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A GENERALIZATION OF THE LEVI–CIVITAE CONNECTION ON RIEMANNIAN MANIFOLDS*

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Ирина Петрова. ОДНО ОБОБЩЕНИЕ СВЯЗНОСТИ ЛЕВИ–ЧИВИТА НА РИМАНОВЫХ МНОГООБРАЗИЯХ

Пусть M римановое многообразие с метричным тензором g. Линейная связность $\tilde{\nabla}$ на M, обладающая свойствами

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = 0, \qquad \underset{X,Y,Z}{\sigma} \left(\tilde{\nabla}_X g\right)(Y,Z) = 0,$$

называется Обобщенной связностью *Леви–Чивита для* g. В статье даны примеры таких связностей. Одна из них это (0)-связность Картана и Схоутена на Ли-группе для каждой левоинвариантной метрики. Описана связь этих связностей с метричными связностями. Доказано, что на каждое римановое многообразие существует Обобщенная Леви–Чивита связность, которая не является Леви–Чивита связностью.

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Let M be a Riemannian manifold with a metric tensor g. A linear connection $\tilde{\nabla}$ on M with the properties

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = 0, \qquad \underset{X,Y,Z}{\sigma} \left(\tilde{\nabla}_X g\right)(Y,Z) = 0$$

is called a Generalised Levi–Civitae connection for g. In the paper are given examples of such connections. One of them is the (0)-connection of Cartan and Schouten on a Lie group for any left invariant metric. A relation is described between these connections and the metric connections. It is proved that there is a Generalized Levi–Civitae connection that is not a Levi–Civitae connection on Riemannian manifolds.

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1. GENERALIZED LEVI–CIVITAE CONNECTIONS. EXAMPLES

Let M be a Riemannian manifold with a metric tensor g. That means M is considered with the tensor field g of type $(0,2)$, which has the properties:

1) g is symmetric, i. e. $g(X,Y) = g(Y,X)$ for $X,Y \in \mathfrak{X}M$;

2) for a point p of M and for $X \in \mathfrak{X}M$, such that $X_p \neq o$, $g(X,X)(p) > 0$ holds.

The metric tensor g is also called a (Riemannian) metric on M.

If M is a Riemannian manifold with a metric tensor g, there is an unique linear connection ∇:

1) ∇ is symmetric, i. e. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$ for $X,Y \in \mathfrak{X}M$;

2) $\nabla g = 0$.

∇ is called a Levi–Civitae connection for g (see [1]).

N o t a t i o n s. Let M be a Riemannian manifold with a metric tensor g, ∇ — the Levi–Civitae connection for g. If $\tilde{\nabla}$ is a linear connection on M, the symbols $\sigma\tilde{\nabla}g$ and $\sigma g\tilde{\nabla}$ are used for a denoting

$$\underset{X,Y,Z}{\sigma}\left(\tilde{\nabla}_X g\right)(Y,Z) \quad \text{and} \quad \underset{X,Y,Z}{\sigma} g\left(\tilde{\nabla}_X Y, Z\right).$$

If t is a tensor field of type $(1,2)$, we denote by the symbols gt and σgt the tensors

$$g(t(X,Y),Z) \quad \text{and} \quad \underset{X,Y,Z}{\sigma} g(t(X,Y),Z)$$

correspondingly.

Definition. Let M be a Riemannian manifold with a metric tensor g. A linear connection $\tilde{\nabla}$ with the properties:

1) $\tilde{\nabla}$ is symmetric;

2) $\sigma\tilde{\nabla}g = 0$,

is called a Generalized Levi–Civitae connection for g.

It is well-known that all linear connections on M are given by $\nabla + T$, where T is a tensor field of type $(1,2)$.

Proposition. *The connection $\tilde{\nabla} = \nabla + \tilde{T}$ is a generalized Levi–Civitae connection iff $\tilde{T}$ is symmetric, $\sigma g\tilde{T} = 0$. We have $\tilde{\nabla} \equiv \nabla$ iff $\tilde{T} = 0$.*

It can be proved by trivial calculations.

