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EQUIVALENT RELIABILITY POLYNOMIALS

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Looking for geometric modeling of reliability polynomials, we discuss three important ideas:
(i) find equivalent reliability polynomials via diffeomorphisms;
(ii) cover a reliability hypersurface by probability straight lines;
(iii) cover a reliability hypersurface by exponential decay curves.
In this paper we shall prove that two reliability polynomials, attached to some electric systems used inside aircrafts, are equivalent via an algebraic diffeomorphism. Also, we introduce the X-loxodromic curves on an equi-reliable hypersurface, which are constrained paths (evolutions) that are equi-reliable.

Keywords: Aircraft designs, reliability polynomials, algebraic diffeomorphism.

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1. BRIDGE STRUCTURE AND RELIABILITY POLYNOMIAL

In some engineering systems [1, 2, 4, 6], units may be connected in a bridge configuration as shown in Figure 1 which represents a three-phase electrical generator, part of the airplane power system, powered by a three-phase electric motor [3].

Theorem 1. *If $R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_7, R_8$ are the reliabilities of the arcs (paths) in the bridge system in Figure1, then the reliability polynomial P of*

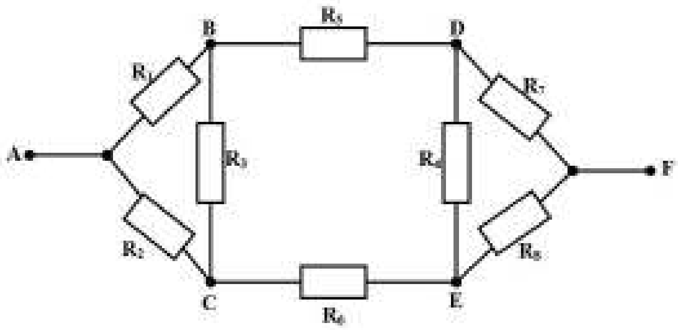

Figure 1: A bridge network

the system is

$$
\begin{aligned}
P =& R_1R_5R_7 + R_2R_6R_8 + R_2R_3R_5R_7 + R_1R_3R_6R_8 + R_1R_4R_5R_8 \\
&+ R_2R_4R_6R_7 - R_1R_2R_3R_5R_7 - R_1R_2R_3R_6R_8 + R_1R_3R_4R_6R_7 \\
&+ R_2R_3R_4R_5R_8 - R_1R_4R_5R_7R_8 - R_2R_4R_6R_7R_8 - R_1R_2R_3R_4R_5R_8 \\
&- R_1R_2R_3R_4R_6R_7 - R_1R_2R_4R_5R_6R_7 - R_1R_2R_4R_5R_6R_8 \\
&- R_1R_3R_4R_5R_6R_7 - R_1R_3R_4R_5R_6R_8 - R_2R_3R_4R_5R_6R_7 \\
&- R_2R_3R_4R_5R_6R_8 - R_1R_2R_5R_6R_7R_8 - R_1R_3R_4R_6R_7R_8 \\
&- R_2R_3R_4R_5R_7R_8 - R_1R_3R_5R_6R_7R_8 - R_2R_3R_5R_6R_7R_8 \\
&+ 2R_1R_2R_3R_4R_5R_6R_7 + 2R_1R_2R_3R_4R_5R_6R_8 \\
&+ R_1R_2R_3R_4R_5R_7R_8 + R_1R_2R_3R_4R_6R_7R_8 + 2R_1R_2R_3R_5R_6R_7R_8 \\
&+ 2R_1R_2R_4R_5R_6R_7R_8 + 2R_1R_3R_4R_5R_6R_7R_8 \\
&+ 2R_2R_3R_4R_5R_6R_7R_8 - 4R_1R_2R_3R_4R_5R_6R_7R_8.
\end{aligned} \tag{1}
$$

This polynomial is very long, and this lead to difficulties in its computation and geometrical interpretation. For these reasons, we shall introduce an equivalent reliability polynomial which is simpler. Perhaps, in engineering judgment, the best way to do this is to use Delta-Star Technique.

