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AN EXTERNAL APPROACH TO ABSTRACT DATA TYPES I: COMPUTABILITY ON ABSTRACT DATA TYPE*

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Алексанра Соскова. ВНЕШНЫЙ ПОДХОД К АБСТРАКТНЫМ ТИПАМ ДАННЫХ I: ВЫЧИСЛИМОСТЬ В АБСТРАКТНОМ ТИПЕ ДАННЫХ

Представлена характеризация эффективных абстрактных типов данных с точки зрения теории рекурсии. Основным инструментом является понятие вычислимости многосортных абстрактных структур. Это понятие имеет некоторые максимальные свойства при естественных условиях.

Рассмотрены связи между специальными свойствами одного определенного класса вычислимых функций в абстрактной структуре и существованием некоторых специальных нумераций.

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A characterization of the effective abstract data types from the recursion theoretical point of view is presented. The main tool is a notion of computability on many-sorted abstract structures. This notion has certain maximal properties under natural conditions.

The relationships between certain special properties of the class of the computable functions in an abstract structure and the existence of some special enumerations of it are considered.

1. INTRODUCTION

An abstract data type (**ADT**) is usually considered as a class of many-sorted first order structures closed with respect to isomorphism [1, 2, 4, 6]. Using only this property, we are going to discuss the following problems:

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- Define the class of computable functions on an **ADT**.
- Characterize those **ADT** which are effective.

It is natural to insist that the notion of effective computability on an **ADT** agrees with the classical notion of computability on the natural numbers. In other words, over all structures on the natural numbers in our class the computable functions should be among the relatively partial recursive functions. We call this property *effectiveness.*

Considering the **ADT** as a class of structures, the second condition is that our notion of computability should be *invariant* with respect to isomorphisms, i.e. the class of computable functions of a given structure is preserved under isomorphisms.

Our third assumption concerns the use of the sorts during the computation. We consider two kinds of sorts — "*effectively enumerable*" and "*general*" ones. During the computation of a function θ we allow a search trough the data of the effectively enumerable sorts while for the general sorts a search is not allowed. This idea is described by the so-called *substructure property* of the computability, defined in the next section.

In the first part of the paper we present a notion of computability having the above properties. From the normal form of the computable functions on an **ADT**, given in Section 4, it will be clear that the so defined functions are effective in the intuitive sense. Moreover, each computability having the above three properties is weaker than our notion.

Having an appropriate notion of computability on an **ADT**, in the second part of the paper we shall define the so-called *effective data types* with respect to this computability. It will be proven that a data type is effective with respect to all programming languages iff it admits an effective enumeration.

2. PRELIMINARIES

Let a many-sorted signature $\Sigma = (\mathbb{S}, \mathbb{E}, \mathbb{F}, \mathbb{P}, \rho)$ with equality be fixed. Here $\mathbb{S} = \{1, \ldots, m\}$ is the set of sorts; $\mathbb{E} \subseteq \mathbb{S}$ is the set of the effectively enumerable sorts; $\mathbb{F} = \{f_1, \ldots, f_n\}$ is the set of functional symbols; $\mathbb{P} = \{T_1, \ldots, T_k\}$ is the set of predicate symbols; and ρ is a mapping which assigns to each f_i of $\mathbb{F}$ a type $\rho(f_i)$ over $\mathbb{S}$ of the form $(s_1, \ldots, s_{a_i}, s)$, where $s_1, \ldots, s_{a_i}$ are the sorts of the arguments and s is the sort of the result, and it assigns to each T_j of $\mathbb{P}$ a type $\rho(T_j)$ over $\mathbb{S}$ of the form $(s_1, \ldots, s_{b_j})$ for some $s_1, \ldots, s_{b_j}$ of $\mathbb{S}$. The equality for each sort is supposed. The only difference from the usual definition is that we include the set $\mathbb{E}$ of the effectively enumerable sorts as a part of Σ.

Let $\mathfrak{A} = (A_1, A_2, \ldots, A_m;\ \theta_1, \theta_2, \ldots, \theta_n;\ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ be a many-sorted structure of signature Σ, where for all $s \in \mathbb{S}$ the initial set A_s of sort s is denumerable and non empty; $\theta_1, \theta_2, \ldots, \theta_n$ are the initial functions, $\theta_i : A_{s_1} \times \ldots \times A_{s_{a_i}} \rightharpoonup A_s$; $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ are the initial predicates, $\Sigma_j : A_{s_1} \times \ldots \times A_{s_{b_j}} \rightharpoonup \{0, 1\}$ (0 for true, 1 for false). If $\theta : A_{s_1} \times \ldots \times A_{s_a} \rightharpoonup A_s$ for some $s_1, \ldots, s_a, s \in \mathbb{S}$, then we shall call the function θ of type $(s_1, \ldots, s_a, s)$ *correctly defined.* By $\mathcal{F}_{\mathfrak{A}}$ we shall denote the set of all correctly defined partial functions on $\mathfrak{A}$, i.e. of a fixed type on $\mathfrak{A}$.

Let $\mathcal{A}$ be a class of many-sorted structures of signature Σ.

Computability on $\mathcal{A}$ we shall call every mapping C on $\mathcal{A}$ such that if $\mathfrak{A} \in \mathcal{A}$, then $C(\mathfrak{A}) \subseteq \mathfrak{F}_{\mathfrak{A}}$, i.e. $C(\mathfrak{A})$ is a set of correctly defined functions on $\mathfrak{A}$.

Denote by N the set of all natural numbers. If the structure $\mathfrak{A}$ is on N, i.e. $A_1 = A_2 = \cdots = A_m = N$, then a function θ of $\mathfrak{F}_{\mathfrak{A}}$ is called *partial recursive in* $\mathfrak{A}$ iff there exists an enumeration operator Γ such that if the graph of θ is G_θ, then $G_\theta = \Gamma(\theta_1, \theta_2, \ldots, \theta_n,\ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ [9].

2.1. Definition. A computability C is called *effective* if whenever $\mathfrak{A} \in \mathcal{A}$ and $\mathfrak{A}$ is a structure on N, then all elements of $C(\mathfrak{A})$ are partial recursive in $\mathfrak{A}$.

