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THE MATHEMATICAL PAPERS OF BLAGOVEST DOLAPTCHIEV

An attempt on an analytical bibliography

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Cum moriar, medium solvar et inter opus
Publius Ovidius Naso

The authors of this paper have been, at various times, assistants of the late Prof. Blagovest Ivanov Dolaptchiev at the Chair of Analytical Mechanics in the Mathematical Faculty of the University of Sofia "St. Kliment Ohridski"; moreover, the second one has been his co-author in several articles and books, mainly in the field of Kármán vortex streets, as well as his successor to the Chair. Therefore it has been quite natural for them to make every endeavour to compose an analytical bibliography of the mathematical works of their teacher. The result is this *talis qualis* paper they propose to the benevolent reader.

Some preliminary work has been done intended to ensure authentic bibliographies of the works in question. We use plural, since we have composed two lists: an alphabetical and a chronological bibliography. At that, the first one is divided into two parts, according to the Cyrillic and the Roman alphabet, respectively. They include all printed papers of Blagovest Dolaptchiev we have been able to trace (original studies, text-books, and popular science works). The citation in this paper is made according to the chronological bibliography.

Blagovest Dolaptchiev was born in 1905, December 16, in the provincial town Sliven, and died in 1974, February 3, in Sofia. In 1925–1929 he studied mathematics in the Physico-Mathematical Faculty of the University of Sofia. Immediately afterwards he was appointed assistant at the Chair of Analytical Mechanics (holder Prof. Ivan Tzenov) in the same Faculty, where he worked until his death. In 1935–1937 he was on a specialization assignement abroad, namely in the Göttingen University, where he studied under the direction of Prof. Ludvig Prandtl. It was Prandtl who gave him the theme *A contribution to the stability of Kármán vortex streets and the trajectories of the particular vortices* for his doctorate thesis. In 1937 Dolaptchiev defended his dissertation and took the first doctor's degree on mathematics in the University of Sofia. In 1943 he was appointed Dotzent (associate professor) on analytical mechanics, in 1947 he was promoted extraordinary professor, and in 1951 he was elected ordinary professor in charge of the Chair of Analytical Mechanics, which post he held until his death. In 1967 Dolaptchiev was elected a corresponding member of the Bulgarian Academy of Sciences.

The scientific interests of Blagovest Dolaptchiev pertain chiefly to two domains of rational mechanics: vortex configurations (until 1965) and non-holonomic dynamics (after 1965). In both of them he has a considerable number of publications which are reviewed briefly below. In the course of some decades he was a leading researcher in the theory of Kármán vortex streets. As a matter of fact, he was the man who put the field on a rigorous mathematical basis. As regards non-holonomic dynamics, the Chair of Analytical Mechanics in the University of Sofia enjoys a tradition more than half a century old: after his specialization in Paris under the direction of Paul Appell in 1910–1912, Ivan Tzenov devoted to non-holonomic dynamics the whole of his professional life, proposing several new forms of non-holonomic equations of motion, called now by his name. Dolaptchiev was the first to call the mathematician's attention to a form of dynamical equations of motion of non-holonomic mechanical systems proposed by J. Nielsen, neglected before and possessing according to Dolaptchiev some advantages in comparison with the remaining versions of equations of motion of Lagrangean type, and he developed this thesis in various directions.

The mathematical performances of Dolaptchiev provided him a lively acceptance among experts in these two domains of fluid and solid mechanics at numerous international scientific activities he paid visit to — congresses, conferences, symposia, sessions, meetings, etc., where he delivered reports on his recent achievements. His articles were published in various international mathematical and mechanical journals of repute, as it is seen from his bibliographies.

As regards the Bulgarian comparatively young mathematical and mechanical science, Dolaptchiev belonged to its third generation of professional mathematicians and to its second generation of mathematicians researchers, and he was a bright figure in its development and history. Besides, he was a brilliant representative of the Bulgarian mathematical educational fellowship.

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In 1932 Dolaptchiev published his article *On a family of curve lines traced on a given surface* [1]. By his words, it seeks "those curve lines on a given surface, in any point of which the angle between the main normal of the curve and the normal to the surface has a constant value" (p. 288), say θ. If $k = \mathrm{tg}\theta$ and

$$\boldsymbol{r} = \boldsymbol{r}(u, v) \tag{1}$$

is a parametric equation of the surface, let by definition

$$(2)\qquad E=\left(\frac{\partial \boldsymbol{r}}{\partial u}\right)^2,\quad F=\frac{\partial \boldsymbol{r}}{\partial u}\cdot\frac{\partial \boldsymbol{r}}{\partial v},\quad G=\left(\frac{\partial \boldsymbol{r}}{\partial v}\right)^2,\quad H=\left|\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\right|,$$

$$(3)\qquad D_1=\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\cdot\frac{\partial^2 \boldsymbol{r}}{\partial u^2},\quad D_1'=\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\cdot\frac{\partial^2 \boldsymbol{r}}{\partial u\partial v},\quad D_1''=\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\cdot\frac{\partial^2 \boldsymbol{r}}{\partial v^2},$$

and let

$$(4)\qquad \left\{{1\,1 \atop 1}\right\}=\frac{1}{2H^2}\left(G\frac{\partial E}{\partial u}-2F\frac{\partial F}{\partial u}+F\frac{\partial E}{\partial v}\right),$$

$$(5)\qquad \left\{{1\,1 \atop 2}\right\}=\frac{1}{2H^2}\left(-F\frac{\partial E}{\partial u}+2E\frac{\partial F}{\partial u}-E\frac{\partial E}{\partial v}\right),$$

$$(6)\qquad \left\{{1\,2 \atop 1}\right\}=\frac{1}{2H^2}\left(G\frac{\partial E}{\partial v}-F\frac{\partial G}{\partial u}\right),\quad \left\{{1\,2 \atop 2}\right\}=\frac{1}{2H^2}\left(E\frac{\partial G}{\partial u}-F\frac{\partial E}{\partial v}\right),$$

$$(7)\qquad \left\{{2\,2 \atop 1}\right\}=\frac{1}{2H^2}\left(-F\frac{\partial G}{\partial v}+2G\frac{\partial F}{\partial v}-G\frac{\partial G}{\partial u}\right),$$

$$(8)\qquad \left\{{2\,2 \atop 2}\right\}=\frac{1}{2H^2}\left(E\frac{\partial G}{\partial v}-2F\frac{\partial F}{\partial v}+F\frac{\partial G}{\partial u}\right)$$

be Christoffel's symbols for (1). Under these conditions the differential equation

$$(9)\qquad \frac{d^2v}{du^2}=\left\{{2\,2 \atop 1}\right\}\left(\frac{dv}{du}\right)^3-\left(\left\{{2\,2 \atop 2}\right\}-2\left\{{1\,2 \atop 1}\right\}\right)\left(\frac{dv}{du}\right)^2+\left(\left\{{1\,1 \atop 1}\right\}-2\left\{{1\,2 \atop 2}\right\}\right)\frac{dv}{du}$$
$$-\left\{{1\,1 \atop 2}\right\}+\frac{k}{H^2}\sqrt{E+2F\frac{dv}{du}+G\left(\frac{dv}{du}\right)^2}\left[D_1+2D_1'\frac{dv}{du}+D_1''\left(\frac{dv}{du}\right)^2\right]$$

of the wanted curves $v=v(u)$ are derived (p. 294).

