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EXAMPLES OF STRUCTURES WHICH DO NOT ADMIT RECURSIVE PRESENTATIONS*

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В статье показано существование класса структур с одной фукцией и с одним предикатом, которые не допускают эффективной нумерации и для которых все Σ_i-определимые подмножества N являются Σ_i^0 $(i \doteq 1,2)$ множествами в арифметической иерархии.

Angel Ditchev. EXAMPLES OF STRUCTURES WHICH DO NOT ADMIT RECURSIVE P-RESENTATIONS

In the paper it is proved that there exists a class of structures with a unary function and a unary predicate which do not admit an effective enumeration, but all Σ_i-definable subsets of N are Σ_i^0 $(i = 1, 2)$ sets in the arithmetical hierarchy.

In the Recursive Model Theory there are a lot of attempts to characterize structures with denumerable domains which admit recursive presentation. First, there are some necessary and some sufficient conditions [1]. Second, in many of them the considerations are restricted to a given class of structures, for example, Boolean algebras, partially ordered sets and so on [1]. Further, there are given definitions which restrict or extend the class of structures satisfying these definitions and attempts to characterize the corresponding classes are made. One of these

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definitions is the well-known strong constructivization (recursive presentation) [1]. In [2] Soskova and Soskov have defined another notion of effective enumeration (recursively enumerable (r.e.) presentation) of a partial structure. So they have succeeded to characterize the structure satisfying their definition by means of REDS [2] with finitely many constants. In connection with this and some other results [3, 4] I. Soskov has stated a conjecture that if in a given partial structure every subset of $\mathbb{N}$, definable by means of SC [3] with finitely many constants, is r.e. then the structure admits an effective enumeration.

In [5] a counter-example of Soskov's conjecture is shown. A necessary and sufficient condition of those structures with unary functions and predicates which admit effective enumeration is obtained as well.

Keeping in mind [6] and the power of $\mathcal{L}_{\omega_1\omega}$ one could suppose that it will be possible to characterize the structures which admit an effective enumeration in the terms of $\mathcal{L}_{\omega_1\omega}$. I. Soskov has made a suggestion that if in a given total structure every subset of $\mathbb{N}$ definable by means of SC with finitely many constants is r.e. and every Σ_2-definable set (cf. the definitions below) is Σ_2^0 in the arithmetical hierarchy, then the structure admits effective enumeration. It has been found that is not true. In any case we could not omit a condition like (iii) of Corollary 2 [5]. We should have an effective way for every termal predicate (conjunction of atomic predicate formulae and their negations) to find a type of an element which satisfies this termal predicate.

In this paper counter-examples of this suggestion are shown. Namely, it is proved that there exists a class of structures with a unary function and a unary predicate which do not admit an effective enumeration, but all Σ_i-definable subsets of $\mathbb{N}$ are Σ_i^0 sets in the arithmetical hierarchy, $i = 1, 2$. And the first order theories of these structures (without constants of the structures) are decidable. The idea of these counter-examples comes from the papers [7, 8, 9]. The problem here (keeping in mind [5]) is to find a propriate family of sets which has no universal r.e. set and which coincides with the family of all types of some structure. And meanwhile such kind of results are obtained.

Let us remember some definitions from [5] which we need.

$\mathbb{N}$ denote the set of all natural numbers.

Let U be a subset of $\mathbb{N}^{n+1}$ and $\mathcal{F}$ be a family of subsets of $\mathbb{N}^{n+1}$. The set U is said to be universal for the family $\mathcal{F}$ iff for any a the set $\{x \mid (a,x) \in U\}$ belongs to the family $\mathcal{F}$ and, conversely, for any A from $\mathcal{F}$ there exists such a, that $A = \{x \mid (a,x) \in U\}$.

We suppose that $E_0, E_1, \ldots$ be a canonical enumeration of all finite sets of natural numbers.