Let $\mathfrak{A} = (A_1, \ldots, A_m;\ \theta_1, \theta_2, \ldots, \theta_n;\ \Sigma_1, \Sigma_2, \ldots, \Sigma_k)$ and $\mathfrak{B} = (B_1, \ldots, B_m;\ \varphi_1, \varphi_2, \ldots, \varphi_n;\ \sigma_1, \sigma_2, \ldots, \sigma_k)$ be many-sorted structures of signature Σ. Consider an one-to-one mapping α_s from B_s onto A_s for all s of $\mathbb{S}$.

The m-tuple $\langle \alpha_1, \ldots, \alpha_m \rangle$ is called Σ-*isomorphism* from $\mathfrak{B}$ to $\mathfrak{A}$ iff the following conditions hold:

(i) $\alpha_s(\varphi_i(x_1, \ldots, x_{a_i})) \simeq \theta_i(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{a_i}}(x_{a_i}))$ for all $x_1 \in B_{s_1}, \ldots, x_{a_i} \in B_{s_{a_i}}$;

(ii) $\sigma_j(x_1, \ldots, x_{b_j}) \simeq \Sigma_j(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{b_j}}(x_{b_j}))$ for all $x_1 \in B_{s_1}, \ldots, x_{b_j} \in B_{s_{b_j}}$.

2.2. Definition. A computability C is called *invariant* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ belong to $\mathcal{A}$, $\langle \alpha_1, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$ and $\theta \in C(\mathfrak{A})$, θ is of type $(s_1, \ldots, s_a, s)$, then there exists a function $\varphi \in C(\mathfrak{B})$ of the same type as θ such that for all $x_1 \in B_{s_1}, \ldots, x_a \in B_{s_a}$

$$(*) \qquad \alpha_s(\varphi(x_1, \ldots, x_a)) \simeq \theta(\alpha_{s_1}(x_1), \ldots, \alpha_{s_a}(x_a)).$$

The structure $\mathfrak{B}$ is called *an extension* of $\mathfrak{A}$ iff the following conditions hold:

(i) $A_s \subseteq B_s$ for all $s \in \mathbb{S}$, but $A_s = B_s$ for all $s \in \mathbb{E}$;

(ii) $\theta_i(t_1, \ldots, t_{a_i}) \simeq \varphi_i(t_1, \ldots, t_{a_i})$ for all $t_1 \in A_{s_1}, \ldots, t_{a_i} \in A_{s_{a_i}}$;

(iii) $\Sigma_j(t_1, \ldots, t_{b_j}) \simeq \sigma_j(t_1, \ldots, t_{b_j})$ for all $t_1 \in A_{s_1}, \ldots, t_{b_j} \in A_{s_{b_j}}$.

By $\mathfrak{A} \subseteq \mathfrak{B}$ we denote the fact that the many-sorted structure $\mathfrak{B}$ is an extension of $\mathfrak{A}$.

Let $|\mathfrak{A}| = A_1 \cup \ldots \cup A_m$.

2.3. Definition. A computability C has *a substructure property* if whenever $\mathfrak{A}$ and $\mathfrak{B}$ are elements of $\mathcal{A}$, $\mathfrak{A} \subseteq \mathfrak{B}$ and $\theta \in C(\mathfrak{A})$, then there exists a function $\varphi \in C(\mathfrak{B})$ of the same type as θ such that for all $t_1, \ldots, t_a$ of $|\mathfrak{A}|$

$$(**) \qquad \theta(t_1, \ldots, t_a) \simeq \varphi(t_1, \ldots, t_a).$$

To explain the last property assume that $\theta \in C(\mathfrak{A})$. Now the above condition follows from the assumption that in the course of the computation of θ if an additional information is needed, then it consists only of elements belonging to some of the effectively enumerable sorts.

Let C_1 and C_2 be two computabilities on $\mathcal{A}$. C_1 is said to be *weaker than* C_2 *on* $\mathcal{A}$ ($C_1 \subseteq_{\mathcal{A}} C_2$) iff $C_1(\mathfrak{A}) \subseteq C_2(\mathfrak{A})$ for all $\mathfrak{A}$ of $\mathcal{A}$.

In the next section we shall present a concept of computability satisfying these properties and such that if the class $\mathcal{A}$ is rich enough, then each computability, which has the above properties, is weaker than ours.

3. A MAXIMAL CONCEPT OF COMPUTABILITY ON MANY-SORTED STRUCTURES

Let $\mathfrak{A} = (\overline{A};\ \overline{\theta};\ \overline{\Sigma})$ be a many-sorted structure of signature Σ.

Combining the assumptions from the previous section, we come to the following technical notion. Suppose that $\mathfrak{A}$ is denumerable and A_s is infinite for each $s \in \mathbb{E}$.

For each sort s consider an one-to-one mapping α_s from a subset of N onto A_s. Let $\mathfrak{B} = (\overline{N};\ \overline{\varphi};\ \overline{\sigma})$ be a partial many-sorted structure of signature Σ on the natural numbers.

3.1. Definition. The tuple $\langle \alpha_1, \ldots, \alpha_m;\ \mathfrak{B} \rangle$ is called *an enumeration of* $\mathfrak{A}$ iff the following conditions hold:

(i) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{a_i} \in \mathrm{dom}(\alpha_{s_{a_i}})$ and $\varphi_i(x_1, \ldots, x_{a_i})$ is defined, then $\varphi_i(x_1, \ldots, x_{a_i}) \in \mathrm{dom}(\alpha_s)$;

(ii) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{a_i} \in \mathrm{dom}(\alpha_{s_{a_i}})$, then $\alpha_s(\varphi_i(x_1, \ldots, x_{a_i})) \simeq \theta_i(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{a_i}}(x_{a_i}))$;

(iii) if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_{b_j} \in \mathrm{dom}(\alpha_{s_{b_j}})$, then $\sigma_j(x_1, \ldots, x_{b_j}) \simeq \Sigma_j(\alpha_{s_1}(x_1), \ldots, \alpha_{s_{b_j}}(x_{b_j}))$;

(iv) for all effectively enumerable sorts $s \in \mathbb{E}$: $\mathrm{dom}(\alpha_s) = N$.

In fact, $\langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from the structure $(\mathrm{dom}(\alpha_1), \ldots, \mathrm{dom}(\alpha_m);\ \overline{\varphi};\ \overline{\sigma})$ to $\mathfrak{A}$.

