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# ON A GENERALIZATION OF THE JACOBI OPERATOR IN THE RIEMANNIAN GEOMETRY*

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*Грозьо Станилов, Веселин Видев.* ОБОБЩЕНИЕ ОПЕРАТОРА ЯКОБИ В РИМАНОВОЙ ГЕОМЕТРИИ

Пусть $(M,g)$ римановое многообразие размерности $n$ с тензором кривизны $R$. Если $X$, $Y$ произвольная пара касательных векторов в точке $p \in M$, мы вводим в рассмотрении оператора кривизны $\lambda_{X,Y}(u) = \frac{1}{2}(R(u,X,Y) + R(u,Y,X))$. Это линейный симметрический оператор. Эго собственные значения зависят от базисе $X$, $Y$ касательного подпространство $E^2(p;X,Y)$ натянутое на векторов $X$, $Y$. Мы доказываем: 1) эйнштейновые многообразия одинственные для которых следа этого оператора не зависит от базисе подпространства $E^2(p;X,Y)$; 2) вещественные пространственные формы размерности четыре одинственные для которых спектр этого оператора не зависит от $E^2(p;X,Y)$. В обоих случаев $p$ произвольная точка многообразии.

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Let $(M,g)$ be a Riemannian manifold of dimension $n$ with curvature tensor $R$. If $X$, $Y$ is an orthonormal pair of tangent vectors at a point $p$ of $M$, we define the curvature operator $\lambda_{X,Y}(u) = \frac{1}{2}(R(u,X,Y) + R(u,Y,X))$. It is a symmetric operator but its eigen values depend on the base $X$, $Y$ of the tangent subspace $E^2(p;X,Y)$ spanned by $X$, $Y$. We prove: 1) the Einstein manifolds are the unique for which the trace of this operator does not depend on the base of

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**$E^2(p;X,Y)$; 2) the real space forms of dimension 4 are the unique for which the spectrum of this operator does not depend on the base of $E^2(p;X,Y)$. Of course in both assertions $p$ is an arbitrary point of $M$.**

## 1. INTRODUCTION

Let $(M,g)$ be an $n$-dimensional Riemannian manifold, $p$ — a point of $M$, $M_p$ — the tangent space in $p$. We denote by $R$ the curvature tensor of the manifold. The well-known Jacobi operator in the Riemannian geometry is defined in the following way: if $X$ is an unit tangent vector of $M_p$, then we consider the linear mapping

$$\lambda_X : M_p \to M_p$$

defined by

$$\lambda_X(u) = R(u,X,X).$$

From the properties of $R$ it follows that $\lambda_X$ is a linear symmetrical operator. Namely, it is the Jacobi operator in respect to $X$. The geometry of this operator is investigated very actively in the last years, see, for example, [1–3].

In this paper we state at first the problem about a generalization of this operator.

Let $X$, $Y$ is an arbitrary pair of unit tangent vectors. Then we define the mapping

$$\lambda_{X,Y} : M_p \to M_p$$

by

$$\lambda_{X,Y}(u) = \frac{1}{2}[R(u,X,Y) + R(u,Y,X)].$$

Evidently, it is a generalization of the Jacobi operator because of

$$\lambda_{X,X} = \lambda_X.$$

Further we propose that $X$, $Y$ is an arbitrary orthonormal pair of tangent vectors. If $x$, $y$ is another pair of tangent vectors in the 2-dimensional tangent subspace $E^2(p;X,Y)$ of $M_p$ spanned by $X$, $Y$, we have the relations

$$\begin{aligned} x &= \cos\varphi.X - \varepsilon.\sin\varphi.Y, \\ y &= \sin\varphi.X + \varepsilon.\cos\varphi.Y, \quad \varepsilon = \pm 1. \end{aligned} \tag{1}$$

By these formulas all orthogonal transformations in $E^2(p;X,Y)$ are represented. Now we establish the relation

$$\lambda_{x,y}(u) = \cos 2\varphi.\lambda_{X,Y}(u) + \varepsilon.\frac{\sin 2\varphi}{2}[R(u,X,X) - R(u,Y,Y)].$$

Hence the operator $\lambda_{X,Y}$ is not invariant under the orthogonal transformations in $E^2(p;X,Y)$.

It is evident that $\lambda_{X,Y} = \lambda_{Y,X}$ and we can take $\varepsilon = 1$.

