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# DIRECT AND INDIRECT METHODS OF PROOF. THE LEHMUS-STEINER THEOREM

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We describe and discuss the different methods of proof of a given statement and illustrate by logical models the essence of specific types of proofs, especially of direct and indirect methods of proof.

Two direct proofs of Lehmus-Steiner's theorem are proposed.

**Keywords:** Direct and indirect methods of proof, logical models, direct proof of Lehmus-Steiner's theorem, Stewart's theorem

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## 1. INTRODUCTION

In most fields of study, knowledge is acquired by way of observations, by reasoning about the results of observations and by studying the observations, methods and theories of other fields and practices.

Ancient Egyptian, Babylonian and Chinese mathematics consisted of rules for measuring land, computing taxes, predicting eclipses, solving equations and so on.

The ancient Greeks found that in arithmetic and geometry it was possible to *prove* that the observation results are true. They found that some truths in mathematics were *obvious* and that many of the others could be shown to follow logically from the obvious ones.

On the other hand, Physics, Biology, Economics and other sciences *discover general truths* relying on observations. Besides, not any general truth can be proved to be true - it can only be tested for contradictions and inconsistencies. If a scientific theory is accepted because observations have agreed with it, there is in principle small doubt that a new observation will not agree with the theory, even if all previous observations have agreed with that theory. However, if a result is proved thoroughly and correctly, that cannot happen.

Under what conditions can we be sure that the steps in our investigations are correct? Are we really sure that what seems to be obvious to us is in fact true? Can we expect *all* mathematical truths to follow from the obvious ones? These questions are not easy to answer.

Disputes and mistakes about what is obvious could be avoided by laying down certain basic notions, relations and statements, called *axioms* (postulates assumed true, but unprovable) for each branch of mathematics, and agreeing that proofs of assertions must be derived from these. To axiomatize a system of knowledge means to show that its claims can be derived from a small, well-understood set of axioms (see also [1]).

Any axiomatic system is subordinated to some conditions.

- The system must be consistent, to lack contradiction, i. e. the ability to derive both a statement and its negation from the system's axioms.

  Consistency is a necessary requirement for the system.

- Each axiom has to be independent, i. e. not to be a theorem that can be derived from other axioms in the system.

  However, independence is not a necessary requirement for the system.

- The system can be complete, i. e. for every statement, either itself or its negation is derivable.

There is no longer an assumption that axioms are *true in any sense*; this allows parallel mathematical theories to be built on alternative sets of axioms (for instance *Axiomatic set theory, Number theory*). *Euclidean* and *Non-Euclidean geometry* have a common basic set of axioms; the differences between these important geometries are based on their alternative axioms of parallel lines.

Another way to avoid mistakes about what is obvious in mathematics could be the use of rules of inference with purely formal content.

In mathematical logic a propositional calculus (also called sentential calculus or sentential logic) is a formal system in which formulas of a formal language may be interpreted to represent propositions.

In [7, 8, 9, 10] we explain methods, based on logical laws, for composition and proof of equivalent and inverse problems.

In [8] we discuss a way of generating groups of equivalent problems. The method we propound is based on the logical equivalences

$$p \wedge \neg q \to r \;\Leftrightarrow\; p \wedge \neg r \to q \;\Leftrightarrow\; p \to q \vee r,$$

where $p, q, r$ are statements.

Using the sentential logic, in [9] and [10] we propose a new problems composing technology as an interpretation of specific logical models. Clarifying and using the logical equivalence (see also [12])

$$(t \wedge p \to r) \ \wedge \ (t \wedge q \to r) \ \Leftrightarrow \ t \wedge (p \vee q) \to r, \qquad (*)$$

we give an algorithm for composition of *inverse* problems with a given logical structure that is based on the steps below.

- Formulating and proving *generating* problems with logical structures of the statements as those at the left hand side of (*).

- Formulating a problem with logical structure $t \wedge (p \vee q) \to r$ of the statement.

- Formulating and proving the *inverse* problem with logical structure $t \wedge r \to p \vee q$.

In [7], besides the generalization of criteria $A$ and $D$ for congruence of triangles, we also illustrate the above algorithm by suitable groups of examples.

In Section 2 of the present paper we describe and discuss different methods of proof of implicative statements and illustrate by logical models the essence of specific types of proofs, especially of direct and indirect proofs.

In Section 3 we propose *direct proofs* of Lehmus-Steiner's theorem that differ from any proofs we have come across.

Our investigations in this field are appropriate for training of mathematics students and teachers.

## 2. TYPES OF PROOFS

Both discovery and proof are integral parts of problem solving. The *discovery* is thinking of possible solutions, and the *proving* ensures that the proposed solution actually solves the problem.