Let $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B} \rangle$ be an enumeration of $\mathfrak{A}$.

3.2. Definition. A function θ is *admissible* in $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B} \rangle$ if there exists a function φ over N, partial recursive in $\mathfrak{B}$, such that if $x_1 \in \mathrm{dom}(\alpha_{s_1}), \ldots, x_a \in \mathrm{dom}(\alpha_{s_a})$, then:

(i) if $\varphi(x_1, \ldots, x_a)$ is defined, then $\varphi(x_1, \ldots, x_a) \in \mathrm{dom}(\alpha_s)$;

(ii) $\alpha_s(\varphi(x_1, \ldots, x_a)) \simeq \theta(\alpha_{s_1}(x_1), \ldots, \alpha_{s_a}(x_a))$.

3.3. Definition. θ is *computable* in $\mathfrak{A}$ iff θ is admissible in every enumeration of $\mathfrak{A}$.

The class of all computable functions in $\mathfrak{A}$ we shall denote by $\mathfrak{C}^*(\mathfrak{A})$.

From the above definitions and the Normal Form Theorem in Section 4 the next proposition follows directly.

3.4. Proposition. *The computability $\mathfrak{C}^*$ on $\mathcal{A}$ is effective, invariant and has the substructure property.* ■

Moreover, we have the following theorem.

The class $\mathcal{A}$ is closed under isomorphisms if whenever $\mathfrak{A} \in \mathcal{A}$ and $\langle \alpha_1, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{A}$.

The class $\mathcal{A}$ is closed with respect to extensions if whenever $\mathfrak{A} \in \mathcal{A}$ and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{B} \in \mathcal{A}$.

3.5. Theorem. *Let $\mathcal{A}$ be a class closed under isomorphisms and with respect to extensions. Every computability C over $\mathcal{A}$ which is effective, invariant and has the substructure property is weaker than $\mathfrak{C}^*$ on $\mathcal{A}$ ($C \subseteq_{\mathcal{A}} \mathfrak{C}^*$).*

Proof. Let C be a computability on $\mathcal{A}$ with the desired properties, $\mathfrak{A} \in \mathcal{A}$ and $\theta \in C(\mathfrak{A})$. Consider an enumeration $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B} \rangle$ on $\mathfrak{A}$, where $\mathfrak{B} = (\overline{N};\ \overline{\varphi};\ \overline{\sigma})$. Let $\mathfrak{B}' = (\mathrm{dom}(\alpha_1), \ldots, \mathrm{dom}(\alpha_m);\ \overline{\varphi};\ \overline{\sigma})$. So $\langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ is a Σ-isomorphism from $\mathfrak{B}'$ to $\mathfrak{A}$. Hence, by the invariant property of C, there exists a function $\varphi' \in C(\mathfrak{B}')$ such that $(*)$ is true. And $\mathfrak{B}' \subseteq \mathfrak{B}$. By the substructure property there exists $\varphi \in C(\mathfrak{B})$ such that $(**)$ holds. But $\mathfrak{B}$ is a structure on the natural numbers and by the effectiveness of C φ is partial recursive in $\mathfrak{B}$. So θ is admissible in $\langle \alpha_1, \alpha_2, \ldots, \alpha_m,\ \mathfrak{B} \rangle$, and hence $\theta \in \mathfrak{C}^*(\mathfrak{A})$. ∎

4. NORMAL FORM THEOREM

The presented approach to the notion of computability is called "external". It was used first by Lacombe in [5]. The equivalence between Lacombe's notion of "∀-admissibility" and search computability on total structures with equality was considered by Moschovakis in [7, 8]. This approach over arbitrary structures (single-sorted) was extended by Soskov in [11] and further external characterizations of other well-known concepts of abstract computability as prime computability [8], computability by means of effective definitional schemes [3, 10] and definability by logic programs were presented in [12, 14, 15]. The idea to consider the behavior of a computability on a class of structures and the concepts of maximal computabilities among these ones satisfying some natural conditions, were introduced in [13]. Our concept of maximal computability on many-sorted structures combines two maximal concepts — of search computability of Moschovakis (over the effectively enumerable sorts) and Friedman's computability (over the general ones).

While the external approach leads usually to maximal concepts of computability, it is necessary to show that the computable functions are "effective" in the intuitive sense. So we need a normal form of the computable functions on $\mathfrak{A}$. From this form it will be clear that these functions are computable by means of some reasonable algorithms.

Suppose that an infinite list of variables of sort s, for each sort s of $\mathbb{S}$, is fixed. *Terms of given sort* in Σ are defined as usual.

4.1. Definition. Let Π be a finite conjunction of atomic formulae and negated atomic formulae, τ — a term of sort s, and $Y_1, \ldots, Y_b$ — variables with sorts of $\mathbb{E}$. The expression of the form $\exists Y_1 \ldots \exists Y_b(\Pi \supset \tau)$ is called an *s-conditional expression*.

Let $Q = \exists Y_1 \ldots \exists Y_b(\Pi \supset \tau)$ be an s-conditional expression with free variables among $X_1, \ldots, X_a$ and $t_1, \ldots, t_a \in |\mathfrak{A}|$. The value $Q_{\mathfrak{A}}(X_1/t_1, \ldots, X_a/t_a)$ of Q is the set

$$\{\tau_{\mathfrak{A}}(Y_1/p_1,\dots,Y_b/p_b,X_1/s_1,\dots,X_a/s_a):$$
$$\Pi_{\mathfrak{A}}(Y_1/p_1,\dots,Y_b/p_b,X_1/t_1,\dots,X_a/t_a)\cong 0,$$
$$\text{for some } p_1,\dots,p_b \text{ with sorts as } Y_1,\dots,Y_b\}.$$

4.2. Proposition. *If $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ is an enumeration of $\mathfrak{A}$, $\tau(X_1,\dots,X_a)$ is a term of sort s, $\Pi(X_1,\dots,X_a)$ is an atomic formula and $x_1\in \mathrm{dom}(\alpha_{s_1}),\dots,x_a\in \mathrm{dom}(\alpha_{s_a})$, then:*

(i) $\alpha_s(\tau_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a))\simeq \tau_{\mathfrak{A}}(X_1/\alpha_{s_1}(x_1),\dots,X_a/\alpha_{s_a}(x_a))$;

