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ON THE EFFECTIVE ENUMERATIONS OF PARTIAL STRUCTURES*

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Ангел В. Дичев. ОБ ЭФФЕКТИВНЫХ НУМЕРАЦИЯХ ЧАСТИЧНЫХ АЛГЕБРАИЧЕСКИХ СИСТЕМ

Проблема существования эффективной нумерации данной частичной алгебраической системы, у которой носитель счетен, является одной из важнейших проблем теории конструктивных моделей.

В этой статье дана характеризация алгебраических систем с унарными функциями и предикатами, допускающих эффективные нумерации. Показано, что существуют только два типа структур, допускающих эффективные нумерации, независимо от конкретного вида функции и предикатов алгебраической системы.

В конце статьи дан пример структуры не допускающей эффективной нумерации, в которой каждое множество, определимое посредством SC с добавлением конечного числа констант, является рекурсивно перечислимым.

Angel V. Ditchev. ON THE EFFECTIVE ENUMERATIONS OF PARTIAL STRUCTURES

The question of the existence of effective enumeration of a given partial structure with denumerable domain is one of the most important problems in the theory of recursively enumerable models.

In this paper a characterization of the unary structures which admit effective enumerations is obtained. In the rest cases the question is open. It is shown also that there exist two sorts of structures only which admit effective enumerations independently of the concrete sort of the functions and predicates in the structure.

An example is given of a structure in which every subset of N definable by means of SC with finitely many constants is recursively enumerable, but the structure does not admit an effective enumeration.

The question of the existence of effective enumerations (recursively enumerable presentations) of a given partial structure with denumerable domain is one of the

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important problems in the theory of recursively enumerable (in the sequel r.e. for the sake of brevity) models.

In section 2 of this paper, a characterization of the unary structures which admit effective enumerations is obtained. In the rest cases the question is open.

Besides it is well known that every structure without predicates and with total functions admits an effective enumeration; every structure without functions and with finitely many unary predicates admits an effective enumerations, as well. It is natural to ask whether these are all partial structures which admit effective enumerations independently of the concrete sort of the functions and predicates in the structures. It turns out that there is only one sort structures more which admit effective enumerations. These are the structures without predicates and with only one partial function. In all another cases the structures admitting effective enumerations depend on the concrete sort of the functions and predicates.

In [4] Soskova and Soskov have defined a notion of effective enumeration and shown that a partial structure admits an effective enumeration in their sense iff every subset of $\mathbb{N}$ definable in this structure by means of REDS with finitely many constants is r.e. Soskov has proved also that if a partial structure admits an effective enumeration then every definable in this structure subset of $\mathbb{N}$ by means of SC (search computability, cf. [3]) with finitely many constants is r.e. In connection with this, I. Soskov stated the conjecture that if in a partial structure every subset of $\mathbb{N}$ definable by means of SC with finitely many constants is r.e., then the structure admits an effective enumeration. Soskov's conjecture is an attempt to characterize via search computability the structures which admit an effective enumeration. Let us remark that there is an example in [1] of a structure which has r.e. $\exists$-theory, but does not admit effective enumeration. In it, however, not all definable subsets of $\mathbb{N}$ are r.e.

In section 3 we give an example of a simple structure, in which every subset of $\mathbb{N}$ definable by means of SC with finitely many constants is r.e., but the structure does not admit an effective enumeration.

1. PRELIMINARIES

In this paper $\mathbb{N}$ denotes the set of all natural numbers $\{0, 1, \dots\}$, and $\mathbb{N}_n$ denotes the set $\{k : k \in \mathbb{N} \,\&\, k < n\}$.

If f is a partial function $Dom(f)$ denotes the domain, and $Ran(f)$ denotes the range of values of the function f. We use W_e to denote the e-th recursively enumerable (r.e.) set. We denote by $\langle .,. \rangle$ an effective coding of all ordered pairs of natural numbers with recursive functions $\lambda x.(x)_0$ and $\lambda x.(x)_1$ such that $(\langle x_0, x_1\rangle)_0 = x_0$ and $(\langle x_0, x_1\rangle)_1 = x_1$.