Let D_1 and D_2 be finite sets. Then we define the relations $\sqsubseteq$, $\bar{\sqsubseteq}$, $\sqsubset$, as follows:

$D_1 \sqsubseteq D_2$ iff $\exists k \forall x (x \in D_1 \iff x + k \in D_2)$;

$D \restriction k = \{x \mid x \in D \,\&\, x \leqq k\}$;

$D_1 \bar{\sqsubseteq} D_2$ iff $\exists k (D_2 \restriction k = D_1)$;

$D_1 \sqsubset D_2$ iff $\exists D' (D_1 \bar{\sqsubseteq} D' \,\&\, D' \sqsubseteq D_2)$.

Besides we define a binary operation "$*$" between two finite sets of natural numbers as follows:

Let E_{v_1} and E_{v_2} be given and $k_i = \max\{x \mid x \in E_{v_i}\}$, $i = 1,2$. Then $E_v = E_{v_1} * E_{v_2}$ iff $E_v \restriction k_1 = E_{v_1}$ and

$$\forall x\big(k_1 + 1 \leqq x \Rightarrow (x \in E_v \iff x - k_1 - 1 \in E_{v_2})\big).$$

Let $\mathfrak{A} = (B; \theta_1, \ldots, \theta_n; F_1, \ldots, F_k)$ be a structure, where B is an arbitrary denumerable set, $\theta_1, \ldots, \theta_n$ are functions of many arguments on B, and $F_1, \ldots, F_k$ are predicates of many arguments on B. We shall consider in the paper that all θ_i $(1 \leqq i \leqq n)$ and all F_j $(1 \leqq j \leqq k)$ are totally defined, so we have in mind only total structures.

Effective enumeration (or r.e. presentation) of the structure $\mathfrak{A}$ is every ordered pair $\langle \alpha, \mathfrak{B} \rangle$, where $\mathfrak{B} = \langle \mathbb{N}; \varphi_1, \ldots, \varphi_n; G_1, \ldots, G_k \rangle$ is a structure of the same relational type as $\mathfrak{A}$ and α is a surjective mapping of $\mathbb{N}$ onto B such that the following conditions hold:

(i) $\varphi_1, \ldots, \varphi_n$; $G_1, \ldots, G_k$ are partial recursive;

(ii) $\alpha(\varphi_i(x_1, \ldots, x_{a_i})) = \theta_i(\alpha(x_1), \ldots, \alpha(x_{a_i}))$ for all natural numbers $x_1, \ldots, x_{a_i}$ $(1 \leqq i \leqq n)$;

(iii) $G_j(x_1, \ldots, x_{b_j}) = F_j(\alpha(x_1), \ldots, \alpha(x_{b_j}))$ for all natural numbers $x_1, \ldots, x_{b_j}$ $(1 \leqq j \leqq k)$.

We say that the structure $\mathfrak{A}$ admits an effective enumeration iff there exists an effective enumeration of the structure $\mathfrak{A}$.

We shall identify the predicates with the mappings which obtain values 0 or 1, taking 0 for true and 1 for false.

Let $\mathfrak{L}$ be the first order language corresponding to the structure $\mathfrak{A}$, i. e. $\mathfrak{L}$ consists of n functional symbols $f_1, \ldots, f_n$ and k predicate symbols $T_1, \ldots, T_k$. We suppose that there is denumerable set of variables.

If τ is a term or a formula in the language $\mathfrak{L}$, then we write $\tau(X_1, \ldots, X_a)$ (or shortly $\tau(\boldsymbol{X})$) to denote that all variables occuring in τ are among $X_1, \ldots, X_a$. If $\tau(X_1, \ldots, X_a)$ is a term or a formula and $s_1, \ldots, s_a$ (or shortly $\boldsymbol{s}$) are arbitrary elements of B then by $\tau_{\mathfrak{A}}(X_1/s_1, \ldots, X_a/s_a)$ (or shortly $\tau_{\mathfrak{A}}(\boldsymbol{X}/\boldsymbol{s})$), we denote the value of τ in $\mathfrak{A}$ over $\boldsymbol{s}$.

Termal predicates in the language $\mathfrak{L}$ are all conjunctions of atomic predicates or their negations. We assume fixed an effective coding of the atomic predicates, the termal predicates and the disjunctions of termal predicates of the language $\mathfrak{L}$.