*Proofs* are logical descriptions of deductive reasoning and are distinguished from inductive or empirical arguments; a proof must demonstrate that a statement is always true (occasionally by listing all possible cases and showing that it holds in each).

An unproven statement that is believed true is known as a *conjecture*.

The *objects* of proofs are premises, conclusions, axioms, theorems (propositions derived earlier from axioms), definitions and evidence from the real world.

The abilities (techniques) to have a working knowledge of these objects include

- Rules of inference: simple valid argument forms. They may be divided into *basic rules*, which are fundamental to logic and cannot be eliminated without losing the ability to express some valid argument forms, and *derived rules*, which can be proven by the basic rules.

  To summarize, the rules of inference are logical rules which allow the deduction of conclusions from premises.

- Laws of logical equivalence.

Different methods of proof combine these objects and techniques in different ways to create valid arguments.

According to *Euclid* a precise proof of a given statement has the following structure:

- *Premises*: These include given axioms and theorems, true statements, strict restrictions for the validity of the given statement, chosen suitable denotations. (*It is given...*)

- *Statement*: Strict formulation of the submitted statement. (*It is to be proved that...*)

- *Proof*: Establishing the truth of the submitted statement using premises, conclusions, rules of inference and logical laws.

Let now $P$ and $Q$ be statements. In order to establish the truth of the implication $P \to Q$, we discuss different methods of proof. Occasionally, it may be helpful first to rephrase certain statements, to clarify that they are really formulated in an implicative form.

If "not" is put in front of a statement $P$, it negates the statement. $\neg P$ is sometimes called the *negation* (or *contradictory*) of $P$. For any statement $P$ either $P$ or $\neg P$ is true and the other is false.

**Formal proofs.** The concept of a proof is formalized in the field of mathematical logic. Purely formal proofs, written in symbolic language instead of natural language, are considered in proof theory. A formal proof is defined as a sequence of formulas in a formal language, in which each formula is a logical consequence of preceding formulas.

In a formal proof the statements $P$ and $Q$ aren't necessarily related comprehensively to each other. Only the structure of the statements and the logical rules that allow the deduction of conclusions from premises are important.

Hence, to prove formally that an argument $Q$ is valid or the conclusion follows logically from the hypotheses $P$, we have to

- assume the hypotheses $P$ are true,

- use the formal rules of inference and logical equivalences to determine that the conclusion $Q$ is true.

The following logical equivalences illustrate a formal proof:

$$\neg(P \to Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow \neg(\neg P) \wedge \neg Q \Leftrightarrow P \wedge \neg Q.$$

**Vacuous proof.** A vacuous proof of an implication happens when the hypothesis of the implication is always false, i. e. if we know one of the hypotheses in $P$ is false then $P \to Q$ is vacuously true.

For instance, in the implication $(P \wedge \neg P) \to Q$ the hypotheses form a contradiction. Hence, $Q$ follows from the hypotheses vacuously.

**Trivial proofs.** An implication is trivially true when its conclusion is always true. Consider an implication $P \to Q$. If it can be shown (independently of $P$) that $Q$ is true, then the implication is always true.

The form of the trivial proof $Q \to (P \to Q)$ is, in fact, a tautology.

**Proofs of equivalences.** For equivalence proofs or proofs of statements of the form $P$ *if and only if* $Q$ there are two methods.

- Truth table.
- Using direct or indirect methods and the equivalence

$$(P \leftrightarrow Q) \Leftrightarrow (P \to Q) \wedge (Q \to P).$$

  Thus, the proposition $P$ *if and only if* $Q$ can be proved if both the implication $P \to Q$ and the implication $Q \to P$ are proved. This is the definition of the *biconditional* statement.

**Proof by cases.** If the hypothesis $P$ can be separated into cases $p_1 \vee p_2 \vee \ldots \vee p_k$, each of the propositions $p_1 \to Q$, $p_2 \to Q$,. . ., $p_k \to Q$, is to be proved separately. A statement $P \to Q$ is true if all possible cases are true.

The logical equivalences in this case are (see also [12], p. 81)

$$p_1 \to Q \wedge p_2 \to Q \wedge \ldots \wedge p_k \to Q \Leftrightarrow p_1 \vee p_2 \vee \ldots \vee p_k \to Q \Leftrightarrow P \to Q.$$

Different methods may be used to prove the different cases.

**Direct proof.** In mathematics and logic, a direct proof is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually existing lemmas and theorems.

The methods of proof of these established facts, lemmas, propositions and theorems are of *no importance*. Their truth or falsehood are to be accepted *without any effort*.

However, it is exceptionally important that the actual proof of the given statement consists of straightforward combinations of these facts *without making any further assumptions*.

Thus, to prove an implication $P \to Q$ directly, we assume that statement $P$ holds and try to deduce that statement $Q$ *must* follow.

