_1 \mathtt{t}'(\tau_2))$, where τ_1 and τ_2 are any terms

$\mathtt{t}'(\neg(\tau = 0)) := \langle U \rangle \mathtt{t}'(\tau)$, where τ is any term

$\mathtt{t}'(\neg\psi) := \neg \mathtt{t}'(\psi)$, where $\neg\psi$ is not as in the above two cases

$\mathtt{t}'((\psi_1 \vee \psi_2)) := (\mathtt{t}'(\psi_1) \vee \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

$\mathtt{t}'((\psi_1 \wedge \psi_2)) := (\mathtt{t}'(\psi_1) \wedge \mathtt{t}'(\psi_2))$ for any ψ_1 and ψ_2

It is easy to see, by induction on PCL terms and PCL formulas, that for any PCL formula ψ, ψ and $\mathtt{t}'(\psi)$ are true in the same models.

We show now how to derive a result from [2] that Sahlqvist formulas have a first-order correspondent as a corollary to the fact that Deterministic SQEMA succeeds on all Sahlqvist $\mathrm{ML}(T, U)$ formulas.

Theorem 2 *The modified translation maps Sahlqvist PCL formulas to Sahlqvist implications from* $\mathrm{ML}(T, U)$.

Proof. An easy induction on PCL terms shows that $\mathtt{t}'(\tau)$ for a positive term τ is a positive $\mathrm{ML}(T, U)$ formula. Similarly, it is simple to show that $\mathtt{t}'(\psi)$ for a positive ψ is a positive $\mathrm{ML}(T, U)$ formula. It remains to show that $\mathtt{t}'$ maps negation-free PCL formulas to $\mathrm{ML}(T, U)$ Sahlqvist antecedents. This again follows from an easy induction, using the definition of $\mathtt{t}'$. □

We use Deterministic SQEMA for the language of Pre-Contact Logic, by translating a pre-contact formula to a formula of $\mathrm{ML}(T, U)$, using $\mathtt{t}'$, and running Deterministic SQEMA on the translation. It immediately follows that Deterministic SQEMA succeeds on the modified translation of any Sahlqvist PCL formula.

It was proved in [1] that: Every pre-contact formula is complete with respect to the class of finite frames defined by it. Hence, every pre-contact formula is complete.

Theorem 3 *Every PCL formula ψ, on whose modified translation Deterministic SQEMA succeeds and produces a FOL formula ψ', is complete on the class of frames defined by ψ'.*

Proof. By the properties of Deterministic SQEMA, $\mathtt{t}'(\psi)$ and ψ' are locally correspondent, therefore globally correspondent. By the properties of $\mathtt{t}'$, ψ and $\mathtt{t}'(\psi)$ define the same class of frames, so ψ and ψ' define the same class of frames, therefore they define the same class of finite frames. By the above-mentioned result in [1], ψ is complete in the class of finite frames, defined by ψ', and therefore is complete in the class of all frames, defined by ψ'. □

9. CONCLUSION

We have shown sufficient conditions for di-persistence and for the existence of first-order correspondents. We have shown that SQEMA can be reduced to a Deterministic SQEMA. We have proven that it always succeeds for Sahlqvist and inductive formulas, and that it always terminates. We have shown the strong completeness of all formulas, on which Deterministic SQEMA succeeds, in the language of ML(T, U). We have extended Deterministic SQEMA so that it succeeds on all Sahlqvist formulas of the pre-contact language. Deterministic SQEMA could be extended via a resolution procedure and a tableaux method in the normalization procedure and via a tableaux method in the *step* function.

It would be interesting to show how Deterministic SQEMA can be modified to succeed on all formulas having only the universal modality.

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10. REFERENCES

[1] Balbiani, P., Tinchev, T., Vakarelov, D.: Modal Logics for Region-based Theories of Space. *Fundamenta Informaticae*, **81**, 2007, IOS Press, 29–82.

[2] Balbiani, P., Kikot, S.: Sahlqvist Theorems for Precontact Logics. *AiML*, **9**, 2014.