In the special cases $\theta=0$ and $\theta=\frac{\pi}{2}$ the geodesic and the asymptotic lines, respectively, of the surface (1) are obtained, namely

$$(10)\quad \frac{d^2v}{du^2}=\left\{{2\,2 \atop 1}\right\}\left(\frac{dv}{du}\right)^3-\left(\left\{{2\,2 \atop 2}\right\}-2\left\{{1\,2 \atop 1}\right\}\right)\left(\frac{dv}{du}\right)^2+\left(\left\{{1\,1 \atop 1}\right\}-2\left\{{1\,2 \atop 2}\right\}\right)\frac{dv}{du}-\left\{{1\,1 \atop 2}\right\}$$

and

$$(11)\qquad D_1+2D_1'\frac{dv}{du}+D_1''\left(\frac{dv}{du}\right)^2=0,$$

respectively.

In the case of rotational surfaces with axis of rotation Oz the equation (10) reduces to an equation of the first order of Bernoulli's type:

$$(12)\qquad \frac{dw}{du}=\left\{{2\,2 \atop 1}\right\}w^3+\left(\left\{{1\,1 \atop 1}\right\}-2\left\{{1\,2 \atop 2}\right\}\right)w,$$

where $w = \frac{dv}{du}$, Christoffel's symbols being functions of u only.

The article *Sur certaines courbes tracées sur une surface donnée* [3], published in 1935, is a complemented French version of paper [1].

In his review of [3] in *Zentralblatt für Mathematik und ihre Grenzgebiete* (Bd. **13**, Hf. 1 (1936), S. 34) Cohn-Vossen states: "Es handelt sich um diejenigen Kurven der Fläche, längst derer der Winkel zwischen Hauptnormale und Flächennormale konstant ist; diese Kurvenklasse enthält also sowohl die geodätischen Linien als auch die Asymptotenlinien. Als ihre allgemeine Gleichung ergibt sich: Es muß ein Bruch konstant sein, dessen Nenner für die asymptotischen Linien und dessen Zähler für die geodätischen Linien verschwindet. Für den Rotationszylinder sind die betrachteten Kurven diejenigen, die bei Abwicklung in einer Ebene entweder in Geraden oder in Kettenlinien übergehen, deren Symmetrieachse einer Zylindererzeugenden entspricht."

The main results of [3] are reproduced in the book *Задачи по высшей геометрии*, часть II: *Дифференциальная геометрия* by *О. К. Житомирский, В. Д. Львовский, В. И. Милинский* (*Ленинград-Москва*, 1937) as problems Nos 265 and 266 (p. 42–43) along with Dolaptchiev's solutions (p. 229–233).

Article [3] is cited in the bibliography of section *Geometry* (p. 963) of the jubilee collection *Математика в СССР за тридцать лет* (*Москва*, 1948).

In 1933 Dolaptchiev published his article *On a manner of division of a plane into domains by means of n straight lines* [2]. As its title suggests and as its author explains in the beginning of the paper, it deals with "the form of the domains generated by intersection of n straight lines 'in a general position' ... under the only restriction that these n lines always surround an n-lateral domain, closed or open", the term *general position* meaning that any two lines intersect and no three lines intersect at the same point; and the terms *open* or *closed* meaning that the domain in question involves or does not involve, respectively, infinite points. The kernel of the paper consists in the following two propositions:

Theorem V. *The configuration formed by the straight lines in the plane, generating an n-lateral domain ($n \geq 4$), contains no other domains save two-, three- or four-lateral.*

Theorem VI. *There is no open four-lateral domain amongst the domains formed by the straight lines generating an open n-lateral domain.*

The *modus operandi* by means of which the author arrives at his conclusions may be qualified as inductive. He starts from a 3-lateral domain and intersects it by means of a fourth straight line, obtaining in this way a 4-lateral domain, investigating afterwards all possible cases. Then he applies the same procedure to an initial 4-lateral domain by means of a new, fifth, straight line, and proceeds further seeking inductive inferences.

The article *On a family of cylindrical curve lines* [8] of 1939 represents, as the author notes, an Addendum to the work [1], partly reflected in its French version [3] of 1935. By his words, "its purpose is ... to find the curve lines on a surface with constant inclination (of their main normals to the surface's normals, say θ) in the case when the surface is a rotational cylinder, and to display that in this special case these lines possess a remarkable property which defines them in another, mechanical way". If the cylinder in question is defined by

$$r = R(\cos v\mathbf{i} + \sin v\mathbf{j}) + u\mathbf{k}, \tag{13}$$

then the differential equation of the said curve lines is

$$\frac{dw}{du} = kw^2\sqrt{1 + R^2w^2} \quad , \tag{14}$$

where $k = \text{tg}\theta$ and $w = \frac{dv}{du}$. Its solutions are

$$u + \alpha = \frac{R}{2k}\left(\frac{1}{\beta}e^{kv} + \beta e^{-kv}\right), \tag{15}$$

α and β denoting constants of integration. If $\alpha = \frac{R}{k}$, $\beta = 1$, then the curve lines are catenaries twisted round the cylinder.

The remaining part of the article is dedicated to mechanical considerations connected with those lines. As the author writes in the *Zusammenfassung* of the paper: "Sucht man die Gleichgewichtskurve eines schweren, homogenen, undehnbaren und biegsamen Fadens, welcher auf einem vertikalen Kreiszylinder liegt, so findet man genau eine Kurve gleicher Neigung auf der Zylinderfläche."

Paper [8] is reviewed by B. Petkantchin in *Jahrbuch über die Fortschritte der Mathematik* (Bd. **65** (1939), S. 1405) and by I. D. Tamarkin in *Mathematical Reviews* (vol. **1**, No. 10, 1940). According to the former, "Verf. stellt die endlichen Gleichungen derjenigen Kurven auf einem Drehzylinder, auf deren Hauptnormalen einen konstanten Winkel mit den jeweiligen Zylindernormalen bilden, und zeigt, daß sie die Gleichgewichtslagen eines schweren, biegsamen, undehnbaren Fadens auf dem vertikal gedachten Zylinder darstellen." According to the latter, the paper proposes a "Proof that the curve $x = R\cos v, y = R\sin v, z = a(e^a + e^{-a})$ gives a figure of equilibrium of a flexible, inextensible, homogeneous string, which it assumes on the surface of a cylider under the action of the force of gravity."

The paper [15] of 1943 represents a Magyar version of the article [8], as it is immediately seen from its summary in German *Eine Art von Flächenkurven. Zylinderkettenlinien*: "Es werden diejenige Flächenkurven untersucht, welche die Eigenschaft besitzen, daß in jedem Punkt der Kurve die Hauptnormale mit der Flächennormale einen konstanten Winkel θ einschließt. Beispiele solcher Flächenkurven 'gleicher Neigung' sind: die Kreise auf der Kugel, die Parallel- und Meridiankurven der Rotationsflächen, die geodätischen Linien ($\theta = 0$), die Asymptotenlinien ($\theta = \frac{\pi}{2}$). Die Differentialgleichung (2) der Flächenkurven gleicher Neigung wird aus der Beziehung $\text{tg}\theta = \frac{\rho_n}{\rho_g}$ abgeleitet, wobei $\frac{1}{\rho_g}$ die geodätische und $\frac{1}{\rho_n}$ die Normalkrümmung der Flächenkurve ist. Bei Rotationsflächen nimmt die Differentialgleichung die einfachere Form (3) an. Sie wird für Rotationszylinder gelöst und es wird gezeigt, daß bei der Abwicklung des Zylinders in die Ebene die Kurven gleicher Neigung in Geraden und Kettenlinien übergehen. Sucht man andererseits die Gleichgewichtslage eines schweren, homogenen, unausdehnbaren und biegsamen

Fadens, welcher auf einen vollkommen glatten Kreiszylinder mit vertikaler Rotationsachse aufgewickelt ist, so findet man eben die Zylinderkurven gleicher Neigung, d. h. die Gleichgewichtskurven sind Zylinderkettenlinien."