Remark. Let s be a symmetric tensor of type $(1,2)$. It is easy to check that the conditions:

1) $\sigma gs = 0$;

2) $g(s(X,X),X) = 0$, $X \in \mathfrak{X}M$,

are equivalent.

We propose two examples of Generalized Levi–Civitae connections.

Example 1. Let M be a Riemannian manifold with a metric tensor g, for which exists $\xi \in \mathfrak{X}M$, not identically equal to zero, globally defined on M. Then

$$\tilde{\nabla} = \nabla + \tilde{T}, \quad \text{where } \tilde{T}(X,Y) = 2g(X,Y)\xi - g(X,\xi)Y - g(Y,\xi)X$$

is a Generalized Levi–Civitae connection, which is not Levi–Civitae connection.

It is so, because $\widetilde{T}$ is a symmetric tensor, $\widetilde{T} \not\equiv 0$, $g(\widetilde{T}(X,X),X) = 0$. By the above remark and the proposition it follows that $\widetilde{\nabla}$ is a Generalized Levi–Civitae connection, $\widetilde{\nabla} \not\equiv \nabla$, and $\widetilde{\nabla}$ has the following properties:

1) if $X, Y \in \mathfrak{X}M$ are such that X, Y, ξ form an orthogonal triple, then $\widetilde{\nabla}_X Y = \nabla_X Y$;

2) on the FM-submodule of $\mathfrak{X}M$, which is orthogonal to ξ, we have $\widetilde{\nabla} g = \nabla g = 0$.

Example 2. Let G be a Lie group and $\mathcal{G}$ be the algebra of the left-invariant vector fields on G. From the theory of the Lie groups is known that G is parallelizable and it has bases of left-invariant vector fields. Hence, we always can construct a Riemannian metric tensor on G. Let we recall that a metric tensor g is left-invariant iff it is invariant for the left translations. It is well-known that a metric g on G is left-invariant iff for an arbitrary chosen base $X_1, \ldots, X_n$, $X_i \in \mathcal{G}$,

$$g(X_i, X_j) = \text{const}.$$

Let $X_1, \ldots, X_n$ be a base on G, $X_i \in \mathcal{G}$. We consider the linear connection

$$\overline{\nabla}_X Y = X(\varphi_i) X_i, \quad Y = \varphi_i X_i.$$

This connection is independent on the choice of the base of left-invariant fields and it is called a left-invariant connection. It is proved in [2] that $\overline{\nabla}$ is complete, i. e. its geodesics can be continued, being defined for all real values of their parameters.

Let $\overline{T}$ be the tensor of torsion for $\overline{\nabla}$. We consider the symmetrization of $\overline{\nabla}$:

$$\widetilde{\nabla} = \overline{\nabla} - \frac{1}{2}\overline{T}.$$

It is also complete, because the geodesics for the both connections coincide. $\widetilde{\nabla}$ is introduced by Cartan and Schouten, and it is called (0)-connection (see [1] or [6]).

Proposition. *Let G be a Lie group:*

1. *For each left-invariant metric g the equation $\sigma\widetilde{\nabla} g = 0$ holds (i. e. $\widetilde{\nabla}$ is a Generalized Levi–Civitae connection for any left-invariant metric).*

2. *$\widetilde{\nabla}$ is the unique linear symmetric connection on G with the property 1.*

3. *For a left-invariant metric g the connection $\widetilde{\nabla}$ is the Levi–Civitae connection iff g is right-invariant.*

Proof. 1. Let g be a left-invariant metric. Since $\sigma\widetilde{\nabla} g$ is a tensor field, to check $\sigma\widetilde{\nabla} g = 0$ it is enough to prove it for the elements of a base on G. G admits a base of left-invariant vector fields, so it is enough to prove $\sigma\widetilde{\nabla} g = 0$ for left-invariant vector fields.