The structure of the direct proof is:

- *Given* - a statement of the form $P \to Q$.
- *Assumption* - the hypotheses in $P$ are true.
- *Proof* - using the rules of inference, axioms, theorems and any logical equivalences to establish in a straightforward way the truth of the conclusion $Q$.

**Indirect proof.** It is often very difficult to give a direct proof to $P \to Q$. The connection between $P$ and $Q$ might not be suitable to this approach.

Indirect proof is a type of proof in which a statement to be proved is assumed false and if the assumption leads to an impossibility, then the statement assumed false has been proved to be true.

There are four possible implications we can derive from the implication $P \to Q$, namely

- *Conversion* (the *converse*): $Q \to P$,
- *Inversion* (the *inverse*): $\neg P \to \neg Q$,
- *Negation*: $\neg(P \to Q)$,
- *Contraposition* (the *opposite*, *contrapositive*): $\neg Q \to \neg P$.

The implications $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent.

The implications $Q \to P$ and $\neg P \to \neg Q$ are logically equivalent too, but they are not equivalent to the implication $P \to Q$.

The two most common indirect methods of proof are called *Proof by Contraposition* and *Proof by Contradiction.* These methods of indirect proof differ from each other in the assumptions we do as premisses.

**Proof by contraposition.** In logic, contraposition is a law that says that a conditional statement is logically equivalent to its contrapositive. This is often called the *law of contrapositive*, or the *modus tollens* (*denying the consequent*) rule of inference.

The structure of this indirect proof is:

- We consider an implication $P \to Q$.
- Its contrapositive (opposite) $\neg Q \to \neg P$ is logically equivalent to the original implication, i.e.

$$\neg Q \to \neg P \;\Leftrightarrow\; P \to Q.$$

- We prove that *if* $\neg Q$ *is true* (the assumption), then $\neg P$ is true.

Therefore, a proof by contraposition is a *direct* proof of the contrapositive.

The proof of Lehmus-Steiner's theorem in [11] is an illustration of a proof by contraposition.

**Proof by contradiction.** In logic, proof by contradiction is a form of proof, and more specifically a form of indirect proof, that establishes the truth or validity of a proposition by showing that the proposition's being false would imply a contradiction. Proof by contradiction is also known as *indirect* proof, *apagogical argument*, proof by assuming the opposite, and *reductio ad impossibility*. It is a particular kind of the more general form of argument known as *reductio ad absurdum*.

We assume the proposition $P \to Q$ is false by assuming the negation of the conclusion $Q$ and the premise $P$ are true, and then using $P \wedge \neg Q$ to derive a contradiction.

Hence, the structure of this indirect proof is:

- We use the equivalence $(P \to Q) \Leftrightarrow (\neg P \vee Q)$.
- The negation of the last disjunction is $P \wedge \neg Q$, i. e. $\neg(P \to Q) \Leftrightarrow (P \wedge \neg Q)$.
- To prove the original implication $P \to Q$, we show that *if its negation* $P \wedge \neg Q$ *is true* (the assumption), then this leads to a contradiction.

In other words, to prove the implication $P \to Q$ by contradiction, we assume the hypothesis $P$ and the negation of the conclusion $\neg Q$ both hold and show that this is a contradiction (see also [12], p. 188).

A logical base of this method are equivalences of the form

$$P \to Q \quad \Leftrightarrow \neg Q \wedge P \to \neg P \quad \Leftrightarrow \neg(P \to Q) \to \neg P;$$

$$P \to Q \quad \Leftrightarrow \neg Q \wedge P \to Q \quad \Leftrightarrow \neg(P \to Q) \to Q.$$

Let now $T$ be a valid theorem, statement, axiom or definition of a notion in the corresponding system of knowledge. The following equivalences can also be logical base of a Proof by Contradiction of the implication $P \to Q$.

$$P \to Q \;\Leftrightarrow \neg Q \wedge P \to \neg T \;\Leftrightarrow \neg(P \to Q) \to \neg T.$$

The theoretical base of this method of proof is the *law of excluded middle* (or the *principle of excluded middle*). It states that for any proposition, either that proposition is true, or its negation is true. The law is also known as the *law* (or *principle*) *of the excluded third*.

Examples of indirect proofs of Lehmus-Steiner's theorem are given in [4].