2. DELTA-STAR TECHNIQUE FOR SIMPLIFIED EQUIVALENT RELIABILITY POLYNOMIAL

The system in Figure 1 can be transformed into its equivalent series and parallel form by using Delta-star technique [5], see Figure 2. The reliability polynomial of the system in Figure 2 is

$$Q = R_5R_AR_BR_DR_F + R_6R_AR_CR_ER_F - R_5R_6R_AR_BR_CR_DR_ER_F. \tag{2}$$

Computationally, this method has some advantages: once a bridge network is transformed to its equivalent parallel and series form, the network reduction approach can be applied to obtain network reliability [7, 8]. Nonetheless, the

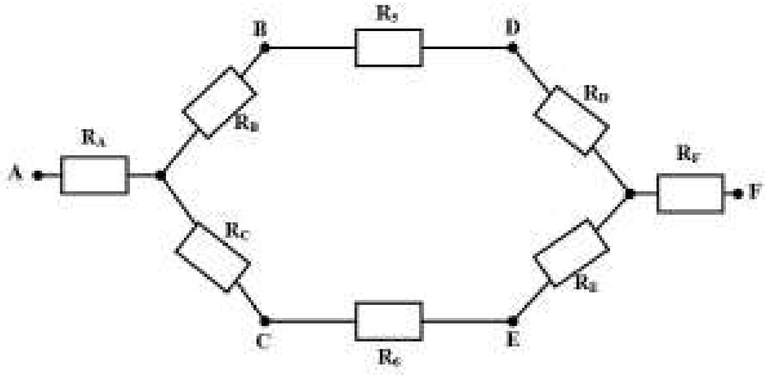

Figure 2: A simplified network

Delta-star method can easily handle networks containing more than one bridge configurations. Furthermore, it can be applied to bridge networks composed of devices having two mutually exclusive failure modes [8, 9].

To obtain the reliability polynomial Q, we observe that the Delta configurations $A, B; B, C; A, C$, respectively $D, F; D, E; E, F$ are replaced by star configurations A, B, C respectively D, E, F. The connection between them (see Figures 1 and 2) is given by the equations

$$R_A R_B = 1 - (1 - R_{AB})(1 - R_{AC} R_{BC}), \tag{3}$$

$$R_B R_C = 1 - (1 - R_{BC})(1 - R_{AC} R_{AB}), \tag{4}$$

$$R_A R_C = 1 - (1 - R_{AC})(1 - R_{AB} R_{BC}) \tag{5}$$

for the first triangle, and similar equations for the second one.

Solving equations (3) – (5), we obtain the following star equivalent reliabilities

$$R_A = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{BC})(1 - R_{AC} R_{AB})}},$$

$$R_B = \sqrt{\frac{[1 - (1 - R_{AB})(1 - R_{AC} R_{BC})][1 - (1 - R_{BC})(1 - R_{AC} R_{AB})]}{1 - (1 - R_{AC})(1 - R_{AB} R_{BC})}},$$

$$R_C = \sqrt{\frac{[1 - (1 - R_{BC})(1 - R_{AC} R_{AB})][1 - (1 - R_{AC})(1 - R_{AB} R_{BC})]}{1 - (1 - R_{AB})(1 - R_{AC} R_{BC})}}.$$

The transformation Delta-star equations applied to R_A, R_B, R_C and R_D, R_E, R_F, gives a simple configuration, so by using the above results, the equivalent to the network complex system in Figure 1 is shown in Figure 2.

In mathematical terms, we use a diffeomorphism to replace the initial reliability polynomial by a simpler ones. This diffeomorphism maps the unit hypercube into itself.

Lemma 1. *The mapping*

$$(R_{BC}, R_{AC}, R_{AB}, R_{EF}, R_{DF}, R_{DE}) \mapsto (R_A, R_B, R_C, R_D, R_E, R_F),$$

defined by formulas (3) - (5) *and the analogous formulae relating* (R_{EF}, R_{DF}, R_{DE}) *and* (R_D, R_E, R_F)*, transforms the unit hypercube into itself.*

Proof. For simplicity, let us denote $R_{AB} = c$, $R_{BC} = a$ and $R_{AC} = b$, where $a, b, c \in [0, 1]$ and $a \neq 0$ or $bc \neq 0$. We introduce the function

$$f(a, b, c) = \frac{(1 - (1 - c)(1 - ba))(1 - (1 - b)(1 - ca))}{1 - (1 - a)(1 - bc)}.$$

In view of the assumptions for a, b and c, both the numerator and denominator in the right-hand side are non-negative, therefore $f \geq 0$. We shall show that $f \leq 1$. After simplification, we obtain

$$f(a, b, c) = \frac{(c + ab - abc)(b + ac - abc)}{a + bc - abc}.$$

We consider separately two cases.