According to the reviewer of [15] H. Gericke, "Flächenkurven 'gleicher Neigung' werden durch die Eigenschaft definiert, daß die Hauptnormale mit der Flächennormalen einen konstanten Winkel θ bildet. Ihre Differentialgleichung wird aus der Beziehung $\mathrm{tg}\theta = \frac{\rho_n}{\rho_g}$ abgeleitet. Sie vereinfacht sich für Rotationsflächen. Sie wird für Rotationszylinder gelöst. Bei Abwicklung des Zylinders gehen die Kurven gleicher Neigung in Geraden und Kettenlinien über. Mechanisch sind sie die Kettenlinien auf dem Zylinder."

The article *A contact transformation in geometry; applications (Nodular parabolae)* [9] of 1940 deals with a plane transformation $[A]$ introduced by Dolaptchiev apropos of the following problem: find the loci of the centres of those circles in a plane π, which pass through a fixed point O of π and touch a given curve line in π; for any such point of contact its $[A]$-image is the centre of the corresponding circle. First and foremost the following proposition is proved.

Theorem I. *The envelope of the symmetrals of the radius vectors of the points of any plane curve line is the locus of the intersecting points of those symmetrals with the normals of the curve line at the corresponding points.*

On the basis of Theorem I any linear element (alias the complex of a point and a direction attached to it) of the original curve may be transformed by means of $[A]$ into the corresponding linear element of the curve-image.

The transformational formulae expressed as relations between the polar co-ordinates ρ, θ of any point of the original curve $\rho = \rho(\theta)$ and the polar co-ordinates r, φ of the corresponding point of the $[A]$-transformed curve $r = r(\varphi)$ are

$$r = \frac{\rho}{2\sin\mu}, \quad \varphi = \theta + \frac{\pi}{2} - \mu, \tag{16}$$

where

$$\mathrm{tg}\,\mu = \frac{\rho\, d\theta}{d\rho}. \tag{17}$$

At that, obviously,

$$\frac{\rho\, d\theta}{d\rho} = \frac{r\, d\varphi}{dr}. \tag{18}$$

These results are generalized for the n-th degree $[A]^n$ of $[A]$ for any natural number n:

$$\rho^{(n)} = \frac{\rho}{2^n \sin^n \mu}, \quad \varphi^{(n)} = \varphi - n\mu + n\frac{\pi}{2}. \tag{19}$$

Some properties of $[A]$ are investigated. The results are organized in the following 4 propositions:

Theorem II. *The succesive positions of any fixed point of the original curve line through the transformations $[A]$, $[A]^2, \ldots, [A]^n, \ldots$ lie on a logarithmic spiral.*

Theorem III. *If the original curve line possesses such points that the angle between their radius-vectors and tangents equals* $\frac{\pi}{6}$, *then these points are restored by* $[A]^6$.

Theorem IV. $[A]^6$ *transforms the logarithmic spiral* $\rho = e^{\sqrt{3}\theta}$ *into itself.*

Theorem V. *The number of the tangents to the original curve line through the origin* O *of the radius-vectors equals the number of the infinite points of its* $[A]$*-transformed curve line.*

The second part of the article is dedicated to various applications. If the original curve is a straight line d not passing through O, then the $[A]$-transformed of d is a common parabola and the $[A]^2$-image of d is Newton's ***parabola nodata***. Taking this into consideration, Dolaptchiev proposes the denomination ***nodular parabolae*** for all $[A]^n$-images of d for $n \geq 2$. If the polar equation of d is

$$(20) \qquad \rho = \frac{p}{\cos\theta}$$

(p denoting the distance between O and d), then the equation of the $[A]^n$-transformed curve of d is

$$(21) \qquad r_n = \frac{p}{2^n \cos^{n+1} \dfrac{\varphi_n}{n+1}},$$

(its polar co-ordinates being r_n and φ_n), and

$$(22) \qquad \theta = \frac{\varphi_n}{n+1}.$$

If the axis Ox of the Cartesian system of reference $Oxyz$ is perpendicular to d and is directed towards d, and if Oy is parallel to d, then the equation of the 5-nodular parabola is

$$(23) \qquad \left(\frac{p-x}{5p}\right)^2 = \frac{x^2+y^2}{4p^2}\left(1 - \frac{x^2+y^2}{4p^2} - \frac{p-x}{p}\left(1 - \frac{p-x}{5p}\right)\right).$$

The following proposition is established inductively.

Theorem VI. $[A]^n$ *transforms any linear relation between* x *and* y *into an algebraic equation of degree* $n+1$.

As regards the inflexion points of the curves (21), Dolaptchiev finds that there are none for any odd n; for any even n the only inflexion point of (21) is its infinite point.

At last, he calls *inverse transformations* those that are formally obtainable from (21) when $n+1$ is a negative whole number. For $n = -2$ and $n = -3$ he obtains a circle and a cardioid through O, respectively.

The article [9] is reviewed by Burau in *Zentralblatt für Mathematik und ihre Grenzgebiete* (Bd. **26**, Heft 2–3 (1942), S. 77): "Verf. führt zu einer gegebenen Kurve C der Ebene eine Transformierte C' als geometrischen Ort der Mittelpunkte aller Kreise ein, die durch einen festen Punkt der Ebene gehen und C berühren.

Die Fortsetzung dieser Transformation definiert eine Kurvenfamilie mit je un eine Einheit wachsender Ordnung. Ist C eine Gerade, so wird man zur Parabel, kubischen Parabel usw. geführt. Die Eigenschaften der so entstandenen Kurven werden rechnerisch nach verschiedenen Richtungen hin untersucht."

The subject of the article *A new manner of investigation of the orthogonal projection of the intersecting curve line of two rotational surfaces of second degree on the plane of their axes* [10] of 1941 becomes clear from its title. As the author emphasizes in the *Conclusion* of the article, "this problem has been investigated repeatedly, each time, however, in such a way that its own, descriptive-geometrical character has almost never been taken into consideration. Those investigations of the problem have been mainly analytical in nature, their starting point not being the projection plane (of the axes of rotation of the surfaces) ... In such a sense our paper represents a new manner of investigation, by means of which old theorems are proved anew and new are discovered, the most important merit being the method of investigation, appropriate for the mathematical nature of the problem" (p. 358). The following central proposition is proved by purely descriptive-geometrical means.