Let $U \subseteq \mathbb{N}^2$ and $\mathfrak{F}$ be a family of subsets of $\mathbb{N}$. The set U is said to be *universal* for the family $\mathfrak{F}$ iff for any n the set $\{x : (n, x) \in U\}$ is in the family $\mathfrak{F}$, and, conversely, for any set A in $\mathfrak{F}$ there exists such an n that $A = \{x : (n, x) \in U\}$.

Let us recall some definitions from [4] and [5].

Let $\mathfrak{A} = (B; \theta_1, \dots, \theta_n; \Sigma_1, \dots, \Sigma_k)$ be a partial structure, where B is an arbitrary denumerable set, $\theta_1, \dots, \theta_n$ are partial functions of many arguments on B, $\Sigma_1, \dots, \Sigma_k$ are partial predicates of many arguments on B and $n \geq 0$ and $k \geq 0$. The relational type of $\mathfrak{A}$ is the ordered pair $\langle (a_1, \dots, a_n), (b_1, \dots, b_k)\rangle$ where each θ_i is a_i-ary and each Σ_j is b_j-ary.

If every θ_i $(1 \leq i \leq n)$ and every Σ_j $(1 \leq j \leq k)$ are totally defined, then we say that the structure $\mathfrak{A}$ is a total one.

Effective enumeration (or r.e. presentation) of the structure $\mathfrak{A}$ is every ordered pair $\langle \alpha, \mathfrak{B} \rangle$ where $\mathfrak{B} = (\mathbf{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k)$ is a partial structure of the same relational type as $\mathfrak{A}$ and α is a partial, surjective mapping of $\mathbf{N}$ onto B such that the following conditions hold:

(i) $Dom(\alpha)$ is recursively enumerable and $\varphi_1, \ldots, \varphi_n$, $\sigma_1, \ldots, \sigma_k$ are partial recursive;

(ii) $\alpha(\varphi_i(x_1, \ldots, x_{a_i})) \cong \theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i}))$ for every natural $x_1, \ldots, x_{a_i}$, $1 \leq i \leq n$;

(iii) $\sigma_j(x_1, \ldots, x_{b_j}) \cong \Sigma_j(\alpha(x_1), \ldots, \alpha(x_{b_j}))$ for every natural $x_1, \ldots, x_{b_j}$, $1 \leq j \leq k$.

We say that the structure $\mathfrak{A}$ admits an effective enumeration iff there exists an effective enumeration of the structure $\mathfrak{A}$.

Remark 1. If the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ then it admits an effective enumeration $\langle \alpha, \mathfrak{B}^* \rangle$ which satisfies the additional condition

(iv) $Dom(\alpha)$ is closed with respect to the partial operations $\varphi_1^*, \ldots, \varphi_n^*$.

Indeed, let the structure $\mathfrak{A}$ admit an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ and φ_i^* be the restriction of the function φ_i to the set $\varphi_i^{-1}(Dom(\alpha))$, $i = 1, \ldots, n$, and $\sigma_j^* = \alpha_j$, $j = 1, \ldots, k$. Then it is easy to verify that $\langle \alpha, \mathfrak{B}^* \rangle$ is an effective enumeration which satisfies (iv).

Remark 2. If the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$, then it admits a totally defined effective enumeration $\langle \alpha, \mathfrak{B}^* \rangle$.

Indeed, let us suppose that the structure $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$. We may assume that $\langle \alpha, \mathfrak{B} \rangle$ satisfies condition (iv). Let f be a total recursive function such that $Ran(f) = Dom(\alpha)$. We define the enumeration $\langle \alpha^*, \mathfrak{B}^* \rangle$ as follows:

$\alpha^* = \lambda x.\alpha(f(x))$;

$\varphi_i^* = \lambda x_1 \ldots \lambda x_{a_i}.f^{-1}(\varphi_i(f(x_1), \ldots, f(x_{a_i})))$, $\quad i = 1, \ldots, n$;

$\sigma_j^* = \lambda x_1 \ldots \lambda x_{b_j}.f^{-1}(\sigma_j(f(x_1), \ldots, f(x_{b_j})))$, $\quad j = 1, \ldots, k$.