[3] Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic*, Cambridge Tracts in Theoretical Computer Science), Cambridge University Press, 2002.

[4] Chagrova, L.: An undecidable problem in correspondance theory. *J. Symb. Logic*, **56**, 1991, 1261–1272.

[5] Conradie, W., Goranko, V., Vakarelov, D.: Elementary Canonical Formulae: A Survey on Syntactic, Algorithmic, and Model–theoretic Aspects. *AiML*, **5**, 2005, 17–51.

[6] Conradie, W., Goranko, V., Vakarelov, D.: Algorithmic correspondence and completeness in modal logic. I. The core algorithm SQEMA. *LMCS* **2** (1:5), 2006, 1–26.

[7] Conradie, W., Goranko, V., Vakarelov, D.: Algorithmic correspondence and completeness in modal logic II. Polyadic and hybrid extensions of the algorithm SQEMA. *J. Logic Comput.*, **16**, no. 5, 2006, 579–612.

[8] Conradie, W., Goranko, V., Vakarelov, D.: Algorithmic Correspondence and Completeness in Modal Logic. III. Extensions of the Algorithm SQEMA with Substitutions. *Fund. Informaticae*, **92**, no. 4, 2009, 307–343.

[9] Conradie, W., Goranko, V., Vakarelov, D.: Algorithmic correspondence and completeness in modal logic. V. Recursive extensions of SQEMA. *J. Appl. Logic*. **8**, no. 4, 2010, 319–333.

[10] Conradie, W., Goranko, V.: Algorithmic correspondence and completeness in modal logic IV. Semantic extensions of SQEMA. *J. Appl. Non-Classical Logics*, **18**, no. 2–3, 2012, 175–211.

[11] Gabbay, D., Ohlbach, H. J.: Quantifier Elimination in Second-Order Predicate Logic. *South African Computer Journal*, **7**, 1992, 35–43.

[12] Gargov, G., Goranko, V.: Modal Logic with Names. *J. Phil. Logic*, **22**, 1993, 607–636.

[13] Georgiev, D.: *An implementation of the algorithm SQEMA for computing first-order correspondences of modal formulas*. Master Thesis, Sofia University, FMI, 2006.

[14] Georgiev, D.: SQEMA with Universal Modality. In: *Proceedings of the* 10^{th} *Panhellenic Logic Symposium*, 2015, pp. 76–81.

[15] Goranko, V., Vakarelov D.: Sahlqvist formulae in Hybrid Polyadic Modal Languages. *J. Logic and Comput.*, **11**, no. 5, 2001, 737–754.

[16] Goranko, V., Vakarelov, D.: Elementary Canonical Formulae: Extending Sahlqvist's Theorem. *Ann. Pure Appl. Logics*, **141**, no. 1-2, 2006, 180–217.

[17] Passy, S., Tinchev, T.: PDL with Data Constants. *Information Processing Letters*, **20**, 1985, 35–41.

[18] Passy, S., Tinchev, T.: An Essay in Combinatory Dynamic Logic. *Information and Computation*, **93**, 1991, 263–332.

[19] Sahlqvist, H.: Completeness and Correspondence in the First and Second Order Semantics for Modal Logics. In: *Proceedings of the* 3^{rd} *Scandinavian Logic Symposium*, 1975, pp. 110–143.

[20] Szalas, A.: On the correspondence between Modal and Classical Logic: an Automated Approach. *J. Logic and Comp.*, **3**, no. 6, 1993, 605–620.

[21] ten Cate, B.: *Model theory for extended modal languages*, PhD Thesis, Institute for Logic, Language and Computation, 2005, chapter 5, pp. 69–89.

[22] Vakarelov D.: A recursive generalization of Ackermann Lemma with applications to modal μ-definability. In: *Proceedings of the 6th Panhellenic Logic Symposium, Volos, Greece, 5-8 July 2007*, pp. 133–137

[23] van Benthem, J.: *Correspondence Theory.* Handbook of Philosophical Logic, Volume II, Synthese Library, Studies in Epistemology, Logic, Methodology, and Philosophy of Science, **165**, 1984, 167–247.

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