Let A be a subset of $\mathbb{N}$. The set A is said to be Σ_1-definable in the structure $\mathfrak{A}$ iff for some recursive sequence $\{\Pi^{i,j}\}$ of termal predicates with free variables among $\boldsymbol{X}$, $\boldsymbol{Y}_j$ and for some fixed elements $\boldsymbol{s}$ of B the following equivalence is true:

$$i \in A \iff \text{there exist } j \text{ and elements } \boldsymbol{t}_j \text{ of } B \text{ such that } \Pi^{i,j}(\boldsymbol{X}/\boldsymbol{s}, \boldsymbol{Y}_j/\boldsymbol{t}_j) = 0.$$

The subset A of $\mathbb{N}$ is said to be Σ_2-definable in the structure $\mathfrak{A}$ iff for some recursive sequence $\{\Pi^{i,j,k}\}$ of disjunctions of termal predicates with free variables among $\boldsymbol{X}$, $\boldsymbol{Y}_j$, $\boldsymbol{Z}_{j,k}$ and for some fixed elements $\boldsymbol{s}$ of B the following equivalence is true:

$$\begin{aligned} i \in A \iff & \text{there exist } j \text{ and elements } \boldsymbol{t}_j \text{ of } B \text{ such that for any } k \\ & \text{and for any elements } \boldsymbol{p}_{j,k} \text{ of } B, \text{ the equality} \\ & \Pi^{i,j,k}(\boldsymbol{X}/\boldsymbol{s}, \boldsymbol{Y}_j/\boldsymbol{t}_j, \boldsymbol{Z}_{j,k}/\boldsymbol{p}_{j,k}) = 0 \text{ holds.} \end{aligned}$$

Here one can see that the values of Σ_1-formula and Σ_2-formula are given in the language $\mathcal{L}_{\omega_1\omega}$ over s in $\mathfrak{A}$. But for our purpose it is not necessary to give these definitions and we omit them.

Theorem 1. *There exists such set $\boldsymbol{V}$ of natural numbers that the set $[\boldsymbol{V}] = \{v \mid \exists w \in \boldsymbol{V}(E_v \sqsubseteq E_w)\}$ is in the class Σ_2^0 of the arithmetical hierarchy and for every $v \in \mathbb{N}$ there exists such $w \in \boldsymbol{V} \subseteq [\boldsymbol{V}]$ that $E_v \sqsubset E_w$ and the family $\{E_v \mid v \in [\boldsymbol{V}]\}$ has not an universal r.e. set.*

Proof. Let U be subset of $\mathbb{N}^3$ which is r.e. and universal for the r.e. subsets of $\mathbb{N}^2$ and $U_{n,x} = \{y \mid (n, x, y) \in U\}$.

The set $\boldsymbol{V}$ we construct by steps, so that on step s we construct V_s, such that $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_s \subseteq \ldots$ and at the end $\boldsymbol{V} = \bigcup_{s \in \mathbb{N}} V_s$. Besides we construct the sequence $V_0, V_1, \ldots, V_s, \ldots$, so that $v \in V_s$ be a Δ_2^0 set relatively v, s. Thus we ensure $\boldsymbol{V}$ (and $[\boldsymbol{V}]$ as well) to be a Σ_2^0 set.

On the other hand, on step s we need some elements which are canonical codes of some finite sets not belonging to $\boldsymbol{V}$. For this purpose we construct set C of prohibited elements. Again we do that on steps so that $C_0 \subseteq C_1 \subseteq \ldots \subseteq C_s \subseteq \ldots$ and $C = \bigcup_{s \in \mathbb{N}} C_s$. The set C plays only an auxiliary role.

When we want to add some elements to C_s we have to know if some $U_{s,x}$ is finite. In this case we find v such that $E_v = U_{s,x}$. For this aim when on some step we don't know if $U_{s,x}$ is finite we put (s, x) in the set $\boldsymbol{I}$. The set $\boldsymbol{I}$ consists of these (s, x) for which $U_{s,x}$ is eventually infinite set. When on step s we understand that some $U_{t,x}$ is finite then we force (t, x) out of $\boldsymbol{I}$.

