ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 109

ANNUAL OF SOFIA UNIVERSITY “ST. KLIMENT OHRIDSKI”
FACULTY OF MATHEMATICS AND INFORMATICS
Volume 109

CHARACTERIZATION OF SOME MINIHYPERS IN PG(4, 3)

IVAN LANDJEV, EMILIYAN ROGACHEV AND ASSIA ROUSSEVA

In this paper we characterize the minihypers with parameters $(66, 21)$ in the geometry $\mathrm{PG}(4, 3)$. These parameters are important because they are instrumental in solving the problem of the existence of several hypothetical ternary Griesmer codes of dimension 6. This classification gives also insight into the classification problem for $(v_r+2v_{r-1}, v_{r-1}+2v_{r-2})$-mimihypers in $\mathrm{PG}(r, q)$ for any $r \geq 3$ and any prime power $q \geq 4$.

Keywords: linear codes, blocking sets, Griesmer bound

2020 Mathematics Subject Classification: 51A20, 51A21, 51A22, 94B65

1. Introduction

In [10] Sawashima and Maruta tackled the main problem in coding theory for ternary linear codes of dimension $k = 6$. For these parameters there exist about 100 unsolved cases all of which are considered to be difficult. In this paper, we provide the classification of the $(66, 21)$-minihypers in $\mathrm{PG}(4, 3)$ which are instrumental in solving three of the open cases for $k = 6$, $q = 3$. Moreover we prove the reducibility of the minihypers with parameters $(67, 21)$ and $(68, 21)$ which provides a characterization also for these two parameter sets. These results are further generalized for blocking sets with parameters $(v_r + 2v_{r-1}, v_{r-1} + 2v_{r-2})$ in the geometries $\mathrm{PG}(r, q)$, $q \geq 4$.

2. Preliminaries

In this section we introduce some basic notions and results on multisets of points in $\mathrm{PG}(r, q)$. A multiset of points is a mapping $\mathcal{K}$ from the pointset $\mathcal{P}$ of $\mathrm{PG}(r, q)$ to the non-negative integers. The multiset $\mathcal{K}$ can be extended additively to the subsets of $\mathcal{P}$. It can be interpreted as an arc or a minihyper depending on whether we put a

DOI: 10.60063/GSU.FMI.109.91-98

lower or an upper bound on the number of points in a hyperplane. So, an (n,w)-arc (resp. (n,w)-minihyper) is a multiset of cardinality n such that any hyperplane is of multiplicity at most (resp. at least) w. A minihyper with point multiplicities 0 and 1 is called a blocking set. The notion of a minihyper was introduced by Hamada. We use it here to denote a blocking set with multiple points.

The next few results have been proved for linear codes, but can be easily reformulated for arcs and blocking sets in finite projective geometries.

Theorem 2.1 ([13]). *Let $\mathcal{K}$ be an (n,w)-arc (resp. (n,w)-minihyper) in* $\mathrm{PG}(r,p)$*, where p is a prime. Let further $w \equiv n \pmod{p^e}$ for some $e \geq 1$. Then for every hyperplane H it holds that $\mathcal{K}(H) \equiv n \pmod{p^e}$.*

An (n,w)-arc in $\mathrm{PG}(r,q)$ is called t-extendable if the multiplicities of some of the points can be increased by a total of t, so that the obtained arc has parameters $(n+t,w)$. Similarly, an (n,w)-minihyper is called t-reducible if the mutiplicities of some of the points can be reduced by a total of t, so that the obtained multiset is an $(n-t,w)$-minihyper. The following result by R. Hill and P. Lizak was proved initially for linear codes.

Theorem 2.2 ([5]). *Let $\mathcal{K}$ be an (n,w)-arc (resp. (n,w)-minihyper) in* $\mathrm{PG}(r,q)$ *with $(n-w,q)=1$, such that the multiplicities of all hyperplanes are n or w modulo q. Then $\mathcal{K}$ is extendable to an $(n+1,w)$-arc (resp. reducible to an $(n-1,w)$-minihyper).*

The next theorem is a more sophisticated extension result by Hitoshi Kanda [6] which applies only for arcs (minihypers) in a geometry over $\mathbb{F}_3$.

