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SMALL MINIMAL (3, 3)-RAMSEY GRAPHS

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We say that G is a $(3,3)$-Ramsey graph if every 2-coloring of the edges of G forces a monochromatic triangle. The $(3,3)$-Ramsey graph G is minimal if G does not contain a proper $(3,3)$-Ramsey subgraph. In this work we find all minimal $(3,3)$-Ramsey graphs with up to 13 vertices with the help of a computer, and we obtain some new results for these graphs. We also obtain new upper bounds for the independence number and new lower bounds for the minimum degree of arbitrary $(3,3)$-Ramsey graphs.

Keywords: Ramsey graph, clique number, independence number, chromatic number

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1. INTRODUCTION

In this work only finite, non-oriented graphs without loops and multiple edges are considered. The following notations are used:

$\mathrm{V}(G)$ - the vertex set of G;

$\mathrm{E}(G)$ - the edge set of G;

$\overline{G}$ - the complement of G;

$\omega(G)$ - the clique number of G;

$\alpha(G)$ - the independence number of G;

$\chi(G)$ - the chromatic number of G;

$N_G(v), v \in \mathrm{V}(G)$ - the set of all vertices of G adjacent to v;

$d(v), v \in \mathrm{V}(G)$ - the degree of the vertex v, i.e. $d(v) = |N_G(v)|$;

$G(v), v \in \mathrm{V}(G)$ - subgraph of G induced by $N_G(v)$;

$G - v, v \in \mathrm{V}(G)$ - subgraph of G obtained from G by deleting the vertex v and all edges incident to v;

$G - e, e \in \mathrm{E}(G)$ - subgraph of G obtained from G by deleting the edge e;

$\Delta(G)$ - the maximum degree of G;

$\delta(G)$ - the minimum degree of G;

K_n - complete graph on n vertices;

C_n - simple cycle on n vertices;

$G_1 + G_2$ - graph G for which $\mathrm{V}(G) = \mathrm{V}(G_1) \cup \mathrm{V}(G_2)$ and $\mathrm{E}(G) = \mathrm{E}(G_1) \cup \mathrm{E}(G_2) \cup E'$, where $E' = \{[x, y] : x \in \mathrm{V}(G_1), y \in \mathrm{V}(G_2)\}$, i.e. G is obtained by connecting every vertex of G_1 to every vertex of G_2.

All undefined terms can be found in [13].

Each partition

$$\mathrm{E}(G) = E_1 \cup \cdots \cup E_r, \quad E_i \cap E_j = \emptyset, \quad i \neq j \tag{1.1}$$

is called an r-coloring of the edges of G. We say that $H \subseteq G$ is a monochromatic subgraph of color i in the r-coloring (1.1), if $\mathrm{E}(H) \subseteq E_i$.

Let p and q be positive integers, $p \geq 2$ and $q \geq 2$. The notation $G \to (p, q)$ means that for every 2-coloring of $\mathrm{E}(G)$ there exists a p-clique of the first color or a q-clique of the second color. If $G \to (p, q)$, we say that G is a (p, q)-Ramsey graph. Similarly, the expression $G \to (p_1, \ldots, p_r)$ is defined for the r-colorings of $\mathrm{E}(G)$.

The smallest possible integer n for which $K_n \to (p, q)$ is called a Ramsey number and is denoted by $R(p, q)$. The Ramsey numbers $R(p_1, p_2, \ldots, p_r)$ are defined similarly. The existence of Ramsey numbers was proved by Ramsey in [32]. Only a few exact values of Ramsey numbers are known (see [30]). In this work we shall use the equality $R(3, 3) = 6$, which means that $K_6 \to (3, 3)$ and $K_5 \not\to (3, 3)$. Clearly, if $\omega(G) \geq 6$, then $G \to (3, 3)$. In [6] Erdös and Hajnal posed the problem:

Is there a graph $G \to (3, 3)$ with $\omega(G) < 6$?

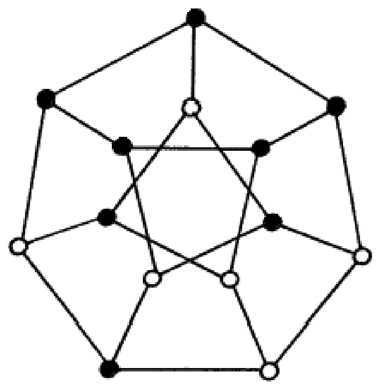

Figure 1: The complement of the van Lint graph from [12]

The first example of a graph which gives an affirmative answer to this question was given by van Lint. The complement of this graph is presented in Figure 1. Van Lint did not publish this result himself, but the graph was included in [12]. Later, Graham [11] constructed the smallest possible example of such a graph, namely $K_3 + C_5$. It is easy to see that the van Lint graph contains $K_3 + C_5$ (it is the subgraph induced by the black vertices in Figure 1).

There exist $(3,3)$-Ramsey graphs which do not contain K_5. These graphs have at least 15 vertices [29]. The first 15-vertex $(3,3)$-Ramsey graph which does not contain K_5 was constructed by Nenov [25]. This graph is obtained from the graph Γ presented in Figure 2 by adding a new vertex which is adjacent to all vertices of Γ.

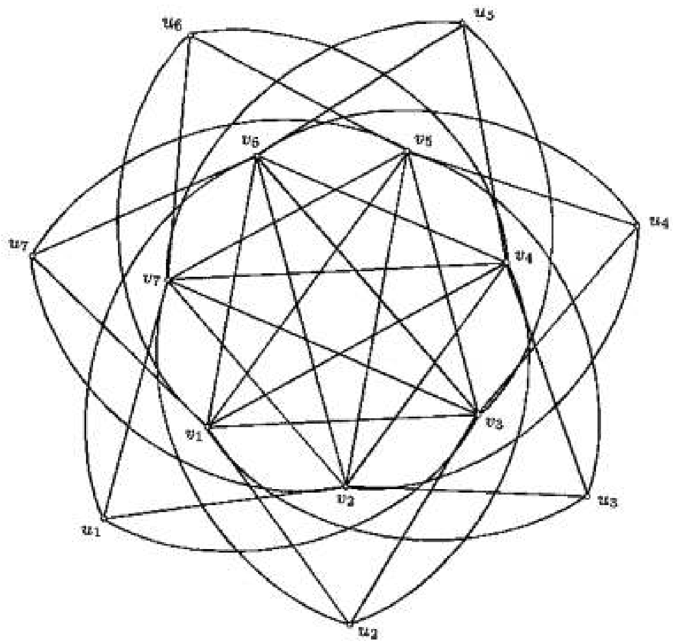

Figure 2: The Nenov graph Γ from [25]

Folkman [7] constructed a graph $G \to (3,3)$ with $\omega(G) = 3$. The minimum number of vertices of such graphs is not known. To date, we only know ([31] and [18]) that this minimum is between 19 and 786.

Obviously, if H is a (p,q)-Ramsey graph, then its every supergraph G is also a (p,q)-Ramsey graph.

Definition 1.1. *We say that* G *is a minimal* (p,q)*-Ramsey graph if* $G \to (p,q)$ *and* $H \not\to (p,q)$ *for each proper subgraph* H *of* G.

It is easy to see that K_6 is a minimal $(3,3)$-Ramsey graph and there are no minimal $(3,3)$-Ramsey graphs with 7 vertices. The only such 8-vertex graph is the Graham graph $K_3 + C_5$, and there is only one such 9-vertex graph, Nenov [22] (see Figure 3).

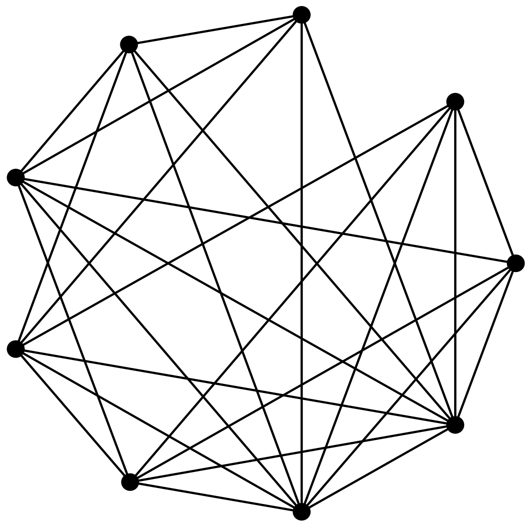

Figure 3: 9-vertex minimal $(3,3)$-Ramsey graph

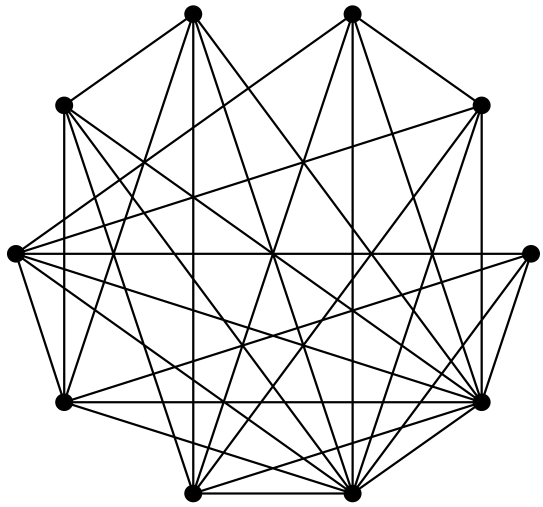

Figure 4: 10-vertex minimal $(3,3)$-Ramsey graph

For each pair of positive integers $p \geq 3$, $q \geq 3$ there exist infinitely many minimal (p,q)-Ramsey graphs [2], [8]. The simplest infinite sequence of minimal $(3,3)$-Ramsey graphs is formed by the graphs $K_3 + C_{2r+1}, r \geq 1$. This sequence contains the already mentioned graphs K_6 and $K_3 + C_5$. This sequence was found by Nenov and Khadzhiivanov in [27]. Later, it was rediscovered in [3], [9], [35].

Three 10-vertex minimal $(3,3)$-Ramsey graphs are known. One of them is $K_3 + C_7$ from the sequence $K_3 + C_{2r+1}, r \geq 1$. The other two were obtained by Nenov in [24] (the second graph is presented in Figure 4 and the third is a subgraph of $K_1 + \overline{C_9}$).

