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ON REITERATION BETWEEN FAMILIES OF BANACH SPACES

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Людмила Николова, Ларс Ерик Персон. О РЕИТЕРАЦИИ В СЕМЕЙСТВЕ БАНАХОВЫХ ПРОСТРАНСТВ

Статья начинается с кратким обзором теории вещественной и комплексной интерполяции между конечным числом и даже в семействе банаховых пространств. Представлены некоторые теоремы реитерации и сделано применение этих теорем, например дано интерполяционное доказательство версии неравенства Гельдера для семейства X^p-пространств и рассмотрена техника помагающая характеризировать интерполяционные пространства в некоторых конкретных семействах банаховых пространств. В конце имеются примеры, нерешенные проблемы и рассматривается отношение представленных результатов к другим связанным с этой тематикой результатов.

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In this paper we briefly discuss some developments concerning real and complex interpolation between finite many or even between general families of Banach spaces. Moreover, we present and apply some reiteration theorems. In particular, we give an interpolation proof of a version of Hölder's inequality for families of X^p-spaces and a technique to characterize the interpolation spaces between some concrete families of Banach spaces. Some examples, open questions, and the relations to other connected results are pointed out.

0. INTRODUCTION

In the theory and applications of interpolation spaces we usually consider a Banach couple (A_0, A_1), i. e. A_0 and A_1 are Banach spaces, which are embedded in a Hausdorff topological vector space U. There are several studied constructions for obtaining interpolation spaces with respect to the couple (A_0, A_1) and the most well-studied and applied such methods are the complex method $[A_0, A_1]_\theta$ $(0 \leqq \theta \leqq 1)$ and the real method $(A_0, A_1)_{\theta,q}$ $(0 < \theta < 1,\ 0 < q \leqq \infty)$. See e. g. the books [1–3, 16, 25, 33] and also the Bibliography of Maligranda [18] (including approximately 2500 references).

Parts of the theory concerning interpolation between two Banach spaces can be generalized to cover also cases where we interpolate between finite many Banach spaces and even between general families of Banach spaces. Here we mention the following developments:

1) A theory for *complex interpolation* between families of Banach spaces was developed by Coifman–Cwikel–Rochberg–Sagher–Weiss (see [6–8]) and, independently, by Krein–Nikolova (see [14, 15]). Another complex interpolation method between n-tuples of Banach spaces was suggested by Lions [17] and studied in detail by Favini [10]. This method of Favini–Lions was extended by Cwikel–Janson [9] to cover also complex interpolation between very general families of spaces.

2) A theory for *real interpolation* between n-tuples of Banach spaces was introduced and studied by Sparr [31]. A similar theory for real interpolation between 2^k-tuples of Banach spaces was studied by Fernandez [11]. In this connection we also mention early works by Yoshikawa, Kerzman and Foias–Lions (cf. the discussion in [31, p. 248]). Moreover, Cobos–Peetre [5] have developed a theory for real interpolation between finite many Banach spaces, which, in particular, covers both Sparr's and Fernandez' constructions for the cases $n = 3$ and $n = 4$, respectively. The construction of Sparr was extended by Cwikel–Janson [9] to cover also real interpolation between a fairly general family $A = \{A_t\}_{t\in\Gamma}$, where A_t are Banach spaces and Γ is a general probability measure space. Some very new constructions have recently been introduced by Carro [4] and the present authors [21] (see also [20] and [22]).

In this paper we discuss and apply some reiteration results for families of Banach spaces. In Section 1 we present some well-known reiteration theorems. In Section 2 we expose an interpolation proof of a version of Hölder's inequality for families of X^p-spaces (cf. [23]). In Section 3 we point out a technique to characterize the interpolation spaces between families of Banach Spaces in several cases of practical interest. Some concrete examples are given and discussed. Finally, Section 4 is reserved for some further examples, concluding remarks and open questions.