Theorem 3. *The orthogonal projection of the curve of intersection of any two rotational surfaces of second degree on the plane, determined by their rotational axes, is a conic section.*

Afterwards Dolaptchiev investigates, sometimes at length, the eight kinds of rotational surfaces of second degree; combined, these lead to 36 cases of mutual intersections (p. 343). He proves four theorems concerning the kind of conic section of Theorem 3. The first of them, *exempli gratia*, reads:

Theorem 9. *The orthogonal projection of the curve of mutual intersection of a sphere and any rotational surface of second degree on the plane, determined by the axis of rotation of the surface and the centre of the sphere, is a parabola the axis of which is normal to the rotational axis of the surface.*

The paper is provided with 22 figures that, besides their auxiliar function, are interesting as self-dependent descriptive performances.

The article *A descriptive-geometrical application of the projective systems of conic sections* [12] of 1942 is an addendum to the previous one. Namely, while Theorem 2 of [10] is proved, in the latter paper, analytically, in [12] it is established purely geometrically, alias by projective means only. In such a manner, the whole exposition of [10] becomes now synthetic-geometrical. The aims of the work are summarized by the author in the following manner: "... the projection of the intersection curve of two rotational surfaces of second degree with intersecting rotational axes on the plane of these axes is determined by means of two systems of similar concentric (co-axial) conic sections. The respective elements of these systems of conic sections (the projections of the level curves with equal elevations) intersect on the projection of the penetrating curve. The aim ... is to deduce all the conclusions concerning the character and the kind of the projection curve on the basis of these namely two systems of conic sections representing the descriptive images of the original intersecting surfaces" (p. 61).

Theorem 2 mentioned above is concerned with those two systems of conic sections. In the present paper it is formulated in the following manner: "A necessary and sufficient condition for the existence of such a correspondence between the curves of the two bundles of conic sections that the locus of the intersection points of any two corresponding conic sections may represent a new conic section is these bundles to possess a common conic section corresponding to itself in the projectivity of the bundles" (p. 60).

The reviewer of article [12] E. Lukacs notes that "J. Steiner's projective rotation (projektive Drehung) (J. Steiner: *Vorlesungen über synthetische Geometrie*, V. 2, 2nd ed., Teubner, Leipzig, 1876, § 39) is used to study projectivities between one-parametric families of conics. The results are applied to examine the intersection of two quadratic surfaces of revolution."

The paper [16] of 1943 is a resumé in German of the article [12], and its main result consists in Theorem 2 of the latter.

The aims of the article *Some extremal cases in the composition of Euclidean-convex polygons and three-edged polyhedra* [11] of 1942 are the answers of the next three questions.

Problem I. Find the maximal number n of straight lines (planes) in an arbitrary general position which constitute a polygon (polyhedron) with this number n of sides (faces).

Problem II. Does the configuration of the lines (planes) of a polygon (polyhedron) with n sides (faces) involve other polygons (polyherda) with n sides (faces)?

If no, then:

Problem IIIa. Find the next maximal number of sides (faces) of polygons (polyhedra).

If yes, then:

Problem IIIb. Find the maximal number of polygons (polyhedra) possessing n sides (faces).

At that (as the author explains in the introduction of the article), if $n \geq 3$ straight lines in a plane or $n \geq 4$ planes in Euclidean space are given, they are said to be in a *general position* when any two lines intersect and no three lines intersect at the same point, as well as any two planes intersect; no plane is parallel to, or incident with, the intersecting line of two other planes; and no four planes intersect at the same point. As regards the term *convex domain*, it is used in its classial sense, "domain" meaning at that an "elementary domain", viz. such that it does not contain subdomains. A domain is *open* or *closed* if it contains or does not contain, respectively, the infinite line or plane, the latter being not included among the sides or faces of the polygon or polyhedron, respectively.

The article being divided into two parts, the first one is covered to a not inconsiderable degree by the previous paper [2], the demonstrations being, however, more informative; as a matter of fact, it serves the purposes of the second part, involving spacial considerations. As regards the latter one, its approach is characterized by Dolaptchiev in the following scholium: "Whereas the problem in the plane is reduced to a simple analysis situs, supported by one and only axiom (of Pasch), the spacial case is brought to the plane one by means of a descriptive-geometrical method. Therefore ... the exposition is accompanied by exact expositions of the constructions leading to conclusions as in the plane problem" (p. 5). Naturally, it is out of the question to reproduce here in details the content of the theorems of the paper numbering 18 propositions (5 for the plane and 13 for the space), so that all of them are formulated at length in the *Zusammenfassung* of the article. The least one may say is that the thoroughness of the paper is impressive.

The article *On the potential of forces under the action of which a mass-point describes the geodesic lines of surfaces* [18] of 1946 deals with the problem: A mass-point P being compelled to move on a smooth surface s under the action of a conservative force $\boldsymbol{F}$, determines the potential of $\boldsymbol{F}$ if the trajectories of P are the geodesic lines on s. The following propositions are formulated.

Theorem 1. *If a mass-point describes a geodesic line $v = const.$ on a smooth surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ (u and v being geodesic co-ordinates) under the action of a conservative force $\boldsymbol{F}$, then the potential of $\boldsymbol{F}$ is a function of the arc of the trajectory only.*

Theorem 2. *Under the conditions of Theorem 1 the force is tangential to the trajectory of the mass-point.*

These theorems are proved anew in the article *Some mechanical considerations of curve lines traced of surfaces* [44] of 1959. Besides, some results of the previous papers [1, 8] are reproduced here, being supplemented with the case of a rotational cone. At last, rotational surfaces are sought, the equilibrium curves (of heavy, homogeneous, flexible, inextensible threads twined round the surface) of which are curves of constant inclination (between the normal of the surface and the main normal to the curve line).

The article [13] of 1942 deals with the following question: "Auf welche Art und Weise sollen n ungleichen Massen $m_1, m_2, m_3, \ldots, m_n$ auf einem Kreis verteilt werden, so daß das Massenzentrum dieses dem Kreise so eingeschriebenen n- Eckes, mit inhomogenen materiellen Ecken, mit dem Kreiszentrum zusammenfalle?"

This problem is obviously mathematically indeterminate. Moreover, a solution does not necessarily exist. Besides, the context does not throw light upon the question whether the numbers $m_1, \ldots, m_n$ are ordered in some manner or not.

As a matter of fact, the author solves the following problem: the first $n-2$ ones of the masses $m_1, \ldots, m_n$ being fixed on the circle, determine the position (if any) of the remaining two of them in such a manner that the mass centre of these n mass-points coincides with the centre of the circle. At that, if (x_ν, y_ν) $(\nu = 1, \ldots, n)$ denote the co-ordinates of the ν-th mass and if by definition

$$X = -\sum_{\nu=1}^{n-2} m_\nu x_\nu, \quad Y = -\sum_{\nu=1}^{n-2} m_\nu y_\nu, \quad U = \sqrt{X^2 + Y^2}, \tag{24}$$

then the inequalities

$$|m_{n-1} - m_n| \leq U \leq |m_{n-1} + m_n| \tag{25}$$

must hold. A simple geometrical construction is proposed. At last, the author announces the following mechanical analogy: "Auf welche Art und Weise sollen n verschiedene (schwere oder gewichtslose) Massen auf der Peripherie einer homogenen Scheibe, die auch als massenlos angenommen werden kann, verteilt werden, so daß bei der Drehung dieser Scheibe um eine vertikale zur Scheibe senkrechte Rotationsachse, durch ihr Zentrum, die Drehachse keinen Druck erfährt?"