Let now $e_1, e_2, \ldots, e_n$ is an arbitrary orthonormal base of the tangent space $M_p$. Then we have ($i = 1, 2, \ldots, n$)

$$\lambda_{x,y}(e_i) = \cos 2\varphi . \lambda_{X,Y}(e_i) + \frac{\sin 2\varphi}{2}[R(e_i, X, X) - R(e_i, Y, Y)].$$

Hence we obtain

$$R(e_i, x, y, e_i) = \cos 2\varphi . R(e_i, X, Y, e_i) + \frac{\sin 2\varphi}{2}[R(e_i, X, X, e_i) - R(e_i, Y, Y, e_i)].$$

From this equalities we get

$$S(x, y) = \cos 2\varphi . S(X, Y) + \frac{\sin 2\varphi}{2}[S(X, X) - S(Y, Y)], \tag{2}$$

where $S$ is the classical Ricci tensor of the manifold $M$.

## 2. A CHARACTERIZATION OF THE EINSTEIN MANIFOLDS BY THE GENERALIZED JACOBI OPERATOR

In this part we prove the following

**Theorem 1.** *Let $(M, g)$ be an n-dimensional Riemannian manifold and $n \geqq 2$. Then the following assertions are equivalent:*

1) *$(M, g)$ is an Einstein manifold;*

2) *The trace of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base $X, Y$ of the tangent subspace $E^2(p; X, Y)$ (and it is equal to zero).*

*Proof.* Let $(M, g)$ be an $n$-dimensional Einstein manifold. Then at every point $p \in M$ and for any tangent vectors $x$, $y$ we have

$$S(x, y) = c'.g(x, y), \quad c' = \text{const},$$

whence it follows that

$$S(X, Y) = 0 \tag{3}$$

for any arbitrary pair of tangent vectors $X$, $Y$. Also from the definition of the operator $\lambda_{X,Y}$ we have

$$c = R(u, X, Y, u),$$

where $u$ is an eigen vector of $\lambda_{X,Y}$ and $c$ is the corresponding eigen value, i. e.

$$\lambda_{X,Y}(u) = \frac{1}{2}[R(u, X, Y) + R(u, Y, X)] = c.u.$$

Hence, if $u_1, u_2, \ldots, u_n$ are the eigen vectors of $\lambda_{X,Y}$ with corresponding eigen values $c_1, c_2, \ldots, c_n$, then

$$\operatorname{tr} \lambda_{X,Y} = \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} R(u_i, X, Y, u_i) = S(X, Y).$$

According to (3) we obtain

$$\operatorname{tr} \lambda_{X,Y} = 0.$$

Conversely, let at any point $p \in M$ and for any orthonormal pair of tangent vectors $X$, $Y$ of $M_p$ the trace of the operator $\lambda_{X,Y}$ not depend on the base of the plane $E^2(p; X, Y)$. If the tangent vectors $x$, $y$ are related to $X$, $Y$ by (1), then

$$S(x, y) = S(X, Y),$$

whence from (2) we have

$$S(X, Y) = \cos 2\varphi . S(X, Y) + \frac{\sin 2\varphi}{2}[S(X, X) - S(Y, Y)],$$

or

$$S(X, Y) = \operatorname{cotg}\varphi[S(X, X) - S(Y, Y)].$$

From the last equality by $\varphi = -\frac{\pi}{4}$ and $\varphi = \frac{\pi}{4}$ we obtain correspondingly

$$S(X, Y) = -S(X, X) + S(Y, Y), \quad S(X, Y) = S(X, X) - S(Y, Y),$$

whence it follows that (3) is satisfied for every orthonormal pair of tangent vectors $X$, $Y$ of $M_p$ and at any point $p \in M$. Now we can apply (3) for the orthonormal pair

$$\frac{u}{|u|}, \quad \frac{v.g(u, u) - u.g(u, v)}{|u|.\sqrt{g(u, u).g(v, v) - g^2(u, v)}},$$

where $u$, $v$ are arbitrary tangent vectors of $M_p$. We have

$$S\left(\frac{u}{|u|}, \frac{v.g(u, u) - u.g(u, v)}{|u|.\sqrt{g(u, u).g(v, v) - g^2(u, v)}}\right) = 0,$$

whence it follows that

$$\frac{S(u, v)}{g(u, v)} = \frac{S(u, u)}{g(u, u)}.$$

From the last equality, if we change $u$ by $v$ and $v$ by $u$, we obtain

$$\frac{S(v, u)}{g(v, u)} = \frac{S(v, v)}{g(v, v)}.$$

Then the last two equalities lead to

$$\frac{S(u, u)}{g(u, u)} = \frac{S(v, v)}{g(v, v)},$$

which is satisfied for arbitrary tangent vectors $u$, $v$ of $M_p$ and at every point $p \in M$. It means that $(M, g)$ is an Einstein manifold.