The main goal of this work is to find new minimal $(3,3)$-Ramsey graphs. To achieve this, we develop computer algorithms which are presented in Section 3. Using Algorithm 3.1, in Section 4 we find all minimal $(3,3)$-Ramsey graphs with up to 12 vertices. In the next Section 5 we find all 13-vertex minimal $(3,3)$-Ramsey graphs using Algorithm 3.11. From the graphs found in Section 4 and Section 5 we obtain interesting corollaries, which are presented in Section 6. With the help of Algorithm 3.8, in Section 7 and Section 8, respectively, we obtain new upper bounds for the independence number and new lower bounds for the minimum degree of minimal $(3,3)$-Ramsey graphs with an arbitrary number of vertices.

Similar computer aided research is made in [17], [29], [4], [5], [31], [36], [18] and [34]. We note that the algorithms from [29] were very useful to us.

This work is an extended version of the author Master Thesis written under the supervision of Prof. Nedyalko Nenov. The most essential new element is Algorithm 3.8, which is obtained jointly with Prof. Nenov.

We will need the following results:

Theorem 2.1. ([2],[8]) *Let G be a minimal (p,p)-Ramsey graph. Then, $\delta(G) \geq (p-1)^2$. In particular, when $p = 3$, we have $\delta(G) \geq 4$.*

Definition 2.2. *We say that G is a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some pair of vertices $u, v \in \mathrm{V}(G)$.*

Proposition 2.3. *If G is a minimal (p,q)-Ramsey graph, then G is not a Sperner graph.*

Proof. Suppose the opposite is true, and let $u, v \in \mathrm{V}(G)$ be such that $N_G(u) \subseteq N_G(v)$. We color the edges of $G - u$ with two colors in such a way that there is no monochromatic p-clique of the first color and no monochromatic q-clique of the second color. After that, for each vertex $w \in N_G(u)$ we color the edge $[u,w]$ with the same color as the edge $[v,w]$. We obtain a 2-coloring of the edges of G with no monochromatic p-cliques of the first color and no monochromatic q-cliques of the second color. □

Theorem 2.4. ([29]) *Let G be a $(3,3)$- Ramsey graph and $G \neq K_6$. If $|V(G)| \leq 14$, then $\omega(G) = 5$.*

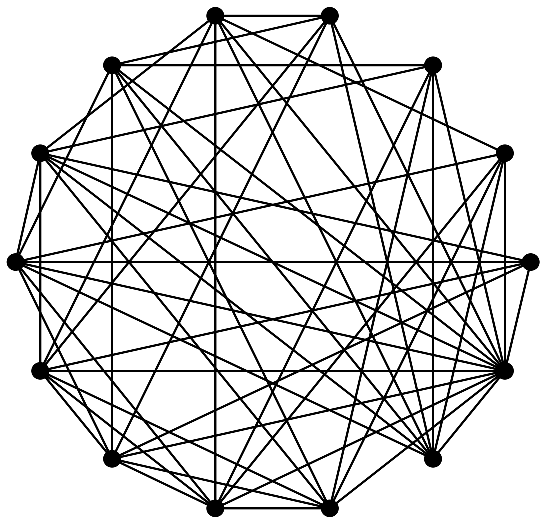

Figure 5: 14-vertex minimal $(3,3)$-Ramsey graph with a single 5 clique

According to Theorem 2.4, every $(3,3)$-Ramsey graph G with at most 14 vertices contains a 5-clique. There exist 14-vertex $(3,3)$-Ramsey graphs containing only a single 5-clique, an example of such a graph is presented in Figure 5. The graph in Figure 5 is obtained with the help of the only 15-vertex bicritical $(3,3)$-Ramsey graph with clique number 4 from [29]. First, by removing a vertex from the bicritical graph, we obtain 14-vertex graphs without 5 cliques. After that, by adding edges to the obtained graphs, we find a 14-vertex $(3,3)$-Ramsey graph with a single 5-clique whose subgraph is the minimal $(3,3)$-Ramsey graph in Figure 5. Let us note that in [29] the authors obtain all 15-vertex $(3,3)$-Ramsey graphs with clique number 4, and with the help of these graphs, one can find more examples of 14-vertex $(3,3)$-Ramsey graphs.

Theorem 2.5. ([19]) *Let G be a graph and $G \to (p,q)$. Then $\chi(G) \geq R(p,q)$. In particular, if $G \to (3,3)$, then $\chi(G) \geq 6$.*

Corollary 2.6. *Let $G \to (3,3)$, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $\chi(H) \geq 5$.*

Theorem 2.7. *Let G be a minimal $(3,3)$-Ramsey graph. Then, for each vertex $v \in \mathrm{V}(G)$ we have $\alpha(G(v)) \leq d(v) - 3$.*

Proof. Suppose the opposite is true, and let $A \subseteq N_G(v)$ be an independent set in $G(v)$ such that $|A| = d(v) - 2$. Let $a, b \in N_G(v) \setminus A$. Consider a 2-coloring of the edges of $G - v$ in which there are no monochromatic triangles. We color the edges $[v,a]$ and $[v,b]$ with the same color in such a way that there is no monochromatic triangle (if a and b are adjacent, we chose the color of $[v,a]$ and $[v,b]$ to be different from the color of $[a,b]$, and if a and b are not adjacent, then we chose an arbitrary color for $[v,a]$ and $[v,b]$). We color the remaining edges incident to v with the other color, which is different from the color of $[v,a]$ and $[v,b]$. Since $N_G(v) \setminus \{a,b\} = A$ is an independent set, we obtain a 2-coloring of the edges of G without monochromatic triangles, which is a contradiction. □

Corollary 2.8. *Let G be a minimal $(3,3)$-Ramsey graph and $d(v) = 4$ for some vertex $v \in \mathrm{V}(G)$. Then, $G(v) = K_4$.*

3. ALGORITHMS

In this section, the computer algorithms used in this work are presented.

The first algorithm is appropriate for finding all minimal $(3,3)$-Ramsey graphs with a small number of vertices.

Algorithm 3.1. *Finding all minimal $(3,3)$-Ramsey graphs with n vertices, where n is fixed and $7 \leq n \leq 14$.*

1. Generate all n-vertex non-isomorphic graphs with minimum degree at least 4, and denote the obtained set by $\mathcal{B}$.

2. Remove from $\mathcal{B}$ all Sperner graphs.

3. Remove from $\mathcal{B}$ all graphs with clique number not equal to 5.

4. Remove from $\mathcal{B}$ all graphs with chromatic number less than 6.

5. Remove from $\mathcal{B}$ all graphs which are not $(3,3)$-Ramsey graphs.

6. Remove from $\mathcal{B}$ all graphs which are not minimal $(3,3)$-Ramsey graphs.

Theorem 3.2. *Fix $n \in \{7, \dots, 14\}$. Then, after executing Algorithm 3.1, $\mathcal{B}$ consists of all n-vertex minimal $(3,3)$-Ramsey graphs.*

Proof. Step 6 guaranties that $\mathcal{B}$ contains only minimal $(3,3)$-Ramsey graphs with n vertices. Let G be an arbitrary n-vertex minimal $(3,3)$-Ramsey graph. We will prove that $G \in \mathcal{B}$. By Theorem 2.1, $\delta(G) \geq 4$, and by Theorem 2.3, G is not a Sperner graph. Since $|V(G)| \leq 14$, by Theorem 2.4 we have $\omega(G) = 5$. By Theorem 2.5, $\chi(G) \geq 6$. Therefore, after step 4, $G \in \mathcal{B}$. □

In Section 4 we apply Algorithm 3.1 to obtain all $(3,3)$-Ramsey graphs with up to 12 vertices. Algorithm 3.1 is not appropriate in the case $n \geq 13$, because the number of graphs generated in step 1 is too big. To find the 13-vertex minimal $(3,3)$-Ramsey graphs, we will apply Algorithm 3.11, which is given below.

In order to present the next algorithms, we shall need the following definitions and auxiliary propositions:

We say that a 2-coloring of the edges of a graph is $(3,3)$-free if it has no monochromatic triangles.

Definition 3.3. *Let G be a graph and $M \subseteq \mathrm{V}(G)$. Let G_1 be a graph which is obtained by adding a new vertex v to G such that $N_{G_1}(v) = M$. We say that M is a marked vertex set in G if there exists a $(3,3)$-free 2-coloring of the edges of G which cannot be extended to a $(3,3)$-free 2-coloring of the edges of G_1.*

It is clear that if $G \to (3,3)$, then there are no marked vertex sets in G. The following proposition is true:

Proposition 3.4. *Let G be a minimal $(3,3)$-Ramsey graph, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $N_G(v_i), i = 1, \dots, s$, are marked vertex sets in H.*

Proof. Suppose the opposite is true, i.e. $N_G(v_i)$ is not a marked vertex set in H for some $i \in \{1, \dots, s\}$. Since G is a minimal $(3,3)$-Ramsey graph, there exists a $(3,3)$-free 2-coloring of the edges of $G - v_i$, which induces a $(3,3)$-free 2-coloring of the edges of H. By supposition, we can extend this 2-coloring to a $(3,3)$-free 2-coloring of the edges of the graph $H_i = G - \{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_s\}$. Thus, we obtain a $(3,3)$-free 2-coloring of the edges of G, which is a contradiction. □

Definition 3.5. *Let $\{M_1, \dots, M_s\}$ be a family of marked vertex sets in the graph G. Let G_i be a graph which is obtained by adding a new vertex v_i to G such that $N_{G_i}(v_i) = M_i, i = 1, \dots, s$. We say that $\{M_1, \dots, M_s\}$ is a complete family of marked vertex sets in G, if for each $(3,3)$-free 2-coloring of the edges of G there exists $i \in \{1, \dots, s\}$ such that this 2-coloring can not be extended to a $(3,3)$-free 2-coloring of the edges of G_i.*

Proposition 3.6. *Let $v_1, \dots, v_s$ be independent vertices of the graph G and $H = G - \{v_1, \dots, v_s\}$. If $\{N_G(v_1), \dots, N_G(v_s)\}$ is a complete family of marked vertex sets in H, then $G \to (3,3)$.*

Proof. Consider a 2-coloring of the edges of G which induces a 2-coloring with no monochromatic triangles in H. According to Definition 3.5, this 2-coloring of the edges of H can not be extended in G without forming a monochromatic triangle. □

It is easy to prove the following strengthening of Proposition 3.4:

Proposition 3.7. *Let G be a minimal $(3,3)$-Ramsey graph, let $v_1, \dots, v_s$ be independent vertices of G and $H = G - \{v_1, \dots, v_s\}$. Then, $\{N_G(v_1), \dots, N_G(v_s)\}$ is a complete family of marked vertex sets in H. What is more, this family is a minimal complete family, in the sense that it does not contain a proper complete subfamily.*

Let G be a minimal $(3,3)$-Ramsey graph and $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$. Let A be an independent set in G such that $|A| = |\mathrm{V}(G)| - k$. Then, $|\mathrm{V}(G - A)| = k$, and therefore the graph G is obtained by adding an independent set of vertices to the k-vertex graph $G - A$. From Proposition 2.3 it is easy to see that for a fixed k there are a finite number of minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$. Below we give an algorithm for finding all minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$, where k is fixed (but $\mathrm{V}(G)$ is not fixed).