There exist also examples of indirect proofs of implications $P \to Q$ in which the statement $\neg Q$ can be separated into cases $q_1 \vee q_2 \vee \ldots \vee q_k$, $k \geq 2$, $k \in \mathbb{N}$. In such a case each of the propositions $P \to q_1$, $P \to q_2, \ldots, P \to q_k$ is to be proved separately to be false. If moreover the premise $P$ is true it follows that all the statements $q_i$, $i = 1, \ldots, k$, are false and the conclusion $Q$ is true, i. e.

$$\neg(\neg Q) \Leftrightarrow \neg(q_1 \vee q_2 \vee \ldots \vee q_k) \Leftrightarrow \neg q_1 \wedge \neg q_2 \wedge \ldots \wedge \neg q_k \Leftrightarrow Q.$$

The logical equivalences in this case are (see also [12], p. 81)

$$P \to \neg q_1 \wedge P \to \neg q_2 \wedge \ldots \wedge P \to \neg q_k \Leftrightarrow P \to \neg q_1 \wedge \neg q_2 \wedge \ldots \wedge \neg q_k \Leftrightarrow P \to Q.$$

The indirect proof of Lehmus-Steiner's theorem given in [3] has in fact logical structure as the described above although this is not mentioned by the authors.

**Proof by construction.** In mathematics, a *constructive* proof is a method of proof that demonstrates the existence of a mathematical object by creating or providing a method for creating the object.

In other words, *proof by construction* (proof by example) is the construction of a concrete example with a property to show that something having that property exists.

A simple constructive proof of Lehmus-Steiner's Theorem is given in [13].

**Nonconstructive proof.** A nonconstructive proof establishes that a mathematical object with a certain property exists without explaining how such an object can be found. This often takes the form of a proof by contradiction in which the nonexistence of the object is proven to be impossible.

**Proof by counterexamples.** We can *disprove* something by showing a single counter example, i. e. one finds an example to show that something is not true.

However, we cannot prove something by example.

**Mathematical induction.** In proof by mathematical induction, a single *base case* is proved, and an *induction rule* is proved, which establishes that a certain case implies the next case. Applying the induction rule repeatedly, starting from the independently proved base case, proves many, often infinitely many, other cases. Since the base case is true, the infinity of other cases must also be true, even if all of them cannot be proved directly because of their infinite number.

The mathematical induction is a method of mathematical proof *typically* used to establish a given statement for all natural numbers. It is a form of direct proof and it is done in three steps.

Let $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ be the set of natural numbers, and $P(n)$ be a mathematical statement involving the natural number $n \geq k$, $k, n \in \mathbb{N}$, $k$ suitably fixed.

- The first step, known as the *base step*, is to prove the given statement for the first possible (admissible) natural number $k$, i.e. to show that $P(k)$ is true for $n = k$.

- The second step, known as the *inductive hypothesis*, is to assume that for a natural number $i \geq k$ the statement $P(i),\ i \in \mathbb{N}$ is true.

- The third step, known as the *inductive step*, is to prove that the given statement $P(i)$ (just assumed to be true) for any one natural number $i$ implies that the given statement for the next natural number $P(i+1)$ is true, i. e. to prove that $P(i) \to P(i+1)$.

From these three steps, mathematical induction is the rule from which we infer that the given statement $P(n)$ is established for all natural numbers $n \geq k$.

## 3. THE LEHMUS-STEINER THEOREM

The Lehmus-Steiner theorem states:.

**Theorem 3.1.** *If the straight line segments bisecting the angles at the base of a triangle and terminating at the opposite sides are equal, then the triangle is isosceles.*

This so–called *equal internal bisectors theorem* was communicated by Professor Lehmus (1780–1863) of Berlin to Jacob Steiner (1796–1867) in the year 1840 with a request for a pure geometrical proof of it. The request was complied with at the time, but Steiner's proof was not published till some years later. After giving his proof, Steiner considered also the case when the angles below the base are bisected; he generalized the theorem somewhat; found an external case where the theorem is not true; finally he discussed the case of the spherical triangle. His solution by the method of *proof by contraposition* [11] is considered to be the most elementary one at that time.

Since then many mathematicians have published analytical and geometrical solutions of this "elementary" theorem.

Does there exist a proof of this theorem which is *direct*? This problem was set in a Cambridge Examination Paper in England around 1850. In 1853, the famous Bristish mathematician James Joseph Sylvester (1814–1897) intended to show that *no direct proof can exist*, but he was not very successful. Since then, there have been a number of direct proofs published, but generally speaking they require some other results which have not been proved directly.

A simple, constructive proof, based mainly on Euclid's Book III, is given in [13].

McBride's paper [5] contains a short history of the theorem, a selection from the numerous other solutions that have been published, some discussion of the logical points raised, and a list of references to the extensive literature on the subject. For the long history of this remarkable theorem see also [6].

Below we propose two strictly direct proofs of Lehmus-Steiner's theorem.