(ii) $\Pi_{\mathfrak{B}}(X_1/x_1,\dots,X_a/x_a)\simeq \Pi_{\mathfrak{A}}(X_1/\alpha_{s_1}(x_1),\dots,X_a/\alpha_{s_a}(x_a))$. ■

4.3. Definition. A function θ is said to be *definable on* $\mathfrak{A}$ iff for some recursively enumerable set $\{Q^v\}_{v\in V}$ of s-conditional expressions with free variables among $Z_1,\dots,Z_r,X_1,\dots,X_a$ and for some fixed elements $q_1,\dots,q_r$ of $|\mathfrak{A}|$ the following equivalence is true:

$$\theta(t_1,\dots,t_a)\simeq t \iff \exists v(v\in V \ \&\ t\in Q^v_{\mathfrak{A}}(Z_1/q_1,\dots,Z_r/q_r,X_1/t_1,\dots,X_a/t_a)).$$

4.4. Theorem (Normal Form Theorem). *The function θ is computable in $\mathfrak{A}$ iff θ is definable on $\mathfrak{A}$.*

Proof. The fact that every definable function on $\mathfrak{A}$ is admissible in all enumerations of $\mathfrak{A}$ follows from the last Proposition 4.2.

To prove the other direction, we will actually prove the contrapositive. Thus we suppose that θ is admissible in $\mathfrak{A}$, but it is not definable on $\mathfrak{A}$. We use this fact to construct an enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ of $\mathfrak{A}$ such that θ is not admissible in $\langle\overline{\alpha},\ \mathfrak{B}\rangle$. The basic idea of the construction is to ensure that the pullback of θ (by $\overline{\alpha}$) is not partial recursive in $\mathfrak{B}$ by diagonalizing over all possible partial recursive in $\mathfrak{B}$ functions.

For ease of exposition we will suppose that all functions and predicates of the signature Σ are unary and $\theta: A_{s_d}\to A_{s_r}$ is of type (s_d,s_r). The general case is a trivial rewriting of the argument below. If $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ is an enumeration of $\mathfrak{A}$, where $\mathfrak{B}=(\overline{N};\ \varphi_1,\varphi_2,\dots,\varphi_n;\ \sigma_1,\sigma_2,\dots,\sigma_k)$, denote by

$$\langle\mathfrak{B}\rangle=\{\langle i,x,\varepsilon\rangle : (1\le i\le n\ \ \&\ \ \varphi_i(x)\simeq\varepsilon)\vee(n+1\le i\le n+k\ \&\ \sigma_{i-n}(x)\simeq\varepsilon)\}.$$

It is clear that a function φ is partial recursive in $\{\varphi_1,\varphi_2,\dots,\varphi_n;\sigma_1,\sigma_2,\dots,\sigma_k\}$ iff φ is partial recursive in $\langle\mathfrak{B}\rangle$.

The enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle$ we shall construct by stages.

On each stage l we find a finite approximation (called *finite part*) Δ_l of $\langle\overline{\alpha},\ \mathfrak{B}\rangle$, so that on even stages we assure that $\mathrm{range}(\alpha_s)=A_s$ for every sort $s\in\mathbb{S}$ and $\mathrm{dom}(\alpha_s)=N$ for $s\in\mathbb{E}$ (effectively enumerable sorts).

On odd stages $l=2n+1$, if Γ_n is the n-th enumeration operator, then for every enumeration $\langle\overline{\alpha},\ \mathfrak{B}\rangle\supseteq\Delta_{l+1}$ there exists $x\in\mathrm{dom}(\alpha_{s_d})$ such that one of the following conditions is not fulfilled:

(A) $\forall y(\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle) \Longrightarrow y \in \mathrm{dom}(\alpha_{s_r}))$;

(B) $\forall t(t \in \theta(\alpha_{s_d}(x)) \Longleftrightarrow \exists y(\alpha_{s_r}(y) \simeq t \ \&\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)))$.

First let us fix some notations.

4.5. Definition. *A finite part* Δ_l is called each tuple $\Delta_l = \langle \overline{\delta}^l, \overline{H}^l, \overline{\varphi}^l, \overline{\sigma}^l\rangle$, where:

(1) δ_s^l is one-to-one partial mapping $N \rightharpoonup A_s$ and $\mathrm{dom}(\delta_s^l)$ is finite for each $s \in \mathbb{S}$;

(2) H_s^l is a finite subset of N, so that $\mathrm{dom}(\delta_s^l) \cap H_s^l = \emptyset$ for each $s \in \mathbb{S}$, but $H_s^l = \emptyset$ for $s \in \mathbb{E}$;

(3) $\varphi_i^l : H_{s_i}^l \rightharpoonup H_s^l \cup \mathrm{dom}(\delta_s^l)$, where $\rho(f_i) = (s_i, s)$ for $1 \leq i \leq n$;

(4) $\sigma_j^l : H_{s_j}^l \rightharpoonup \{0, 1\}$, where $\rho(T_j) = (s_j)$ for $1 \leq j \leq k$.

Given Δ_l and Δ_q — finite parts, denote by $\Delta_l \subseteq \Delta_q$ the fact that $\delta_s^l \leq \delta_s^q$, $H_s^l \subseteq H_s^q$ for $s \in \mathbb{S}$ and $\varphi_i^l \leq \varphi_i^q$, $i = 1 \ldots n$, $\sigma_j^l \leq \sigma_j^q$, $j = 1 \ldots k$. Here by $\varphi \leq \psi$ we mean that if $\varphi(x)$ is defined, then $\psi(x)$ is also defined and $\varphi(x) \simeq \psi(x)$.

4.6. Definition. If Δ_l is a finite part and $\langle \overline{\alpha}, \mathfrak{B}\rangle$ is an enumeration, then $\Delta_l \subseteq \langle \overline{\alpha}, \mathfrak{B}\rangle$ iff:

(1) $\delta_s^l \leq \alpha_s$ for $s \in \mathbb{S}$;

(2) $\mathrm{dom}(\alpha_s) \cap H_s^l = \emptyset$ for each $s \in \mathbb{S}$;

(3) $\varphi_i^l \leq \varphi_i$ for $1 \leq i \leq n$;

(4) $\sigma_j^l \leq \sigma_j$ for $1 \leq j \leq k$.