Construction. Step $s = 0$. Fix $V_0 = C_0 = I_0 = \varnothing$.

Step $s+1$. Let $m_v = \max\{x \mid x \in E_v\}$, $k_{s+1} = \min\{m_v \mid \forall w \in V_s(E_v \not\sqsubset E_w)\}$ and $\underline{v}_{s+1}$ be the least positive integer v such that $m_v = k_{s+1}$ $\forall w \in V_s (E_v \not\sqsubset E_w)$. We verify if there exists $(t, x_t) \in I_s$ such that $\forall y (y \leqq k_{s+1} + s + 3 \vee y \notin U_{t,x_t})$. If that is so let all of them be $U_{t_1,x_{t_1}}, \ldots, U_{t_l,x_{t_l}}$. We find $w_1, \ldots, w_l$ such that $E_{w_1} = U_{t_1,x_{t_1}}, \ldots, E_{w_l} = U_{t_l,x_{t_l}}$. We fix

$$C_{s+1} = C_s \cup \{w_1, \ldots, w_l\}, \quad I'_{s+1} = I_s \setminus \{(t_1, x_{t_1}), \ldots, (t_l, x_{t_l})\}.$$

Let

$$r_{s+1} = \min\{m_v \mid E_{\underline{v}_{s+1}} \sqsubset E_v \;\&\; \forall w \in C_{s+1}(E_w \not\sqsubseteq E_v)\}$$

and v_{s+1} be the least positive integer v such that

$$m_v = r_{s+1} \;\&\; E_{\underline{v}_{s+1}} \sqsubset E_v \;\&\; \forall w \in C_{s+1} (E_w \not\sqsubseteq E_v).$$

We denote such v by v_{s+1} and fix $V_{s+1} = V_s \cup \{v_{s+1}\}$. At the end we verify whether there exists x such that

$$\forall v \in V_{s+1} \forall k \leqq v \exists y (y \in U_{s+1,x} \;\&\; y + k \notin E_v \;\&\; |U_{s+1,x}| \geqq 2).$$

If it is so let x_{s+1} be the least such x. In this case we fix

$$I_{s+1} = I'_{s+1} \cup \{(s + 1, x_{s+1})\},$$

otherwise $I_{s+1} = I'_{s+1}$. The construction is completed.

First, it is easy to check by induction on s that $\forall s (v \in V_s \Rightarrow |E_v| \geqq 2)$.

One could see easily that for each s the sets V_s and I_s contain at most s elements, as well.

Lemma. *For every $s > 0$ and for every finite sets $E_{p_1}, \ldots, E_{p_s}$, each of them containing at least 2 elements, there exists such finite set E_p that*

$$\max\{x \mid x \in E_p\} \leqq s+1$$

and for all v, $E_{p_i} \not\sqsubseteq E_v * E_p$, $i = 1, \ldots, s$.

The proof of this lemma one could easily do by induction on s and it is more or less combinatorial one.

Now we can see that there exists such v that

$$E_{\underline{v}_{s+1}} \sqsubset E_v \;\&\; \forall w \in C_{s+1}(E_w \not\sqsubseteq E_v).$$

Indeed, according to the above lemma there exists such p that

$$\forall w \in C_{s+1}(E_w \not\sqsubseteq E_{\underline{v}_{s+1}} * E_p) \text{ and } \max\{x \mid x \in E_p\} \leqq s+2,$$

i.e. v_{s+1} exists on every step $s+1$ and $\max\{x \mid x \in E_{v_{s+1}}\} \leqq k_{s+1} + s + 3$. Obviously, for all v there exists $w \in V$, such that $E_v \sqsubset E_w$.

Let us prove that $\forall v \in V\, \forall w \in C(E_w \not\sqsubseteq E_v)$. For this purpose it is enough to prove that $\forall s\, \forall v \in V_s\, \forall t_{t \leq s}(U_{t,x_t} \not\sqsubseteq E_v)$.