Article [13] is revied by Rehbock in *Zentralblatt für Mathematik* (Bd. **27** (1943), S. 123) in the following manner: "Behandelt werden die Aufgaben: I. n ungleiche Massen sind auf einem Kreise so zu verteilen, daß ihr Massenzentrum in dem Kreismittelpunkt fällt. II. n verschiedene Massen sind auf der Peripherie einer homogenen Scheibe so zu verteilen, daß die durch den Mittelpunkt gehende Drehachse keinen Druck erfährt."

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Dolaptchiev's debut in the mathematical theory of Kármán vortex streets consists in the papers *A contribution to the stability of Kármán vortex streets and the trajectories of the particular vortices* [4, 5] of 1937 and [7] of 1938, the first of which is an enlarged version in Bulgarian of the German publications.

The main purpose of [5], as well as the contemporary state of affairs in that domain, is formulated by Dolaptchiev in the introduction of the article (S. 313): "Die von Th. v. Kármán 1912 angegebene Stabilitätsbedingungen der Wirbelkonfiguration, welche aus zwei parallelen Reihen unsymmetrisch geordneter Wirbel mit gleichen und entgegengesetzten Zirkulationen besteht, enthält, wie bekannt, eine gewisse Unvollständigkeit in sich, die bis auf den heutigen Tag eine ganze Reihe von Betrachtungen veranlaßt hat. Es handelt sich um die Tatsache, daß die theoretische Bedingung $\mathrm{ch}\kappa\pi = \sqrt{2}$, welche sich auch experimentell genügend bestätigt, sich nur als eine notwendige, aber nicht hinreichende Bedingung des stabilen, genauer gesagt, des indifferenten Gleichgewichts erweist ($\kappa = \frac{h}{l}$ ist das Verhältnis der Abmessungen in der Anordnung eines Wirbelpaares). Neuere Untersuchungen des Problems haben darüber hinaus zu der Schlußfolgerung geführt, daß die Kármánsche Wirbelstraße überhaupt instabil ist. Diese Instabilität wird bei einem speziellen Störungsgesetz und einer speziellen Art von Anfangsstörungen gesucht, sowohl bei Störungen erster Ordnung[1)], als auch dann, wenn Störungen höherer Ordnung[2)] herangezogen werden. ‚Mit diesem Artikel[3)] wollen wir eine Bestätigung der Stabilität der Kármánschen Wirbelstraße geben und zeigen, daß die bekannte Stabilitätsbedingung die einzige bleibt, und zwar bei allen von den unten zitierten Autoren angenommenen Voraussetzungen. Sowohl die von G. Durand gefundene Ergänzung, ohne die Instabilität herrschen soll, als auch die Instabilität, an die sich C. Schmieden stößt, werden sich dadurch als scheinbar erweisen."

The footnotes quoted read as follows: [1)] G. Durand. Sur la stabilité des fils tourbillonaires, *C. R. des Séances de l'Ac. des Sciences*, T. **196**, 1, 1933, 382–385; [2)] C. Schmieden. Zur Theorie der Kármánschen Wirbelstraße, I u. II, *Ing.-Archiv*, Bd. **VII**, 1936, S. 215 u., S. 387; [3)] Vorliegender Aufsatz soll ein kurzer Auszug meiner Doktor-Dissertation sein, die ich über die weiteren Untersuchungen der Kármánschen Wirbelstraße anfertige. Meinem hochverehrten Lehrer Prof. Dr. Prandtl, nach dessen Idee (die spezielle Art der Störung) und unter dessen Leitung ich gearbeitet habe, möchte ich auch an dieser Stelle meinen tiefempfundenen Dank aussprechen.

The aims of the article [7] are described by the author in the following manner (S. 263): "Der vorliegende Aufsatz ist der zweite Teil einer Untersuchung . . . , deren erster Teil ich in dieser Zeitschrift (Bd. **17**, Heft 6, 1937) unter dem Titel 'Über die Stabilität der Kármánschen Wirbelstraße' veröffentlicht habe. Jetzt handelt es sich um die genaue Berechnung und Konstruktion der stabilen und instabilen Bahnkurven, welche die einzelnen Wirbelpunkte einer gestörten Kármánstraße beschreiben. Da ich einerseits den Einfluß auf die Ergebnisse zeige, wenn noch quadratische Störungen berücksichtigt werden, und andererseits die Zerstörung der Wirbelstraße verfolge, falls die Anordnung nicht genau die Kármánsche ist ($\frac{h}{l} \neq 0{,}281$), so kann diese Betrachtung, sozusagen, als ein mathematisches Experiment gelten. Durch die nachfolgenden theoretishen Untersuchungen glaube ich in das Verhalten der Wirbelkonfigurationen etwas tiefer einzudringen."

The results obtained by Dolaptchiev in [5] and [7] are summarized in the following manner in the *Conclusion* of the Bulgarian version [4] (p. 210–211): "Assuming disturbances of second order, we have treated Kármán's first problem, using special, namely group disturbances. The hypothesis of disturbances of first order we have begun our considerations with has led us to Kármán's stability again, only that, depending on the initial dislocations, the effect of a translational deflection of the whole of the vortex configuration from the initial direction is observed. The hypothesis of disturbances of second order implies a condition, depending on the initial translations, whose interpretation displays an ostensible instability. The latter consists in the phenomenon that, under a certain kind of initial disturbances, the vortex street undergoes also some extension or narrowing that does not influence the stability ... The latter phenomenon had to be eliminated. After this elimination a disturbance takes place (an alternative one, from the first kind, in our case) that leads to Kármán's condition. In other words, the establishment of the unambiguousness of the initial configuration had to be undertaken, on which the disturbance is accomplished. In the stability case it is shown that: 1) the trajectories of second order are essentially different from those of disturbances of the first order; 2) whereas for any initial translation r, however gross, the stability paths of first order are always circles (ellipses), for disturbances of second order these paths are different for different r. Moreover, in the instability case we have found that 3) whereas the linear trajectories converge toward straight lines, the quadratic paths display also a longitudinal evasion, along with the large transversal amplitudes of the oscillations with irregular periods."

The article [4] is reviewed by Garten (*Jahrbuch über die Fortschritte der Mathematik*, Bd. **64**, II, Jahrgang 1938, Hf. 4, *Hydrodynamik*, S. 1447) in the following manner: "Der erste Teil der Arbeit liefer einen Beitrag zu den Stabilitätsuntersuchungen der Kármánschen Wirbelstraßen, der zweite Teil liefert die Berechnung und Konstruktion der Bahnen der einzelnen Wirbel, um insbesondere den Einfluß der nichtlinearen, speziell quadratische Glieder der Störungen auf die Trajektorien zu ermitteln. Den Stabilitätsuntersuchungen werden die beiden folgenden speziellen Störungsformen zugrunde gelegt: 1) 'gleiche' Störung (sämtliche Wirbel einer jeden Reihe erfahren zwar dieselbe Störung, aber die Störung ist in beiden Reihen verschieden). Wie Verf. bemerkt, liefert diese Störungsform lediglich eine schräge Fortbewegung der Konfiguration, aber keine Stabilitätsbedingung; die von Durand gefundene Ibstabilität tritt nicht auf; 2) die 'alternative' Störung (die Wirbel von geradzahligem Index in einer Reihe erfahren alle dieselbe Verschiebung, ebenso die