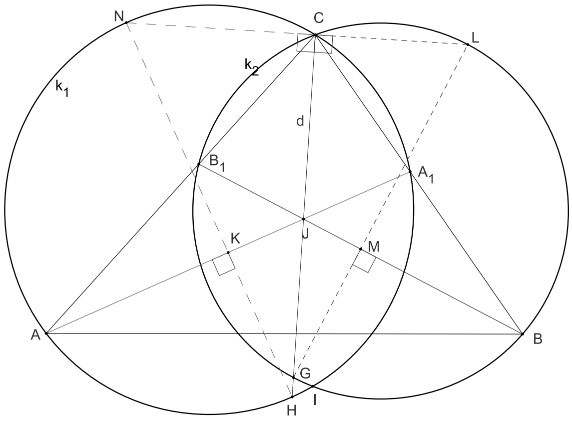


Figure 1.

3.1. FIRST PROOF OF THE THEOREM OF LEHMUS-STEINER

Let $AA_1$ ($A_1 \in BC$) and $BB_1$ ($B_1 \in AC$) be the internal bisectors in $\triangle ABC$, $AA_1 = BB_1$ and $AA_1 \cap BB_1 = J$. Then $CJ$ is the internal bisector of $\angle ACB$. We use the denotation $\gamma := \angle ACJ = \angle BCJ$.

Let also $k_1$ be the circumscribing circle of $\triangle ACA_1$, and $k_2$ the circumscribing circle of $\triangle BCB_1$ (Figure 1). First we need the following

**Proposition 3.2.** *The cut loci of points, from which two equal segments appear under the same angle, are equal arcs of congruent circles.*

*Proof of Proposition* 3.2. Consider $\triangle ACA_1$ and $\triangle BC_1B_1$, where $\angle ACA_1 = \angle BC_1B_1 = 2\gamma$ and $AA_1 = BB_1$. Let $k_1$ with center $O_1$ be the circumscribing circle of $\triangle ACA_1$, and $k_2$ with center $O_2$ the circumscribing circle of $\triangle BC_1B_1$ (Figure 2).

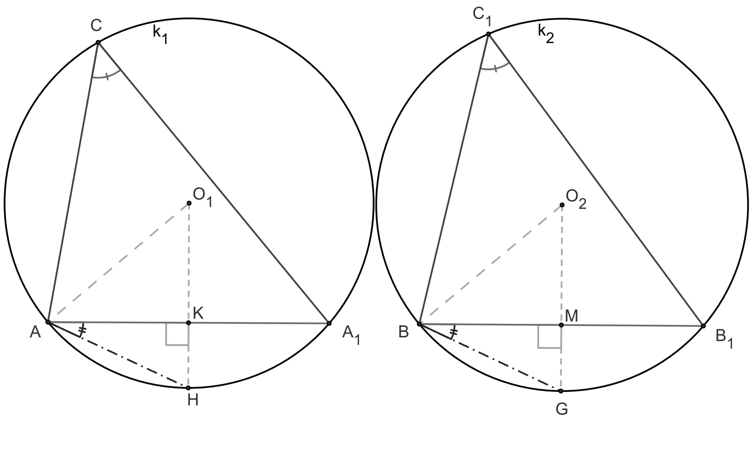


Figure 2.

The cut loci of points, from which the equal segments $AA_1$ and $BB_1$ appear under the same angle $2\gamma$, are respectively the arcs $\widehat{ACA_1}$ in $k_1$ and $\widehat{BC_1B_1}$ in $k_2$.

The perpendicular line $O_1K$ ($K \in AA_1$) from $O_1$ to the chord $AA_1$ cuts the arc $\widehat{AA_1}$ in $k_1$ at its midpoint $H$, the perpendicular line $O_2M$ ($M \in BB_1$) from $O_2$ to the chord $BB_1$ cuts the arc $\widehat{BB_1}$ in $k_2$ at its midpoint $G$.

The right angled triangles $\triangle AKH$ and $\triangle BMG$ are congruent, because of $AK = BM$ (as a half of equal chords) and $\angle KAH = \angle MBG = \gamma$. Hence, $AH = BG$ and $\angle AHK = \angle BGM$.

Then, the isosceles triangles $\triangle AO_1H$ and $\triangle BO_2G$ are congruent and the circles $k_1$ and $k_2$ have equal radii.

This proves the assertion of the proposition. $\square$

Since the equal segments $AA_1$ and $BB_1$ in $\triangle ABC$ (fig. 1) appear under the same angle $2\gamma$ from $C$, the circles $k_1$ and $k_2$ have equal radii (Proposition 3.2).

Let now $CJ \cap k_1 = H$ and $CJ \cap k_2 = G$.

The points $H$ and $G$ lie on the same ray $CJ^{\rightarrow}$. Since $CJ$ bisects the angles $\angle ACA_1$ and $\angle BCB_1$, the point $H$ is midpoint of the arc $\widehat{AA_1}$ in $k_1$, and the point $G$ is midpoint of $\widehat{BB_1}$ in $k_2$.