It is obvious that $\forall v \in V_0\, \forall t_{t \leq 0}(U_{t,x_t} \not\sqsubseteq E_v)$.

Let us assume that $\forall v \in V_s\, \forall t_{t \leqq s}(U_{t,x_t} \not\sqsubseteq E_v)$. On step $s+1$ we have $V_{s+1} = V_s \cup \{v_{s+1}\}$. Let us remember that we chose v_{s+1} such that $\forall t_{t \leqq s}\left(U_{t,x_t} \not\sqsubseteq E_{v_{s+1}}\right)$. Besides, we chose $(s+1, x_{s+1})$ such that

$$\forall v \in V_{s+1}\, \forall k_{k \leqq v}\, \exists y \left(y \in U_{s+1,x_{s+1}} \;\&\; y+k \notin E_v \;\&\; \left|U_{s+1,x_{s+1}}\right| \geqq 2\right).$$

Therefore, $U_{s+1,x_{s+1}} \not\sqsubseteq E_v$, i.e. $\forall v \in V_{s+1}\, \forall t_{t \leqq s+1}(U_{t,x_t} \not\sqsubseteq E_v)$.

So it is proved that $\forall v \in V\, \forall w \in C\,(E_w \not\sqsubseteq E_v)$.

Let us note that the construction is Δ^0_2, since the conditions which we verify are Π^0_1 or Σ^0_1. Note that we define w such that $E_w = U_{s,x_s}$, and that w is a function of s. So $\boldsymbol{V}$ is Σ^0_2 set and since $[\boldsymbol{V}]$ is Σ^0_2 set, as well.

At the end let us assume that the family $\{E_v \mid v \in [\boldsymbol{V}]\}$ has an universal r.e. set. Let it be $\boldsymbol{A}$. Denote by $A_x = \{y \mid (x,y) \in \boldsymbol{A}\}$ and $U_x = \{(x,y) \mid (s,x,y) \in U\}$. Then $\Delta = U_s$ for some s.

Let us consider step $s+1$. It is clear that

$$\exists x\, \forall v \in V_{s+1}\, \forall k_{k \leqq v}\, \exists y\,(y \in U_{s+1,x} \;\&\; y+k \notin E_v \;\&\; |U_{s+1,x}| \geqq 2).$$

Then

$$U_{s+1,x_{s+1}} = A_{x_{s+1}} \in \{E_v \mid v \in [\boldsymbol{V}]\} \text{ and } U_{s+1,x_{s+1}} = E_w \text{ for some } w,$$

i.e. there exist $w \in C$ and $v \in [\boldsymbol{V}]$ such that $E_w \sqsubseteq E_v$, which is a contradiction.

So, the family $\{E_v \mid v \in [\boldsymbol{V}]\}$ has not a universal r.e. set and Theorem 1 is proved.

One can prove in the same way the following relativized version of Theorem 1.

Theorem 1′. *For any set of natural numbers A there exists such subset $\boldsymbol{V}_A$ of $\mathbb{N}$ that the set $[\boldsymbol{V}_A] = \{v \mid \exists w \in \boldsymbol{V}_A[E_v \sqsubseteq E_w]\}$ is in the class $\Sigma^0_2[A]$ and for every*

v there exists such $w \in \boldsymbol{V_A} \subseteq [\boldsymbol{V_A}]$ that $E_v \sqsubset E_w$ and the family $\{E_v \mid v \in \boldsymbol{V_A}\}$ has not a universal r.e. in $\boldsymbol{A}$ set.