Let $X, Y, Z \in \mathcal{G}$. We have

$$\overline{\nabla}_X Y = X(\text{const}).Y = 0,$$

$$\overline{T}(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = -[X,Y],$$

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{2}\overline{T}(X,Y) = \frac{1}{2}[X,Y],$$

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = X \circ g(Y,Z) - g\left(\widetilde{\nabla}_X Y, Z\right) - g\left(Y, \widetilde{\nabla}_X Z\right).$$

If $X, Y \in \mathcal{G}$, then $g(X,Y) = \text{const}$. Hence

$$\left(\widetilde{\nabla}_X g\right)(Y,Z) = \frac{1}{2}\left(g([Z,X],Y) - g([X,Y],Z)\right).$$

Now it can be checked easy that

$$\underset{X,Y,Z}{\sigma} \left(\widetilde{\nabla}_X g\right)(Y,Z) = 0,$$

which proves 1.

2. Let $\hat{\nabla}$ be a linear symmetric connection on G such that for each left-invariant metric g

$$\sigma \hat{\nabla} g = 0.$$

Let $X_1, \ldots, X_n$, $X_i \in \mathcal{G}$, be a base on G. Then

$$\hat{\nabla}^{X_j}_{X_i} = \Gamma^k_{ij} X_k, \quad \Gamma^k_{ij} \in FM.$$

We shall prove that Γ^k_{ij} are defined uniquely and hence $\widetilde{\nabla} \equiv \hat{\nabla}$. Let we consider only the left-invariant metrics, for which the base $X_1, \ldots, X_n$ is orthogonal, i. e.

$$g(X_i, X_j) = 0, \quad i \neq j.$$

Let Ω is the set, formed by them. From the identity $\sigma \hat{\nabla} g = 0$ we obtain after some transformations:

$$\underset{i,j,k}{\sigma} \Gamma^k_{ij} g(X_k, X_k) = \underset{i,j,k}{\sigma} g\left(\hat{\nabla}^{X_j}_{X_i}, X_k\right),$$

for each $g \in \Omega$ and each i, j, k from 1 to n. We set

$$F_{g(i,j,k)} = \underset{i,j,k}{\sigma} g\left(\hat{\nabla}^{X_j}_{X_i}, X_k\right), \quad g_{kk} = g(X_k, X_k).$$

Then

$$\Gamma^k_{ij} g_{kk} + \Gamma^i_{jk} g_{ii} + \Gamma^j_{ki} g_{jj} = F_{g(i,j,k)}.$$

2.1. Let $i = j = k$. We have $3\Gamma^i_{ii} g_{ii} = F_{g(i,i,i)}$. That means Γ^i_{ii} are uniquely defined.

2.2. Let $i \neq j \neq k \neq i$. We consider three left-invariant Ω-metrics g^1, g^2, g^3, for which

$$\begin{pmatrix} g^1_{kk} & g^1_{ii} & g^1_{jj} \\ g^2_{kk} & g^2_{ii} & g^2_{jj} \\ g^3_{kk} & g^3_{ii} & g^3_{jj} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then we have

$$1.\Gamma^k_{ij} + 1.\Gamma^i_{jk} + 1.\Gamma^j_{ki} = F_{g^1(i,j,k)},$$
$$1.\Gamma^k_{ij} + 2.\Gamma^i_{jk} + 1.\Gamma^j_{ki} = F_{g^2(i,j,k)},$$
$$1.\Gamma^k_{ij} + 1.\Gamma^i_{jk} + 2.\Gamma^j_{ki} = F_{g^3(i,j,k)}.$$

These relations can be considered like a system of linear equations about Γ^k_{ij}, Γ^i_{jk}, Γ^j_{ki} with determinant different from zero. Hence Γ^k_{ij}, Γ^i_{jk}, Γ^j_{ki} are uniquely defined.