Let $K$ be the midpoint of the chord $AA_1$, $M$ be the midpoint of the chord $BB_1$, $HK \cap k_1 = N$ and $GM \cap k_2 = L$. Hence, the segments $HN$ and $GL$ are diameters of the circles $k_1$ and $k_2$ respectively. The triangles $\triangle CHN$ and $\triangle CGL$ are right angled with right angles at the vertex $C$.

The quadrilateral $CJKN$ can be inscribed in a circle and it follows that

$$|HK||HN| = |HJ||HC|. \tag{1}$$

The quadrilateral $CJML$ can be inscribed in a circle and it follows that

$$|GM||GL| = |GJ||GC|. \tag{2}$$

*Remark* 3.3. The equalities (1) and (2) are also a consequence of the similarities $\triangle HKJ \sim \triangle HCN$ and $\triangle GMJ \sim \triangle GCL$.

Since the circles $k_1$ and $k_2$ have equal radii and the chords $AA_1$ and $BB_1$ are equal, then $HK = GM$ and $HN = GL$. If we put $d = |CJ| > 0$, $x = |HJ| > 0$, $y = |GJ| > 0$, then $|HC| = x + d$ and $|GC| = y + d$.

The left-hand sides of equalities (1) and (2) are equal, so are their right hand sides. Hence

$$x(x+d) = y(y+d) \;\Leftrightarrow\; (x-y)(x+y+d) = 0. \tag{3}$$

Since $x + y + d \neq 0$, equality (3) is equivalent to the equality

$$x - y = 0 \,.\, \frac{1}{x+y+d} = 0,$$

which directly implies $x = y$.

*Remark* 3.4. If we denote the equal positive left-hand sides of equalities (1) and (2) by $a^2$, we get respectively the quadratic equations

$$
\begin{aligned}
x^2 + dx - a^2 = 0 \quad &\Leftrightarrow \ \left(x + \frac{d}{2}\right)^2 - \left(\frac{\sqrt{4\,a^2 + d^2}}{2}\right)^2 = 0 \\
&\Leftrightarrow \ \left(x + \frac{\sqrt{4\,a^2 + d^2} + d}{2}\right)\left(x - \frac{\sqrt{4\,a^2 + d^2} - d}{2}\right) = 0 \\
&\Leftrightarrow \ x - \frac{\sqrt{4\,a^2 + d^2} - d}{2} = 0\,.\left(x + \frac{\sqrt{4\,a^2 + d^2} + d}{2}\right)^{-1} = 0,
\end{aligned}
$$

and, analogously,

$$y^2 + dy - a^2 = 0,$$

with the same solution

$$x = y = \frac{1}{2}\left(\sqrt{4a^2 + d^2} - d\right).$$

Hence, the points $H$ and $G$, which lie on the same ray, coincide and $CG$ is the common chord of the circles $k_1$ and $k_2$.

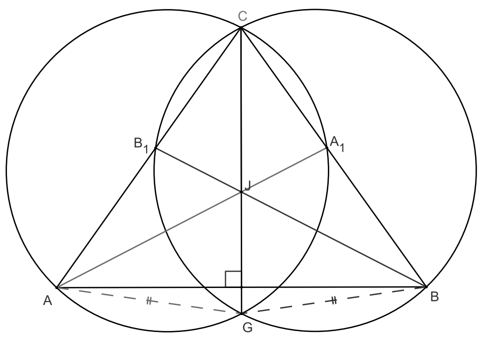


Figure 3.

As a consequence of the conditions

- $CG$ is a common side,
- $\angle ACG = \angle BCG$ ($CG$ is the bisector of $\angle ACB$),
- $\angle CAG = \angle CBG$ ($CG$ is the common chord of two circles with equal radii, hence $\widehat{CA_1G} = \widehat{CB_1G}$),

the triangles $\triangle AGC$ and $\triangle BGC$ are congruent (Figure 3).

Thus, $CA = CB$ and $\triangle ABC$ is isosceles. The direct proof of Theorem 3.1 is complete. $\square$

*Remark* 3.5. In this proof, the condition that the segments $AA_1$ and $BB_1$ are internal bisectors of the angles based at $AB$ in $\triangle ABC$ is not necessary.

It is only of importance that they are equal by length cevians and their intersection point lies on the bisector of $\angle ACB$.

We recall that a *cevian* is a line segment which joins a vertex of a triangle with a point on the opposite side (or its extension).