From the theorem we get the following

**Corollary.** *Let $(M, g)$ be a 3-dimensional Riemannian manifold. Then $(M, g)$ is a real space form iff the trace of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base $X$, $Y$ of the plane $E^2(p; X, Y)$.*

*Proof.* Let $(m, g)$ be a 3-dimensional real space form, i. e. Riemannian manifold of constant sectional curvature $\mu$. Then

$$R(x, y, z) = \mu.[g(y, z).x - g(x, z).y].$$

Hence

(4) $$\lambda_{X,Y}(u) = -\frac{1}{2}\mu[g(u,X).Y + g(u,Y).X]$$

for any orthonormal pair $X$, $Y$ of tangent vectors of $M_p$ and at any point $p \in M$. Then for the spectrum $\Omega_{X,Y}$ of $\lambda_{X,Y}$ we have

$$\Omega_{X,Y} = \left\{-\frac{\mu}{2}, \frac{\mu}{2}, 0\right\},$$

whence it follows that $\operatorname{tr}\lambda_{X,Y} = 0$.

Conversely, if $\operatorname{tr}\lambda_{X,Y}$ does not depend on the plane $E^2_\bullet(p;X,Y)$, then from Theorem 1 follows that $(M,g)$ is an Einstein manifold and, since $\dim M = 3$, $(M,g)$ is a real space form [6].

## 3. A CHARACTERIZATION OF THE REAL SPACE FORMS BY THE SPECTRUM OF THE GENERALIZED JACOBI OPERATOR

Now we investigate the Riemannian manifolds with the following properties: at every point $p \in M$ and for any orthonormal pair $X$, $Y$ of tangent vectors of $M_p$ the spectrum of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p;X,Y)$.

At first we remark that from this property it follows that the trace of $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p;X,Y)$ and according to Theorem 1 we get that $(M,g)$ is an Einstein manifold. Also from this property follows that the spectrum of the operator $\lambda_{X,Y}$ has the form

(5) $$\Omega_{X,Y} = \{c_1, -c_1, c_2, -c_2, \dots, c_{2k}, -c_{2k}, 0, \dots, 0\}, \quad 2k \leqq n.$$

Indeed, let $u$ be an arbitrary eigen vector of $\lambda_{X,Y}$ with a corresponding eigen value $c$. It follows that $u$ is also an eigen vector of the operator $-\lambda_{X,Y}$ with a corresponding eigen value $-c$. But from the definition of $\lambda_{X,Y}$ we have

$$\lambda_{-X,Y} = -\lambda_{X,Y},$$

whence we obtain that $-c$ is also an eigen value of $\lambda_{-X,Y}$. Since the spectra of $\lambda_{X,Y}$ and $\lambda_{-X,Y}$ coincide, then $-c$ is also an eigen value of the operator $\lambda_{X,Y}$. That means that $c$ and $-c$ are eigen values of $\lambda_{X,Y}$, and $\Omega_{X,Y}$ has the form (5).

Let now $(M,g)$ be an $n$-dimensional Riemannian manifold of a constant sectional curvature $\mu$. Then from (4) we directly obtain that

(6) $$\Omega_{X,Y} = \left\{-\frac{\mu}{2}, \frac{\mu}{2}, 0, \dots, 0\right\},$$

which is of the form (5). Our conjecture is that the converse is also true. In this paper we can prove it only in the four-dimensional case.