1. SOME REITERATION RESULTS

In the sequel we let D and $P(z,t)$, $z \in D$, $0 \leqq t < 2\pi$, denote the open unit disc and the Poisson kernel, respectively. Moreover, we let $\alpha = \alpha(e^{it})$, $p = p(e^{it})$ and $q = q(e^{it})$ denote measurable functions on $[0, 2\pi)$ such that $0 < \alpha(e^{it}) \leqq 1$ and

$p(e^{it}), q(e^{it}) \geqq 1$. The functions $\alpha(z)$, $p(z)$ and $q(z)$, $z \in D$, are defined by

$$\text{(1.1)} \qquad \alpha(z) = \frac{1}{2\pi}\int_0^{2\pi} \alpha(e^{it})P(z,t)\,dt, \quad \frac{1}{p(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(e^{it})}P(z,t)\,dt,$$

$$\frac{1}{q(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{q(e^{it})}P(z,t)\,dt,$$

respectively. $B = \{B_t\}$, $t \in [0, 2\pi]$, denotes an *interpolation family* in the sense of R. R. Coifman et al. [6] (see also [7, 8]).

First we consider the complex interpolation spaces $[B]_z$ and $\{B\}_z$ (see [6–8]). The following reiteration theorem ([8, Theorem 5.1]) may be regarded as a genuine generalization of the "complex" reiteration theorem (see e. g. [2, Theorem 4.6.1]):

Theorem A. *Let* (A_0, A_1) *be a compatible Banach couple and let* $B_t = [A_0, A_1]_{\alpha(e^{it})}$. *Then* $B = \{B_t\}$, $t \in [0, 2\pi]$, *is an interpolation family and, for any* $z \in D$, $[B]_z \equiv [A_0, A_1]_{\alpha(z)}$ *(with equality of norms).*

Remark. If there exist $t_0, t_1 \in [0, 2\pi]$ such that $\alpha(e^{it_0}) = \sup \alpha(e^{it})$ and $\alpha(e^{it_1}) = \inf \alpha(e^{it})$, then it is well-known that $[B]_z \equiv \{B\}_z$.

Next we consider the following "mixed" reiteration result of Hernandez [12, Main Theorem] (cf. [2, Theorem 4.7.2]):

Theorem B. *Assume that* (A_0, A_1) *is a compatible Banach couple and let* $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$. *Then* $B = \{B_t\}$, $t \in [0, 2\pi]$, *is an interpolation family. Moreover, if*

$$\text{(1.2)} \quad \int_0^{2\pi} \frac{1}{q(e^{it})}\,dt > 0, \ \int_0^{2\pi} \log \alpha(e^{it})\,dt > -\infty, \ \textit{and} \ \int_0^{2\pi} \log\left(1 - \alpha(e^{it})\right)dt > -\infty,$$

then, for any $z \in D$, $[B]_z = (A_0, A_1)_{\alpha(z), q(z)}$ *(with equivalence of norms).*

Now we consider the real interpolation spaces $\{B\}^S_{z,p,q}$ (the K-method) and $(B)^S_{z,p,q}$ (the J-method) recently introduced by Carro [4] and state her reiteration result (cf. [2, Theorems 3.5.1 and 3.5.4]).

Theorem C. *Let* $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$, *where* (A_0, A_1) *is a compatible Banach couple. If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbb{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then, for any* $z \in D$,

$$\{B\}^S_{z,q,1} = (A_0, A_1)_{\alpha(z),q} = (B)^S_{z,q,r}.$$

Remark. The method of proof in [4] ensures the first equality only with $q = 1$, but it is true also for any $q \geqq 1$ whenever $\{B\}^S_{z,p,q}$ is continuously imbedded in $A_0 + A_1$.

Remark. In [4] there is also stated a complex variant of Theorem C, where $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$ is replaced by $B_t = [A_0, A_1]_{\alpha(e^{it})}$.

2. REITERATION AND HÖLDER'S INEQUALITY FOR FAMILIES OF X^p SPACES

Let X denotes a Banach lattice on a σ-finite complete measure space Ω. The space X^p, $p \geqq 1$, consists of all $x = x(t)$, $t \in \Omega$, such that $|x|^p \in X$ and equipped with the norm $\|x\|_{X^p} = (\| |x(t)|^p \|_X)^{1/p}$. First we recall the following result (see [23]):

Proposition 2.1. *Assume that the Banach lattice X has the dominated convergence property. Then*

$$\left[X^{p(e^{it})}\right]_z \equiv X^{p(z)}.$$

For the case $X = L_1(\mu)$ see [8, Corollary 5.2]. Proposition 2.1 is proved in [23, p. 96] and here we present an alternative proof based upon Theorem A.