Theorem 2.3 ([6]). *Let $\mathcal{K}$ be an (n,w)-arc (resp. (n,w)-minihyper) in* $\mathrm{PG}(r,3)$. *Assume further that the multiplicity of every hyperplane H is congruent to n, $n+1$, or $n+2$ modulo 9. Then $\mathcal{K}$ is extendable to an $(n+2,w)$-arc (resp. reducible to an $(n-2,w)$-minihyper).*

3. The classification of (66, 21)-blocking sets in PG(4, 3)

In this section, we denote by $\mathcal{B}$ a $(66,21)$-minihyper in $\mathrm{PG}(4,3)$. Let us note that $2-\mathcal{B}$ is then a $(176,59)$-arc in $\mathrm{PG}(4,3)$, which is associated with a $[176,5,117]_3$ Griesmer code. By Ward's Theorem this code is divisible and hence, in turn, $\mathcal{B}$ is also divisible. Thus for each hyperplane S (in this case S is a solid) in $\mathrm{PG}(4,3)$ one has $\mathcal{B}(S) \equiv 0 \pmod 3$.

Furthermore, the plane multiplicities with respect to a $(21,6)$-blocking set in $\mathrm{PG}(3,3)$ are in the set $\{6,9,12,15,18,21\}$ (cf. [7]). By an easy counting argument, we can prove that for $\mathcal{B}$ solids of multiplicities $22,\dots,27$, $31,\dots,36,40,\dots,45,49,\dots,$ $54,58,\dots,63$ do not exist. This observation is stated in following lemma.

Lemma 3.1. *Let $\mathcal{B}$ be a* $(66,21)$*-minihyper in* $\mathrm{PG}(4,3)$. *For every solid S in* $\mathrm{PG}(4,3)$ *it holds* $\mathcal{B}(S) \in \{21,30,39,48,57,66\}$.

Proof. By Ward's divisibility theorem, solids of multiplicity not divisible by 3 are impossible. Hence we have to rule out the existence of solids of multiplicity 24, 27, 33, 36, 42, 45, 51, 54, 60, 63.

We shall demonstrate that solids of multiplicity 24 are impossible. The remaining multiplicities are ruled out analogously. Let us assume that S_0 is a solid of multiplicity 24. Obviously $\mathcal{B}|_{S_0}$ is a $(24,7)$-blocking set. Let π be a 7-plane in S_0 and let S_i, $i=1,\dots,3$, be the other solids through π. Since a $(21,6)$-minihyper in $\mathrm{PG}(3,3)$ does not have 7-planes all S_i are of multiplicity at least 24. Now we have

$$|\mathcal{B}| = \sum_{i=0}^{3} \mathcal{B}(S_i) - 3\mathcal{B}(\pi) \geq 4\cdot 24 - 3\cdot 7 = 75,$$

a contradiction. □

Since $\mathcal{B}$ is a minihyper with respect to the lines (i.e. the multiplicity of each line is at least 1), a solid of multiplicity ≥ 48 does not have 0-points. Otherwise $|\mathcal{B}| \geq 48 + 27\cdot 1 > 66$. This implies that if $\mathcal{B}$ has a solid of multiplicity at least 48 it is the sum of a solid and $(26,8)$-blocking set which in turn is the sum of two planes [8].

It remains to classify the $(66,21)$-minihypers for which the hyperplane multiplicities are only 21, 30, and 39. For the spectrum of such a blocking set we have

$$\begin{array}{rcrcrcr} a_{21} & + & a_{30} & + & a_{39} & = & 121 \\ 21a_{21} & + & 30a_{30} & + & 39a_{39} & = & 66\cdot 40 \\ 210a_{21} & + & 435a_{30} & + & 741a_{39} & = & 2145\cdot 13 + 27\lambda_2 + 81\lambda_3 \end{array}$$

whence $3a_{39} = \lambda_2 + 3\lambda_3$.