Now we can prove the following

Theorem 2. *There exists a structure $\mathfrak{A} = \langle B; \theta; F\rangle$ with unary θ, F, which does not admit an effective enumeration but all Σ_1-definable in $\mathfrak{A}$ subsets of $\mathbb{N}$ are r.e. and all Σ_2-definable in $\mathfrak{A}$ subsets of $\mathbb{N}$ are Σ_2^0 sets in the arithmetical hierarchy.*

Proof. Let $\boldsymbol{V}$ be such a set which exists according to Theorem 1 and $\boldsymbol{V} = \{v_0, v_1, \ldots\}$. Let in addition B be the set $\{b_{k,n} \mid k, n \in \mathbb{N}\}$, where all $b_{k,n}$ are distinct, $k, n \in \mathbb{N}$. We define θ and F as follows:

$$\theta(b_{k,n}) = b_{k,n+1} \quad \text{and} \quad F(b_{k,n}) = 0 \iff n \in E_{v_k} \qquad k, n \in \mathbb{N}.$$

It would be useful to give some intuitive explanations about the above structure. For each fixed k we can consider the set $\{b_{k,n} \mid n \in \mathbb{N}\}$ as a copy of the set of natural numbers and the function F over this set as the succesor function.

If s is an arbitrary element of B then we use $[s]$ to denote the type $\{k \mid F(\theta^k(s)) = 0\}$ of the element s. Thus $v \in [V] \iff E_v$ is a type of some element of B. So for any element s there exist finitely many k such that $F(\theta^k(s)) = 0$ and infinitely many k such that $F(\theta^k(s)) = 1$.

In the case which we consider the language is with a single unary functional symbol f and a single unary predicate symbol T. We use $f^k(X)$ to denote the term $f(\ldots(f(X))\ldots)$, where the symbol f occurs k times in the term. If Y is a variable and Π is a termal predicate, then we denote by $[Y;\Pi]$ the set

$$\{k \mid T(f^k(Y)) \text{ join in } \Pi \text{ as a conjuctive member }\}$$

and by $\neg[Y;\Pi]$ the set

$$\{k \mid \neg T(f^k(Y)) \text{ join in } \Pi \text{ as a conjuctive member }\}.$$

It is obvious that for any fixed element $s \in B$, any variable Y and any termal predicate Π there exists effective way to verify whether $[Y;\Pi] \subseteq [s]$ or not and whether $\neg[Y;\Pi] \cap [s] = \varnothing$ or not.

First, we prove that every closed consistent existential formula in this language is true in $\mathfrak{A}$.

It is obvious, that is enough to show that every formula of the kind $\exists X \Pi(X)$ is true in $\mathfrak{A}$, where $\Pi(X)$ is a termal predicate on the form

$$T\left(f^{k_1}(X)\right) \,\&\, \ldots \,\&\, T\left(f^{k_s}(X)\right) \,\&\, \neg T\left(f^{l_1}(X)\right) \,\&\, \ldots \,\&\, \neg T\left(f^{l_t}(X)\right)$$

and $\{k_1, \ldots, k_s\} \cap \{l_1, \ldots, l_t\} = \varnothing$.

We can consider that $k_s = \max\{k_1, \ldots, k_s, l_1, \ldots, l_t\}$ and let $\underline{v}$ be such that $E_{\underline{v}} = \{k_1, \ldots, k_s\}$. We find v_k such that $E_{\underline{v}} \sqsubset E_{v_k}$. This means that for some E', $E_{\underline{v}} \mathrel{\bar{\sqsubset}} E' \sqsubseteq E_{v_k}$, i. e. $E_{\underline{v}} = E' \lceil k_s$ and $\forall x (x \in E' \iff x + m \in E_{v_k})$ for some m. Now it is easy to check that $\Pi(X/\theta^m(b_{k,0})) = 0$, i. e. $\exists X\, \Pi(X)$ is true in $\mathfrak{A}$.

It is obvious now that every closed universal formula $\forall \boldsymbol{X} \Pi(\boldsymbol{X})$, where $\Pi(\boldsymbol{X})$ is a termal predicate, is not true in $\mathfrak{A}$.

Second, we show that every Σ_1-definable subset of $\mathbb{N}$ in $\mathfrak{A}$ is r.e.

