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FACTORIZATIONS OF THE GROUPS $PSU_6(q)$ *

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Цанко Генчев, Еленка Генчева. ФАКТОРИЗАЦИИ ГРУПП $PSU_6(q)$

Доказан следующий результат:

Пусть $G = PSU_6(q)$ и $G = AB$, где A, B — собственные неабелевые простые подгруппы G. Тогда имеет место одно из следующих:

(1) $q = 2$ и $A \cong M_{22}$, $B \cong PSU_5(2)$;
(2) $q = 2$ и $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;
(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ и $A \cong PSU_5(q)$, $B \cong G_2(q)$;
(4) $q \not\equiv -1 \pmod 5$ и $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

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The following result is proved.

Let $G = PSU_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:

(1) $q = 2$ and $A \cong M_{22}$, $B \cong PSU_5(2)$;
(2) $q = 2$ and $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;
(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ and $A \cong PSU_5(q)$, $B \cong G_2(q)$;
(4) $q \not\equiv -1 \pmod 5$ and $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

1. INTRODUCTION

In [4], the first author determined all the factorizations (with two proper simple subgroups) of the groups of Lie type of Lie rank 1 or 2. In the present paper we extend this investigation to groups of Lie type of Lie rank 3. Let $PSU_6(q)$ be

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the simple group of Lie type $(^2A_5)$ over the finite field $GF(q^2)$. Assuming the classification of the finite simple groups, we prove the following result.

Theorem. *Let $G = PSU_6(q)$ and $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 2$ *and* $A \cong M_{22}$, $B \cong PSU_5(2)$;

(2) $q = 2$ *and* $A \cong PSU_4(3)$, $B \cong PSU_5(2)$;

(3) $q = 2^n > 2$, $n \not\equiv 2 \pmod 4$ *and* $A \cong PSU_5(q)$, $B \cong G_2(q)$;

(4) $q \not\equiv -1 \pmod 5$ *and* $A \cong PSU_5(q)$, $B \cong PSp_6(q)$.

The factorizations (1) – (4) *exist.*

The factorizations of the groups $PSU_6(q)$ into the product of two maximal subgroups have been determined in [9].

Throughout this paper we use standard group-theoretic notation. Simple group means non-Abelian simple group. $|G|_p$ denotes the order of a Sylow p-subgroup of a group G and $M(G)$ denotes the Schur multiplier of G. Next, A_n is the alternating group of degree n and $L_n(q)$, $U_n(q)$ stand for $PSL_n(q)$, respectively $PSU_n(q)$. Notation and basic information of the (known) simple groups can be found in [1], [2].

The factorizations of the groups $U_6(2)$ and $U_6(3)$ are determined in [5]. This gives (1), (2) and (4) (with $q = 2,3$) in the theorem. Thus we can assume that $q \geqq 4$.

The following lemmas are needed in the proof of the theorem.

Lemma 1.1 ([7]). *If q is odd, then $SL_6(q)$ does not contain an elementary Abelian subgroup of order 8 such that its involutions are conjugate.*

Lemma 1.2. *The group $U_6(q)$ contains a subgroup isomorphic to $U_5(q)$ if and only if $q \not\equiv -1 \pmod 5$.*

Proof. If $q \equiv -1 \pmod 5$, the group $U_5(q)$ has an elementary Abelian subgroup of order 25 all of whose non-identity elements are conjugate. On the other hand, it is easily checked that this is impossible in $U_6(q)$, so $U_6(q)$ cannot contain $U_5(q)$. If $q \not\equiv -1 \pmod 5$, the statement is clear.

If a, b are positive integers and $(a, b) = 1$, then $\mathrm{Ord}_a(b)$ denotes the multiplicative order of b modulo a (i. e. the least positive integer s with $b^s \equiv 1 \pmod a$).

Lemma 1.3 (see [8]). *Let q be a prime power and s a positive integer. Then there exists a prime r such that $\mathrm{Ord}_r(q) = s$ unless $s = 6$ and $q = 2$ or $s = 2$ and q a Mersenne prime.*

2. PROOF OF THE THEOREM

The group $G = PSU_6(q)$, $q = p^n$, p any prime, has order

$$q^{15}(q^2-1)(q^3+1)(q^4-1)(q^5+1)(q^6-1)/(6, q+1).$$

Using Lemma 1.3, choose a prime r such that $\mathrm{Ord}_r(p) = 10n$. Then $r = 10nt + 1$ for some $t \geqq 1$. Now $r \mid q^5 + 1$ and hence $r \mid |G|$. We shall suppose that $r \nmid |A|$. We next discuss the possibilities for A.