2.3. Let $i = j \neq k$. We have

$$\Gamma^k_{ii} g_{kk} + (\Gamma^i_{ik} + \Gamma^i_{ki}) g_{ii} = F_{g(i,i,k)}, \quad g \in \Omega.$$

Analogously to 2.2 we can prove that Γ^k_{ii}, $\Gamma^i_{ik} + \Gamma^i_{ki}$ are uniquely defined. But $\hat{\nabla}$ is symmetrical. Hence, for each $g \in \Omega$:

$$g\left(\hat{\nabla}^{X_k}_{X_i} - \hat{\nabla}^{X_i}_{X_k}, X_i\right) = g([X_i, X_k], X_i),$$

$$(\Gamma^i_{ik} - \Gamma^i_{ki}) g_{ii} = g([X_i, X_k], X_i).$$

That means $\Gamma^i_{ik} - \Gamma^i_{ki}$ are uniquely defined too. Hence Γ^i_{ik}, Γ^i_{ki} are uniquely defined; that proves 2.

3. Let g be a left-invariant metric, ∇ — the Levi–Civitae connection for g. Since G has bases of left-invariant vector fields, the condition $\nabla \equiv \tilde{\nabla}$ is equivalent to

$$g(\nabla_X Y, Z) = g\left(\tilde{\nabla}_X Y, Z\right), \quad X, Y, Z \in \mathcal{G}.$$

But from the proof of 1 we know that for each $X, Y, Z \in \mathcal{G}$

$$2g\left(\tilde{\nabla}_X Y, Z\right) = g([X,Y], Z),$$

$$2g(\nabla_X Y, Z) = g([X,Y], Z) - g([X,Z], Y) - g([Y,Z], X).$$

Hence $g(\nabla_X Y, Z) = g\left(\tilde{\nabla}_X Y, Z\right)$ is equivalent to

$$-g([X,Z],Y) - g([Y,Z],X) = 0, \quad g([Z,X],Y) + g(X,[Z,Y]) = 0,$$

i. e. for each $Z \in \mathcal{G}$ the operator

$$\mathrm{ad}_Z : \mathcal{G} \to \mathcal{G}$$
$$X \to [Z, X]$$

is antisymmetric.

So we have got that if g is left-invariant then $\nabla \equiv \tilde{\nabla}$ iff ad_Z is antisymmetric for each $Z \in \mathcal{G}$. But in [3] is proved if g is left-invariant then ad_Z is antisymmetric for each $Z \in \mathcal{G}$ iff g is right-invariant. That proves 3.

From the proposition we receive the following

Corollary. *Let G be a Lie group that admits a bi-invariant metric:*

1. *All bi-invariant metrics induce the same Levi–Civitae connection. (Let we call this common connection bi-invariant connection.)*

2. *The symmetrization of the left-invariant connection, the symmetrization of the right-invariant connection and the bi-invariant connection coincide.*

2. A RELATION BETWEEN THE METRIC CONNECTIONS AND THE GENERALIZED LEVI–CIVITAE CONNECTIONS. AN EXISTENCE THEOREM

Let M be a Riemannian manifold with metric tensor g. We recall the following

Definition. A linear connection D on M is metric, if $Dg = 0$.

Notations. M_1 is the set of the metric connections on M, M_2 — the set of the Generalized Levi–Civitae connections on M, (D,τ) — a metric connection D with tensor of torsion τ, $M_1' = \{(D,\tau) \mid \sigma g\tau = 0\}$.

Proposition 1. *Let $D \in M_1$, τ is the torsion for D. Then $\tilde{\nabla} = D - \frac{1}{2}\tau \in M_2$, $\tilde{\nabla}$ and D have the same geodesics.*

The proposition can be proved by trivial calculations.

Remark 1. Let G be a Lie group, $\overline{\nabla}$ is the left-invariant connection on G. It is easy to see that $\overline{\nabla}$ is a metric connection for any left-invariant metric g, i. e. Proposition 1 is a generalization of Example 2 from Section 1.

Remark 2. Let $D \in M_1$, τ is the torsion of D. Let T be the tensor, $D = \nabla + T$, aT is the antisymmetric part of T, $\tilde{T}$ — the symmetric part of T. Then

$$aT = \frac{1}{2}\tau, \quad \tilde{\nabla} = D - \frac{1}{2}\tau = \nabla + \tilde{T}.$$

Proposition 2. *Let $D \in M_1$, τ is the torsion of D, $\tilde{\nabla} = D - \frac{1}{2}\tau$. Then $\nabla \equiv \tilde{\nabla}$ iff $g\tau$ is antisymmetric.*

It is easy to check the assertion of the proposition.