Indeed, let A be Σ_1-definable subset of $\mathbb{N}$ in the structure $\mathfrak{A}$, i. e. there exists a recursive sequence $\{\Pi^{i,j}\}$ of termal predicates with free variables among $\boldsymbol{X}$, $\boldsymbol{Y}_j$ and for some fixed elements $\boldsymbol{s}$ of B the following equivalence is true:

$$i \in A \iff \text{there exist } j \text{ and elements } \boldsymbol{t}_j \text{ of } B \text{ such that } \Pi^{i,j}(\boldsymbol{X}/\boldsymbol{s}, \boldsymbol{Y}_j/\boldsymbol{t}_j) = 0.$$

Let us represent the formula $\Pi^{i,j}(\boldsymbol{X}, \boldsymbol{Y}_j)$ in the form $P^{i,j}(\boldsymbol{X}) \,\&\, Q^{i,j}(\boldsymbol{Y}_j)$, where some of them could be empty. There is an effective way to find $P^{i,j}(\boldsymbol{X})$ and $Q^{i,j}(\boldsymbol{Y}_j)$ from $\Pi^{i,j}(\boldsymbol{X}, \boldsymbol{Y}_j)$. Then keeping in mind that every closed consistent existential formula is true in $\mathfrak{A}$, we obtain

$$\begin{aligned} i \in A \iff & \text{ there exists } j \text{ such that } P^{i,j}(\boldsymbol{X}/\boldsymbol{s}) = 0 \\ & \,\&\, \text{ cotradictory conjunctive members do not join in } Q^{i,j}, \end{aligned}$$

i. e.

$$\begin{aligned} i \in A \iff & \exists j([X_1; P^{i,j}] \subseteq [s_1] \,\&\, \neg[X_1; P^{i,j}] \cap [s_1] = \varnothing \,\&\, \ldots \\ & \,\&\, [X_a; P^{i,j}] \subseteq [s_a] \,\&\, \neg[X_a; P^{i,j}] \cap [s_a] = \varnothing \\ & \,\&\, \text{contradictory members do not join in } Q^{i,j}). \end{aligned}$$

Therefore the set A is r.e. (Σ_1^0 set).

If Π is a disjunction of termal predicates with free variables among $X_1, \ldots, X_a$ ($\boldsymbol{X}$), $Y_1, \ldots, Y_b$ ($\boldsymbol{Y}$), $Z_1, \ldots, Z_c$ ($\boldsymbol{Z}$), and $s_1, \ldots, s_a$ ($\boldsymbol{s}$) are elements of B and $E_{v_1}, \ldots, E_{v_b}$ are finite sets, then by $\{\Pi; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\}$ we denote the formula with free variables among $\boldsymbol{Z}$, obtained from Π as follows:

a) If Π ia a termal predicate and $\Pi = P(\boldsymbol{X}, \boldsymbol{Y}) \,\&\, Q(\boldsymbol{Z})$, then $\{\Pi; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\} = Q'(\boldsymbol{Z})$, if

$$[X_1; \Pi] \subseteq [s_1] \,\&\, \neg[X_1; \Pi] \cap [s_1] = \varnothing \,\&\, \ldots \,\&[X_a; \Pi] \subseteq [s_a] \,\&\, \neg[X_a; \Pi] \cap [s_a] = \varnothing$$

$$\&\, [Y_1; \Pi] \subseteq E_{v_1} \,\&\, \neg[Y_1; \Pi] \cap E_{v_1} = \varnothing \,\&\, \ldots \,\&[Y_b; \Pi] \subseteq E_{v_b} \,\&\, \neg[Y_b; \Pi] \cap E_{v_b} = \varnothing$$

and $\{\Pi; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\} = T(Z_1) \,\&\, \neg T(Z_1)$, otherwise. Here $Q'(\boldsymbol{Z}) = Q(\boldsymbol{Z})$ if $Q(\boldsymbol{Z})$ is nonempty, and $Q'(\boldsymbol{Z}) = T(Z_1) \vee \neg T(Z_1)$, otherwise.

b) If $\Pi = (\Pi^1 \vee \Pi^2)$, then

$$\{\Pi; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\} = \{\Pi^1; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\} \vee \{\Pi^2; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\}.$$

There is an effective way to find $\{\Pi; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_b}\}$ from Π, $\boldsymbol{s}$ and $E_{v_1}, \ldots, E_{v_b}$.