Let us first assume that the $(66,21)$-minihyper $\mathcal{B}$ contains a 21-solid S_0 with a full plane. Let us fix a 6-line L which is contained in a 6-plane π in S_0. Note that L is of type $(3,1,1,1)$ or $(2,2,1,1)$. Consider a projection φ from L in this solid onto some plane in $\mathrm{PG}(4,3)$ disjoint from L. The image of S_0 is then a line L_0 of one of the following types: (a) $(15,0,0,0)$, (b) $(12,3,0,0)$, (c) $(9,6,0,0)$, (d) $(9,3,3,0)$ (cf. [7] for the classification of the $(21,6)$-minihypers in $\mathrm{PG}(3,3)$).

Denote by S_i the solids in $\mathrm{PG}(4,3)$ through π and set $L_i = \varphi(S_i)$. By Lemma 3.1 all lines in the projection plane must have multiplicity which is congruent to 6 (mod 9) and is at least 15.

The induced minihyper $\psi = \frac{1}{3}(\mathcal{B}^{\varphi} \pmod 9)$ is a plane $(8,2)$-blocking set. It is clear that, 0- and 9-points with respect to $\mathcal{B}^{\varphi}$ become 0-points with respect to ψ, 3- and 12-points become 1-points and 6-points become 2-points. Now ψ is clearly a double line or a sum of two different lines.

The first possibility leads to a contradiction since in this case all four lines L_i are of type $(15,0,0,0)$, or $(9,6,0,0)$. In this case the points of multiplicity $\equiv 6 \pmod 9$ are collinear. If at least two of the lines are of type $(15,0,0,0)$, then we have a line of multiplicity at least 42, which is the image of a 48-solid. This case has been already settled. If exactly one of L_i is of type $(15,0,0,0)$ we have a line of type $(15,9,9,9)$, again a 42-line. If all the lines L_i are of type $(9,6,0,0)$, then there exist three collinear 0-points, and hence a solid of multiplicity at most 15, a contradiction.

The second possibility is the case when ψ is the sum of two different lines. Then one of the lines L_i, L_0 say, is of type $(15,0,0,0)$ or $(9,6,0,0)$, while the remaining three lines L_1, L_2, L_3 are of type $(12,3,0,0)$ or $(9,3,3,0)$. If L_0 is of type $(15,0,0,0)$ it is easily checked that either there is a line of multiplicity at least 42 or a line of multiplicity less than 15. The former has already been treated, and the latter is contradiction. If L_0 is of type $(9,6,0,0)$, the only possibility which does not lead to a contradiction is when L_1, L_2, L_3 are all of type $(9,3,3,3)$. In this case, a $(66,21)$-blocking set can be constructed as follows.

Denote by Q the point of multiplicity 6 on the line L_0. The lines through Q are L_0 itself, a line of type $(6,9,9,9)$, and two lines of type $(6,3,3,3)$. The point Q is the image of a 12-plane δ in $\mathrm{PG}(4,3)$, which is the sum of three lines in δ, G_0, G_1, G_2 say, not necessarily different. One of these lines coincides with the line L from which the projection φ is defined. The preimage of the line in the projection plane, which is of type $(6,9,9,9)$, is a solid that contains δ and that has all points outside δ with multiplicity 1. The remaining three solids (whose image is of type $(6,3,3,3)$) contain δ and have nine further 1-points, which are the points of a plane through G_0, G_1 and G_2 respectively, and are not in δ.

Figure 1 represents the structure of this minihyper when the lines G_i in δ are all different and non-concurrent. The gray points represent 1-points, and the black points are 2-points. The two small pictures on the right represent the 30-solid, and any of the 21-solids through δ.