Algorithm 3.8. (A. Bikov and N. Nenov) *Finding all minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < q$ and $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$, where q and k are fixed positive integers.*

1. Denote by $\mathcal{A}$ the set of all k-vertex graphs H for which $\omega(H) < q$ and $\chi(H) \geq 5$. The obtained minimal $(3,3)$-Ramsey graphs will be output in the set $\mathcal{B}$, let $\mathcal{B} = \emptyset$.

2. For each graph $H \in \mathcal{A}$:

2.1. Find all subsets M of $\mathrm{V}(H)$ which have the properties:

(a) $K_{q-1} \not\subseteq H[M]$, i.e. M is a $K_{(q-1)}$-free subset.

(b) $M \not\subseteq N_H(v), \forall v \in \mathrm{V}(H)$.

(c) M is a marked vertex set in H (see Definition 3.3).

Denote by $\mathcal{M}(H)$ the family of subsets of $\mathrm{V}(H)$ which have the properties (a), (b) and (c). Enumerate the elements of $\mathcal{M}(H)$: $\mathcal{M}(H) = \{M_1, \dots, M_t\}$.

2.2. Find all minimal complete subfamilies of $\mathcal{M}(H)$ (see Definition 3.5). For each such found subfamily $\{M_{i_1}, \dots, M_{i_s}\}$ construct the graph $G = G(M_{i_1}, \dots, M_{i_s})$ by adding new independent vertices $v_1, v_2, \dots, v_s$ to $\mathrm{V}(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \dots, s$. Add G to $\mathcal{B}$. If there are no complete subfamilies of $\mathcal{M}(H)$, then no supergraphs of H are added to $\mathcal{B}$.

3. Remove the isomorphic copies of the graphs from $\mathcal{B}$.

4. Remove from $\mathcal{B}$ all non-minimal $(3,3)$-Ramsey graphs.

Remark 3.9. *It is clear that if G is a minimal $(3,3)$-Ramsey graph and $\omega(G) \geq 6$, then $G = K_6$. Obviously there are no $(3,3)$-Ramsey graphs with clique number less than 3. Therefore, we shall use Algorithm 3.8 only for $q \in \{4,5,6\}$.*

Theorem 3.10. *After executing Algorithm 3.8, the set* $\mathcal{B}$ *coincides with the set of all minimal* (3,3)*-Ramsey graphs* G *for which* $\omega(G) < q$ *and* $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$.

Proof. From step 2.2 it becomes clear that every graph G which is added to $\mathcal{B}$ is obtained by adding independent vertices $v_1, \ldots, v_s$ to a graph $H \in \mathcal{A}$. Therefore, $\alpha(G) \geq s = |\mathrm{V}(G)| - |\mathrm{V}(H)| = |\mathrm{V}(G)| - k$. From $\omega(H) < q$ and $K_{q-1} \not\subseteq H[N_G(v_i)], i = 1, \ldots, s$, it follows that $\omega(G) < q$. According to Proposition 3.6, after step 2.2 $\mathcal{B}$ contains only $(3,3)$-Ramsey graphs, and after step 4 $\mathcal{B}$ contains only minimal $(3,3)$-Ramsey graphs.

In order to prove that $\mathcal{B}$ contains all minimal $(3,3)$-Ramsey graphs which fulfill the conditions, consider an arbitrary minimal $(3,3)$-Ramsey graph G for which $\omega(G) < q$ and $\alpha(G) \geq |\mathrm{V}(G)| - k \geq 1$. We will prove that $G \in \mathcal{B}$.

Denote $s = |\mathrm{V}(G)| - k \geq 1$. Let $v_1, \ldots, v_s$ be independent vertices of G and $H = G - \{v_1, \ldots, v_s\}$. By 2.6, $\chi(H) \geq 5$. Therefore, after executing step 1, $H \in \mathcal{A}$.

From $\omega(G) < q$ it follows $\omega(G(v_i)) < q - 1$. By Proposition 2.3, G is not a Sperner graph, and therefore $N_G(v_i) \not\subseteq N_H(v), \forall v \in \mathrm{V}(H)$. According to Proposition 3.4, $N_G(v_i)$ are marked vertex sets in H. Therefore, after executing step 2.1, $N_G(v_i) \in \mathcal{M}(H), i = 1, \ldots, s$.

From Proposition 3.7 it becomes clear that $\{N_G(v_1), ..., N_G(v_s)\}$ is a minimal complete subfamily of $\mathcal{M}(H)$. Therefore, in step 2.2 the graph G is added to $\mathcal{B}$.

Thus, the theorem is proved. □

In order to find the 13-vertex minimal $(3,3)$-Ramsey graphs we shall use the following modification of Algorithm 3.8 in which $n = |\mathrm{V}(G)|$ is fixed:

Algorithm 3.11. *Modification of Algorithm 3.8 for finding all n-vertex minimal* $(3,3)$*-Ramsey graphs* G *for which* $\omega(G) < q$ *and* $\alpha(G) \geq n - k \geq 1$*, where* q*,* k *and* n *are fixed positive integers.*

In step 2.2 of Algorithm 3.8 add the condition to consider only minimal complete subfamilies $\{M_{i_1}, ..., M_{i_s}\}$ *of* $\mathcal{M}(H)$ *in which* $s = n - k$.

4. MINIMAL $(3,3)$-RAMSEY GRAPHS WITH UP TO 12 VERTICES

We execute Algorithm 3.1 for $n = 7, 8, 9, 10, 11, 12$, and we find all minimal $(3,3)$-Ramsey graphs with up to 12 vertices except K_6. In this way, we obtain the known results: there is no minimal $(3,3)$-Ramsey graph with 7 vertices, the Graham graph $K_3 + C_5$ is the only such 8-vertex graph, and there exists only one such 9-vertex graph, the Nenov graph from [22] (see Figure 3). We also obtain the following new results:

Theorem 4.1. *There are exactly 6 minimal 10-vertex* $(3,3)$*-Ramsey graphs. These graphs are given in Figure 14, and some of their properties are listed in Table 2.*

Theorem 4.2. *There are exactly 73 minimal 11-vertex* $(3,3)$*-Ramsey graphs. Some of their properties are listed in Table 3. Examples of 11-vertex minimal* $(3,3)$*-Ramsey graphs are given in Figure 15 and Figure 16.*

Theorem 4.3. *There are exactly 3041 minimal 12-vertex* $(3,3)$*-Ramsey graphs. Some of their properties are listed in Table 4. Examples of 12-vertex minimal* $(3,3)$*-Ramsey graphs are given in Figure 17 and Figure 18.*

We will use the following enumeration for the obtained minimal $(3,3)$-Ramsey graphs:
- $G_{10.1}, \dots, G_{10.6}$ are the 10-vertex graphs;
- $G_{11.1}, \dots, G_{11.73}$ are the 11-vertex graphs;
- $G_{12.1}, \dots, G_{12.3041}$ are the 12-vertex graphs;

The indices correspond to the order of the graphs' canonical labels defined in *nauty* [20].

Detailed data for the number of graphs obtained at each step of the execution of Algorithm 3.1 is given in Table 1.

Step of Algorithm 3.1	$n = 8$	$n = 9$	$n = 10$	$n = 11$	$n = 12$
1	424	15 471	1 249 973	187 095 840	48 211 096 031
2	59	2 365	206 288	33 128 053	9 148 907 379
3	9	380	41 296	8 093 890	2 763 460 021
4	1	7	356	78 738	44 904 195
5	1	3	126	23 429	11 670 079
6	1	1	6	73	3041

Table 1: Steps in finding all minimal $(3,3)$-Ramsey graphs with up to 12 vertices

5. MINIMAL $(3,3)$-RAMSEY GRAPHS WITH 13 VERTICES

The method we apply for findning all 13-vertex minimal $(3,3)$-Ramsey graphs consists of two parts:

1. First, we find the 13-vertex minimal $(3,3)$-Ramsey graphs with independence number 2. We use that (see [30]) $R(3,6) = 18$, and that all graphs G for which $\alpha(G) < 3$ and $\omega(G) < 6$ are known [21]. Among them, the 13-vertex graphs are 275 086. By computer check, we find that exactly 13 of these graphs are minimal $(3,3)$-Ramsey graphs.