Wirbel von ungeradem Index; für jede der beiden Reihen ist aber diese Verschiebung eine andere; demzufolge erfahren benachbarte Wirbel verschiedene Störungen). Für die Störungen 1. Ordnung ergibt sich: Die in t linearen Glieder, die mit der Zeit unendlich werden, besitzen nur den Charakter einer Translation des Wirbelsystems. Für die Kármánsche Stabilitätsbedingung werden mit dem Anwachsen der Zeit t keine Glieder undendlich groß. Bei den Störungen 2. Ordnung ist nur relative Gleichgewicht möglich (d. h. das Gleichgewicht der Straße ist nur gegen gewiße ganz spezielle Störungen gesichert). Es bestehen die beide Möglichkeiten: a) Kármánsche Stabilität ('Störung 1. Art' nach Schmieden), b) Instabilität ('Storung 2. Art' nach Schmieden). Es wird jedoch gezeigt, daß die Störungen zweiter Art sich stets auf solche erster Art zurückführen lassen, und somit nur die Kármánsche Stabilität resultiert."

The stability investigation [5] provoked an immediate reaction on the part of C. Schmieden, whose paper *Zur Theorie der Kármánschen Wirbelstraße* has been quoted and discussed in [5]. In Bd. **18**, Hf. 4 (1938) of the *Zeitschrift für angewandte Mathematik und Mechanik* (*ZAMM*) he published, in the rubric *Zuschriften an den Herausgeber*, his considerations under the title *Zu der Arbeit von Herrn B. Dolaptschiew* (S. 261). Immediately bellow is published Dolaptchiev's *Erwiderung* [6]. The controversies cannot be discussed here mainly since they involve considerable technicalities. And yet, some words concerning the mechanical ideology laid down in the foundations of the investigations treating the stability problem of Kármán vortex streets and the resistance experienced by the obstacle in the fluid will be said below.

As a matter of fact, one of the leading mechanicians in those times, Georg Hamel, takes a stand on this question in Bd. **64**, I, Hf. 6 (Jahrgang 1938) of *Jahrbuch über die Fortschritte der Mathematik* (*Mechanik der Kontinua*, S. 853). Apropos of the disputation Schmieden–Dolaptchiev he writes: "Auseinandersetzung daruber, ob das Verhalten der Wirbelstraße gegen Grupenstörungen als stabil oder instabil zu bezeichnen sei. Nach der exakten Definition Kleins ist sie sicher instabil. Aber gefühlsmäßig gibt man dem zweiten Verfassers [Dolaptchiev] recht. Nach Ansicht des Referenten ist die Kleinische Definition für Systeme von unendlichen Freiheitsgrad unzweckmäßig. Es genügt nicht, daß die Störungen alle hinreichend klein sind, man wird sinnvoll verlangen müssen, daß die zu ihrer Erzeugung notwendige Energie noch klein sei. Das ist bei den benutzten Gruppenstörungen wohl nicht der Fall. Es würde sich dann um einen ähnlichen Unterschied wie den zwischen Stetigkeit und Vollstetigkeit im Hilbertschen Raum handeln."

There is behind Hamel's words "Es genügt nicht, daß die Störungen alle hinrechend klein sind, man wird sinnvoll verlangen müssen, daß die zu ihrer Erzeugung notwendige Energie noch klein sei" something much more important than the researchers of the field, by whom this statement has been underestimated, have ever seen: an important mechanical idea due to a memorable perspicacity *in rerum natura*. In its essence it is rooted in the eternal problem of mathematical existence. In order to fix the ideas let us first say that the transition from finite to infinite cases in the mathematical theory of stability of motion is hiding all those logical risks that the very idea of infinity conceals. On top of it things are aggravated by the problem of mechanical possibility. Let the stability of motion of a system V of vortices V_ν ($\nu = 1, \dots, n$) be investigated and let W be the system of the perturbed positions W_ν ($\nu = 1, \ldots, n$) of V_ν ($\nu = 1, \ldots, n$) respectively. No objections can be made against the hypothesis that W is the perturbed state of V whichever this W may be: the conviction that there always exist forces capable of transforming V into W is a question of primary physical instinct. Let us now regard the infinity case of a system V^∞ of infinite many vortices V_ν ($\nu = 1, 2, \ldots$) and the system W^∞ of their perturbed positions W_ν ($\nu = 1, 2, \ldots$). Again no objections can be made against the transition from V^∞ to W^∞ as far as one remains in the domains of *geometry* or *analysis*. If, however, the elements of V^∞ and W^∞ are treated as *mechanical* entities, then the hypothesis of this transition comes within the provisions of the mechanical laws and cannot be accepted without reservation unless the mathematically vital question "Is this possible?" is answered in the affirmative. And "Is this possible?" means: "Do there exist forces capable of carrying into effect such a transition?" In Hamel's opinion the answer is in the negative. The authors of these notes share this opinion most ardently. Moreover, they give explicit utterance to their firm conviction that many cases of infinite motions of fluid masses in the whole plane or in the whole space may prove to be mechanically wholly illusionary when put under the scope of the existence problem, turning out to be dynamical impossibilities. As regards the disturbing displacements of the vortices of V^∞ in order to obtain W^∞, the objection is that infinitely great forces are needed to this end (ergo infinitely great work of those forces, alias energy); as regards the infinite bulk of fluid masses in the cases of motions in the whole plane or in the whole space, such hypotheses are inconsistent with Euler's fundamental dynamical axioms (or laws of momentum and of moment of momentum) tolerating only finite values of the mechanical entities therein involved (masses, momenta and moments of momenta, bases and moments of systems of forces, etc.). A mechanician dares never forget that not every motion *kinematically* possible is possible also *dynamically*.

Hamel is reviewer in *Jahrbuch* of the articles [5] and [7] too. Apropos of [5] he states: "Die Stabilität der Kármánschen Wirbelstraße ist von G. Durand (*C. R. Acad. Sci. Paris*, **196**, 1933, 382–385; *F. d. M.*, **59**, I, 761) und C. Schmieden (*Ingenieur-Arch.*, **7**, 1936, 215–221, 337–341; *F. d. M.*, **62**, I, 973) angezweifelt worden. Verf. untersucht die Sache nocheinmal und kommt zu dem Schluße, daß die behaupteten Instabilitäten nur in eine Änderung der Wirbelstraße bestehen, die sich als Ganzes verschieben, erweitern und drehen kann. Sie bleibt aber Wirbelstraße. Verf. will das nicht als Instabilität gelten lassen."

Apropos of [7] Hamel states: "Verf. setzt seine Untersuchung 'Über die Stabilität der Kármánschen Wirbelstraße' (*ZAMM*, **17**, 1937, 313–323; *F. d. M.*, **63**, II) fort. Insbesondere werden sowohl für den stabilen wie für den instabilen Fall die Bahnen berechnet und gezeichnet. Bemerkenswert ist, daß im zweiten Fall die Bahnen zunächst noch ziemlich lange Zeit um die alte Lage pendeln, um sich dann erst ins Unendliche zu entfernen. Bei der ganzen Theorie sind noch Störungen zweiter Ordnung berücksichtigt."