Again, it is easy to verify that $\langle \alpha^*, \mathfrak{B}^* \rangle$ is a totally defined effective enumeration.

We shall identify the partial predicates with the partial mappings which obtain values in $\{0, 1\}$, taking 0 for true and 1 for false.

Let $\mathfrak{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i.e. $\mathfrak{L}$ consists of n functional symbols $f_1, \ldots, f_n$ and k predicate symbols $T_1, \ldots, T_k$ where each f_i is a_i-ary and each T_j is b_j-ary. Let T_0 be a new unary predicate symbol which is intended to represent the unary total predicate $\Sigma_0 = \lambda s.0$.

Let $\{X_1, X_2, \ldots\}$ be a denumerable set of variables. We use capital letters X, Y, Z to denote variables.

If τ is a term in the language $\mathfrak{L}$, then we write $\tau(X_1, \ldots, X_a)$ to denote that all of the variables in τ are among $X_1, \ldots, X_a$. If $\tau(X_1, \ldots, X_a)$ is a term and $s_1, \ldots, s_a$ are arbitrary elements of B, then by $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ we denote the value, if it exists, of the term τ in the structure $\mathfrak{A}$ over the elements $s_1, \ldots, s_a$.

Termal predicates in the language $\mathfrak{L}$ are defined by the following inductive clauses:

If $T \in \{T_0, \ldots, T_k\}$, T is b-ary and $\tau^1, \ldots, \tau^b$ are terms, then $T(\tau^1, \ldots, \tau^b)$ and $\neg T(\tau^1, \ldots, \tau^b)$ are atomic termal predicates;

If Π_1 and Π_2 are termal predicates, then $(\Pi_1 \& \Pi_2)$ is a termal predicate.

Let $\Pi(X_1,\ldots,X_a)$ be a termal predicate with variables among $X_1, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of B. The value $\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$ of Π over $s_1, \ldots, s_a$ in $\mathfrak{A}$ is defined by the inductive clauses:

If $\Pi = T_j(\tau^1,\ldots,\tau^{b_j})$, $0 \leq j \leq k$, then

$$\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong \Sigma_j(\tau^1_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a),\ldots,\tau^{b_j}_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a));$$

If $\Pi = \neg\Pi^1$, where Π^1 is a termal predicate, then

$$\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong (\Pi^1_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) = 0 \supset 1,\ 0).$$

Here each expressions of the form "$\Pi^1 = 0 \supset \Pi^2, \Pi^3$" should be read as "if $\Pi^1 = 0$ then Π^2, else Π^3".

If $\Pi = (\Pi^1 \& \Pi^2)$, where Π^1 and Π^2 are termal predicates, then

$$\Pi_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong$$
$$(\Pi^1_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) = 0 \supset \Pi^2_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a), 1).$$

If Π is a termal predicate and n is a natural number, then $\exists Y_1 \ldots \exists Y_r(\Pi \supset n)$ is called *conditional expression.*

Let $Q = \exists Y_1 \ldots \exists Y_r(\Pi \supset n)$ be a conditional expression with free variables among $X_1, \ldots, X_a$ and let $s_1, \ldots, s_a$ be arbitrary elements of B. Then the value $Q_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a)$ of Q is defined by the equivalence

$$Q_{\mathfrak{A}}(X_1/s_1,\ldots,X_a/s_a) \cong l \iff \text{there exist } p_1, \ldots, p_r \text{ in } B \text{ such that}$$
$$\Pi_{\mathfrak{A}}(Y_1/p_1,\ldots,Y_r/p_r,X_1/s_1,\ldots,X_a/s_a) \cong 0 \quad \text{and} \quad l = n.$$

We assume fixed an effective coding of the atomic predicates, the termal predicates and the conditional expressions of the language $\mathfrak{L}$.