Proof. It is well-known that $[X, L_\infty]_\theta \equiv X^{1/(1-\theta)}$. Therefore we have $X^{p(e^{it})} \equiv [X, L_\infty]_{\alpha(t)}$, where $\alpha(t) = 1 - 1/p(e^{it})$, and, thus, according to Theorem A,

$$\left[X^{p(e^{it})}\right]_z \equiv \left[[X, L_\infty]_{\alpha(t)}\right]_z \equiv [X, L_\infty]_{\alpha(z)} \equiv [X, L_\infty]_{1-1/p(z)} \equiv X^{p(z)}.$$

The proof is complete.

Next we recall the following generalization of Hölder's inequality (see [29]): If $x_j \in X^{p_j}$, $1 \leqq p_j < \infty$, $j = 0, 1, \ldots, N$, then

$$\prod_0^N x_j \in X^r, \quad \text{where} \quad \frac{1}{r} = \sum_0^N x_j \quad \text{and} \quad \|\prod_0^N x_j\|_{X^{p_i}} \leqq \prod_0^N \|x_j\|_{X^{p_i}}. \tag{2.1}$$

By using the interpolation result in Proposition 2.1 we can prove the following extention of this inequality to the case with families of X^pspaces:

Theorem 2.2. *Let X be a Banach lattice having the dominated convergence property. If $x_t \in X^{p(e^{it})}$ and $\log|x_t(s)| \in L[0, 2\pi)$ for $s \in \Omega$, $x_t(s)$ is measurable on $[0, 2\pi] \times \Omega$ and $\log \|x_t\|_{X^{p(e^{it})}} \in L[0, 2\pi)$, then*

$$x = x_z(s) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|x_t(s)| P(z,t)\, dt\right) \in X^{p(z)} \quad \textit{and}$$

$$\|x\|_{X^{p(z)}} \leqq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\|x_t(s)\|_{X^{p(e^{it})}} P(z,t)\, dt\right). \tag{2.2}$$

Remark. We note that (2.2) coincides with (2.1) if $z = 0$ and, for $0 \leqq a_0 < \ldots < a_N = 1$, $E_j = [a_{j-1}, a_j]$, $p(e^{it}) = p_j/m(E_j)$, and $x_t(s) = |y_t(s)|^{1/m(E_j)}$, $t \in 2\pi E_j$, $j = 0, 1, \ldots, N$.

Remark. A somewhat different version of Theorem 2.2 was proved in [23] (cf. also [24]) but the proof we present below is simpler and completely different.

Proof (by interpolation). We adopt the notations and definitions of [23]. In particular $[B]_z$ and $[B]^z$ denote the interpolation spaces of Coifman–Cwikel–Rochberg–Sagher–Weiss and the generalized Calderon construction for the family $B = \{B_t\}$,

$t \in [0, 2\pi]$, of Banach spaces, respectively. In view of Proposition 2.1 we have $\left[X^{p(e^{it})}\right]^{\cdot}_{z} \equiv X^{p(z)}$. Moreover, according to Hernandez [13, Theorem 6.1] it yields that $\left[X^{p(e^{it})}\right]_{z} \equiv \left[X^{p(e^{it})}\right]^{z}$. Now we consider

$$y_z(s) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log|y_t(s)| P(z,t)\,dt\right), \quad \text{where } \|y_t(s)\|_{X^{p(e^{it})}} \leqq 1.$$

Then

$$\|y_z(s)\|_{X^{p(z)}} = \|y_z(s)\|_{[X^{p(e^{it})}]_z} = \|y_z(s)\|_{[X^{p(e^{it})}]^z} \leqq 1. \tag{2.3}$$

Finally, we obtain (2.2) by inserting

$$y_t(s) = x_t(s)/\|x_t(s)\|_{X^{p(e^{it})}} \quad \text{and} \quad x_z(s) = \exp\left(\frac{1}{2\pi}\log|x_t(s)| P(z,t)\,dt\right)$$

into (2.3).