Remark 3. Let T be a tensor of type $(1,2)$, $D = \nabla + T$. Then D is metric iff $g(T(X,Y),Z) + g(T(X,Z),Y) = 0$.

Proposition 3. *The correspondence*

$$M_1' \to M_2,$$
$$(D,\tau) \to \tilde{\nabla} = D - \frac{1}{2}\tau$$

is bijective. By it ∇ (considered as a generalized Levi–Civitae connection) is generated by itself (considered as a metric connection).

Proof. Let $\tilde{\nabla} = \nabla + \tilde{T}$ be a Generalized Levi–Civitae connection. We search for the metric connections D with torsion τ, such that

$$(s) \qquad \left| \begin{array}{l} D - \frac{1}{2}\tau = \tilde{\nabla}, \\ \sigma g\tau = 0. \end{array} \right.$$

(We show below there is an unique connection D with the property (s).) It is enough to find all tensors τ of type $(1,2)$:

$$(s') \qquad \left| \begin{array}{l} \tau \ \text{ is antisymmetric}, \\ g(T(X,Y),Z) + g(T(X,Z),Y) = 0, \ \text{where } T = \frac{1}{2}\tau + \tilde{T}, \\ \sigma g\tau = 0. \end{array} \right.$$

Then $D = \nabla + T$, $T = \frac{1}{2}\tau + \tilde{T}$, will give all the metric connections with the property (s).

It is easy to see the conditions (s') are equivalent to

$$(s'') \quad \left| \begin{array}{l} \tau(X,Y) = -\tau(Y,X), \\ g(\tau(X,Y),Z) + g(\tau(X,Z),Y) = 2g(\widetilde{T}(Y,Z),X), \\ \sigma g \tau = 0, \end{array} \right.$$

and (s'') are equivalent to

$$(s''') \qquad g(\tau(X,Y),Z) = \frac{2}{3}(g(\widetilde{T}(Y,Z),X) - g(\widetilde{T}(Z,X),Y).$$

Hence there is an unique tensor τ, for which holds (s).

That proves the proposition.

Remark 4. If τ is an antisymmetric tensor of type $(1,2)$, there is a (unique) metric connection D with tensor of torsion τ (see [5]).

From this remark and Proposition 3 we receive

Corollary. *Let M be a Riemannian manifold with metric tensor g. Then the following conditions are equivalent:*

1. *On M there is a symmetric tensor $\widetilde{T} \neq 0$ of type $(1,2)$, for which holds $\sigma g \widetilde{T} = 0$.*

2. *On M there is an antisymmetric tensor $\tau \neq 0$ of type $(1,2)$, for which holds $\sigma g \tau = 0$.*

Theorem. *Let M be a Riemannian manifold with metric tensor g. Then:*

1. *On M there is a Generalized Levi–Civitae connection that is not a Levi–Civitae one.*

2. *On M there is a metric connection that is not a Levi–Civitae one.*

If M is a Hausdorff space, for each $p \in M$ can be found a Generalized Levi–Civitae connection and a metric connection, which coincide with the Levi–Civitae connection locally around p.

Proof. In [1, 4] is proved the following

Lemma. *Let M be a manifold, p is an arbitrary chosen point of M, U is a co-ordinate neighbourhood of p. There is a function $f \in FM$, such that $f(p) = 1$, $f(M \setminus U) = 0$.*

From the proof of the lemma in [4] is clear that we can choose $f \neq$ const even on M. If f is such a function, then $\omega(X) = X \circ f$ is a differential 1-form, not identically equal to zero. We set

$$\tau(X,Y) = \omega(X)Y - \omega(Y)X.$$

Obviously, τ is an antisymmetric tensor of type $(1,2)$ that is not identically equal to zero. It is easy to check $\sigma g \tau = 0$. From the last corollary we receive the theorem.

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