From now on let $(M,g)$ be a 4-dimensional manifold with the property given at the beginning of this section: $p$ is an arbitrary point of $M$, $X$, $Y$ is an arbitrary orthonormal pair of tangent vectors of $M_p$. Let $x$, $y$ be another orthonormal base of the plane $E^2(p;X,Y)$. Further, if $e_1$, $e_2$, $e_3$, $e_4$ is an arbitrary orthonormal base of the tangent space $M_p$, then every eigen vector $v$ of the operator $\lambda_{x,y}$ can be represented in the form

$$v = v^1.e_1 + v^2.e_2 + v^3.e_3 + v^4.e_4.$$

From the definition of $\lambda_{x,y}$ we get the relation

$$R(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4, \cos\varphi.X - \sin\varphi.Y)$$
$$+ R(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4, \sin\varphi.X + \cos\varphi.Y) = c(v^1e_1 + v^2e_2 + v^3e_3 + v^4e_4),$$

whence after scalar multiplying by $e_1$, $e_2$, $e_3$, $e_4$ we obtain

$$a_{i1}v^1 + a_{i2}v^2 + a_{i3}v^3 + a_{i4}v^4 = 0, \tag{7}$$

where

$$a_{ii} = \sin 2\varphi.[R(e_i, X, X, e_i) - R(e_i, Y, Y, e_i)] + 2.\cos 2\varphi.R(e_i, X, Y, e_i) - 2c,$$
$$a_{kj} = \sin 2\varphi.[R(e_k, X, X, e_j) - R(e_k, Y, Y, e_j)]$$
$$+ \cos 2\varphi.[R(e_k, X, Y, e_j) + R(e_k, Y, X, e_j)], \quad k \neq j, \quad i, j, k = 1, 2, 3, 4.$$

Since $v$ is a non-zero vector, then

$$\det(a_{ij}) = 0, \quad i, j = 1, 2, 3, 4. \tag{8}$$

From here, by $X = e_1$, $Y = e_2$, $\varphi = 0$, we obtain the following equation:

$$\begin{vmatrix} -2c & -K_{12} & R_{3121} & R_{4121} \\ -K_{12} & -2c & R_{2123} & R_{2124} \\ R_{3121} & R_{2123} & 2R_{3123} - 2c & R_{4123} + R_{4213} \\ R_{4121} & R_{2124} & R_{4123} + R_{4213} & 2R_{4124} - 2c \end{vmatrix} = 0,$$

or

$$16c^4 - 4I_1c^2 - 2I_2c + I_3 = 0,$$

where

$$I_1 = 4R_{4124}^2 + (R_{4123} + R_{3124})^2 + R_{2123}^2 + R_{2124}^2 + K_{12}^2 + R_{3121}^2 + R_{4121}^2,$$
$$I_2 = 2R_{2123}(R_{4123} + R_{3124})R_{2124} - 2R_{2124}^2R_{3123} - 2R_{2123}R_{4124} - 2K_{12}R_{4121}R_{2124}$$
$$- 2K_{12}R_{3121}R_{2123} - R_{3121}(R_{4123} + R_{3124})R_{4121} + 2R_{3121}R_{4124} - 2R_{4121}R_{3123},$$
$$I_3 = 2K_{12}^2R_{4124} + K_{12}R_{4121}R_{2123}(R_{4123} + R_{3124}) + K_{12}R_{3121}(R_{4123} + R_{3124})R_{2124}$$
$$- 4K_{12}R_{4121}R_{2124}R_{3123} + K_{12}^2(R_{4123} + R_{3124})^2 - 2K_{12}R_{3121}R_{2123}R_{4124}$$
$$+ R_{3121}^2R_{2124}^2 - 2R_{4121}R_{2123}R_{2124}R_{3121} + K_{12}R_{4121}R_{2123}(R_{4123} + R_{3124})$$
$$+ R_{4121}^2R_{2123}^2.$$

Here

$$R_{ijkt} = R(e_i, e_j, e_k, e_t), \quad i, j, k, t = 1, 2, 3, 4,$$

are the components of the curvature tensor $R$ in respect to the base $e_1$, $e_2$, $e_3$, $e_4$.

Also from (7) and (8) by $X = e_1$, $Y = e_2$ and $\varphi = \frac{\pi}{4}$ we have

$$\begin{vmatrix} -K_{12} - 2c & 0 & -R_{3221} & -R_{4221} \\ 0 & K_{12} - 2c & R_{3112} & R_{4112} \\ -R_{3221} & R_{3112} & K_{13} - K_{23} - 2c & 2R_{4113} \\ -R_{4221} & R_{4112} & 2R_{4113} & K_{14} - K_{24} - 2c \end{vmatrix} = 0,$$