W. Tollmien's reviews of the papares [5] and [7] in *Zentralblatt für Mechanik* (Bd. **7**, Hf. 3 (1938), S. 134, and Bd. **8**, Hf. 4 (1939), S. 177, respectively) are also auspicious. Apropos of [5] he writes: "Th. von Kármán hat gezeigt, daß die nach ihm benannte Wirbelstraße für ein ganz bestimmes Verhältnis des Langs- und Querabstandes der Wirbel beider Reihen im indifferenten Gleichgewicht ist, wenn die Störungen nach der Methode der kleinen Schwingungen unter Beschränkung auf lineare Störungsglieder diskutiert werden. Der Verf. will zu der Frage einen Beitrag liefern, ob die Kármánsche Stabilitätsbedingung noch genügt, wenn man auch Störungsglieder 2. Ordnung in Betracht zieht. Dabei nimmt der Verf. als Anfangsstörungen solche, wie sie von G. Durand (1933) und C. Schmieden (1936) in ihren Untersuchungen benutzt wurden, die ein Versagen der Kármánschen Stabilitätsbedingungen festgestellt haben wollen. Der Verf. kommt dagegen zu dem Schluß, daß in diesen Fällen die Kármánsche Stabilitätsbedingung gültig bleibt. Dabei muß allerdings bei der von Schmieden untersuchten Störung die Breite der Wirbelstraße sinngemäß definiert werden, nämlich als Abstand der Schwerlinien der beiden Wirbelreihen, nachdem die Anfangsstörung angebracht ist."

Apropos of [7] Tollmien writes: "In dieser Arbeit werden die Bahnen berechnet, welche die Wirbel in einer Kármánschen Wirbelstraße unter dem Einfluß der früher von dem Verf. (vgl. dies. *Zbl.*, **7**, 134) angegebenen Störungen ausführen. Grundsätzlich wichtig ist dabei der Einfluß der quadratischen Störungsgliedern. Im Anfang wird eine bemerkenswerte Ergänzung zu der früheren Veröffentlichung des Verf. gegeben, indem eine dort definitionsmäßig eingeführte Bedingung hier bewiesen wird."

Dolaptchiev's articles [5] and [7] have been quoted favourably over and over again by many authors of articles and books treating various aspects of Kármán vortex streets theory. For instance, they are reflected in § 32.1 of A. Sommerfeld's *Vorlesungen über theoretische Physik. Mechanik der deformierbaren Medien* (Bd. II, 2. Aufl., Leipzig, 1949, p. 225; see also A. Sommerfeld, *Mechanics of Deformable Bodies. Lectures on Theoretical Physics*, vol. II, New York, 1950, p. 234; as well as A. Зоммерфельд: *Механика деформируемых сред*, Москва, 1954, p. 292) in the following manner: "Die Kármánsche Untersuchung ist von Bl. Dolaptschiew durch Berücksichtigung der Glieder zweiter Ordnung vervollständigt worden. *Z. angew. Mathem. u. Mech.*, **17** (1937), 313, **18** (1938), 263."

The same is made in the *Anhang zur deutschen Ausgabe: Zusätze der Wissenschaftlichen Redaktion* (zum Kapitel V, § 21) of the German translation *Theoretische Hydromechanik* (Bd. 1, Berlin, 1954, S. 190) of the book *Теоретическая гидромеханика* (Москва, 1948) of N. J. Kotschin, I. A. Kibel, N. W. Rose: "Eine Stabilitätsuntersuchung unter Berücksichtigung der Glieder zweiter Ordnung findet sich auch bei Bl. Dolaptschiew, Über die Stabilitt der Kármámschen Wirbelstraße, *Ztschr. für angew. Math. u. Mech.*, **17**, 313 (1937) und **18**, 263 (1938)."

Apropos of the same papers of Dolaptchiev, in his article *Ein Beitrag zur Stabilitätstheorie der Wirbelstraßen unter Berücksichtigung endlicher und zeitlich wachsender Wirbeldurchmesser* (*Ingenieur-Archiv*, **22**, Heft 6, S. 400) U. Domm writes: "Später zeigte *Bl. Dolaptschiev*, daß diese Stabilitätsbedingung (Kármán's $\frac{h}{l} = 0{,}281$) bei Berücksichtigung von Störungen zweiter Ordnung erhalten bleibt."

Again these papers are mentioned in the article *Некоторые вопросы аэродинамической теории сопротивления* (*Ученные записки МГУ*, **46** (1940), 43) of A. A. Kosmodemjanski: "В 1937 году Долапшиев пришел к аналогичному результату, проведя вычисления более строго (удерживая члены второго порядка малости — квадраты и произведения налагаемых возмущений)", i.e. "In 1937 Dolaptchiev arrived at a similar result by means of more exact calculations (concerving the terms of second order — the squares and the products of the imposed disturbances)".

For the sake of completeness let us mention Dolaptchiev's article *Математическа постройка на учението за летението* (Mathematical upbuilding of the doctrine of flying) printed in the *Юбилеен сборник на Физико-математическото дружество в София по случай 40-годишния му юбилей* (София, 1939, 48–55). As its reviewer Höffding (*Jahrbuch*, **66**, Jahrgang, 1940, S. 1392) notes, " Verf. gibt einen kurzen Überblick über die mathematischen Methoden, die der modernen Aerodynamik zugrunde liegen (Stromfunktion, Geschwindigkeitspotential, Quellen und Senken, Zirkulation, konforme Abbildung)."

The article *Mathematical solutions in the aerodynamics of flying* [17] of 1943 is Dolaptchiev's inaugural lecture and has a surveying character.

In the paper *Two-parametric vortex streets* [14] of 1942 Dolaptchiev subjects to a systematic investigation those non-traditional from the view-point of the classical Kármán vortex streets theory vortex configurations which he calls *two-parametric*. By his own words, "wenn ich unter 'Parameter' einer Kármánstraße eine solche Zahl κ verstehe, welche das Verhältnis der Breite $2h$ der Straße zu dem Abstand $2l$ zwischen zwei aufeinander folgenden Wirbeln einer und derselben Reihe darstellt ($\kappa = \frac{h}{l}$), ... zu Bestimmung einer nicht-Kármánstraße führe ich ein zweites Parameter λ ein, nämlich das Verhältnis der Größe $2d$, um welche die eine Wirbelreihe in Bezug auf die andere, verschoben ist, zu dem Abstand $2l$ ($\lambda = \frac{d}{l}$)" (cf. [14], 315–316). As Dolaptchiev notes (p. 288), his interest in such "verschobene Straßen" has been arisen by the investigation *Double row of vortices with arbitrary stagger* of L. Rosenhead (*Proceedings of the Cambridge Philosophical Society*, **XXV**, part II, 1928). The problem he rises in [14] is "Auf welche Art und Weise können nun Wirbelstraßen mit zwei Parametern κ, λ festgestellt werden, welche sich doch als stabil erweisen?" (p. 316). The stability condition he arrives at is

$$\cosh^2 \kappa\pi + \cos^2 \lambda\pi = 2 \tag{26}$$

or, just the same,

$$\mathrm{sh}\kappa\pi = \sin \lambda\pi. \tag{27}$$

In his article *On Coddington's stability theory of vortex streets* (*Journal of Matematics and Physics*, **44**, 1, 1965) O. Bils (quoting Dolaptchiev's article [14], the name and the title of the journal being misprinted) writes: "As far as the instability of the non-linear system [of perturbed equations] is concerned, we believe that the pioneering work was done by Kochin [5] and Dolaptschien [6]."