Let $A \subseteq \mathbb{N}$. The set A is said to be definable in the structure $\mathfrak{A}$ iff for some r.e. set of conditional expressions $\{Q^v\}_{v\in V}$ with free variables among $X_1, \ldots, X_a$ and for some fixed elements $t_1, \ldots, t_a$ of B the following equivalence is true

$$l \in A \iff \exists v(v \in V \;\&\; Q^v_{\mathfrak{A}}(X_1/t_1,\ldots,X_a/t_a) \cong l).$$

If s is an element of B, then $T_{\mathfrak{A}}[s]$ (the type of s) is the set of natural numbers $\{v : \Pi_v$ is an atomic termal predicate with a single variable X_1 occuring in Π_v and $\Pi^v_{\mathfrak{A}}(X_1/s) \cong 0\}$.

2. THE CONNECTION BETWEEN THE EFFECTIVE ENUMERATION OF A GIVEN STRUCTURE AND THE EXISTENCE OF AN R.E. SET UNIVERSAL FOR THE FAMILY OF THE TYPES OF THE STRUCTURE

Suppose a partial structure $\mathfrak{A} = (B;\theta_1,\ldots,\theta_n;\Sigma_1,\ldots,\Sigma_k)$ be given, where all functions and predicates are unary and B is a denumerable set. Then the following theorem holds

Theorem. *A partial structure $\mathfrak{A}$ admits an effective enumeration iff the family of all types of the elements of the structure $\mathfrak{A}$ has an universal r.e. set.*

P r o o f. Suppose that the partial structure $\mathfrak{A}$ admits an effective enumeration $(\alpha, \mathfrak{B})$, where $\mathfrak{B} = (\mathbb{N};\ \varphi_1,\ldots,\varphi_n;\ \sigma_1,\ldots,\sigma_k)$. Clearly, we can consider that α is totally defined over $\mathbb{N}$. A routine construction shows that there exists a primitive recursive in $\{\varphi_1,\dots,\varphi_n;\sigma_1,\dots,\sigma_k\}$ function Ψ, such that for each termal predicate Π with code v, $\Psi(v,x) = \Pi_{\mathfrak{A}}(\underline{X}_1/\alpha(x))$ for all x of $\mathbb{N}$. Then, it is obvious that the set

$$U = \{(x,v) : \Psi(v,x) = 0 \ \&\ v \text{ is a code of an atomic predicate }\}$$

is r.e. and universal for the family of all types of the elements of the structure $\mathfrak{A}$.

Suppose now that all types of the elements of the structure $\mathfrak{A}$ are r.e. and that the family of all these types has an universal r.e. set U^1. Denote by U^2 the set $\{(x,v) : ((x)_0, v) \in U_1\}$ and by U^2_x the set $\{v : (x,v) \in U^2\}$. It is obvious that U^2 is r.e. and also universal for the family of all types of the elements of the structure $\mathfrak{A}$. For any natural x, U^2_x is a type of some element of B. Moreover, for every x there exist infinitely many y such that $U^2_x = U^2_y$.

Let φ_i be the function $\lambda x.\langle i,x\rangle$, $i = 1,\dots,n$, $\mathbb{N}_0 = \mathbb{N}\backslash(Ran(\varphi_1)\cup\dots\cup Ran(\varphi_n))$ and g be monotonically increasing function such that $Ran(g) = \mathbb{N}_0$. We denote by U the set $\{(g(x),v) : (x,v)\in U^2\}$ and by U_x the set $\{v : (x,v)\in U\}$. For any natural number x, let B_x be the set $\{s : s\in B \ \&\ T_{\mathfrak{A}}[s] = U_x\}$ and α^0 be arbitrary mapping of $\mathbb{N}_0$ onto B, satisfying the equalities $\alpha^0(\{y : U_x = U_y\}) = B_x$, $x\in\mathbb{N}$. Evidently, α^0 is surjective and $Dom(\alpha^0) = \{x : \exists x((x,v)\in U)\} = \mathbb{N}_0$.

We define the partial mapping α of $\mathbb{N}$ onto B by the inductive clauses:

If $x\in\mathbb{N}_0$, then $\alpha(x) = \alpha^0(x)$;

If $x = \langle i,y\rangle$, $1\leqq i\leqq n$, $\alpha(y) = s$ and $\theta_i(s) = t$, then $\alpha(x) = t$.