Construction:

Stage $l = 0$. $H_s^0 = \emptyset$ and $\delta_s^0, \varphi_i^0, \sigma_j^0$ are totally undefined.

Stage $l = 2n$, $n > 0$. For each sort s consider the first $x_s \notin \mathrm{dom}(\delta_s^l) \cup H_s^l$ and $t_s \notin \mathrm{range}(\delta_s^l)$. If there is no such t_s, do nothing. Otherwise, let $\delta_s^{l+1}(x_s) = t_s$ and $\Delta^{l+1} = \langle \overline{\delta}^{l+1}, \overline{H}^l, \overline{\varphi}^l, \overline{\sigma}^l\rangle$.

Stage $l = 2n + 1$. Let $\Delta_l = \langle \overline{\delta}^l, \overline{H}^l, \overline{\varphi}^l, \overline{\sigma}^l\rangle$, where $\mathrm{dom}(\delta_s^l) = \{w_1^s, \ldots, w_{k_s}^s\}$ and $\mathrm{range}(\delta_s^l) = \{t_1^s, \ldots, t_{k_s}^s\}$.

Let Γ_n be the n-th enumeration operator. So, if $R \subseteq N$, then

$$z \in \Gamma_n(R) \Longleftrightarrow \exists v(\langle v, z\rangle \in W_n \ \& \ E_v \subseteq R),$$

where W_n is the recursively enumerable set with code n and E_v is the finite set with code v.

Let $x \in N$ and $x \notin H_{s_d}^l$. Then

$$\forall y(\langle x, y\rangle \in \Gamma_n(R) \Longleftrightarrow \exists v(\langle v, x, y\rangle \in W_n \ \& \ E_v \subseteq R)).$$

Denote by $U_{n,x} = \{\langle v, y\rangle \ : \ \langle v, x, y\rangle \in W_n\}$.

The main tool is the construction of a definable function $\xi : A_{s_d} \rightharpoonup A_{s_r}$, based on Γ_n, using the following translation. By $\overline{t}$ we denote the list of all elements of $\mathrm{range}(\delta_1^l) \cup \ldots \cup \mathrm{range}(\delta_m^l)$. For each sort s consider an one-to-one mapping var_s from

N onto the set of all variables of sort s. Let $\mathrm{var}_{s_d}(x) = X$ and $\overline{W} = \{W_1, \ldots, W_a\}$ be the corresponding variables to the elements of the set $\mathrm{dom}(\delta_1^l) \cup \ldots \cup \mathrm{dom}(\delta_m^l)$. Let c_{s_r} be a new constant of the sort s_r.

4.7. Lemma. *Let $p \in A_{s_d}, t \in A_{s_r}$. There exists an effective way, given $v, y \in N$, to define an s_r-conditional expression $Q^{\langle v,y \rangle}(X, \overline{W})$ such that:*

(1) *if $t \in Q_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t})$, then there exists a finite part $\Lambda \supseteq \Delta_l$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Lambda$: $E_v \subseteq \langle \mathfrak{B} \rangle$ & $\alpha_{s_d}(x) = p$ & $\alpha_{s_r}(y) = t$;*

(2) *if $t \notin Q_{\mathfrak{A}}^{\langle v,y \rangle}(X/p, \overline{W}/\overline{t})$, then one of the following is true:*
   (a) *there exists $\Lambda \supseteq \Delta_l$ such that for every $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Lambda$*
   $(E_v \subseteq \langle \mathfrak{B} \rangle$ & $\alpha_{s_d}(x) = p \Longrightarrow y \notin \mathrm{dom}(\alpha_{s_r}))$, or
   (b) *for every $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l$ $(E_v \subseteq \langle \mathfrak{B} \rangle$ & $\alpha_{s_d}(x) = p \Longrightarrow \alpha_{s_r}(y) \not\simeq t)$.*

Proof. Let $v, y \in N$ be fixed. Consider the finite set E_v with the code v.

4.8. Definition. The finite set E_v is called *correct* if the following conditions hold:

(1) each element u of E_v is of the form $u = \langle i, z, \varepsilon \rangle$ and $1 \leq i \leq n$ or $(n+1 \leq i \leq k+n$ and $\varepsilon \in \{0,1\})$;
(2) if $\langle i, z, \varepsilon_1 \rangle \in E_v$ and $\langle i, z, \varepsilon_2 \rangle \in E_v$, then $\varepsilon_1 = \varepsilon_2$;
(3) if $\langle i, z, \varepsilon \rangle \in E_v$, $1 \leq i \leq n$, $z \in \mathrm{dom}(\delta_{s_i}^l)$, where $\rho(f_i) = (s_i, s)$, and $\varphi_i^l(z)$ is defined, then $\varphi_i^l(z) \simeq \varepsilon$;
(4) if $\langle j, z, \varepsilon \rangle \in E_v$, $n+1 \leq j \leq n+k$, $z \in \mathrm{dom}(\delta_{s_j}^l)$, where $\rho(T_{j-n}) = (s_j)$, and $\sigma_{j-n}^l(z)$ is defined, then $\sigma_{j-n}^l(z) \simeq \varepsilon$.

If E_v is not correct, then put $Q^{\langle v,y \rangle} = (X \neq X \supset c_{s_r})$.
Let E_v is correct. Denote by

$$\begin{aligned} M_s = &\{z : \exists i, \varepsilon(\langle i, z, \varepsilon \rangle \in E_v \,\&\, (i \leq n \,\&\, \rho(f_i) = (s, s_1)) \vee (n < i \,\&\, \rho(T_{i-n}) = (s)\} \\ &\cup \{z : \exists i, z_1(\langle i, z_1, z \rangle \in E_v \,\&\, i \leq n \,\&\, \rho(f_i) = (s_i, s)\}. \end{aligned}$$

$$D_s = \begin{cases} N & \text{if } s \in \mathbb{E}, \\ \mathrm{dom}(\delta_s^l) \cup \{x\} & \text{if } s = s_d \,\&\, s_d \notin \mathbb{E}, \\ \mathrm{dom}(\delta_s^l) & \text{otherwise.} \end{cases}$$

Let $K_s^0 = M_s \cap D_s$. Suppose that for each s of $\mathbb{S}$ the set K_s^q is defined. Then

$$\begin{aligned} K_s^{q+1} = \{z : \exists i, z_1(\langle i, z_1, z \rangle \in E_v \,\&\, z \notin K_s^0 \cup \cdots \cup K_s^q \,\&\, i \leq n \\ \&\, \rho(f_i) = (s_i, s) \,\&\, z_1 \in K_{s_i}^q)\}. \end{aligned}$$

Since the set E_v is finite, there exists an r so that $K_s^{r+1} = \emptyset$.