Let $A \cong A_l$ ($l \geqq 5$). Then $l \geqq r \geqq 11$. Hence A contains a subgroup X isomorphic to $L_2(8)$. As $M(L_2(8)) = 1$, X must embed into $SU_6(q^2)$ and hence into $SL_6(q^2)$. This contradicts Lemma 1.1 if p is odd. Thus $p = 2$. Set $n = 3^\alpha n_1$, where $3 \nmid n_1$. It is directly checked that if $3^\nu \mid |G|$ then $\nu \leqq 5\alpha + 7$. On the other hand, $|A|$ is a multiple of 3^μ, where

$$\mu \geqq \left[\frac{l}{3}\right] + \left[\frac{l}{9}\right] \geqq \left[\frac{10nt+1}{3}\right] + \left[\frac{10nt+1}{9}\right] \geqq 4.3^\alpha.n_1 t \geqq 4(2\alpha+2) > 5\alpha + 7$$

unless $n = t = 1$ or $n = 3$, $t = 1$. This forces $G = U_6(2)$ and $A \cong A_l$ ($l \geqq 11$) or $G = U_6(8)$ and $A \cong A_l$ ($l \geqq 31$). But $|A_{11}| \nmid |U_6(2)|$ and $|A_{31}| \nmid |U_6(8)|$, whence $|A| \nmid |G|$. This contradiction shows that $A \not\cong A_l$.

Let A be a sporadic group (${}^2F_4(2)'$ is excluded by $r \equiv 1 \pmod{10}$). If $p = 2$ then the choice of r implies $r = 11$ (and $n = 1$), $r = 41$ (and $n = 2$), $r = 61$ (and $n = 6$), or $r > 71$. Now it is easily verified that there is no sporadic group of order divisible by at least one of these primes and dividing the order of the corresponding $U_6(2^n)$ (recall $q = 2^n > 2$). So $p > 2$. Now if $A \not\cong M_{11}, M_{12}, M_{22}, HS, McL, Suz$ then (recall $|A|$ is divisible by a prime $r \equiv 1 \pmod{10}$). A contains a subgroup Y (possibly $Y = A$) with $|M(Y)|$ prime to 6 and Y contains an E_8 subgroup all of whose involutions are conjugate in Y (see [2]). This contradicts Lemma 1.1. Thus, we have proved that if A is sporadic then $A \cong M_{11}, M_{12}, M_{22}, HS, McL$ or Suz; hence $r = 11$ and consequently $n = 1$, that is, $G = U_6(p)$ for odd p.

Further, let A be a group of Lie type of characteristic $\neq p$. Now [6] leads to $A \cong L_2(11)$. Hence $r = 11$, so $G = U_6(p)$.

Finally, let A be a group of Lie type over the field $GF(q')$, $q' = p^m$ (see [1] for the orders of these groups). If $A \cong A_l(q')$ ($\cong L_{l+1}(q')$), $l \geqq 1$, then $r \mid |A|$ yields $r \mid q'^k - 1$ for some k, $2 \leqq k \leqq l+1$. Our choice of r then implies $10n \mid mk$. Choose a prime r_1 with $\mathrm{Ord}_{r_1}(p) = mk$. This is possible by Lemma 1.3 and $r_1 \mid |G|$ (as obviously $r_1 \mid |A|$). Since $mk \geqq 10n$, necessarily $r_1 \mid q^5 + 1$, i. e. $r_1 \mid q^{10} - 1$ and then the choice of r_1 leads to $mk \mid 10n$. Thus $10n = mk$, $2 \leqq k \leqq l+1$. Now as $|A|_p \leqq |G|_p = p^{15n}$, we have $ml(l+1)/2 \leqq 15n = 3mk/2 \leqq 3m(l+1)/2$. It follows that $l = 1$, $m = 5n$, or $l = 2$, $m = 5n$ or $m = 10n/3$ ($3 \mid n$), or $l = 3$, $m = 5n/2$ ($2 \mid n$), that is, $A \cong L_2(q^5)$, $L_3(q^5)$, $L_3(q^{10/3})$, or $L_4(q^{5/2})$. However, then $|A| \nmid |G|$.

If $A \cong B_l(q')$ ($\cong P\Omega_{2l+1}(q')$) or $A \cong C_l(q')$ ($\cong PSp_{2l}(q')$), $l \geqq 2$, similar arguments produce $A \cong PSp_4(q^{5/2})$ ($2 \mid n$), $PSp_6(q^{5/6})$, or $P\Omega_7(q^{5/3})$ ($3 \mid n$) and again $|A| \nmid |G|$.