In fact we proved directly the following

**Theorem 3.6.** *If in a $\triangle ABC$ the segments $AA_1$ $(A_1 \in BC)$ and $BB_1$ $(B_1 \in AC)$ interesct at a point on the bisector of $\angle ACB$ and are equal by length, then $\triangle\, ABC$ is isosceles.*

### 3.2. SECOND PROOF OF THE THEOREM OF LEHMUS-STEINER

The idea for this proof comes from Problem 2.1–2.16 in [2]: *Find a direct proof of Lehmus-Steiner's theorem as a consequence of Stewart's theorem.*

We need the notion *algebraic measure* (*relative measure*) of a line segment.

On any straight line there are two (opposite to each other) directions. The *axis* is a couple of a straight line and a fixed (positive) direction on it.

Let $g^+$ denotes any axis. For any non zero line segment $MN$ on $g^+$ we can define its *relative* (*algebraic*) *measure* by $\overline{MN} = \varepsilon|MN|$, where $\varepsilon = +1$ in case $\overrightarrow{MN}$ has the same direction as $g^+$, and $\varepsilon = -1$ in case $\overrightarrow{MN}$ has the opposite direction with respect to $g^+$.

*Stewart's theorem* yields a relation between the lengths of the sides of a triangle and the length of a cevian.

Let in $\triangle\, ABC$ the line segment $CP$, $P \in AB$, be a cevian (more general, let $\{C; A, B, P\}$ be a quadruple of points such that $A, B, P$ are collinear).

**Theorem 3.7** (Theorem of Stewart)**.** *If $A, B$, $P$ are three collinear points and $C$ is any point then*

$$|CA|^2 \cdot \overline{BP} + |CB|^2 \cdot \overline{PA} + |CP|^2 \cdot \overline{AB} + \overline{BP} \cdot \overline{PA} \cdot \overline{AB} = 0.$$

*Remark* 3.8. Using the Pythagoras theorem, the proof of Steward's theorem is a simply verification.

In what follows we prove the *equal internal bisectors theorem* in the following formulation.

**Theorem 3.9.** *The straight line segments bisecting the angles at the base of a triangle and terminating at the opposite sides are equal if and only if the triangle is isosceles.*

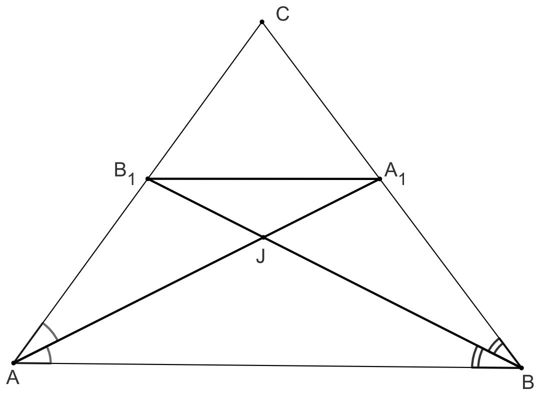


Figure 4.

Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in a triangle $ABC$ (Figure 4).

Since the triples $\{B, A_1, C\}$ and $\{A, B_1, C,\}$ consist of collinear points there exist integers $\alpha$ and $\beta$ such that

$$\begin{aligned} \overline{BA_1} &= \alpha\,\overline{BC}, \qquad \overline{A_1C} = (1-\alpha)\,\overline{BC}, \quad 0<\alpha<1; \\ \overline{AB_1} &= \beta\,\overline{AC}, \qquad \overline{B_1C} = (1-\beta)\,\overline{AC}, \quad 0<\beta<1. \end{aligned} \tag{4}$$

Using the fact that $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ are the internal bisectors of $\angle CAB$ and $\angle CBA$ in a triangle $ABC$, i. e. that

$$\frac{\overline{CA_1}}{\overline{A_1B}} = \frac{|CA|}{|AB|}, \quad \frac{\overline{CB_1}}{\overline{B_1A}} = \frac{|CB|}{|BA|},$$

from relations (4) we obtain

$$\begin{aligned} \alpha &= \frac{|AB|}{|AB|+|AC|}\,, \qquad 1-\alpha = \frac{|AC|}{|AB|+|AC|}\,, \\ \beta &= \frac{|AB|}{|AB|+|BC|}\,, \qquad 1-\beta = \frac{|BC|}{|AB|+|BC|}\,. \end{aligned} \tag{5}$$