The following simple lemmas are proved in [6].

Lemma 1. *For every $x\in\mathbb{N}$ and i, $1\leqq i\leqq n$,*

$$\alpha(\varphi_i(x)) \cong \alpha(\langle i,x\rangle) \cong \theta_i(\alpha(x)).$$

Let us denote by $\mathfrak{B}^*$ the partial structure $(\mathbb{N};\varphi_1,\dots,\varphi_n)$.

Corollary. *Let $\tau(Y)$ be a term and $y\in\mathbb{N}$. Then,*

$$\alpha(\tau_{\mathfrak{B}^*}(Y/y)) \cong \tau_{\mathfrak{A}}(Y/\alpha(y)).$$

Lemma 2. *There exists an effective way to define for every x of $\mathbb{N}$ an element y of $\mathbb{N}_0$ and a term $\tau(\underline{X}_1)$ such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$.*

Lemma 3. *There exists an effective way to define for every x of $\mathbb{N}$ an element y of $\mathbb{N}_0$ and a term $\tau(\underline{X}_1)$ such that $\alpha(x)\cong\tau_{\mathfrak{A}}(\underline{X}_1/\alpha(y))$.*

We need also the following auxiliary lemma.

Lemma 4. *$Dom(\alpha)$ is recursively enumerable.*

P r o o f. Let for an arbitrary natural number x, $y = y(x)\in\mathbb{N}_0$ and the term $\tau(\underline{X}_1)$ be such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$. Let in addition $v = v(x)$ be the code of the atomic predicate $T_0(\tau(\underline{X}_1))$. Then it is clear that

$$x\in Dom(\alpha) \iff \tau_{\mathfrak{A}}(\underline{X}_1/\alpha(y)) \text{ is defined } \iff v\in T_{\mathfrak{A}}[\alpha_0(y)]$$
$$\iff (y(x),v(x))\in U.$$

Therefore, $Dom(\alpha)$ is r.e.

Finally, let us define the partial predicates $\sigma_1,\dots,\sigma_k$ on $\mathbb{N}$ using the conditional equalities $\sigma_j(x)\cong\Sigma_j(\alpha(x))$, $j=1,\dots,k$. Again, for arbitrary x let $y = y(x)\in\mathbb{N}$ and the term $\tau(\underline{X}_1)$ be such that $x = \tau_{\mathfrak{B}^*}(\underline{X}_1/y)$. Let in addition $v_1 = v_1(x)$ $(v_2 = v_2(x))$ be the code of the atomic predicate $T_j(\tau(\underline{X}_1))$ $(\neg T_j(\tau(\underline{X}_1)))$. Then it is obvious that

$$\sigma_j(x) = 0 \iff \Sigma_j(\alpha(x))\cong 0 \iff (y(x),v_1(x))\in U;$$
$$\sigma_j(x) = 1 \iff \Sigma_j(\alpha(x))\cong 1 \iff (y(x),v_2(x))\in U;$$

Therefore, the predicates $\sigma_1, \ldots, \sigma_k$ are partial recursive. So, we have proved that $\langle \alpha, (\mathbb{N}; \varphi_1, \ldots, \varphi_n; \sigma_1, \ldots, \sigma_k) \rangle$ is an effective enumeration of the structure $\mathfrak{A}$.

Corollary 1. *Let $\mathfrak{A} = (B; \theta)$ be a partial structure, where B is a denumerable set and θ is a partial unary function such that $Dom(\theta) \subseteq B$ and $Ran(\theta) \subseteq B$. Then the structure $\mathfrak{A}$ admits an effective enumeration.*

P r o o f. If s is an arbitrary element of B then we use $[s]$ to denote the type $\{k : \theta^k(s)$ is defined $\}$ of the element s. So, if $s \in B$, then either $[s] = \mathbb{N}_n$ for some natural number n or $[s] = \mathbb{N}$.