Let $A \cong D_l(q')$ or ${}^2D_l(q')$, $l \geqq 4$. It follows (just as above) that $10n = mk$ for some k, $2 \leqq k \leqq 2l$. But then $ml(l-1) \leqq 15n = 3mk/2 \leqq 3ml$ (as $|A|_p \leqq |G|_p$) which forces $l = 4$, $k = 8$, i. e. $A \cong D_4(q^{5/4})$ or ${}^2D_4(q^{5/4})$ ($4 \mid n$) and $|A| \nmid |G|$.

If $A \cong G_2(q')$, we have similarly $6m \leqq 15n$ and $|A|$ then shows that $r \mid q'^6 - 1$ which yields $10n \mid 6m$. Thus $m = 5n/3$ ($3 \mid n$) and $A \cong G_2(q^{5/3})$. But then $|A| \nmid |G|$.

If $A \cong Sz(q')$ (m odd > 1), we have $2m \leqq 15n$ and hence r must divide $q'^2 + 1$ which yields $10n \mid 4m$. Thus $m = 5n$, $5n/2$ or $15n/2$ ($2 \mid n$), that is, $A \cong Sz(q^5)$, $Sz(q^{5/2})$ or $Sz(q^{15/2})$ and again $|A| \nmid |G|$.

Let $A \cong {}^3D_4(q')$ or $A \cong {}^2F_4(q')$ (and m is odd > 1). Then $12m \leq 15n$ and the choice of r implies that $r \mid q'^8+q'^4+1$ if $A \cong {}^3D_4(q')$ and $r \mid q'^6+1$ if $A \cong {}^2F_4(q')$. In either case $r \mid q'^{12}-1$ whence $10n \mid 12m$. Thus $m = 5n/6$ $(6 \mid n)$ and $A \cong {}^3D_4(q^{5/6})$ or ${}^2F_4(q^{5/6})$. But then $|A| \nmid |G|$.

Let $A \cong {}^2G_2(q')$ $(p = 3,\ m > 1,\ 2 \nmid m)$. Then $3m \leq 15n$ and hence r must divide q'^3+1 which yields $10n \mid 6m$. Thus $m = 5n$ and $A \cong {}^2G_2(q^5)$ or $m = 5n/3$ $(3 \mid n)$ and $A \cong {}^2G_2(q^{5/3})$. However, then $|A| \nmid |G|$.

Let $A \cong F_4(q'),\ E_6(q'),\ E_7(q'),\ E_8(q')$, or ${}^2E_6(q')$. Then $|A|_p \leq p^{15n}$ means $15n \geq 24m,\ 36m,\ 63m,\ 120m$, or $36m$, respectively. But $r \mid |A|$ implies $r \mid q'^k - 1$ for some k, where $k \leq 12, 12, 18, 30$, or 18, respectively. This leads to $10n \leq km$, i. e. $15n \leq 3mk/2 \leq 18m,\ 18m,\ 27m,\ 45m$, or $27m$, respectively, a contradiction.

Let $A \cong {}^2A_l(q')$ $(\cong U_{l+1}(q'))$, $l \geq 2$. Now $r \mid q'^k + (-1)^{k-1}$ for some k, $2 \leq k \leq l+1$. If k is even then $r \mid q'^k - 1$ which leads to $10n \mid mk$. As in the case $A \cong A_l(q')$, we have also $mk \mid 10n$. Now $ml(l+1)/2 \leq 15n = 3mk/2 \leq 3m(l+1)/2$ whence necessarily $l = 2$, or $l = 3$ and $k = 2$ or $k = 4$, respectively. But then $A \cong U_3(q^5)$ or $A \cong U_4(q^{5/2})$ $(2 \mid n)$ and again $|A| \nmid |G|$. Thus k is odd and $r \mid q'^k+1$ whence $r \mid q'^{2k} - 1$. This leads to $2mk = 10n$ and then $ml(l+1)/2 \leq 15n = 3mk \leq 3m(l+1)$ yields $l = 2$, $m = 5n/3$ $(3 \mid n)$, or $l = 3$, $m = 5n/3$ $(3 \mid n)$, or $l = 4$, $m = n$, or $l = 5$, $m = n$, or $l = 6$, $m = 5n/7$ $(7 \mid n)$. Accordingly, $A \cong U_3(q^{5/3}),\ U_4(q^{5/3}),\ U_5(q),\ U_6(q)$, or $U_7(q^{5/7})$. In each case either $|A| \nmid |G|$, or $A = G$, or $A \cong U_5(q)$.