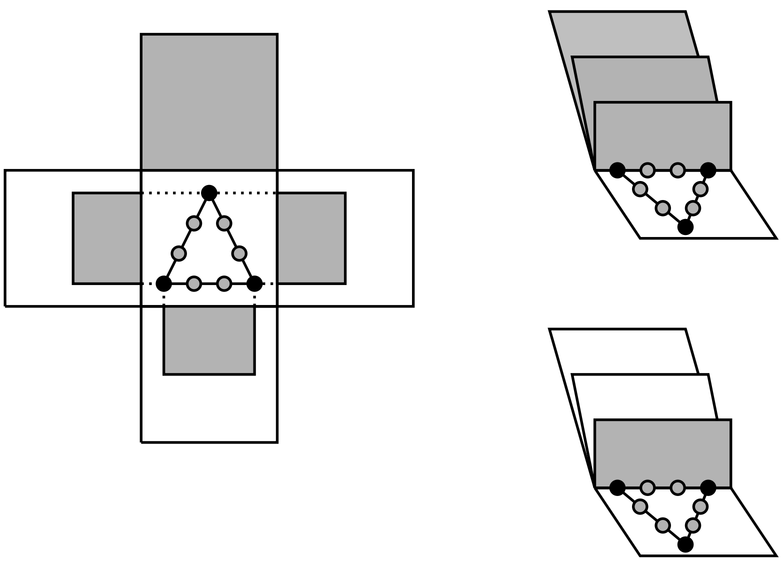

Figure 1. $(66,21)$-blocking set of type (B)

Assume there exists a solid S_0 of type (2) [7, Theorem 6]. Denote by L the 5-line of type $(2,1,1,1,0)$ and by π one of the two 6-planes through L. Let the other three solids through π be S_1, S_2, S_3. All solids are of multiplicity 21 and the multisets $\mathcal{B}|_{S_i}$ are all $(21,6)$-blocking sets (in $\mathrm{PG}(3,3)$) of the same type as S_0. This is because solids of type (1) have already been dealt with above, and type (3) is projective (note that L has a 2-point).

Consider a projection φ from L onto some plane disjoint from L. The image of π is a 1-point, P say, and the images of the solids S_i are lines through P, say L_i. All they have type $(7,7,1,1)$ with respect to the induced blocking set $\mathcal{B}^\varphi$. Now it is clear that in the projection plane there exists a line which is incident with at least three 1-points. It is the preimage of a solid of multiplicity at most $3\cdot 1+7+5=15$, which is a contradiction.

It remains to consider the case when every 21-solid is a $(21,6)$-minihyper of type described in [7, Theorem 6(3)], i.e. all 21-solids are projective. Since every point is contained in a minimal hyperplane the minihyper $\mathcal{B}$ is itself projective (i.e. it is a blocking set). Hence we have $\lambda_2=\lambda_3=0$ and $a_{39}=0$. This implies that

$$a_{21}=110, \quad a_{30}=11.$$

Define a point set in the dual space in which the 30-solids are 1-points and the 21-solids are 0-ponts. Obviously, this is an $(11,5)$-arc in $\mathrm{PG}(4,3)$, which is associated with an $[11,5,6]_3$ code. This code is unique since it is equivalent to the dual of the ternary Golay code with parameters $[11,6,5]$ that is known to be unique. Both codes are near-MDS codes (cf. [1]). This implies that there exists exactly one projective $(66,21)$-blocking set in $\mathrm{PG}(4,3)$.

It is possible to give an explicit geometric description of this blocking set. The 21-solids are $(21,6)$-blocking sets projectively equivalent to the blocking sets described in [7, Theorem 6(3)]. Every 30-solid is equivalent to the complement of a 10-cap (since it is projective). Now consider a 12-plane π in a 21-solid and denote by S_i, $i=0,1,2,3$, the solids through π. Without loss of generality, S_0 and S_1 are 21-solids and S_2, S_3 are 30-solids. The three special lines in each of the planes S_0, S_1 meet π in collinear points that are incident with two different lines through the 0-point in π.

Figure 2 represents the structure of the $(66,21)$-blocking set we just described.

Thus we have proved the following theorem which provides a characterization of the $(66,21)$-blocking sets in $\mathrm{PG}(4,3)$.