2. It remains to find the 13-vertex minimal $(3,3)$-Ramsey graphs with independence number at least 3. To do this, we execute Algorithm 3.11 with $n = 13$; $k = 10$; $q = 6$. First, in step 1 of Algorithm 3.11 we find all 1 923 103 graphs H with 10 vertices for which $\omega(H) \leq 5$ and $\chi(H) \geq 5$. After that, in step 2 of Algorithm 3.11 we add 3 independent vertices to the obtained 10-vertex graphs,

$\lvert \mathrm{E}(G) \rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G) \rvert$	#
30	1	4	1	9	6	2	3	6	6	4	2
31	1	5	4			3	3			8	2
32	2	6	1							16	1
33	1									84	1
34	1										

Table 2: Some properties of the 10-vertex minimal $(3,3)$-Ramsey graphs

$\lvert \mathrm{E}(G) \rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G) \rvert$	#
35	6	4	5	8	1	2	4	6	73	1	20
36	13	5	58	10	72	3	66			2	29
37	23	6	10			4	3			4	14
38	25									6	1
39	5									8	4
41	1									12	1
										16	3
										24	1

Table 3: Some properties of the 11-vertex minimal $(3,3)$-Ramsey graphs

$\lvert \mathrm{E}(G) \rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G) \rvert$	#
38	5	4	129	8	43	2	124	6	3 041	1	1 792
39	27	5	2 178	9	1 196	3	2 431			2	851
40	144	6	611	11	1 802	4	485			4	286
41	418	7	123			5	1			6	1
42	1 014									8	67
43	459									12	16
44	224									16	18
45	351									24	6
46	299									32	1
47	84									36	1
48	16									96	1
										108	1

Table 4: Some properties of the 12-vertex minimal $(3,3)$-Ramsey graphs

$\lvert \mathrm{E}(G) \rvert$	#	$\delta(G)$	#	$\Delta(G)$	#	$\alpha(G)$	#	$\chi(G)$	#	$\lvert Aut(G) \rvert$	#
41	4	4	13 725	8	16	2	13	6	306 622	1	251 976
42	44	5	191 504	9	61 678	3	218 802	7	13	2	46 487
43	220	6	85 932	10	175 108	4	86 721			3	10
44	1 475	7	15 391	12	69 833	5	1 097			4	6 851
45	7 838	8	83			6	2			6	83
46	28 805									8	916
47	33 810									12	129
48	26 262									16	106
49	39 718									24	44
50	62 390									32	12
51	59 291									36	3
52	34 132									40	1
53	10 878									48	11
54	1 680									72	3
55	86									96	2
56	2									144	1

Table 5: Some properties of the 13-vertex minimal $(3,3)$-Ramsey graphs

and thus, we obtain all 306 622 minimal $(3,3)$-Ramsey graphs with 13-vertices and independence number at least 3.

Finally, we obtain the following

Theorem 5.1. *There are exactly 306 635 minimal 13-vertex* $(3,3)$*-Ramsey graphs. Some of their properties are listed in Table 5. Examples of 13-vertex minimal* $(3,3)$*-Ramsey graphs are given in Figure 6, Figure 20 and Figure 21.*

We denote the obtained 13-vertex $(3,3)$-Ramsey graphs by $G_{13.1}, \ldots, G_{13.306635}$.

As was noted, all graphs G for which $\alpha(G) < 3$ and $\omega(G) < 6$ are known and from $R(3,6) = 18$ it follows that these graphs have at most 17 vertices. By computer check we find that there are no minimal $(3,3)$-Ramsey graphs with independence number 2 and more than 13 vertices. Thus, we prove the following

Theorem 5.2. *Let* G *be a minimal* $(3,3)$*-Ramsey graph and* $\alpha(G) = 2$*. Then,* $|\mathrm{V}(G)| \leq 13$*. There are exactly 145 minimal* $(3,3)$*-Ramsey graphs for which* $\alpha(G) = 2$*:*

- *8-vertex: 1 (*$K_3 + C_5$*);*
- *9-vertex: 1 (see Figure 3);*
- *10-vertex: 3 (*$G_{10.3}$*,* $G_{10.5}$*,* $G_{10.6}$*, see Figure 14);*
- *11-vertex: 4 (*$G_{11.46}$*,* $G_{11.47}$*,* $G_{11.54}$*,* $G_{11.69}$*, see Figure 16);*
- *12-vertex: 124;*
- *13-vertex: 13 (see Figure 21);*

By executing Algorithm 3.11($n = 10, 11, 12; k = 7, 8, 9; q = 6$), we find all minimal $(3,3)$-Ramsey graphs with 10, 11 and 12 vertices and independence number greater than 2. In this way, with the help of Theorem 5.2, we obtain a new proof of Theorem 4.1, Theorem 4.2 and Theorem 4.3.

6. COROLLARIES FROM THE OBTAINED RESULTS

6.1. MINIMUM AND MAXIMUM DEGREE

By Theorem 2.1, if G is a minimal $(3,3)$-Ramsey graph, then $\delta(G) \geq 4$. Via very elegant constructions, in [2] and [8] it is proved that the bound $\delta(G) \geq (p-1)^2$ in Theorem 2.1 is exact. However, these constructions are not very economical in the case $p = 3$. For example, the minimal $(3,3)$-Ramsey graph G from [8] with $\delta(G) = 4$ is not presented explicitly, but it is proved that it is a subgraph of a graph with 17577 vertices. From the next theorem we see that the smallest minimal $(3,3)$-Ramsey graph G with $\delta(G) = 4$ has 10 vertices:

Theorem 6.1. *Let* G *be a minimal* $(3,3)$*-Ramsey graph and* $\delta(G) = 4$*. Then,* $|\mathrm{V}(G)| \geq 10$*. There is only one 10-vertex minimal* $(3,3)$*-Ramsey graph* G *with* $\delta(G) = 4$*, namely* $G_{10.2}$ *(see Figure 14). What is more,* G *has only a single vertex of degree 4. For all other 10-vertex minimal* $(3,3)$*-Ramsey graphs* G*,* $\delta(G) = 5$*.*

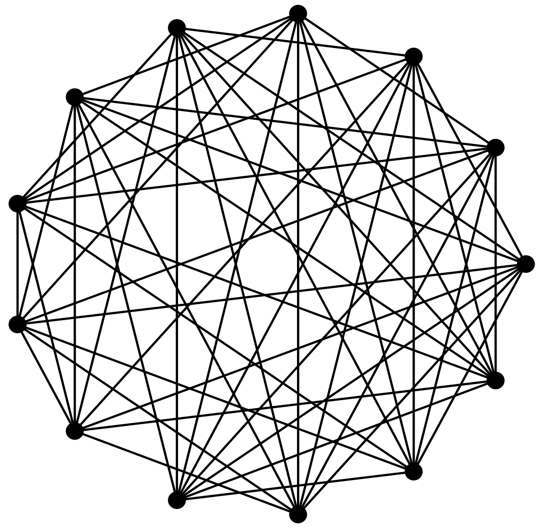

Figure 6: 8-regular 13-vertex minimal $(3,3)$-Ramsey graph

Let G be a $(3,3)$-Ramsey graph. By Theorem 2.5, $\chi(G) \geq 6$ and from the inequality $\chi(G) \leq \Delta(G) + 1$ (see [13]) we obtain $\Delta(G) \geq 5$. From the Brooks' Theorem (see [13]) it follows that if $G \neq K_6$, then $\Delta(G) \geq 6$. The following related question arises naturally:

Are there minimal $(3,3)$*-Ramsey graphs which are 6-regular?*
(i.e. $d(v) = 6, \forall v \in \mathrm{V}(G)$*)*

From the obtained minimal $(3,3)$-Ramsey graphs we see that the following theorem is true:

Theorem 6.2. *Let* G *be a regular minimal* $(3,3)$*-Ramsey graph and* $G \neq K_6$*. Then,* $|\mathrm{V}(G)| \geq 13$*. There is only one regular minimal* $(3,3)$*-Ramsey with 13 vertices, and this is the graph presented in Figure 6, which is 8-regular.*

Regarding the maximum degree of the minimal $(3,3)$-Ramsey graphs, we obtain the following result:

Theorem 6.3. *Let* G *be a minimal* $(3,3)$*-Ramsey graph. Then:*
(a) $\Delta(G) = |\mathrm{V}(G)| - 1$, *if* $|\mathrm{V}(G)| \leq 10$.
(b) $\Delta(G) \geq 8$, *if* $|\mathrm{V}(G)| = 11$, 12 *or* 13.

6.2. CHROMATIC NUMBER

By Theorem 2.5, if G is a $(3,3)$-Ramsey graph, then $\chi(G) \geq 6$.

From the obtained minimal $(3,3)$-Ramsey graphs we derive the following results:

Theorem 6.4. *Let G be a minimal $(3,3)$-Ramsey graph and $|\mathrm{V}(G)| \leq 12$. Then $\chi(G) = 6$.*

Theorem 6.5. *Let G be a minimal $(3,3)$-Ramsey graph and $|V(G)| \leq 14$. Then $\chi(G) \leq 7$. The smallest 7-chromatic minimal $(3,3)$-Ramsey graphs are the 13 minimal $(3,3)$-Ramsey graph with 13 vertices and independence number 2, given in Figure 21.*

Proof. Suppose the opposite is true, i.e. $\chi(G) \geq 8$. Then, according to [26], $G = K_1 + Q$, where $\overline{Q}$ is the graph presented in Figure 7. The graph $K_1 + Q$ is a $(3,3)$-Ramsey graph, but it is not minimal. By Theorem 6.4, there are no 7-chromatic minimal $(3,3)$-Ramsey graphs with less than 13 vertices. The graphs in Figure 21 are 13-vertex minimal $(3,3)$-Ramsey graphs with independence number 2, and therefore these graphs are 7-chromatic. By computer check, we find that among the 13-vertex $(3,3)$-Ramsey graphs with independence number greater than 2 there are no 7-chromatic graphs. □

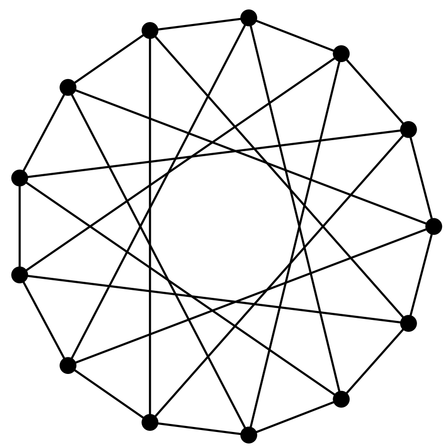

Figure 7: Graph $\overline{Q}$

6.3. MULTIPLICITIES

Definition 6.6. *Denote by $M(G)$ the minimum number of monochromatic triangles in all 2-colorings of $\mathrm{E}(G)$. The number $M(G)$ is called a K_3-multiplicity of the graph G.*

In [10] the K_3-multiplicities of all complete graphs are computed, i.e. $M(K_n)$ is computed for all positive integers n. Similarly, the K_p-multiplicity of a graph is defined [14]. The following papers are dedicated to the computation of the multiplicities of some concrete graphs: [15], [16], [33], [1], [28].