Thus we are reduced to the possibilities $G = U_6(p)$, p odd, $A \cong L_2(11)$, M_{11}, M_{12}, M_{22}, HS, McL, or Suz and $G = U_6(q)$, $A \cong U_5(q)$. In the first case $\mathrm{Ord}_{11}(p) = 10$ implies that $p \neq 3, 5, 11$. It follows that in any case p^{14} divides $|B|$. We shall prove that there is no possibility for B.

Indeed, $B \not\cong A_l$ as otherwise l must be too large and then we reach a contradiction just as in the case $A \cong A_l$ above. B is not isomorphic to a sporadic group because if p is odd, $p \neq 3, 5, 11$, then p^{14} does not divide the order of any sporadic group. Further, [6] shows that B cannot be a group of Lie type of characteristic $\neq p$. Finally, if B is of Lie type of characteristic p then (checking the orders of these groups) we conclude that $B \cong L_2(p^{14}),\ L_2(p^{15}),\ L_3(p^5),\ L_6(p),\ U_3(p^5)$, or $U_6(p)$. However, then either $|B| \nmid |G|$ or $B = G$, an impossibility.

In the second case $A \cong U_5(q)$. Choose a prime r_1 such that $\mathrm{Ord}_{r_1}(p) = 6n$ (the choice is possible as $q \geq 4$). As $r_1 \mid q^6 - 1$ and as $q^6 - 1$ divides $|B : A \cap B| = |G : A| = q^5(q^6-1).(5, q+1)/(6, q+1)$, it follows that $r_1 \mid |B|$. Now we again consider the possibilities for B (as for A above) taking also into account that $q^5(q^6-1) \mid |B|$. This leads to $B \cong L_3(q^2),\ PSp_6(q),\ P\Omega_7(q)$ $(2 \nmid q)$, or $G_2(q)$.

Note that if q is odd then G does not contain a subgroup isomorphic to $G_2(q)$ and consequently a subgroup isomorphic to $P\Omega_7(q)$ (as $P\Omega_7(q)$ contains $G_2(q)$). Indeed, this follows from Lemma 1.1, as $B \cong G_2(q)$ (q odd) has Schur multiplier of order prime to 6, 2-rank three and only one conjugacy class of involutions.

Now, if $B \cong G_2(q)$ $(q = 2^n > 2)$ or $B \cong PSp_6(q)$ we reach (3) or (4) of the theorem.

Lastly, let $A \cong U_5(q)$, $B \cong L_3(q^2)$. Denote $D = A \cap B$; then $|D| = q(q^4-1).(6,q+1)/(3,q^2-1)$ (recall $(5,q+1)=1$). By the known subgroup structure of $L_3(q^2)$, it follows that D is contained in a subgroup of B isomorphic to

$$H = \left\{ \left(\begin{array}{c|cc} a & b & c \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a,b,c \in GF(q^2);\ A \in GL_2(q^2),\ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle,$$

where ω is an element of order $(3, q^2-1)$ in $GF(q^2)$. Further, $H = FK$ and $F \triangleleft H$, $F \cap K = 1$ where

$$F = \left\{ \left(\begin{array}{c|cc} 1 & b & c \\ \hline 0 & & \\ & & E \\ 0 & & \end{array} \right) \mid b,c \in GF(q^2) \right\} \cong E_{q^4},$$

$$K = \left\{ \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & & \\ & & A \\ 0 & & \end{array} \right) \mid a \in GF(q^2);\ A \in GL_2(q^2),\ a.\det A = 1 \right\} \Big/ \langle \omega E \rangle$$

$$\cong GL_2(q^2)/\mathbb{Z}_{(3,q^2-1)}.$$

Suppose that $T = D \cap F \neq 1$. Then $T \triangleleft D$ and $T \cong E_{p^k}$, where $p \leqq p^k \leqq q$. The centralizer of any non-identity p-element in $L_3(q^2)$ has order dividing $q^6(q^2-1)$. Hence $|C_D(T)|$ divides $q(q^2-1).(6,q+1)/(3,q^2-1)$. Then $|D/C_D(T)|$ is divisible by q^2+1. However, $D/C_D(T)$ is a subgroup of $\mathrm{Aut}(T) \cong GL_k(p)$, so

$$|GL_k(p)| = p^{k(k-1)/2}(p-1)\ldots(p^k-1)$$

must be divisible by q^2+1 which (in view of $p^k \leqq q$) contradicts Lemma 1.3.