Let us note that if for some element $s \in B$, $[s] = \mathbb{N}_n$ and $k < n$ then there exists such an element $t \in B$ that $[t] = \mathbb{N}_k$. Indeed, let t be the element $\theta^{n-k}(s)$. Therefore, for any structure $\mathfrak{A}$ we have the following possibilities for the families $\{[s] : s \in B\}$ of all types of the elements of B:

(i) $\{\mathbb{N}\}$;

(ii) $\{\mathbb{N}_k : k < n\}$ for some natural number n;

(iii) $\{\mathbb{N}\} \cup \{\mathbb{N}_k : k < n\}$ for some natural number n;

(iv) $\{\mathbb{N}_k : k \in \mathbb{N}\}$;

(v) $\{\mathbb{N}\} \cup \{\mathbb{N}_k : k \in \mathbb{N}\}$;

It is obvious that everyone of these families have an universal r.e. set. Thus, we obtain that the structure $\mathfrak{A}$ admits an effective enumeration.

If $\Pi^v = \Pi^{v_1} \& \ldots \& \Pi^{v_k}$ is a termal predicate where $\Pi^{v_1}, \ldots, \Pi^{v_k}$ are atomic predicates, then we use $\{\Pi^v\}$ to denote the set $\{v_1, \ldots, v_k\}$.

If all types of the structure $\mathfrak{A}$ are finite, then using Theorem 1 and Theorem 2 from [2] one can obtain more sophisticated necessary and sufficient conditions for the existence of an effective enumeration of the structure $\mathfrak{A}$. Namely, the following two theorems hold.

Corollary 2. *Let all function and predicates of the partial structure $\mathfrak{A}$ be unary and all types of the structure $\mathfrak{A}$ be finite. Then the structure $\mathfrak{A}$ admits an effective enumeration iff the following three conditions hold:*

(i) *The set $V = \{v : \exists s \in B(\Pi^v_{\mathfrak{A}}(s) = 0)\}$ is recursively enumerable;*

(ii) *The set $I = \{v : \{\Pi^v\} = T_{\mathfrak{A}}[s]$ for some $s \in B\}$ is a Σ^0_2 set in the arithmetical hierarchy;*

(iii) *There exists a partial recursive function f such that $V \subseteq Dom(f)$ and for every $v \in V$, $W_{f(v)}$ is a type of some element of the structure $\mathfrak{A}$ and $\{\Pi^v\} \subseteq W_{f(v)}$.*

Corollary 3. *Let all functions and predicates of the partial structure $\mathfrak{A}$ be unary, all types of the structure $\mathfrak{A}$ be finite and for any type t there exist at most finitely many types containing t. Then the structure $\mathfrak{A}$ admits an effective enumeration iff the following two conditions hold:*

(i) *The set $V = \{v : \exists s \in B(\Pi^v_{\mathfrak{A}}(s) = 0)\}$ is recursively enumerable;*

(ii) *The set $I = \{v : \{\Pi^v\} = T_{\mathfrak{A}}[s]$ for some $s \in B\}$ is a Σ^0_2 set of the arithmetical hierarchy.*

3. SOME COUNTER EXAMPLES

Example 1. *There exists a structure $\mathfrak{A} = (B; \theta_1; \Sigma_1)$ with unary θ_1 and Σ_1 which does not admit effective enumeration, but all definable in $\mathfrak{A}$ subsets of $\mathbb{N}$ are r.e.*

Let $\mathbb{Z}$ be the set of all integers and E be a set of natural numbers which is not Σ_2^0, $E = \{1 = p_0 < p_1 < \ldots\}$. Let in addition $B = \{b_{k,n} : k, n \in \mathbb{N}\} \cup \{a_m : m \in \mathbb{Z}\}$ where all $b_{k,n}$ and a_m are distinct for any $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

Define θ_1 and Σ_1 as follows:

$$\theta_1(b_{k,n}) = b_{k+1,n}, \quad k, n \in \mathbb{N};$$

$$\theta_1(a_m) = a_{m+1}, \quad m \in \mathbb{Z};$$

$$\Sigma_1(b_{k,n}) \cong 0 \iff k = 0 \vee k = p_n;$$

$$\Sigma_1(a_m) \cong 0 \iff m \leqq 0.$$

First, we show that every definable subset of $\mathbb{N}$ in the structure $\mathfrak{A}$ is r.e.