With the help of a computer, we check the K_3-multiplicities of the obtained minimal $(3,3)$-Ramsey graphs and we derive the following results:

Theorem 6.7. *If G is a minimal $(3,3)$-Ramsey graph, $|\mathrm{V}(G)| \leq 13$ and $G \neq K_6$, then $M(G) = 1$.*

We suppose that the following hypothesis is true:

Hypothesis 6.8. *If G is a minimal $(3,3)$-Ramsey graph and $G \neq K_6$, then $M(G) = 1$.*

In support to this hypothesis we prove the following:

Proposition 6.9. *If G is a minimal $(3,3)$-Ramsey graph, $G \neq K_6$ and $\delta(G) \leq 5$, then $M(G) = 1$.*

Proof. Let $v \in \mathrm{V}(G)$ and $d(v) \leq 5$. Consider a 2-coloring of $\mathrm{E}(G - v)$ without monochromatic triangles. We will color the edges incident to v with two colors in such a way that we will obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle. To achieve this, we consider two cases:

Case 1: $d(v) = 4$. By Corollary 2.8, $G(v) = K_4$. Let $N_v = \{a, b, c, d\}$ and suppose that $[a, b]$ is colored with the first color. Then, $[c, d]$ is also colored with the first color (otherwise, by coloring $[v, a]$ and $[v, b]$ with the second color and $[v, c]$ and $[v, d]$ with the fist color, we would obtain a 2-coloring of $\mathrm{E}(G)$ without monochromatic triangles). Thus, $[a, b]$ and $[c, d]$ are colored in the first color. We color $[v, a]$ and $[v, b]$ with the first color and $[v, c]$ and $[v, d]$ with the second color. We obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle $[v, a, b]$.

Case 2: $d(v) = 5$. Since $\omega(G) \leq 5$, in $N_G(v)$ there are two non-adjacent vertices a and b. From $G \to (3, 3)$ it follows easily that in $G(v) - \{a, b\}$ there is an edge of the first color and an edge of the second color. Therefore, we may assume that in $G(v) - \{a, b\}$ there is exactly one edge of one of the colors, say the first color. We color $[v, a]$ and $[v, b]$ with the second color and the other three edges incident to v with the first color. We obtain a 2-coloring of $\mathrm{E}(G)$ with exactly one monochromatic triangle. □

In the end, we note that, according to [27], $M(K_3 + C_{2r+1}) = 1,\ r \geq 2$, which also supports our hypothesis.

6.4. AUTOMORPHISM GROUPS

Denote by $Aut(G)$ the automorphism group of the graph G. We use the *nauty* programs [20] to find the number of automorphisms of the obtained minimal $(3,3)$-Ramsey graphs with 10, 11, 12 and 13 vertices. Most of the obtained graphs have small automorphism groups (see Table 2, Table 3, Table 4 and Table 5). We list the graphs with at least 60 automorphisms:

- The graphs of the form $K_3 + C_{2r+1}$: $|Aut(K_3 + C_5)| = 60$. $|Aut(K_3 + C_7)| = 84$, $|Aut(K_3 + C_9)| = 108$;

- $|Aut(G_{12.2240})| = 96$ (see Figure 18);

- $|Aut(G_{13.255653})| = 144$, $|Aut(G_{13.248305})| = 96$, $|Aut(G_{13.304826})| = 96$, $|Aut(G_{13.113198})| = 72$, $|Aut(G_{13.175639})| = 72$, $|Aut(G_{13.302168})| = 72$ (see Figure 20);

7. UPPER BOUNDS ON THE INDEPENDENCE NUMBER OF THE MINIMAL (3,3)-RAMSEY GRAPHS

In regard to the maximal possible value of the independence number of the minimal $(3,3)$-Ramsey graphs, the following theorem holds:

Theorem 7.1. ([23]) *If G is a minimal $(3,3)$-Ramsey graph, $G \neq K_6$ and $G \neq K_3 + C_5$, then $\alpha(G) \leq |V(G)| - 7$. There is a finite number of graphs for which equality is reached.*

From Theorem 7.1 it follows that by executing Algorithm 3.8($q = 6; k = 8$) we obtain all minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) = |\mathrm{V}(G)| - 7$ or $\alpha(G) = |\mathrm{V}(G)| - 8$. Hence, we derive the following supplements to Theorem 7.1:

Theorem 7.2. *There are exactly 11 minimal $(3,3)$-Ramsey graphs G, for which $\alpha(G) = |\mathrm{V}(G)| - 7$:*

- 9-vertex: 1 (Figure 3);
- 10-vertex: 3 ($G_{10.1}$, $G_{10.2}$, $G_{10.4}$, see Figure 14);
- 11-vertex: 3 ($G_{11.1}$, $G_{11.2}$, $G_{11.21}$, see Figure 15);
- 12-vertex: 1 ($G_{12.163}$, see Figure 17);
- 13-vertex: 2 ($G_{13.}$, $G_{13.}$, see Figure 19);
- 14-vertex: 1 (see Figure 8).

Theorem 7.3. *There are exactly 8633 minimal $(3,3)$-Ramsey graphs G for which $\alpha(G) = |\mathrm{V}(G)| - 8$. The largest of these graphs has 26 vertices, and it is given in Figure 9. There is only one minimal $(3,3)$-Ramsey graph G for which $\alpha(G) = |\mathrm{V}(G)| - 8$ and $\omega(G) < 5$, and it is the 15-vertex graph $K_1 + \Gamma$ from [25] (see Figure 2).*

Corollary 7.4. *Let G be a minimal $(3,3)$-Ramsey graph and $|\mathrm{V}(G)| \geq 27$. Then, $\alpha(G) \leq |\mathrm{V}(G)| - 9$.*

According to Theorem 7.3, if G is a minimal $(3,3)$-Ramsey graph, $\omega(G) < 5$, and $G \neq K_1 + \Gamma$, then $\alpha(G) \leq |\mathrm{V}(G)| - 9$. From Theorem 2.4 it follows that by executing Algorithm 3.8($q = 5; k = 9$) we obtain all minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < 5$ and $\alpha(G) = |\mathrm{V}(G)| - 9$, and the graph $K_1 + \Gamma$. As a result of the execution of this algorithm we derive:

Theorem 7.5. *There are exactly 8903 minimal $(3,3)$-Ramsey graphs G for which $\omega(G) < 5$ and $\alpha(G) = |\mathrm{V}(G)| - 9$. The largest of these graphs has 29 vertices, and it is given in Figure 10.*

```
0 0 0 0 0 0 0 1 1 0 0 0 1 1
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0 1 1 0 1 0 1 0 0 0 1 1 1 1
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0 0 1 1 1 1 0 0 0 1 1 0 1 1
1 1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 0
```

Figure 8: 14-vertex minimal
$(3,3)$-Ramsey graph
with independence number 7

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 1 1 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 0 1 1
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1 1 0 0 1 1 1 1 0 0 0 1 0 1 1 1 1 1 0 0 0 0 1 0 1 1
0 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0
```

Figure 9: 26-vertex minimal
$(3,3)$-Ramsey graph
with independence number 18

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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 0 1
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0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 1 1 1 0 1 1 1 1 0 0 1
1 1 1 1 1 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 1 1 1 1 0 1 1 0
```

Figure 10: 29-vertex minimal $(3,3)$-Ramsey graph
with clique number 4 and independence number 20

Corollary 7.6. *Let G be a minimal* $(3,3)$*-Ramsey graph such that* $\omega(G) < 5$ *and* $|\operatorname{V}(G)| \geq 30$*. Then* $\alpha(G) \leq |\operatorname{V}(G)| - 10$*.*

8. LOWER BOUNDS ON THE MINIMUM DEGREE OF THE MINIMAL $(3,3)$-RAMSEY GRAPHS

According to Proposition 3.4, if G is a minimal $(3,3)$-Ramsey graph, then for each vertex v of G, $N_G(v)$ is a marked vertex set in $G - v$, and therefore $N_G(v)$ is a marked vertex set in $G(v)$.

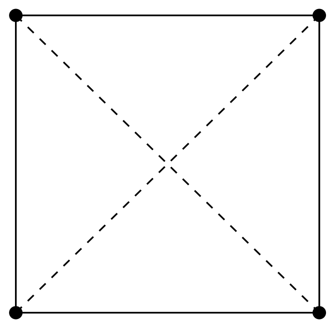
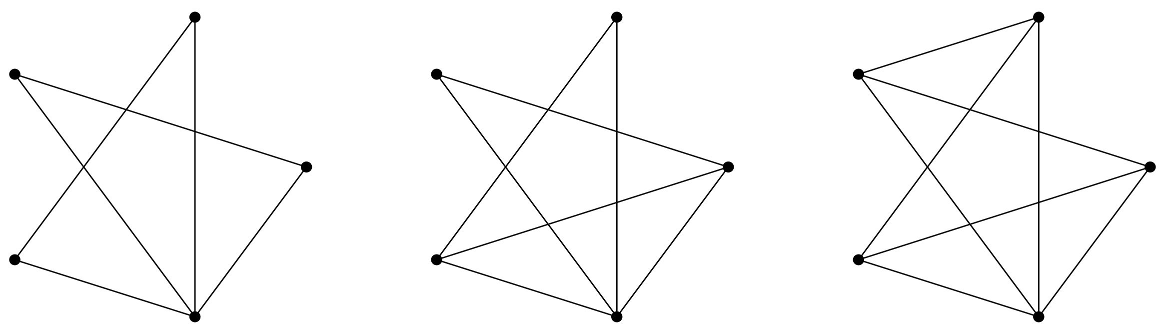


Figure 11: $(3,3)$-free 2-coloring of the edges of K_4

It is easy to see that if $W \subseteq \operatorname{V}(G)$ and $|W| \leq 3$, or $|W| = 4$ and $G[W] \neq K_4$, then W is not a marked vertex set in G. A $(3,3)$-free 2-coloring of K_4 which cannot be extended to a $(3,3)$-free 2-coloring of K_5 is shown in Figure 11. Therefore, the only 4-vertex graph N such that $\operatorname{V}(N)$ is a marked vertex set in N is K_4.