Let $K_s = K_s^0 \cup \cdots \cup K_s^r$ and if $z \in K_s^q$, define $|z| = q$. For each element z of K_s we define a term τ^z by induction on $|z|$.

Let $|z| = 0$, then $\tau^z = \mathrm{var}_s(z)$. Let $|z| = q+1$, $\langle i, z_1, z \rangle \in E_v$ for some z_1, so that $|z_1| = p < q+1$. Then $\tau^z = f_i(\tau^{z_1})$.

Some of the elements of E_v we shall call *appropriate*. Namely, $u = \langle i, z, z_1 \rangle$ from E_v is appropriate if:

(1) $i \leq n$ and $\rho(f_i) = (s, s_1)$, $z \in K_s$, $z_1 \in K_{s_1}$;

(2) $n < i$ and $\rho(T_{i-n}) = (s)$, $z \in K_s$.

So we are ready to define the desired conditional expression $Q^{(v,y)}(X, \overline{W})$. If for some $s \in \mathbb{S}$ $H^l_s \cap K_s \neq \emptyset$, then we define $Q^{(v,y)} = (X \neq X \supset c_{s_r})$ as in the case where E_v is not correct. We keep in mind that the initial functions of an enumeration on $\mathfrak{A}$ should be closed under the domain of the enumeration (Definition 3.1 (i)).

Let for all $s \in \mathbb{S}$ $H^l_s \cap K_s = \emptyset$. Then we check if $y \in K_{s_r}$. If not, we put again $Q^{(v,y)} = (X \neq X \supset c_{s_r})$.

Otherwise, for each appropriate element $u = \langle i, z, \varepsilon \rangle \in E_v$ we find the corresponding formula Π^u under the following rule:

(1) if $i \leq n$, then Π^u is $f_i(\tau^z) = \tau^\varepsilon$;

(2) if $n < i$, then Π^u is $T_{i-n}(\tau^z)$ for $\varepsilon = 0$ and Π^u is $\neg T_{i-n}(\tau^z)$ for $\varepsilon = 1$.

Let $u_1, \ldots, u_q$ be all appropriate elements of E_v. Denote by Π the conjunction $\Pi^{u_1} \,\&\, \cdots \,\&\, \Pi^{u_q} \,\&\, V$, where

$$V = \begin{cases} \mathop{\&}\limits_{\substack{z_i \neq z_j, s \in \mathbb{S} \\ z_i, z_j \in K_s}} \mathrm{var}_s(z_i) \neq \mathrm{var}_s(z_j) \,\&\, \mathop{\&}\limits_{z \in K_{s_d} \setminus \{x\}} X \neq \mathrm{var}_{s_d}(z) & \text{if } x \notin \mathrm{dom}(\delta^l_{s_d}); \\ \mathop{\&}\limits_{\substack{z_i \neq z_j, s \in \mathbb{S} \\ z_i, z_j \in K_s}} \mathrm{var}_s(z_i) \neq \mathrm{var}_s(z_j) \,\&\, X = \mathrm{var}_{s_d}(w) & \text{if } x = w \in \mathrm{dom}(\delta^l_{s_d}). \end{cases}$$

Let $y_1, \ldots, y_b$ be all elements of K_s for those $s \in \mathbb{E}$ (effectively enumerable sorts) not belonging to $\mathrm{dom}(\delta^l_1) \cup \ldots \cup \mathrm{dom}(\delta^l_m) \cup \{x\}$ and $\mathrm{var}_{s_1}(y_1) = Y_1, \ldots, \mathrm{var}_{s_b}(y_b) = Y_b$. From the construction it follows that the variables of Π are among $X, W_1, \ldots, W_a, Y_1, \ldots, Y_b$.

Define

$$Q^{(v,y)}(X, \overline{W}) = \exists Y_1 \ldots \exists Y_b (\Pi \supset \tau^y).$$

Let consider some properties of the constructed in this way conditional expression $Q^{(v,y)}$. From the above construction and Proposition 4.2 follows:

4.9. Proposition. *Let $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_l$, $\alpha_{s_d}(x) = p$, $E_v \subseteq \langle \mathfrak{B} \rangle$ and $\alpha_{s_i}(y_i) = q_i$, $i = 1, \ldots, b$. Then:*

(1) E_v is correct;

(2) $K_s \subseteq \mathrm{dom}(\alpha_s)$ and $H^l_s \cap K_s = \emptyset$ for each $s \in \mathbb{S}$;

(3) $\forall z \in K_s (\alpha_s(z) \simeq \tau^z_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b))$ for $s \in \mathbb{S}$;

(4) $\Pi_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b) \simeq 0$. ∎

We are ready to prove that $Q^{(v,y)}$ satisfies the conditions of Lemma 4.7.

Case 1. Let $t \in Q^{(v,y)}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})$. So E_v is correct, $H^l_s \cap K_s = \emptyset$ for each sort s and $y \in K_{s_r}$. Thus there exist different elements $q_1, \ldots, q_b$ of $|\mathfrak{A}|$ with sorts $s_1, \ldots, s_b$ from $\mathbb{E}$, so that

$$\Pi_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b) \simeq 0, \; t \simeq \tau^y_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}, Y_1/q_1, \ldots, Y_b/q_b).$$

Define a finite part $\Lambda \supseteq \Delta^l$, where $\Lambda = \langle\overline{\lambda},\ \overline{H},\ \overline{\varphi},\ \overline{\sigma}\rangle$, as follows: Let $\lambda_{s_d}(x) = p$, $\lambda_{s_1}(y_1) = q_1, \ldots, \lambda_{s_b}(y_b) = q_b$ and $\lambda_s(z) \simeq \delta^l_s(z)$ for all other z, $s \in \mathbb{S}$. Let $H_s = H^l_s \cup (M_s \setminus K_s)$ for each sort s.