Thus $D \cap F = 1$ and hence D is isomorphic to a subgroup of $H/F \cong K$. Of course, K contains a subgroup $L \cong SL_2(q^2)$ of index $(q^2-1)/(3,q^2-1)$ and then $D \cap L$ is a proper subgroup of L of order divisible by $q(q^2+1).(6,q+1)$. It follows that $L_2(q^2)$ has a proper subgroup of order divisible by $q(q^2+1)$ which (for $q \geqq 4$) contradicts the structure of $L_2(q^2)$. ∎

It remains to show that the factorizations in (3) and (4) actually exist. From [7, Proposition 3.3] we have

$$SU_6(q^2) = SU_5(q^2).Sp_6(q)$$

with "natural" embeddings of $SU_5(q^2)$ and $Sp_6(q)$ in $SU_6(q^2)$. Factoring out by $Z(SU_6(q^2))$, we obtain the factorization in (4), as $SU_5(q^2) \equiv U_5(q)$ (by Lemma 1.2).

Now we prove the existence of the factorization in (3) of the theorem. We use the following two realizations of the group $SU_6(q^2)$:

(i) $$SU_6(q^2) = \left\{X \in GL_6(q^2) \mid \overline{X}^t I X = I,\ \det X = 1\right\},$$

$$I = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix};$$

(ii) $$SU_6(q^2) = \left\{Y \in GL_6(q^2) \mid \overline{Y}^t Y = E,\ \det Y = 1\right\}.$$

(Here, if $D = (d_{ij})$ is a matrix with entries in $GF(q^2)$ then $\overline{D} = (d_{ij}^q)$ and D^t is the transpose of D.)

Let X, $Y \in GL_6(q^2)$ and $Y = T^{-1}XT$, where

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 & 0 & t^q \\ 0 & t & 0 & 0 & t^q & 0 \\ 0 & 0 & t & t^q & 0 & 0 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 0 & 0 & t^q & 1 & 0 & 0 \\ 0 & t^q & 0 & 0 & 1 & 0 \\ t^q & 0 & 0 & 0 & 0 & 1 \\ t & 0 & 0 & 0 & 0 & 1 \\ 0 & t & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 1 & 0 & 0 \end{pmatrix}$$

$$(t \in GF(q^2),\ t + t^q = 1).$$

Then $\overline{Y}^t Y = E$ if and only if $\overline{X}^t I X = I$.

Now, with respect to (ii), we have

$$\left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & 1 \end{array} \right) \in SU_6(q^2) \right\} \cong SU_5(q^2).$$

On the other hand (see [3]), with respect to (i), a $G_2(q)$ subgroup of $SU_6(q^2)$ is generated by the matrices $X_{\pm r}(t)$, $r \in \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$, $t \in GF(q)$, where

$$X_a(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_b(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{a+b}(t) = \begin{pmatrix} 1 & 0 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{2a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$X_{3a+b}(t) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad X_{3a+2b}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & t & 0 \\ 0 & 1 & 0 & 0 & 0 & t \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix $X_{-r}(t)$ is the transpose of $X_r(t)$. Now a direct computation shows that the common elements of the above $SU_5(q^2)$ and $G_2(q)$ subgroups are exactly as follows:

$$T^{-1}.\begin{pmatrix} v & 0 & 0 & 0 & v^{-1}s & 0 \\ 0 & v & 0 & 0 & 0 & v^{-1}s \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & v^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & v^{-1} \end{pmatrix}.T \quad (v \in GF(q)^*,\ s \in GF(q)),$$

$$T^{-1}.\begin{pmatrix} u^{-1}l & 0 & 0 & 0 & u+u^{-1}lk & 0 \\ 0 & u^{-1}l & 0 & 0 & 0 & u+u^{-1}lk \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ u^{-1} & 0 & 0 & 0 & u^{-1}k & 0 \\ 0 & u^{-1} & 0 & 0 & 0 & u^{-1}k \end{pmatrix}.T$$

$$(u \in GF(q)^*;\ l, k \in GF(q)).$$

Hence $|SU_5(q^2) \cap G_2(q)| = q(q^2-1)$ (in fact, $SU_5(q^2) \cap G_2(q) \cong L_2(q)$). Now order considerations imply

$$SU_6(q^2) = SU_5(q^2).G_2(q)$$

whence (again by Lemma 1.2) the factorization in (3) follows.

This completes the proof of the theorem.

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