We use $f_1^k(Y)$ to denote the term $f_1(\ldots(f_1(Y))\ldots)$, where the symbol f_1 occurs k times in the term.

If s is an arbitrary element of B then we use $[s]$ to denote the type $\{k : \Sigma_1(\theta_1^k(s)) \cong 0\}$ of the element s; if Y is a variable and $Q = \exists Y_1 \ldots, \exists Y_r(\Pi \supset m)$ is a conditional expression, then we denote by $[Y; Q]$ the finite set

$$\{k : T_1(f_1^k(Y)) \text{ join in } \Pi \text{ as a conjunctive member }\}.$$

It is obvious that for any fixed element $s \in B$, any variable Y and conditional expression Q, there exists an effective way to verify whether or not $[Y; Q] = [s]$.

On the other hand, if $Q = \exists Y_1 \ldots \exists Y_r(\Pi \supset m)$ is a conditional expression, then for every conjunctive member $T_1(f_1^k(Y_i))$ of Π, $1 \leqq i \leqq r$, we can find an element s such that the conditional equality $\Sigma_1(\theta_1^k(s)) \cong 0$ is true. Therefore, in the structure $\mathfrak{A}$ the value $Q_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ of every conditional expression Q does not depend on these conjunctive members which have no free variables.

Now, let A be a definable subset of $\mathbb{N}$, $\{Q^v\}_{v \in V}$ be an r.e. set of conditional expressions with free variables among $X_1, \ldots, X_a$, and $s_1, \ldots, s_a$ be elements of B, such that the following equivalence is true:

$$l \in A \iff \exists v(v \in V \,\&\, Q^v_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a) \cong l).$$

Therefore, if $Q^v = \exists Y_1 \ldots \exists Y_{r_v}(\Pi^v \supset m^v)$, then for arbitrary $v \in V$ the following equivalence hold:

$$m^v \in A \iff [X_1; Q^v] = [s_1] \,\&\, \ldots \,\&\, [X_a; Q^v] = s_a.$$

Therefore, there exists an effective way to verify whether or not $m^v \in A$, i.e. the set A is r.e.

Thus we have established that every definable subset of $\mathbb{N}$ is r.e. Now we shall see that the structure $\mathfrak{A}$ does not admit an effective enumeration.

Assume that $\mathfrak{A}$ admits an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = (\mathbb{N}; \varphi_1; \sigma_1)$ is a partial structure of the same relational type as $\mathfrak{A}$ and α is a partial, surjective mapping of $\mathbb{N}$ onto B such that the following conditions hold:

(i) $Dom(\alpha)$ is r.e. and φ_1, σ_1 are p.r.;

(ii) $\alpha(\varphi_1(x)) \cong \theta_1(\alpha(x))$ for every natural x;

(iii) $\sigma(x) \cong \Sigma_1(\alpha(x))$ for every natural x.

Then $\{\{k : \sigma_1(\varphi_1^k(n)) \cong 0\} : n \in \mathbb{N}\} = \{[s] : s \in B\}$. Therefore, the family of finite sets $\{[s] : s \in B\}$ has an universal set $U = \{(n, k) : \sigma_1(\varphi_1^k(n)) \cong 0\}$.

Let F be the set $\{v : \exists s(s \in B \,\&\, [s] = E_v)\}$. According to Theorem 1 the set F is Σ_2^0. Then, the set $F = \{v : v \in F \,\&\, E_v$ has exactly two elements $\}$ is Σ_2^0, as well. Thus, the set $E = \{x : x \neq 0 \,\&\, \exists v(v \in F_1 \,\&\, x \in E_v)\}$ is Σ^0_2, contrary to the choice of the set E. So we prove that $\mathfrak{A}$ does not admit effective enumeration.