Theorem 3.2. *Let $\mathcal{B}$ be a $(66,21)$-blocking set in $\mathrm{PG}(4,3)$. Then $\mathcal{B}$ is one of the following:*

(A) *the sum of a solid and two planes*;

(B) *the sum of an affine space of dimension 3 and three affine planes contained in the four solids through a common 12-plane which is the sum of three (not necessarily different lines)*;

(C) *the dual of the $(11,5)$-arc in $\mathrm{PG}(4,3)$.*

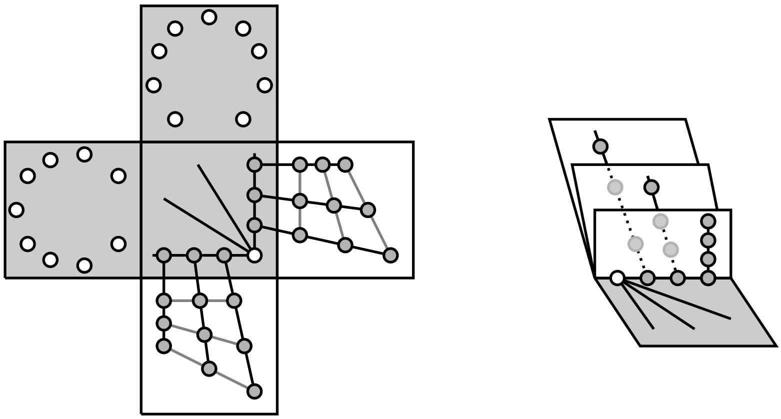

Figure 2. $(66, 21)$-blocking set of type (C)

Theorem 3.3. *Every* $(68, 21)$*-blocking set in* PG(4, 3) *is reducible to a* $(66, 21)$*-blocking set.*

Proof. Let $\mathcal{B}$ be a $(68, 21)$-blocking set in PG(4, 3). Assume there exists a solid of multiplicity at least 48. This solid does not have 0-points and $\mathcal{B}$ is the sum of a solid and a $(28, 8)$-blocking set. The latter is the sum of two planes and two points. Hence in this case $\mathcal{B}$ is reducible.

Now assume that all solids have multiplicity smaller than 48. Since a $(21, 6)$-blocking set has only planes of multiplicity $\equiv 0 \pmod 3$, we get by an easy counting that the only multiplicities of a solid are 21, 22, 23, 30, 31, 32, 39, 40, 41. Now $\mathcal{B}$ is extendable by Theorem 2.3. $\square$

Remark 3.4. In order to solve the main problem in coding theory for $d = 343$, 344, and 345 one needs blocking sets with maximal point multiplicity 2. So, in the case of $(68, 21)$-blocking sets reducible to $(66, 21)$-blocking sets of type (A) we have exactly one admissible multiset in which both planes meet the solid in two skew lines and hence intersect in a point which is outside the solid. Such a blocking set is unique since it is determined by the skew lines and the common point of the two planes. This minihyper is presented in Figure 3. In the case of blocking sets

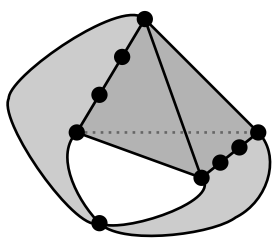

Figure 3. $(66, 21)$-blocking set of type (A) without 3-points

reducible to a $(66, 21)$-blocking sets of type (B) the exceptional 12-plane should be the sum of three different lines that are non-concurrent.

4. Characterization of the $(v_r + 2v_{r-1}, v_{r-1} + 2v_{r-2})$-blocking sets in $\mathrm{PG}(r, q)$ for $q \geq 5$

It was proved in [7] that every $(v_3 + 2v_2, v_2 + 2v_1)$-blocking set in $\mathrm{PG}(3, q)$, $q \geq 5$, is the sum of a plane and two lines. This result can be generalized to the following theorem.

Theorem 4.1. *Every $(v_r+2v_{r-1}, v_{r-1}+2v_{r-2})$-blocking set in $\mathrm{PG}(r, q)$, $q \geq 5$, is the sum of one $(r-1)$-dimensional subspace and two $(r-2)$-dimensional subspaces.*

Proof. The proof is by induction on r. The basis is provided by [7, Theorem 4]. Assume the result is proved for all geometries of dimension $r - 1$ or less. An easy counting gives that a minimal subspace of codimension i, $i = 0, \dots, r-1$ is a blocking set with parameters $(v_{r-i} + 2v_{r-i-1}, v_{r-i-1} + 2v_{r-i-2})$.