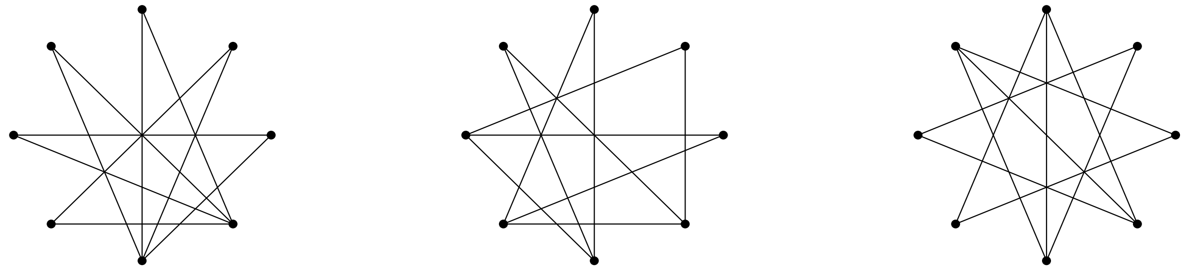
$N_{5.1}$ $N_{5.2}$ $N_{5.3}$

Figure 12: The graphs $N_{5.1}$, $N_{5.2}$, $N_{5.3}$

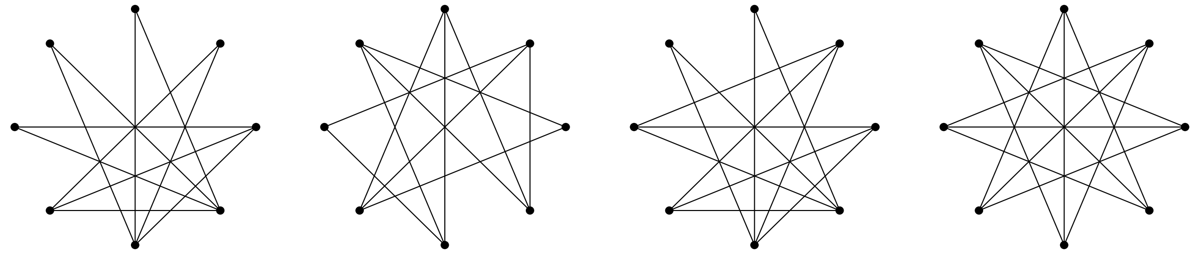
With the help of a computer, we obtain that there are exactly 3 graphs N with 5 vertices such that $K_4 \not\subset N$ and $\operatorname{V}(N)$ is a marked vertex set in N. Namely, they are the graphs $N_{5.1}$, $N_{5.2}$ and $N_{5.3}$ given in Figure 12. Note that $N_{5.1} \subset N_{5.2} \subset N_{5.3}$. From these results we derive

Theorem 8.1. *Let G be a minimal $(3,3)$-Ramsey graph and $\omega(G) \leq 4$. Then $\delta(G) \geq 5$. If $v \in \mathrm{V}(G)$ and $d(v) = 5$, then $G(v) = N_{5.i}$ for some $i \in \{1,2,3\}$ (see Figure 12).*

The bound $\delta(G) \geq 5$ in Theorem 8.1 is exact. For example, the graph $G = K_1 + \Gamma$ from [25] (see Figure 2) has 7 vertices v such that $d(v) = 5$ and $G(v) = N_{5.3}$.



$N_{8.1}$ $N_{8.2}$ $N_{8.3}$



$N_{8.4}$ $N_{8.5}$ $N_{8.6}$ $N_{8.7}$

Figure 13: The graphs $N_{8.i}$, $i = 1, \dots, 7$

With the help of a computer, we also obtain that the smallest graphs N such that $K_3 \not\subset N$ and $\mathrm{V}(N)$ is a marked vertex set in N have 8 vertices, and there are exactly 7 such graphs. Namely, they are the graphs $N_{8.i}, i = 1, \dots, 7$ presented in Figure 13. Among them, the minimal graphs are $N_{8.1}$, $N_{8.2}$ and $N_{8.3}$, and the remaining 4 graphs are their supergraphs. Thus, we derive the following

Theorem 8.2. *Let G be a minimal $(3,3)$-Ramsey graph and $\omega(G) = 3$. Then, $\delta(G) \geq 8$. If $v \in \mathrm{V}(G)$ and $d(v) = 8$, then $G(v) = N_{8.i}$ for some $i \in \{1, \dots, 7\}$ (see Figure 13).*

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```
0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 1 1 1 1
0 0 0 1 1 1 0 1 1 1
0 0 1 0 1 1 1 0 1 1
1 0 1 1 0 0 1 0 1 1
0 1 1 1 0 0 0 1 1 1
1 1 0 1 1 0 0 0 1 1
1 1 1 0 0 1 0 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.1}$

```
0 0 0 0 0 1 1 0 1 1
0 0 1 0 0 1 0 1 1 1
0 1 0 0 0 1 0 1 1 1
0 0 0 0 1 0 1 1 1 1
0 0 0 1 0 0 1 1 1 1
1 1 1 0 0 0 1 0 1 1
1 0 0 1 1 1 0 0 1 1
0 1 1 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.2}$

```
0 0 1 1 0 0 1 0 1 1
0 0 0 1 1 0 1 1 1 1
1 0 0 1 1 0 1 1 1 1
1 0 1 0 0 1 1 1 1 1
0 1 1 0 0 1 1 1 1 1
0 1 0 1 1 0 1 1 1 1
1 0 1 1 1 1 0 1 0 1
0 1 1 1 1 1 1 0 0 1
1 1 1 1 1 1 0 0 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.3}$

```
0 1 1 0 0 0 0 1 1 1
1 0 0 0 0 1 0 1 1 1
1 0 0 0 0 0 1 1 1 1
0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 1 1 1 1
0 1 0 1 0 0 0 1 1 1
0 0 1 0 1 0 0 1 1 1
1 1 1 1 1 1 1 0 1 1
1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.4}$

```
0 1 1 0 0 0 0 1 1 1
1 0 0 1 0 0 1 1 1 1
1 0 0 0 1 1 0 1 1 1
0 1 0 0 1 1 1 1 0 1
0 0 1 1 0 1 1 0 1 1
0 0 1 1 1 0 1 0 1 1
0 1 0 1 1 1 0 1 0 1
1 1 1 1 0 0 1 0 1 1
1 1 1 0 1 1 0 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.5}$

```
0 1 1 1 0 0 1 1 0 1
1 0 0 0 1 1 1 0 1 1
1 0 0 1 0 0 1 1 1 1
1 0 1 0 0 0 1 1 1 1
0 1 0 0 0 1 1 1 1 1
0 1 0 0 1 0 1 1 1 1
1 1 1 1 1 1 0 0 0 1
1 0 1 1 1 1 0 0 1 1
0 1 1 1 1 1 0 1 0 1
1 1 1 1 1 1 1 1 1 0
```

$G_{10.6}$

Figure 14: 10-vertex minimal $(3,3)$-Ramsey graphs

```
0 0 0 0 0 0 1 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 1 1 1 0 0 1 1
0 1 1 1 0 0 0 1 0 1 1
0 0 1 1 0 0 1 0 1 1 1
1 0 0 1 0 1 0 0 1 1 1
1 1 1 0 1 0 0 0 1 1 1
1 1 0 0 0 1 1 1 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.1}$

```
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 1 1 1 1 1
0 0 0 0 0 1 1 1 0 1 1
1 0 0 0 0 1 1 0 1 1 1
0 1 0 1 1 0 1 0 0 1 1
0 0 1 1 1 1 0 0 0 1 1
1 1 1 1 0 0 0 0 1 1 1
1 1 1 0 1 0 0 1 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.2}$

```
0 0 0 1 0 0 0 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1
1 0 0 0 0 0 1 1 0 1 1
0 1 0 0 0 1 1 0 1 1 1
0 0 1 0 1 0 1 0 1 1 1
0 0 0 1 1 1 0 1 0 1 1
1 1 1 1 0 0 1 0 0 1 1
1 1 1 0 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.21}$

Figure 15: 11-vertex minimal $(3,3)$-Ramsey graphs with independence number 4

```
0 0 1 0 1 1 1 0 1 0 1
0 0 0 1 1 1 0 1 0 1 1
1 0 0 1 0 0 1 0 1 1 1
0 1 1 0 0 0 0 1 1 1 1
1 1 0 0 0 1 1 1 0 1 1
1 1 0 0 1 0 1 1 1 0 1
1 0 1 0 1 1 0 0 1 1 1
0 1 0 1 1 1 0 0 1 1 1
1 0 1 1 0 1 1 1 0 0 1
0 1 1 1 1 0 1 1 0 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.46}$

```
0 0 1 1 1 0 0 0 1 0 1
0 0 0 0 0 1 1 1 0 1 1
1 0 0 1 1 1 1 0 1 1 1
1 0 1 0 1 1 0 1 1 1 1
1 0 1 1 0 0 1 1 1 1 1
0 1 1 1 0 0 1 1 1 1 1
0 1 1 0 1 1 0 1 1 1 1
0 1 0 1 1 1 1 0 1 1 1
1 0 1 1 1 1 1 1 0 1 0
0 1 1 1 1 1 1 1 1 0 0
1 1 1 1 1 1 1 1 0 0 0
```