3.REITERATION AND CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN FAMILIES OF BANACH SPACES

Let $B = \{B_t\}$ be an interpolation family of Banach spaces. Inspired by the proof of Proposition 2.1 we find that it is possible to characterize $[B]_z$ for several concrete interpolation families of Banach spaces by using the following technique:

(a) Write B_t, if possible, as $B_t = [A_0, A_1]_{\alpha(e^{it})}$ or as $B_t = (A_0, A_1)_{\alpha(e^{it}), q(e^{it})}$, where (A_0, A_1) denotes a compatible Banach couple.

(b) Calculate $\alpha(z)$, or $\alpha(z)$ and $q(z)$ (see (1.1)).

(c) Calculate $[A_0, A_1]_{\alpha(z)}$ or $(A_0, A_1)_{\alpha(z), q(z)}$.

(d) Apply Theorem A or Theorem B, respectively.

By using steps (a)–(c) together with Theorem C we can describe the real spaces $\{B\}^S_{z,p,1}$ and $(B)^S_{z,p,q}$ in a similar way.

Nowadays we have very good knowledge concerning the possibilities to carry out the crucial steps (a) and (c) in several cases of practical importance (see e. g. [1–3, 16, 25, 33] and the Bibliography of Maligranda [18]). Here we only give some examples of applications of this technique (cf. also [4, 8]).

3.1. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF LORENTZ $L_{p,q}$-SPACES

Proposition 3.1. *Let* $1 < p(e^{it}) < \infty$ *and* $1 \leqq q(e^{it}) \leqq \infty$. *Then* $B = \left\{L_{p(e^{it}), q(e^{it})}\right\}$, $t \in [0, 2\pi]$, *is an interpolation family of Banach spaces.*

(a) *If the condition* (1.2) *is satisfied with* $\alpha(e^{it}) = 1 - 1/p(e^{it})$, *then* $[B]_z = L_{p(z), q(z)}$.

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = L_{p(z), q}$.

Proof. We use the well-known equality

$$L_{p(e^{it}),q(e^{it})} = (L_1, L_\infty)_{\alpha(e^{it}),q(e^{it})}, \quad \text{where } \alpha(e^{it}) = 1 - 1/p(e^{it}).$$

We also note that $\alpha(z) = 1 - 1/p(z)$ (see (1.1)). Therefore, according to Theorem B,

$$[B]_z = [L_1, L_\infty]_{\alpha(z),q(z)} = [L_1, L_\infty]_{1-1/p(z),q(z)} = L_{p(z),q(z)}.$$

Moreover, by using Theorem C, we find that

$$\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = (L_1, L_\infty)_{\alpha(z),q} = L_{p(z),q},$$

and the proof is complete.

Remark. The first part of Proposition 3.1 is due to Hernandez [12, p. 81]. For the case $p(e^{it}) = q(e^{it})$ we can use Theorem A and a complex variant of Theorem C to find that Proposition 3.1 (b) holds also if we permit that $p(e^{it}) = 1$ or $p(e^{it}) = \infty$. In particular, for any $p(e^{it})$, $q(e^{it}) \geqq 1$ and $r \geqq 1$, it holds that

$$\{\{L_{p(e^{it})}\}\}^S_{z,p(z),1} = (\{L_{p(e^{it})}\})^S_{z,p(z),r} = L_{p(z)}.$$

3.2. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF OPERATOR IDEAL SPACES

Let A_0 and A_1 denote Banach spaces and let $L(A_0, A_1)$ denotes the space of bounded linear operators between A_0 and A_1. We can give more precize information about the operator T by considering the approximation numbers $a_n(T)$, $n = 1, 2, \ldots$, where

$$a_n(T) = \inf\{\|T - T_n\| \mid \operatorname{rank} T_n < n,\ T_n \in L(A_0, A_1)\}.$$

We say that $T \in \sigma_{p,q}$, $1 \leq p, q \leqq \infty$, if

$$\|T\|_{\sigma_{p,q}} = \left(\sum_1^\infty \left(a_n(T) n^{1/p}\right)^q \frac{1}{n}\right)^{1/q} < \infty,$$

with the usual supremum interpretation of the sum when $q = \infty$. The space $\sigma_{p,p}$ coincides with the usual Schatten–von Neumann class σ_p.

Proposition 3.2. *Let* $1 \leq p(e^{it}) \leqq \infty$. *Then* $B = \{\sigma_{p(e^{it})}\}$, $t \in [0, 2\pi)$, *is an interpolation family of Banach spaces and* $[B]_z \equiv \sigma_{p(z)}$.