Now let us note that if we have a disjunction of termal predicates Π with free variables among $\boldsymbol{X}$, $\boldsymbol{Z}$, then the formula $\forall \boldsymbol{Z}(\Pi(\boldsymbol{X}, \boldsymbol{Z}))$ is true in $\mathfrak{A}$ over $\boldsymbol{s}$ iff the formula $\{\Pi; \boldsymbol{s}\}$ (here $b = 0$) is a tautology in propositional logic, taking the atomic predicates as atoms. Here one has to keep in mind that for every atomic predicate there is an element which it is true in $\mathfrak{A}$ for and there is an element which it is not true in $\mathfrak{A}$ for. Besides there is an effective way to verify if given disjunction of conjunctions of atoms and their negations is a tautology or not.

Now let A be Σ_2-definable subset of $\mathbb{N}$ in the structure $\mathfrak{A}$, i. e. there exists a recursive sequence $\{\Pi^{i,j,k}\}$ of disjunctions of termal predicates with free variables among $\boldsymbol{X}$, $\boldsymbol{Y}_j$, $\boldsymbol{Z}_{j,k}$ and for some fixed elements $\boldsymbol{s}$ of B the following equivalence is true:

$$i \in A \iff \text{there exist } j \text{ and elements } \boldsymbol{t}_j \text{ of } B \text{ such that for any } k \text{ and for any elements } \boldsymbol{p}_{j,k} \text{ of } B\text{, the equation } \Pi^{i,j,k}(\boldsymbol{X}/\boldsymbol{s}, \boldsymbol{Y}_j/\boldsymbol{t}_j, \boldsymbol{Z}_{j,k}/\boldsymbol{p}_{j,k}) = 0 \text{ holds.}$$

Hence,

$$i \in A \iff \exists j \, \exists \text{ finite sequence } v_1, \ldots, v_{l_j} (v_1 \in [\boldsymbol{V}] \,\&\, \ldots \,\&\, v_{l_j} \in [\boldsymbol{V}] \,\&\, \forall k(\{\Pi^{i,j,k}; \boldsymbol{s}; E_{v_1}, \ldots, E_{v_{l_j}}\} \text{ is a tautology})).$$

Now it is obvious that A is Σ_2^0 set.

At the end let us assume that the structure $\mathfrak{A}$ admits an effective enumeration. Then the partial structure $\overline{\mathfrak{A}} = \langle B; \theta, \boldsymbol{F} \rangle$, where θ is the same as in $\mathfrak{A}$ and $\boldsymbol{F}$ is the restriction of F over those elements which obtain value 0, admits an effective enumeration too. Let us note again that $v \in [\boldsymbol{V}] \iff E_v$ is a type of some element of B in the structure $\overline{\mathfrak{A}}$. According to the main theorem in [5], the family $\{E_v \mid v \in [\boldsymbol{V}]\}$ has a universal r.e. set, which contradict to Theorem 1. So, $\mathfrak{A}$ does not admit effective enumerations.

Theorem 2′. *For any $A \subseteq \mathbb{N}$ there exists a structure $\mathfrak{A}_A = \langle B; \theta; F \rangle$ with unary θ, F, which does not admit an effective enumeration (even effective in A) but all Σ_1-definable in $\mathfrak{A}_A$ subsets of $\mathbb{N}$ are r.e. in A and all Σ_2-definable in $\mathfrak{A}_A$ subsets of $\mathbb{N}$ are $\Sigma_2^0[A]$ sets in the arithmetical hierarchy.*

Let us note that in fact we obtain also the following

Theorem 3. *There exists an infinite class of structures*

$$\{\mathfrak{A}_A \mid A \subseteq \mathbb{N} \,\&\, \mathfrak{A}_A = \langle B; \theta; F \rangle\}$$

with unary θ, F, and decidable first order theory (without constants of the structure) such that for any A, $\mathfrak{A}_A$ does not admit an effective (even effective in A) enumeration.

So, the problem of characterization of those structures which admit an effective enumeration is still open.

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