or

$$16c^4 - 4J_1c^2 - 2J_2c + J_3 = 0,$$

where

$$J_1 = K_{12}^2 + (K_{13} - K_{23})^2 + R_{4112}^2 + 4R_{4113}^2 + R_{3112}^2 + R_{3221}^2 + R_{4221}^2,$$

$$J_2 = K_{12}(R_{3221}^2 + R_{4221}^2 - R_{3112}^2 - R_{4112}^2) + (K_{13} - K_{23})(R_{4112}^2 + R_{4221}^2 - R_{3112}^2 - R_{3221}^2) - 4R_{3114}(R_{4112}R_{3112} + R_{4221}R_{3221}),$$

$$J_3 = -K_{12}[4R_{3112}R_{4112}R_{4113} - R_{4112}^2(K_{13}-K_{23}) - 4R_{4113}K_{12} - R_{3112}^2(K_{14}-K_{24})] + K_{12}^2(K_{13} - K_{23})^2 + 4K_{12}R_{3221}R_{4221}R_{4113} - 2R_{3221}R_{4112}R_{4221}R_{3112} + R_{4112}^2R_{3221}^2 + K_{12}(K_{14} - K_{24})^2R_{3221}^2 - R_{4221}^2K_{12}(K_{13} - K_{23}) + R_{3112}^2R_{4221}^2.$$

Since the spectrum of $\lambda_{X,Y}$ does not depend on the orthonormal base of the plane $E^2(p;X,Y)$, we have

$$I_1 = J_1, \quad I_2 = J_2 = 0, \quad I_3 = J_3.$$

Further we use the first of these equalities. Then

$$4R_{4124}^2 + (R_{4123} + R_{3124})^2 + R_{2123}^2 + R_{2124}^2 + R_{3121}^2 + R_{4121}^2 = (K_{13} - K_{23})^2 + R_{4112}^2 + 4R_{4113}^2 + R_{3112}^2 + R_{3221}^2 + R_{4221}^2. \tag{9}$$

Since $(M,g)$ is an Einstein manifold, then:

a) the sectional curvature of every 2-dimensional subspace of the tangent space $M_p$ is equal to the sectional curvature of its orthogonal complements [4];

b) $R_{issj} + R_{ittj} = 0$ (cf. [3]), $i \neq j$, $i \neq s$, $i \neq t$, $j \neq s$, $j \neq t$, $s \neq t$; $i,j,s,t = 1,2,3,4$.

Now from (8) we obtain

$$(K_{13} - K_{23})^2 + 4R_{4113}^2 = 4R_{1442}^2 + (R_{4123} + R_{4213})^2, \tag{10}$$

which is hold for any orthonormal base $e_4$, $e_3$, $e_2$, $e_1$ of $M_p$, i.e.

$$(K_{42} - K_{32})^2 + 4R_{1442}^2 = 4R_{4113}^2 + (R_{1432} + R_{1342})^2,$$

or

$$(K_{13} - K_{23})^2 + 4R_{1442}^2 = 4R_{4113}^2 + (R_{4123} + R_{4213})^2. \tag{11}$$

Now from (10) we get

$$R_{1442}^2 = R_{4113}^2,$$

and whence from (9) it follows that

$$R_{4123} + R_{4213} = \varepsilon(K_{13} - K_{23}), \quad \varepsilon = \pm 1. \tag{12}$$

From (12) and the first Bianchi identity we obtain

$$2R_{4123} + R_{4312} = \varepsilon(K_{13} - K_{23}). \tag{13}$$

If we change $e_3$ by $e_4$, we get

$$2R_{3124} + R_{3412} = \varepsilon(K_{14} - K_{24}).$$

Hence

$$(14) \qquad 2R_{4213} - R_{4312} = \varepsilon(K_{23} - K_{13}).$$

Now from (13) and (14) it follows that

$$R_{4123} + R_{4213} = 0,$$

whence

$$K_{13} - K_{23} = 0.$$

This equality can be written in the form

$$K(X \wedge Z) = K(Y \wedge Z).$$

It holds for every orthonormal tripple of tangent vectors $X$, $Y$, $Z$ in $M_p$ and at any point $p \in M$. We conclude that $(M, g)$ is a real space form [6].

Thus we have proved the following

**Theorem 2.** *Let $(M, g)$ be a 4-dimensional Riemannian manifold. Then the following assertions are equivalent:*

1) *$(M, g)$ is a real space form;*

2) *The spectrum of the curvature operator $\lambda_{X,Y}$ does not depend on the orthonormal base $X$, $Y$ of the plane $E^2(p; X, Y)$ for every plane $E^2$ and at any point $p \in M$.*

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