Proof. We use the formula $[\sigma_1, \sigma_\infty]_\theta \equiv \sigma_{1/(1-\theta)}$, $0 \leqq \theta \leqq 1$, (see [30]) and find that

$$\sigma_{p(e^{it})} \equiv [\sigma_1, \sigma_\infty]_{1-1/p(e^{it})}.$$

If $\alpha(e^{it}) = 1 - 1/p(e^{it})$, then $\alpha(z) = 1 - 1/p(z)$. Hence, by using Theorem A, we obtain that

$$[B]_z \equiv [\sigma_1, \sigma_\infty]_{\alpha(z)} \equiv [\sigma_1, \sigma_\infty]_{1-1/p(z)} \equiv \sigma_{p(z)}.$$

Proposition 3.3. *Let* $1 < p(e^{it}) < \infty$ *and* $1 \leqq q(e^{it}) \leqq \infty$. *Then* $B = \{\sigma_{p(e^{it}),q(e^{it})}\}$, $t \in [0, 2\pi)$, *is an interpolation family of Banach spaces.*

(a) *If the condition* (2.1) *is satisfied with* $\alpha(e^{it}) = 1 - 1/p(e^{it})$, *then* $[B]_z = \sigma_{p(z),q(z)}$.

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbb{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,q,1} = (B)^S_{z,q,r} = \sigma_{p(z),q}$.

Proof. It is well-known that $(\sigma_1, \sigma_\infty)_{\theta,q} \equiv \sigma_{1/(1-\theta),q}$, $0 < \theta < 1$, $1 \leqq q \leqq \infty$ (see [32]),

$$\sigma_{p(e^{it}),q(e^{it})} = (\sigma_1, \sigma_\infty)_{\alpha(e^{it}),q(e^{it})}, \quad \text{where } \alpha(e^{it}) = 1 - 1/p(e^{it}).$$

Therefore the proof follows by arguing exactly as in the proof of Proposition 3.1.

3.3. CHARACTERIZATION OF INTERPOLATION SPACES BETWEEN SOME FAMILIES OF BESOV SPACES

In this section we assume that the reader is acquainted with Peetre's abstract definitions of the Sobolev spaces H^s_p and the Besov spaces $B^s_{p,q}$ (see [2, 25]). It is a difficult task to describe the real interpolation spaces between Besov spaces for the general (off-diagonal) case (see [25, p. 110] and [26, p. 228]). However, for some special cases we have suitable such interpolation formulas (see [2, Theorem 6.4.5]), e. g. the following ones: Let $-\infty < s_0 < s_1 < \infty$, $0 < \alpha < 1$, $s = (1-\alpha)s_0 + \alpha s_1$ and $1 \leqq p, q \leqq \infty$, then

$$B^s_{p,p} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha,p}, \quad \alpha = 1 - 1/p, \tag{3.1}$$

$$B^s_{p,q} = \left(H^{s_0}_p, H^{s_1}_p\right)_{\alpha,q}. \tag{3.2}$$

We also introduce another measurable function $s(e^{it})$ on $[0, 2\pi)$ and define

$$s(z) = \frac{1}{2\pi}\int_0^{2\pi} s(e^{it})P(z,t)\,dt.$$

Proposition 3.4. *Assume that* $1 < p(e^{it}) < \infty$, *and* $-\infty < s_0 < s(e^{it}) < s_1 < \infty$. *Then* $\left\{B^{s(u)}_{p(u),p(u)}\right\}$, $u = e^{it}$, $t \in [0, 2\pi)$, *is an interpolation family of Banach spaces.*

(a) *If the condition* (1.2) *is satisfied with* $\alpha(e^{it}) = 1 - 1/p(e^{it})$ *and* $q(e^{it}) = p(e^{it})$, *then* $[B]_z = B^{s(z)}_{p(z),p(z)}$.

(b) *If* $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbb{Z}\right\}$, $1 \leqq q < \infty$ *and* $r \geqq 1$, *then* $\{B\}^S_{z,p(z),1} = (B)^S_{z,p(z),r} = B^{s(z)}_{p(z),p(z)}$.