Example 2. *There exists a structure $\mathfrak{A} = (B; \theta_1, \theta_2)$ with unary θ_1, θ_2 which does not admit effective enumeration.*

Let $B = \mathbb{N}$ and E be a set which is not r.e. and define the functions θ_1, θ_2 as follows:

$$\theta_1(k) = k + 1,\ k \in \mathbb{N};$$

$$\theta_2 \text{ be arbitrary function such that } Dom(\theta_2) = E.$$

It is easy to check that $E = \{k : \theta_2(\theta_1^k(0)) \text{ is defined }\}$. Therefore, the structure $\mathfrak{A}$ does not admit effective enumeration.

Example 3. *There exists a structure $\mathfrak{A} = (B; \Sigma_1)$ with a single binary predicate, which does not admit effective enumeration.*

Let E be a set of natural numbers which is not r.e., $p_0 < p_1 < \ldots$ be the sequence of all prime numbers, $E_1 = \{k : \exists l(l \in E \,\&\, p_l = k)\}$ and $E_2 = \{k : \exists l \exists m(m \in E_1 \,\&\, k = m.l)\}$.

It is obvious that the sets E_1 and E_2 are not r.e. too.

If $\mathfrak{A} = (B; \Sigma_1)$ is a structure where Σ_1 is a binary predicate, we say that there exists k-cycle $(k > 0)$ for Σ_1 if for some elements $t_0, t_1, t_2, \ldots, t_{k-1}$ of B

$$\Sigma_1(t_0, t_1) = 0 \,\&\, \Sigma_1(t_1, t_2) = 0 \,\&\ldots\&\, \Sigma_1(t_{k-1}, t_0) = 0.$$

Let $\mathfrak{A} = (B; \Sigma_1)$ be a structure where B is infinite denumerable set and Σ_1 be a binary predicate totally defined over B such that there exists a k-cycle for Σ_1 iff $k \in E_2$. It is obvious that one can construct such a structure $\mathfrak{A}$.

Suppose that there exists an effective enumeration $\langle \alpha, \mathfrak{B} \rangle$ of the structure $\mathfrak{A}$, where $\mathfrak{B} = (\mathbb{N}; \sigma_1)$. Then, there exists a k-cycle for Σ_1 iff for some natural numbers $x_0, x_1, x_2, \ldots, x_{k-1}$, $\sigma_1(x_0, x_1) = 0 \,\&\, \sigma_1(x_1, x_2) = 0 \,\&\ldots\&\, \sigma_1(x_{k-1}, x_0) = 0$, i.e. $k \in E_2$ iff $\exists x_0 \exists x_1 \ldots \exists x_{k-1}(\sigma_1(x_0, x_1) = 0 \& \sigma_1(x_1, x_2) = 0 \& \ldots \& \sigma_1(x_{k-1}, x_0) = 0)$. We obtain that E_2 is r.e. which contradicts the choice of the set E_2. Therefore, the structure $\mathfrak{A}$ does not admit effective enumeration.

From this example one can easily obtain that there exists a structure $\mathfrak{A} = (B; \theta_1)$ with binary θ_1 which does not admit effective enumeration.

It is well known the following

Proposition 1. *If the structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ is a total one and* $[k = 0$ or $(n = 0 \,\&\, \Sigma_1, \ldots, \Sigma_k$ are unary $)]$ *then the structure $\mathfrak{A}$ admits an effective enumeration.*

From the above mentioned examples we obtain also the following

Proposition 2. a) *If the structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; \Sigma_1, \ldots, \Sigma_k)$ is such that*

$$[(k = 0 \,\&\, n = 1 \,\&\, \theta_1 \text{ is unary }) \text{ or } (n \neq 0 \,\&\, \Sigma_1, \ldots, \Sigma_k \text{ are unary })]$$

then the structure $\mathfrak{A}$ admits an effective enumeration.

b) *If* $[(k \neq 0 \,\&\, n \neq 0)$ or $(k \neq 0 \,\&$ at least one of $\Sigma_1, \ldots, \Sigma_k$ is not unary$)$ or $n \geq 2$ or $(n \neq 0 \,\&$ at least one of $\theta_1, \ldots, \theta_n$ is not unary $)]$ *then there exists a structure $\mathfrak{A}$ which does not admit effective enumeration.*

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