Case 1: $bc = 0$**.** In this case

$$f(a, b, c) = \frac{(c + ab)(b + ac)}{a} = \frac{ac^2 + ab^2}{a} = b^2 + c^2 \leq 1,$$

since either b or c is zero, and the other summand does not exceed 1.

Case 2: $bc > 0$**.** In this case the denominator of f is $a + (1 - a)bc > 0$, and the inequality $f(a, b, c) \leq 1$ is equivalent to $(c + ab - abc)(b + ac - abc) - (a + bc - abc) \leq 0$, or, after simplification, to

$$a\Big[abc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1\Big] \leq 0.$$

Since $a \geq 0$, we need to show that the expression in the brackets in non-positive. The latter is seen as follows:

$$\begin{aligned} &abc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1 \\ &\qquad \leq bc(1 - b)(1 - c) + c^2(1 - b) + b^2(1 - c) + bc - 1 \\ &\qquad = (1 - b)(1 - c)(bc - b - c - 1) \leq 0. \end{aligned}$$

The lemma is proved. □

Theorem 2. *The reliability polynomials* P *and* Q *are equivalent via the algebraic diffeomorphism defined by formulas* (3) - (5) *and their analogues relating* (R_D, R_E, R_F) *and* (R_{EF}, R_{DF}, R_{DE})*.*

Generally, a convenient algebraic diffeomorphism is the one which possesses the following three properties: (i) it transforms the unit hypercube into a subset of the unit hypercube; (ii) the number of terms in Q is smaller than the number of terms in P; (iii) the degree of Q is smaller than or equal to the degree of P.

2.1. EQUI-RELIABLE LOXODROMIC CURVES

For some geometrical concepts, we consider the reliability polynomial (2).

First, we rewrite this reliability polynomial (2) by replacing the indices A, B, C, D, E, F by numbers: $R_1 = R_A$, $R_2 = R_B$, $R_3 = R_C$, $R_4 = R_D$, $R_7 = R_E$, $R_8 = R_F$. In this way we obtain the polynomial

$$Q = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8. \tag{6}$$

In $\mathbb{R}^8$, let us consider the constant level algebraic hypersurfaces

$$c = R_1R_2R_4R_5R_8 + R_1R_3R_6R_7R_8 - R_1R_2R_3R_4R_5R_6R_7R_8,$$

which will be called *equi-reliable hypersurfaces* [4]. The normal vector field is

$$N = \left(\frac{\partial Q}{\partial R_1}, \frac{\partial Q}{\partial R_2}, \frac{\partial Q}{\partial R_3}, \frac{\partial Q}{\partial R_4}, \frac{\partial Q}{\partial R_5}, \frac{\partial Q}{\partial R_6}, \frac{\partial Q}{\partial R_7}, \frac{\partial Q}{\partial R_8}\right).$$

Consequently, the vector field

$$X = \left(-\frac{\partial Q}{\partial R_2}, \frac{\partial Q}{\partial R_1}, 0, 0, 0, 0, 0, 0\right)$$

is tangent to equi-reliable hypersurfaces.

Let Y be a significant vector field tangent to equi-reliable hypersurfaces, i.e., $< N, Y >= 0$. A curve $\gamma(t) = (R_1(t), R_2(t), R_3(t), R_4(t), R_5(t), R_6(t), R_7(t), R_8(t))$ in an equi-reliable hypersurface is called Y*-loxodroma* if

$$< \dot{\gamma}(t), Y(\gamma(t)) >= const.$$

For example, the X-loxodromic curves satisfy the first order ODE

$$-(R_1R_4R_5 - R_1R_3R_4R_5R_6R_7)(t)\dot{R}_1(t)$$

$$+(R_2R_4R_5 + R_3R_6R_7 - R_2R_3R_4R_5R_6R_7)(t)\dot{R}_2(t) = 0.$$

Along each Y-loxodroma the reliability is constant. Consequently, the previous Y-loxodromic curves are locally constrained paths (evolutions) that are equi-reliable.

Let $\gamma(t)$ be an X*-loxodroma*. The curve $\gamma(t)\exp(-\lambda t)$, $\lambda > 0$ is a decay curve that is necessary when we built the pullback reliability (to compute mean time to failure (MTTF)).

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