Proof. According to (3.1) we have

$$B^{s(u)}_{p(u),p(u)} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(u),p(u)}, \quad u = e^{it},$$

where $\alpha(e^{it}) = 1 - 1/p(e^{it})$ and $s(e^{it}) = (1-\alpha(e^{it}))s_0 + \alpha(e^{it})s_1$. We note that $\alpha(z) = 1 - 1/p(z)$, $s(z) = (1-\alpha(z))s_0 + \alpha(z)s_1$ and, thus, by using Theorem B and (3.1) once again, we find that

$$[B]_z = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(z),p(z)} = B^{s(z)}_{p(z),p(z)}.$$

Furthermore, according to Theorem C and (3.1),

$$\{B\}^S_{z,p(z),1} = (B)^S_{z,p(z),q} = \left(B^{s_0}_{1,1}, B^{s_1}_{\infty,\infty}\right)_{\alpha(z),p(z)} = B^{s(z)}_{p(z),p(z)},$$

and the proof is complete.

Remark. By using the formula (3.2) instead of (3.1) and arguing exactly as in the proof of Proposition 3.4 we find that $B = \left\{B^{s(u)}_{p,q(u)}\right\}$, $u = e^{it}$, $t \in [0, 2\pi)$, $1 \leqq p \leqq \infty$, is an interpolation family of Banach spaces and that the (Hernandez) formula (see [13])

$$[B]_z = B^{s(z)}_{p,q(z)}$$

and the formula

$$\{B\}^S_{z,q(z),1} = (B)^S_{z,q(z),r} = B^{s(z)}_{p,q(z)}$$

hold under the corresponding restrictions on $s(e^{it})$, $q(e^{it})$ and S, respectively.

4. CONCLUDING REMARKS AND EXAMPLES

First we discuss the special case where $B = \{B_t\}$ is a finite family of Banach spaces, for example that $B_t = A_k$, $2\pi(k-1)/N \leqq t \leqq 2\pi k/N$, $k = 1, 2, \ldots, N$. By applying the statements in Section 3 we obtain concrete descriptions of the interpolation spaces between N Banach spaces. Alternatively, we can use the technique in Section 3 and the well-known reiteration results for finite families of Banach spaces (instead of Theorems A–C). We illustrate this idea by considering the Sparr spaces (see [31])

$$\overline{A}_{\overline{\lambda},q}, \quad \overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N), \; 0 \leqq \lambda_i \leqq 1, \; \sum \lambda_i = 1, \; 1 \leqq q \leqq \infty,$$

and $\overline{A} = (A_1, A_2, \ldots, A_N)$ is a N-tuple of compatible Banach spaces. We use the reiteration theorem of Sparr and the arguments used in Section 3.1 to present just one example of this technique:

Example 1. Let $1 \leqq q, p_i, q_i \leqq \infty$, $0 \leqq \lambda_i \leqq 1$ ($i = 1, 2, \ldots, N$), and $\sum \lambda_i = 1$.

(a) If $\dfrac{1}{p} = \sum\limits_1^N \dfrac{\lambda_i}{p_i}$ and not all of p_i are equal, then $(\sigma_{p_1,q_1}, \sigma_{p_2,q_2}, \ldots, \sigma_{p_N,q_N})_{\overline{\lambda},q} = \sigma_{p,q}$.

(b) If $1 \leqq p \leqq \infty$ and $\dfrac{1}{q} = \sum\limits_1^N \dfrac{\lambda_i}{q_i}$, then $(\sigma_{p,q_1}, \sigma_{p,q_2}, \ldots, \sigma_{p,q_N})_{\overline{\lambda},q} = \sigma_{p,q}$.

Question 1. Let $1 \leqq q, p, p_i, q_i \leqq \infty$, $0 \leqq \lambda_i \leqq 1$ ($i = 1, 2, \ldots, N$), and $\sum \lambda_i = 1$. Describe the spaces $(\sigma_{p,q_1}, \sigma_{p,q_2}, \ldots, \sigma_{p,q_N})_{\overline{\lambda},q}$ when $\dfrac{1}{q} \neq \sum\limits_1^N \dfrac{\lambda_i}{q_i}$.

This question is non-trivial also in the case $N = 2$ (see e. g. [26, 28]).