Define $\varphi_i : \varphi^l_i \leq \varphi_i$ for $i = 1, \ldots, n$ and $\sigma_j : \sigma^l_j \leq \sigma_j$ for $j = 1, \ldots, k$, so that the extensions are defined under the rule that for each element $u \in E_v$, which is not appropriate:

if $u = \langle i, z_1, z_2\rangle$, $i \leq n$, then $\varphi_i(z_1) \simeq z_2$, and,

if $u = \langle j, z, \varepsilon\rangle$, $n < j$, then $\sigma_{j-n}(z) \simeq \varepsilon$.

Let $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Lambda$. From the definition of Λ and Proposition 4.2 we have $E_v \subseteq \langle\mathfrak{B}\rangle$, $\alpha_{s_d}(x) = p$, and since $y \in K_{s_r}$, then from Proposition 4.9 it follows that $\alpha_{s_r}(y) = t$. So the condition (1) from Lemma 4.7 is satisfied.

Case 2. Let $t \notin Q^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})$.

We have the following possibilities:

(a) E_v is not correct. Then, since for every $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Delta_l : E_v \not\subseteq \langle\mathfrak{B}\rangle$, the condition (2)(b) from Lemma 4.7 is trivially satisfied.

(b) Let E_v is correct:

(b.1) If $H^l_s \cap K_s \neq \emptyset$ for some s, then there is $z \in K_s$ and $z \in H_s$. Therefore for some $i \in \{1 \ldots n\}, z_1 \in K_{s_1}$, $\langle i, z_1, z\rangle \in E_v$. If $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Delta_l$, then it is not possible that $E_v \subseteq \langle\mathfrak{B}\rangle$, since $H_s \cap \mathrm{dom}(\alpha_s) = \emptyset$. Then (2)(b) is fulfilled because of the same argument as in (a).

(b.2) Let $H^l_s \cap K_s = \emptyset$ for all $s \in \mathbb{S}$. Denote by $C^{\langle v,y\rangle} = \exists Y_1 \ldots \exists Y_b(\Pi)$.

(b.2.1) $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \not\simeq 0$. Then suppose that $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Delta_l$, $E_v \subseteq \langle\mathfrak{B}\rangle$ and $\alpha_{s_d}(x) = p$. Let

$$\{y_1, \ldots, y_b\} = (K_{s_1} \cup \ldots \cup K_{s_b}) \setminus (\mathrm{dom}(\delta^l_1) \cup \ldots \cup \mathrm{dom}(\delta^l_m) \cup \{x\}),$$

$s_1, \ldots, s_b \in \mathbb{E}$. From the definition of enumeration of $\mathfrak{A}$ we have that $\alpha_{s_i}(y_i) = q_i$ for $i = 1, \ldots, b$. Then from Proposition 4.9: $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \simeq 0$, which is a contradiction. So (2)(b) holds trivially.

(b.2.2) Let $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \simeq 0$:

(b.2.2) (i) $y \notin K_{s_r}$. Then we construct a finite part $\Lambda \supseteq \Delta_l$ in the same way as in the Case 1, but with only one difference — we put $y \in H_{s_r}$. If $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Lambda$, then we have again that $E_v \subseteq \langle\mathfrak{B}\rangle, \alpha_{s_d}(x) = p$, but since $H_{s_r} \cap \mathrm{dom}(\alpha_{s_r}) = \emptyset$, then $y \notin \mathrm{dom}(\alpha_{s_r})$. Thus the condition (2)(a) from Lemma 4.7 is fulfilled.

(b.2.2) (ii) Let $y \in K_{s_r}$. Suppose that $\langle\overline{\alpha},\ \mathfrak{B}\rangle \supseteq \Delta_l$ and $E_v \subseteq \langle\mathfrak{B}\rangle, \alpha_{s_d}(x) = p$. Then by Proposition 4.9 it follows that

$$y \in \mathrm{dom}(\alpha_{s_r}) \quad \text{and} \quad \alpha_{s_r}(y) \in Q^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}).$$

So $\alpha_{s_r}(y) \not\simeq t$, i.e. the condition (2)(b) of Lemma 4.7 is true. ∎

We return to the proof of Theorem 4.4. For each $u \in U_{n,x}$ denote by $Q^u(X, \overline{W})$ the conditional expression found effectively in Lemma 4.7.

Let fix an $x \notin \mathrm{dom}(\delta^l_{s_d}) \cup H^l_{s_d}$ and $\mathrm{dom}(\delta^l_{s_d}) = \{w_1, \ldots, w_r\}$. Consider the function ξ with the following definition:

$$
\begin{aligned}
t \in \xi(p) \iff & \exists u \in U_{n,x}(t \in Q^u_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})) \\
& \vee \exists u_1 \in U_{n,w_1}(t \in Q^{u_1}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})) \\
& \dots \\
& \vee \exists u_r \in U_{n,w_r}(t \in Q^{u_r}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})).
\end{aligned}
$$

Then ξ is definable and therefore $\xi \neq \theta$. Using this fact we ensure that one of the conditions (A) or (B) is not satisfied. There are two possibilities:

(a) There exist $p \in A_{s_d}$ and $t \in A_{s_r}$ such that $\xi(p) \simeq t$, but $\theta(p) \not\simeq t$.

Let $p \notin \text{range}(\delta^l_{s_d})$. From $\xi(p) \simeq t$ it follows that for some $u \in U_{n,x}$ we have $t \in Q^u_{\mathfrak{A}}(X/p\overline{W}/\overline{t})$. Let $u = \langle v, y\rangle$. From Lemma 4.7 we know that there exists a finite part $\Lambda \supseteq \Delta_l$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Lambda$: $E_v \subseteq \langle \mathfrak{B}\rangle$ & $\alpha_{s_d}(x) = p$ & $\alpha_{s_r}(y) = t$. Define $\Delta_{l+1} = \Lambda$. Then if $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$, there are $x \in \text{dom}(\alpha_{s_d})$ and $y \in \text{dom}(\alpha_{s_r})$ such that $\alpha_{s_r}(y) = t$, $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$, but $\theta(\alpha_{s_d}(x)) \not\simeq t$. So the condition (B) is not valid.

If $p = \delta^l_{s_d}(w)$, then for some $u \in U_{n,w}$ we have $t \in Q^u_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})$. Then using Lemma 4.7 we prove that (B) is not valid analogously.