$G_{11.47}$

```
0 1 0 0 1 1 0 0 1 1 1
1 0 0 0 1 1 0 0 1 1 1
0 0 0 1 0 0 1 1 1 1 1
0 0 1 0 0 0 1 1 1 1 1
1 1 0 0 0 1 1 1 1 0 1
1 1 0 0 1 0 1 1 0 1 1
0 0 1 1 1 1 0 1 0 1 1
0 0 1 1 1 1 1 0 1 0 1
1 1 1 1 1 0 0 1 0 0 1
1 1 1 1 0 1 1 0 0 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.54}$

```
0 1 1 0 1 1 1 0 0 0 1
1 0 0 1 0 1 1 0 0 1 1
1 0 0 1 1 0 0 1 1 0 1
0 1 1 0 0 0 0 1 1 1 1
1 0 1 0 0 1 1 1 1 0 1
1 1 0 0 1 0 1 1 0 1 1
1 1 0 0 1 1 0 0 1 1 1
0 0 1 1 1 1 0 0 1 1 1
0 0 1 1 1 0 1 1 0 1 1
0 1 0 1 0 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 0
```

$G_{11.69}$

Figure 16: 11-vertex minimal $(3, 3)$-Ramsey graphs
with independence number 2

```
0 0 0 0 1 0 0 1 0 1 1 1
0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 1 0 0 0 1 1 1 1
0 0 0 0 0 1 0 1 0 1 1 1
1 0 1 0 0 1 1 0 0 0 1 1
0 1 0 1 1 0 1 0 0 0 1 1
0 0 0 0 1 1 0 1 1 0 1 1
1 0 0 1 0 0 1 0 1 1 1 1
0 1 1 0 0 0 1 1 0 1 1 1
1 1 1 1 0 0 0 1 1 0 1 1
1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{12.163}$

Figure 17: 12-vertex minimal
$(3, 3)$-Ramsey graph
with independence number 5

```
0 1 0 0 0 0 1 1 0 1 1 1
1 0 0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 1 0 1 1 1 1
0 0 1 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 0 1 1 1 1 1
0 0 0 0 1 0 0 1 1 1 1 1
1 1 1 1 0 0 0 1 1 0 0 1
1 1 0 0 1 1 1 0 1 0 0 1
0 0 1 1 1 1 1 1 0 0 0 1
1 1 1 1 1 1 0 0 0 0 1 1
1 1 1 1 1 1 0 0 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{12.2240}$

Figure 18: 12-vertex minimal
$(3, 3)$-Ramsey graph
with 96 automorphisms

```
0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 0 0 1 0 1 1 0 1 1
1 0 1 0 0 1 0 1 0 0 1 1 1
0 1 0 1 1 0 1 0 0 0 1 1 1
1 0 0 1 1 1 0 0 0 1 0 1 1
0 1 1 0 1 1 0 0 1 0 0 1 1
1 1 1 1 0 0 1 1 0 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.1}$

```
0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 0 1 0 1 0 1 1 1
0 0 0 0 0 0 1 0 0 1 1 1 1
0 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 0 0 0 1 1 1 1 1
1 1 1 0 0 0 0 0 0 0 1 1 1
1 0 0 1 1 0 0 0 1 1 0 1 1
0 1 0 1 0 1 0 1 0 1 0 1 1
0 0 1 0 1 1 0 1 1 0 0 1 1
1 1 1 1 1 1 1 0 0 0 0 1 1
1 1 1 1 1 1 1 1 1 1 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.2}$

Figure 19: 13-vertex minimal $(3,3)$-Ramsey graphs with independence number 6

```
0 0 0 1 0 0 1 0 0 1 1 1 1
0 0 0 0 1 0 1 0 0 1 1 1 1
0 0 0 0 0 1 1 0 0 1 1 1 1
1 0 0 0 1 1 1 1 1 0 0 0 1
0 1 0 1 0 1 1 1 1 0 0 0 1
0 0 1 1 1 0 1 1 1 0 0 0 1
1 1 1 1 1 1 0 0 0 0 0 0 1
0 0 0 1 1 1 0 0 1 1 1 1 0
0 0 0 1 1 1 0 1 0 1 1 1 0
1 1 1 0 0 0 0 1 1 0 1 1 1
1 1 1 0 0 0 0 1 1 1 0 1 1
1 1 1 0 0 0 0 1 1 1 1 0 1
1 1 1 1 1 1 1 0 0 1 1 1 0
```

$G_{13.113198}$

```
0 0 0 0 0 0 1 1 1 0 0 0 1
0 0 1 1 0 0 0 0 0 1 1 1 1
0 1 0 1 0 0 0 0 0 1 1 1 1
0 1 1 0 0 0 0 0 0 1 1 1 1
0 0 0 0 0 1 1 1 1 1 1 1 0
0 0 0 0 1 0 1 1 1 1 1 1 0
1 0 0 0 1 1 0 1 1 1 1 0 1
1 0 0 0 1 1 1 0 1 1 0 1 1
1 0 0 0 1 1 1 1 0 0 1 1 1
0 1 1 1 1 1 1 1 0 0 0 0 1
0 1 1 1 1 1 1 0 1 0 0 0 1
0 1 1 1 1 1 0 1 1 0 0 0 1
1 1 1 1 0 0 1 1 1 1 1 1 0
```

$G_{13.175639}$

```
0 1 0 0 0 0 0 1 1 0 1 1 1
1 0 0 0 0 0 0 1 1 0 1 1 1
0 0 0 1 0 0 0 1 0 1 1 1 1
0 0 1 0 0 0 0 1 0 1 1 1 1
0 0 0 0 0 1 1 0 1 1 0 1 1
0 0 0 0 1 0 1 0 1 1 0 1 1
0 0 0 0 1 1 0 0 1 1 0 1 1
1 1 1 1 0 0 0 0 1 1 1 0 0
1 1 0 0 1 1 1 1 0 1 1 0 0
0 0 1 1 1 1 1 1 1 0 1 0 0
1 1 1 1 0 0 0 1 1 1 0 1 1
1 1 1 1 1 1 1 0 0 0 1 0 1
1 1 1 1 1 1 1 0 0 0 1 1 0
```

$G_{13.248305}$

```
0 1 0 0 0 0 0 0 1 1 1 1 0
1 0 0 0 0 0 0 0 1 1 1 0 1
0 0 0 1 1 0 0 0 0 1 0 1 1
0 0 1 0 1 0 0 0 0 1 0 1 1
0 0 1 1 0 0 0 0 0 1 0 1 1
0 0 0 0 0 0 1 1 0 0 1 1 1
0 0 0 0 0 1 0 1 0 0 1 1 1
0 0 0 0 0 1 1 0 0 0 1 1 1
1 1 0 0 0 0 0 0 0 1 1 1 1
1 1 1 1 1 0 0 0 1 0 1 1 1
1 1 0 0 0 1 1 1 1 1 0 1 1
1 0 1 1 1 1 1 1 1 1 1 0 0
0 1 1 1 1 1 1 1 1 1 1 0 0
```

$G_{13.255653}$

```
0 1 1 0 0 0 1 1 1 1 0 0 1
1 0 1 0 0 0 0 1 1 1 1 0 1
1 1 0 0 0 0 0 1 1 1 0 1 1
0 0 0 0 1 1 1 1 0 0 1 1 1
0 0 0 1 0 1 1 0 1 0 1 1 1
0 0 0 1 1 0 1 0 0 1 1 1 1
1 0 0 1 1 1 0 1 1 1 0 0 1
1 1 1 1 0 0 1 0 0 0 1 1 1
1 1 1 0 1 0 1 0 0 0 1 1 1
1 1 1 0 0 1 1 0 0 0 1 1 1
0 1 0 1 1 1 0 1 1 1 0 0 1
0 0 1 1 1 1 0 1 1 1 0 0 1
1 1 1 1 1 1 1 1 1 1 1 1 0
```

$G_{13.302168}$

```
0 1 1 0 0 0 0 1 1 1 1 0 0
1 0 1 1 1 0 0 0 0 0 1 1 1
1 1 0 0 0 1 1 0 0 0 1 1 1
0 1 0 0 1 0 0 1 1 1 0 1 1
0 1 0 1 0 0 0 1 1 1 0 1 1
0 0 1 0 0 0 1 1 1 1 0 1 1
0 0 1 0 0 1 0 1 1 1 0 1 1
1 0 0 1 1 1 1 0 1 1 1 0 0
1 0 0 1 1 1 1 1 0 1 1 0 0
1 0 0 1 1 1 1 1 1 0 1 0 0
1 1 1 0 0 0 0 1 1 1 0 1 1
0 1 1 1 1 1 1 0 0 0 1 0 1
0 1 1 1 1 1 1 0 0 0 1 1 0
```

$G_{13.304826}$

Figure 20: 13-vertex minimal $(3,3)$-Ramsey graphs with a large number of automorphisms

```
0 0 0 1 1 1 0 0 1 1 1 1 0
0 0 1 0 0 0 1 1 1 1 0 1 1
0 1 0 0 0 0 1 1 1 1 1 0 1
1 0 0 0 1 1 0 0 1 1 1 1 1
1 0 0 1 0 1 0 1 0 1 1 1 1
1 0 0 1 1 0 1 1 0 0 1 1 1
0 1 1 0 0 1 0 1 1 0 1 1 1
0 1 1 0 1 1 1 0 0 0 1 1 1
1 1 1 1 0 0 1 0 0 1 0 1 1
1 1 1 1 1 0 0 0 1 0 1 1 0
1 0 1 1 1 1 1 1 0 1 0 0 1
1 1 0 1 1 1 1 1 1 1 0 0 0
0 1 1 1 1 1 1 1 1 0 1 0 0
```

$G_{13.193684}$

```
0 0 0 1 1 1 1 1 0 0 0 1 1
0 0 1 0 0 0 1 1 1 1 1 1 0
0 1 0 0 0 0 1 1 1 1 1 1 0
1 0 0 0 1 1 0 1 1 1 0 1 1
1 0 0 1 0 1 1 0 1 0 1 1 1
1 0 0 1 1 0 1 0 0 1 1 1 1
1 1 1 0 1 1 0 0 0 0 1 1 1
1 1 1 1 0 0 0 0 1 1 0 1 1
0 1 1 1 1 0 0 1 0 1 1 0 1
0 1 1 1 0 1 0 1 1 0 1 0 1
0 1 1 0 1 1 1 0 1 1 0 0 1
1 1 1 1 1 1 1 1 0 0 0 0 0
1 0 0 1 1 1 1 1 1 1 1 0 0
```

$G_{13.193760}$