Next we remark that it is well-known that if we interpolate between two Besov spaces, then for the general (off-diagonal) case we need not obtain a Besov space (see the classical question by Peetre [25, p. 110]). A concrete (but somewhat curious) description of these spaces is given in [26, p. 228]. On the other hand, by making reiteration of these (Beurling type) spaces it may happen that we return to the Besov scale of spaces. We illustrate this fact with the following examples:

Example 2. Let $0 < \theta < 1$ and consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$, $q(e^{it}) \neq 1/(1-\theta)$, $t \in [0, 2\pi)$, but $q(z) = 1/(1-\theta)$. Then each space

$$B_t = \left(B^s_{p,1}, B^s_{p,\infty}\right)_{\theta, q(e^{it})}, \quad p \geqq 1, \quad t \in [0, 2\pi), \tag{4.1}$$

in the interpolation family $B = \{B_t\}$ is *not* a Besov space (see [25]). On the other hand, according to Theorems B and C, it yields respectively that

a) If $1 < p < \infty$ and $\int\limits_0^{2\pi} \frac{1}{q(e^{it})}\, dt > 0$, then $[B]_z = B^s_{p,q(z)}$;

b) If $S = \left\{\alpha_n(t) = 2^{n\alpha(e^{it})} \mid n \in \mathbf{Z}\right\}$ and $r \geqq 1$, then $\{B\}^S_{z,q(z),1} = (B)^S_{z,q(z),r}$ $= B^s_{p,q(z)}$.

Question 2. Consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$ and $B = \{B_t\}$, where B_t is defined by (4.1). Describe $[B]_z$ for the case when $q(z) \neq 1/(1-\theta)$.

Question 3. Describe the spaces $[B]_z$ when $B = \{B_t\}$, $B_t = B^{s(u)}_{p(u),q(u)}$, $u = e^{it}$, $t \in [0, 2\pi]$, without e. g. the restrictions $p(e^{it}) = q(e^{it})$ or $p(e^{it}) = p$ (cf. Proposition 3.4 or the remark after that proposition, respectively).

The observation above, concerning off-diagonal interpolation between Besov spaces, is a special case of the results obtained in [26]. In our next example we present some similar situations with interpolation in off-diagonal cases.

Example 3. Let $1 \leqq p, q_0, q_1, q(e^{it}) \leqq \infty$, $0 < \theta < 1$, and $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

(a) $B_t = (L_{p,q_0}, L_{p,q_1})_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a Lorentz $L_{p,q}$-space.

(b) $B_t = (L_{q_0}(\omega_0), L_{q_1}(\omega_1))_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a weighted L_p-space.

(c) $B_t = (l_{q_0}(\{A_k\}), l_{q_1}(\{B_k\}))_{\theta, q(e^{it})}$, $q(e^{it}) \neq q_\theta$, is *not* a (strong) sequence space of the type $l_q(\{C_k\})$ (A_k, B_k, C_k, $k = 1, 2, \ldots$, are Banach spaces).

These statements are all special cases of the descriptions given in [19] and [26]. Moreover, by arguing as in Example 2 and considering the case $q(z) = q_\theta$, we find that the reiteration space $[B]_z$, in fact, is a $L_{p,q}$-space, a L_p-space, and a (strong) sequence space of the type $l_q(\{C_k\})$, respectively.

Question 4. Consider $q(e^{it})$ such that $1 \leqq q(e^{it}) \leqq \infty$ and $B = \{B_t\}$, where B_t is defined as in Example 3 (a), (b) or (c). Describe $[B]_z$ if $q(z) \neq 1/(1-\theta)$ in each of these cases.

For several reasons it should be interesting to generalize Theorems A–C in various ways. In particular, by using such generalized forms of Theorems A–C and the technique described in Section 3 we get a possibility to describe interpolation spaces between more general families of Banach spaces than those studied in this paper. For example, by replacing the spaces $(A_0, A_1)_{\theta,q}$ with more general parameter function spaces $(A_0, A_1)_{\lambda,q}$ (see [27]) in Theorems B–C it is possible to obtain some weighted versions of the results obtained in Propositions 3.2–3.4.

Finally we remark that the problems posed in Questions 2–4 above are open also for the case when the spaces $[B]_z$ are replaced by $\{B\}^S_{z,q(z),1}$ or $(B)^S_{z,q(z),r}$.

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