Consider a minimal subspace T of codimension 2. By the induction hypothesis, T is the sum of a subspace of codimension 3 and two subspaces of codimension 4. Hence there exists a hyperplane U of T in which all points are of multiplicity at least 1. In fact, this hyperplane is of multiplicity $v_{r-2}+\varepsilon v_{r-3}+(2-\varepsilon)v_{r-4}$, $\varepsilon = 0, 1, 2$.

Consider a projection from U onto some plane π with $U \cap \pi = \varnothing$. In the case $\varepsilon = 0$ the projection plane has $q + 1$ points of multiplicity $q^{r-2} + 2q^{r-4}$. These points have to be collinear which implies that the blocking set is the sum of a $(r-1)$-dimensional subspace and a $(2v_{r-1}, 2v_{r-2})$-blocking set, which, in turn, is the sum of two $(r - 2)$-dimensional subspaces.

The remaining two cases are treated in a similar way. □

Remark 4.2. In the case $q = 4$, a $(v_r + 2v_{r-1}, v_{r-1} + 2v_{r-2})$-blocking set in $\mathrm{PG}(r, 4)$, $r \geq 3$, is either the sum of a hyperplane and two hyperlines, or is lifted from a Baer subplane (cf. [7, Theorem 8]).

Acknowledgments

The first and the third authors were supported by the Research Fund of Sofia University under Contract 80-10-72/25.04.2023. The second author was supported by the Research Fund of Sofia University under Contract 80-10-99/26.04.2023.

References

[1] S. Dodunekov and I. Landjev, Near MDS codes of small length, Discrete Math. 213 (2000) 55–65.

[2] N. Hamada, A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry, Discrete Math. 116 (1993) 229–268.

[3] R. Hill, Optimal linear codes, in: Cryptography and Coding II, ed. by C. Mitchell, Oxford Univ. Press, Oxfod, 1992, 75–104.

[4] R. Hill, An extension theorem for linear codes, Des. Codes Cryptogr. 17 (1999) 151–157.

[5] R. Hill and P. Lizak, Extensions of linear codes, in: Proc. IEEE Int. Symposium on Inf. Theory, Whistler, Canada, 1995, 345.

[6] H. Kanda, A new extension theorem for ternary linear codes and its application, Finite Fields and Appl. 67 (2020) 1017111, https://doi.org/10.1016/j.ffa.2020.101711.

[7] I. Landjev, E. Rogachev and A. Rousseva, On a class of minihypers in the geometries PG(r, q), in: Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunication Engineering ed. by T. Zlateva and R. Goleva, vol. 450, 2022, 142–153.

[8] I. Landjev and P. Vandendriesche, A study of (xv_t, xv_{t-1})-minihypers in PG(t, q), J. Combin. Theory Ser. A 119 (2012) 1123–1131.

[9] T. Maruta and Y. Oya, On optimal ternary linear codes of dimension 6, Adv. Math. Comm. 5(3) (2011) 505–520.

[10] T. Sawashima and T. Maruta, Nonexistence of some ternary linear codes with minimum weight -2 modulo 9, Adv. Math. Comm. 17(6) (2023) 1338–1357.

[11] M. Takenaka, K. Okamoto and T. Maruta, On optimalnon-projective termary linear codes, Discrete Math. 308 (2008) 842–854.

[12] Y. Yoshida and T. Maruta, Ternary linear codes and quadrics, Electronic J. Combin. 16 (2009) #R9.

[13] H. N. Ward, Divisibility of codes meeting the Griesmer bound, J. Comb. Theory Ser. A 83(1) (1998) 79–93.

Received on September 5, 2023
Accepted on September 11, 2023

Ivan Landjev

New Bulgarian University
Department of Informatics
21 Montevideo Str.
1618 Sofia
BULGARIA

E-mail: i.landjev@nbu.bg

Emiliyan Rogachev and Assia Rousseva

Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5 James Bourchier Blvd.
1164 Sofia
BULGARIA

E-mails: rogachev@uni-sofia.bg
assia@fmi.uni-sofia.bg