```
0 0 0 1 1 1 1 0 1 1 1 0 0
0 0 1 0 0 0 1 1 1 1 0 1 1
0 1 0 0 0 1 0 1 1 1 0 1 1
1 0 0 0 1 1 1 1 0 0 1 1 1
1 0 0 1 0 1 1 0 0 1 1 1 1
1 0 1 1 1 0 0 1 1 0 1 0 1
1 1 0 1 1 0 0 0 0 1 1 1 1
0 1 1 1 0 1 0 0 1 0 1 1 1
1 1 1 0 0 1 0 1 0 1 1 1 0
1 1 1 0 1 0 1 0 1 0 1 1 0
1 0 0 1 1 1 1 1 1 1 0 0 1
0 1 1 1 1 0 1 1 1 1 0 0 1
0 1 1 1 1 1 1 1 0 0 1 1 0
```

$G_{13.193988}$

```
0 1 0 0 1 0 0 1 1 1 1 0 1
1 0 0 0 0 1 1 0 1 1 1 0 1
0 0 0 1 1 1 1 1 1 0 1 1 0
0 0 1 0 1 1 1 1 0 1 1 1 0
1 0 1 1 0 1 0 1 1 0 0 1 1
0 1 1 1 1 0 1 0 1 0 0 1 1
0 1 1 1 0 1 0 0 0 1 1 1 1
1 0 1 1 1 0 0 0 1 1 1 1 0
1 1 1 0 1 1 0 1 0 0 0 1 1
1 1 0 1 0 0 1 1 0 0 1 1 1
1 1 1 1 0 0 1 1 0 1 0 0 1
0 0 1 1 1 1 1 1 1 1 0 0 1
1 1 0 0 1 1 1 0 1 1 1 1 0
```

$G_{13.265221}$

```
0 1 0 0 1 1 0 0 1 1 1 0 1
1 0 0 0 0 0 1 1 1 1 1 0 1
0 0 0 1 1 1 1 1 0 1 0 1 1
0 0 1 0 1 1 1 1 0 0 1 1 1
1 0 1 1 0 1 1 0 0 1 0 1 1
1 0 1 1 1 0 0 1 0 0 1 1 1
0 1 1 1 1 0 0 1 1 1 0 1 0
0 1 1 1 0 1 1 0 1 0 1 1 0
1 1 0 0 0 1 1 0 1 1 1 1 1
1 1 1 0 1 0 1 0 1 0 0 1 1
1 1 0 1 0 1 0 1 1 0 0 1 1
0 0 1 1 1 1 1 1 1 1 1 0 0
1 1 1 1 1 1 0 0 1 1 1 0 0
```

$G_{13.265299}$

```
0 1 1 0 1 1 0 0 1 1 1 1 0
1 0 0 1 0 1 0 1 1 1 1 0 1
1 0 0 1 1 0 1 0 1 1 1 1 0
0 1 1 0 0 0 1 1 1 1 1 0 1
1 0 1 0 0 1 1 1 0 1 0 1 1
1 1 0 0 1 0 1 1 0 0 1 1 1
0 0 1 1 1 1 0 1 0 0 1 1 1
0 1 0 1 1 1 1 0 0 1 0 1 1
1 1 1 1 0 0 0 0 0 1 1 1 1
1 1 1 1 1 0 0 1 1 0 0 1 1
1 1 1 1 0 1 1 0 1 0 0 1 1
1 0 1 0 1 1 1 1 1 1 1 0 0
0 1 0 1 1 1 1 1 1 1 1 0 0
```

$G_{13.299797}$

```
0 1 1 1 1 1 0 0 0 0 1 1 0
1 0 0 1 1 0 0 1 1 0 1 1 1
1 0 0 1 0 1 1 0 1 1 1 0 1
1 1 1 0 1 1 0 0 0 1 0 1 1
1 1 0 1 0 1 0 1 1 1 0 1 0
1 0 1 1 1 0 1 0 0 1 1 1 0
0 0 1 0 0 1 0 1 1 1 1 1 1
0 1 0 0 1 0 1 0 1 1 1 1 1
0 1 1 0 1 0 1 1 0 1 1 0 1
0 0 1 1 1 1 1 1 1 0 0 0 1
1 1 1 0 0 1 1 1 1 0 0 1 1
1 1 0 1 1 1 1 1 0 0 1 0 1
0 1 1 1 0 0 1 1 1 1 1 1 0
```

$G_{13.301368}$

```
0 1 1 0 0 0 1 0 1 1 0 1 1
1 0 1 0 0 0 0 0 1 1 1 1 1
1 1 0 0 0 0 0 1 1 1 1 0 1
0 0 0 0 1 1 1 1 1 0 1 1 1
0 0 0 1 0 1 1 1 0 1 1 1 1
0 0 0 1 1 0 1 1 1 1 1 1 0
1 0 0 1 1 1 0 1 0 1 0 1 1
0 0 1 1 1 1 1 0 0 1 1 0 1
1 1 1 1 0 1 0 0 0 0 1 1 1
1 1 1 0 1 1 1 1 0 0 0 1 0
0 1 1 1 1 1 0 1 1 0 0 1 1
1 1 0 1 1 1 1 0 1 1 1 0 0
1 1 1 1 1 0 1 1 1 0 1 0 0
```

$G_{13.302151}$

```
0 1 1 0 0 1 0 0 1 1 1 0 1
1 0 1 0 0 0 0 0 1 1 1 1 1
1 1 0 0 0 0 1 1 0 1 0 1 1
0 0 0 0 1 1 1 1 1 0 1 1 1
0 0 0 1 0 1 1 1 1 1 1 1 0
1 0 0 1 1 0 1 1 1 0 1 0 1
0 0 1 1 1 1 0 1 0 1 0 1 1
0 0 1 1 1 1 1 0 0 0 1 1 1
1 1 0 1 1 1 0 0 0 1 1 1 0
1 1 1 0 1 0 1 0 1 0 1 1 0
1 1 0 1 1 1 0 1 1 1 0 0 1
0 1 1 1 1 0 1 1 1 1 0 0 1
1 1 1 1 0 1 1 1 0 0 1 1 0
```

$G_{13.302764}$

```
0 1 1 1 1 1 0 0 0 0 1 1 0
1 0 1 0 0 0 1 1 0 0 1 1 1
1 1 0 0 0 0 1 1 0 0 1 1 1
1 0 0 0 1 1 1 0 1 1 1 0 0
1 0 0 1 0 1 0 1 1 1 0 1 0
1 0 0 1 1 0 0 0 1 1 1 1 1
0 1 1 1 0 0 0 1 1 1 1 0 1
0 1 1 0 1 0 1 0 1 1 0 1 1
0 0 0 1 1 1 1 1 0 1 1 0 1
0 0 0 1 1 1 1 1 1 0 0 1 1
1 1 1 1 0 1 1 0 1 0 0 1 1
1 1 1 0 1 1 0 1 0 1 1 0 1
0 1 1 0 0 1 1 1 1 1 1 1 0
```

$G_{13.305857}$

```
0 1 1 1 1 1 1 0 0 0 1 0 0
1 0 1 1 1 0 1 0 1 0 0 1 1
1 1 0 0 0 1 1 0 0 1 1 1 1
1 1 0 0 1 0 1 1 1 0 1 1 0
1 1 0 1 0 1 0 1 1 0 1 0 1
1 0 1 0 1 0 0 1 0 1 1 1 1
1 1 1 1 0 0 0 0 1 1 1 1 0
0 0 0 1 1 1 0 0 1 1 1 1 1
0 1 0 1 1 0 1 1 0 1 0 1 1
0 0 1 0 0 1 1 1 1 0 1 1 1
1 0 1 1 1 1 1 1 0 1 0 0 1
0 1 1 1 0 1 1 1 1 1 0 0 1
0 1 1 0 1 1 0 1 1 1 1 1 0
```

$G_{13.306448}$

```
0 1 1 1 1 1 1 0 0 0 1 0 0
1 0 1 1 1 1 0 0 0 0 1 1 1
1 1 0 0 1 0 1 1 0 0 1 1 1
1 1 0 0 0 1 1 0 1 1 0 1 1
1 1 1 0 0 1 0 1 0 1 1 0 1
1 1 0 1 1 0 0 0 1 1 1 0 1
1 0 1 1 0 0 0 1 1 1 1 1 0
0 0 1 0 1 0 1 0 1 1 1 1 1
0 0 0 1 0 1 1 1 0 1 1 1 1
0 0 0 1 1 1 1 1 1 0 0 1 1
1 1 1 0 1 1 1 1 1 0 0 1 0
0 1 1 1 0 0 1 1 1 1 1 0 1
0 1 1 1 1 1 0 1 1 1 0 1 0
```

$G_{13.306460}$

```
0 1 1 1 0 0 0 0 0 1 1 1 1
1 0 1 1 0 0 0 0 0 1 1 1 1
1 1 0 1 0 0 0 1 0 1 1 0 1
1 1 1 0 0 0 0 0 1 1 0 1 1
0 0 0 0 0 1 1 1 1 1 1 1 0
0 0 0 0 1 0 1 1 1 1 1 1 0
0 0 0 0 1 1 0 1 1 0 1 1 1
0 0 1 0 1 1 1 0 1 0 1 0 1
0 0 0 1 1 1 1 1 0 0 0 1 1
1 1 1 1 1 1 0 0 0 0 1 1 0
1 1 1 0 1 1 1 1 0 1 0 0 1
1 1 0 1 1 1 1 0 1 1 0 0 1
1 1 1 1 0 0 1 1 1 0 1 1 0
```

$G_{13.306470}$

Figure 21: 13-vertex minimal $(3, 3)$-Ramsey graphs
with independence number 2

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