Applying Stewart's theorem for the quadruple $\{A; B, A_1, C\}$

$$|AB|^2.\overline{A_1C} + |AA_1|^2.\overline{CB} + |AC|^2.\overline{BA_1} + \overline{A_1C}.\overline{CB}.\overline{BA_1} = 0,$$

and for the quadruple $\{B; A, B_1, C\}$

$$|BA|^2.\overline{B_1C} + |BB_1|^2.\overline{CA} + |BC|^2.\overline{AB_1} + \overline{B_1C}.\overline{CA}.\overline{AB_1} = 0,$$

from (4) and (5) we get

$$|AA_1|^2 = \frac{|AB||AC|}{(|AB|+|AC|)^2}\,\{(|AB|+|AC|)^2 - |BC|^2\},$$

$$|BB_1|^2 = \frac{|AB||BC|}{(|AB|+|BC|)^2}\{(|AB|+|BC|)^2 - |AC|^2\},$$

and finally

$$(|AA_1| - |BB_1|)\,\frac{(|AA_1|+|BB_1|)}{|AB|} = (|AC| - |BC|)$$
$$\times\left\{1+\frac{|AC||BC|(|AB|^2+|AC|^2+|BC|^2+2|AB|(|AC|+|BC|)+|AC||BC|)}{(|AB|+|BC|)^2(|AB|+|AC|)^2}\right\}.$$

Using the denotations

$$X := \frac{(|AA_1|+|BB_1|)}{|AB|}$$

and

$$Y := \left\{1+\frac{|AC||BC|(|AB|^2+|AC|^2+|BC|^2+2|AB|(|AC|+|BC|)+|AC||BC|)}{(|AB|+|BC|)^2(|AB|+|AC|)^2}\right\},$$

we rewrite the last equation in the form

$$(|AA_1| - |BB_1|)\,X = (|AC| - |BC|)\,Y.$$

Since $X \neq 0$ and $Y \neq 0$, the latter equation is equivalent to the equation

$$(|AA_1| - |BB_1|)\,\frac{X}{Y} = |AC| - |BC|. \tag{6}$$

Now, from (6) we see that $|\,AA_1| = |BB_1| \;\Leftrightarrow\; |AC| = |BC|$, which completes this direct proof of Lehmus-Steiner's theorem. □

*Remark* 3.10.
- In this proof, the condition that the segments $AA_1$ and $BB_1$ are internal bisectors of the angles based at $AB$ in $\triangle ABC$ is necessary.
- Using equalities (5) we compute

$$\alpha - \beta = \frac{|AB|}{(|AB|+|AC|)(|AB|+|BC|)}\,(|BC| - |AC|)$$

and obtain

$$A_1B_1 \parallel AB \quad\Leftrightarrow\quad \alpha = \beta \quad\Leftrightarrow\quad |AC| = |BC|.$$

The following statement is easily proved directly.

**Proposition 3.11.** *Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in $\triangle\,ABC$. Then $\triangle\,ABC$ is isosceles if and only if $A_1B_1 \parallel AB$.*

*Proof.* Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be the internal bisectors of $\angle CAB$ and $\angle CBA$, respectively, in $\triangle\, ABC$.

- Let $A_1B_1 \parallel AB$ (Figure 5).

  It follows that $\triangle\, AA_1B_1$ and $\triangle\, BB_1A_1$ are isosceles and the quadrilateral $ABA_1B_1$ is a trapezium with $|AB_1| = |BA_1|\,(= |A_1B_1|)$.

  Hence, $\triangle\, ABC$ is isosceles.

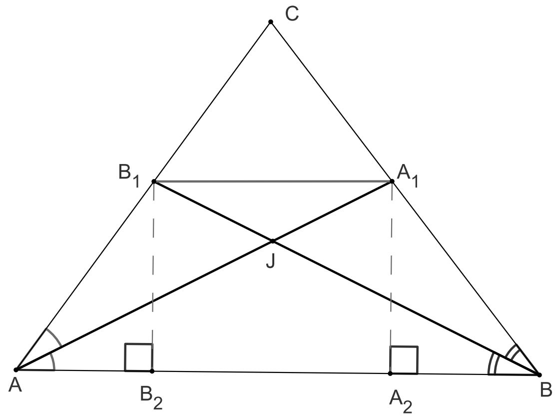


Figure 5.

- Let now $\triangle\, ABC$ be isosceles and $B_1B_2 \perp AB$ $(B_2 \in AB)$, $A_1A_2 \perp AB$ $(A_2 \in AB)$.

  Since $\triangle\, AA_1B \cong \triangle\, BB_1A$ (Figure 5), then $|AA_1| = |BB_1|$.

  Hence, $\triangle\, AA_1A_2 \cong \triangle\, BB_1B_2$, $|A_1A_2| = |B_1B_2|$ and $A_1B_1 \parallel AB$.

$\square$

In view of this proposition we can reformulate the Lehmus-Steiner theorem in the following form:

**Theorem 3.12.** *Let $AA_1\,(A_1 \in BC)$ and $BB_1\,(B_1 \in AC)$ be respectively the internal bisectors of $\angle CAB$ and $\angle CBA$ in $\triangle\, ABC$. If $|AA_1| = |BB_1|$, then $A_1B_1 \parallel AB$.*

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