(b) Let $\xi(p) \not\simeq t$, but $\theta(p) \simeq t$, $p \in A_{s_d}$, $t \in A_{s_r}$. We shall consider the case when $p \notin \text{range}(\delta^l_{s_d})$.

Let have the following situation:

There exist v and y such that $\langle v, y\rangle \in U_{n,x}, Q^{\langle v,y\rangle}(X, \overline{W}) = \exists Y_1 \dots \exists Y_b(\Pi \supset \tau^y)$, $C^{\langle v,y\rangle} = \exists Y_1 \dots \exists Y_b(\Pi)$, $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \simeq 0$ and $y \notin K_{s_r}$. In this case we choose $\Delta_{l+1} = \Lambda$ from the proof of Lemma 4.7, Case 2, (b.2.2) (i). We know that if $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$, then $x \in \text{dom}(\alpha_{s_d})$, $E_v \subseteq \langle \mathfrak{B}\rangle$ and therefore $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$, but $y \notin \text{dom}(\alpha_{s_r})$, i.e. the condition (A) is not valid.

Otherwise,

$$\forall u \in U_{n,x}(C^u_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \simeq 0 \Longrightarrow y \in K_{s_r}).$$

Put $\delta^{l+1}_{s_d}(x) = p$ and $\delta^{l+1}_s(z) \simeq \delta^l_s(z)$ for all other z and s and $\Delta_{l+1} = \langle \overline{\delta}^{l+1}, \overline{H}^l, \overline{\varphi}^l, \overline{\sigma}^l\rangle$. We shall prove that in this case the condition (B) is not valid. Let $\langle \overline{\alpha}, \mathfrak{B}\rangle \supseteq \Delta_{l+1}$. So $\alpha_{s_d}(x) = p$. Suppose that there exists $y \in \text{dom}(\alpha_{s_r})$ such that $\alpha_{s_r}(y) = t$ and $\langle x, y\rangle \in \Gamma_n(\langle \mathfrak{B}\rangle)$. Then for some v: $\langle v, x, y\rangle \in W_n$, i.e. $\langle v, y\rangle \in U_{n,x}$ and $E_v \subseteq \langle \mathfrak{B}\rangle$. From Proposition 4.9 we have that E_v is correct, $H^l_s \cap K_s = \emptyset$ for each $s \in \mathbb{S}$ and $C^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t}) \simeq 0$. Hence $y \in K_{s_r}$. Since $t \notin Q^{\langle v,y\rangle}_{\mathfrak{A}}(X/p, \overline{W}/\overline{t})$, from the proof of Lemma 4.7 it follows that the condition (2)(b) is valid, i.e. $\alpha_{s_r}(y) \neq t$, which is a contradiction.

The case when $p \in \text{range}(\delta^l_{s_d})$ is considered similarly.

End of construction.

Consider an enumeration $\langle \alpha_1, \alpha_2, \dots, \alpha_m, \mathfrak{B}\rangle$, where $\mathfrak{B} = (\overline{N}; \varphi_1, \dots, \varphi_n; \sigma_1, \dots, \sigma_k)$, defined as follows:

$$\alpha_s = \bigcup_{l=0}^{\infty} \delta^l_s \quad \text{for each } s \in \mathbb{S},$$

$$\varphi_i^* = \bigcup_{l=0}^{\infty} \varphi_i^l \text{ for } 1 \le i \le n, \quad \sigma_j^* = \bigcup_{l=0}^{\infty} \sigma_j^l \text{ for } 1 \le j \le k.$$

From the construction it is clear that range$(\alpha_s) = A_s$ for all sorts, dom$(\alpha_s) = N$ for every $s \in \mathbb{E}$ and dom$(\alpha_s) \cap H_s^l = \emptyset$ for $s \in \mathbb{S}$ and $l \in N$.

Let

$$\varphi_i(x) \simeq \begin{cases} \theta_i(\alpha_{s_i}(x)) & \text{if } x \in \text{dom}(\alpha_{s_i}), \\ \varphi_i^*(x) & \text{otherwise,} \end{cases}$$

and

$$\sigma_j(x) \simeq \begin{cases} \Sigma_j(\alpha_{s_j}(x)) & \text{if } x \in \text{dom}(\alpha_{s_j}), \\ \sigma_j^*(x) & \text{otherwise.} \end{cases}$$

Suppose that the function θ is admissible in $\langle \overline{\alpha}, \mathfrak{B} \rangle$. Then for some n, if Γ_n is the enumeration operator with number n, for each $x \in \text{dom}(\alpha_{s_d})$ we have:

(A) $\forall y(\langle x, y \rangle \in \Gamma_n(\langle \mathfrak{B} \rangle) \Longrightarrow y \in \text{dom}(\alpha_{s_r}))$;

(B) $\forall t(t \in \theta(\alpha_{s_d}(x)) \Longleftrightarrow \exists y(\alpha_{s_r}(y) \simeq t \;\&\; \langle x, y \rangle \in \Gamma_n(\langle \mathfrak{B} \rangle)))$.

Let $l = 2n + 1$. Then there exists $x \in \text{dom}(\delta_{s_d}^{l+1})$ such that for every enumeration $\langle \overline{\alpha}, \mathfrak{B} \rangle \supseteq \Delta_{l+1}$ the condition (A) or (B) is not valid. Hence, θ is not admissible in $\langle \overline{\alpha}, \mathfrak{B} \rangle$. ■

4.10. Corollary. *If the structure $\mathfrak{A}$ is single-sorted, then:*

(i) *if the only sort is effectively enumerable, then $\mathcal{C}^*(\mathfrak{A})$ is the class of search computable functions of Moschovakis on $\mathfrak{A}$;*

(ii) *otherwise $\mathcal{C}^*(\mathfrak{A})$ is the class of computable functions of Friedman.* ■

In [17, 18] several notions of computability on **ADT** were considered. A generalized variant of Church-Turing thesis for deterministic and non-deterministic computation is announced. It is easy to see that their notion for deterministic computability — *star computability*, coincides with our in case that all sorts are not effectively enumerable, i.e. $\mathbb{E} = \emptyset$. If $\mathbb{E} = \mathbb{S}$, then our computability coincides with the *projective star-computability*.

Remark. All results could be generalized for **ADT** without equality, considering another algebraic transformations — special homomorphisms instead of isomorphisms [16].

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