These investigations are continued and extended in the chronologically following articles [19–21].

The paper *On the oblique flow of vortex streets* [19] has a recapitulative, unifying, and systematizing character.

The paper *On the stabilization of vortex streets* [20] makes an attempt at clarification of the mechanism of stabilization of the vortex streets by means of a transitional process through the symmetric, alternating, and two-parametric configurations: "Mittels gleichzeitigen Operationen der gleichartigen Störung, d. h. der gleichen und entgegengesetzten Verschiebung und Verengerung, die man auf eine stabile Ausgangsorderung einer schachmaten Straße ausübt, wurde für die Bahn irgendeines Wirbels diejenige Kurve gefunden, welche als Stabilitätskurve gilt. Diese Kurven sind ausführlich betrachtet und konstruiert. Im Anschluß an einer Wirbelkonfiguration, lassen diese Kurven ersehen, auf welch einen Weg die einzelnen Wirbel der Konfiguration sich bewegen, bevor sie von irgendeiner schmaleren assymetrischen Anordnung bis zur stabilen Kármáschen Lage herkommen; oder von derletzten ausgehend, bis zur instabilen einreihigen Wirbelanordnung. Bei ihrer Stabilisierung erhält die Straße, allmählig eine kleinste Fortschreitungsgeschwindigkeit; wird dies überschritten, so wird die Straße labil" (S. 178).

The note [21] is an abridged French version of [20]. Its reviewer D. Gilbarg (*Mathematical Reviews*, 1951) states: "The author considers two infinite parallel rows of equally spaced point vortices, calling the general configuration of this type 'asymmetric' to distinguish it from the well-known symmetric and alternating vortex streets. The asymmetric configuration, which is known to be in equilibrium, has been shown by Godefroy (*Comment. Math. Helv.*, **II**, 1939, 293–320), Maue (*Z. angew. Math. Mech.*, **20**, 1940, 129–137; this *Rev.*, **2**, 170), and the author, to be stable (with respect to a certain wide class of small perturbations), provided the configuration satisfies the relation

$$(+) \qquad \operatorname{sh}\frac{h}{l}\pi = \sin\frac{d}{l}\pi,$$

where h is the distance between the rows, l the spacing of the vortices in each row, and d the distance along the axis that each row is displaced from the symmetrical configuration. The author shows that if a stable Kármán vortex street is perturbed through a sequence of asymmetric configuration, the velocity of the system being that of the instantaneous asymmetric configuration, then the motion of the vortices is such that (+) is satisfied at each instant. The author uses this result as basic for a qualitative explanation of why the Kármán street is the only stable configuration observed in nature."

Maruhn's review of [21] in *Zentralblatt für Mathematik* (Bd. **37**, Hf. 4/6, 1951, S. 276) reads: "Löst man die Differentialgleichungen für die Bewegung eines Wirbels, der einem System zweier paralleler Wirbelstraßen, auf denen die Wirbel mit entgegengesetzt gleicher Zirkulation äquidistant angeordnet sind, angehört, so ergibt sich für die Bahn die Beziehung

$$(1) \qquad \operatorname{sh}^2(\psi\pi) = \sin^2(\varphi\pi)$$

(φ und ψ die dimensionslosen Koordinaten des Wirbels). $\psi = 0$ entspricht zwei zusammenfallenden Straßen, $\psi = \frac{1}{2}$ der von Kármánschen stabilen Konfiguration. Die Beziehung (1) wird andererseits als verallgemeinerte Stabilitätsbedingung erhalten, wenn man Bewegungen der beiden Wirbelstraßen schräg zur Flüssigkeitsbewegung noch als 'stabil' zuläßt."

In 1939 N. E. Kotchin published in *Comptes rendus de l'Académie des Sciences d'USSR* (new series, vol. **XXIV**, No 1, 19–23) the article *On the instability of von Karman's vortex streets*, the title of which speaks about the author's thesis. It has been reviewed by Schubert in *Jahrbuch über die Fortschritte der Mathematik* (Bd. **65**, Jahrgang 1939, S. 981) and by Tollmien in *Zentralblatt für Mechanik* (Bd. **10**, 1940, S. 228). Both of reviewers place special emphasis upon the necessity of a comparative analysis of Kotchin's and Dolaptchiev's results, as follows:

Schubert: "Verf. beweist die Instabilität der Kármánschen Wirbelstraße bei Annahme spezieller periodischer Störungen, indem er in den Differentialgleichungen für die gestörte Bewegung alle Glieder bis einschließlich vierter Ordnung beibehält und ein Stabilitätskriterium von A. Liapounoff anwendet. Auf die Frage, wieweit das gefundene Ergebnis mit den Untersuchungen desselben Problems von C. Schmieden (*Ingenieur-Arch.*, **7**, 1936, 215–221, 337–341; *F. d. M.*, Bd. **62**, I, 973) und Bl. Dolaptschiew (*Z. angew. Math. Mech.*, **17**, 1937, 313–323; *F. d. M.*, **63**, II, 1353) in Einklang steht, wird nicht eingegangen."

Tollmien: "Die Kármánsche Bedingung für die Stabilität einer Wirbelstraße ist notwendig, aber nicht hinreichend. In der vorliegenden Arbeit wird der Versuch unternommen, Klarheit über die Stabilitätsverhältnisse bei kleinen, aber endlichen Störungen zu gewinnen. Dabei wird jede der beiden Wirbelreihen mit gerader und ungerader Wirbelnummer unterteilt. Die Wirbel in jeder der so entstandenen 4 Reihen werden jeweils derselben Anfangsstörung unterworfen. Nach Stabilitätskriterien von Liapounoff schließt der Verf. auf Instabilität der Kármánschen Anordnung gegenüber der angenommenen Störung. Eine Auseinandersetzung mit der hier nicht berücksichtigten Untersuchungen von Dolaptschiew (vgl. dies. *Zbl.*, **7**, 134, u. **8**, 177), der ähnlichen Störungen der Kármánschen Wirbelstraßen untersucht hat und zu einem anderen Ergebnis gelangt, wäre sehr erwünscht."

An attempt at such a juxtaposing is launched by Dolaptchiev himself in his article *An application of Kotchin's method for studing the equilibrium state of the two-parametric vortex streets* [23] and its Russian version [25] of 1950, with regard to the two-parametric vortex streets at that. He recognizes that Kármán streets are instable in the sense of Liapounoff's definition of stability, namely "Man nennt ein Wirbelsystem dann stabil, wenn man für jedes beliebig kleine, positive ε, ein solches positives δ finden kann, so daß sich das Wirbelsystem nach jeder Wirbelversetzung, die δ nicht überschreitet, so fortbewegt, daß die Differenz zwischen den Abständen je zweier Wirbel des Systems in Bezug auf die entsprechenden Wirbelabstände in ihrer ungestörten Zuständen, nicht die Zahl ε übersteigt" [23, S. 366], and claims that "im vorliegenden Arbeit wird also die volle Analogie festgestellt, welche im Verhalten zwischen den schachmaten und asymmetrischen Wirbelstraßen besteht. Diese Analogie stellt uns einerseits die zweiparametrigen Wirbelkonfigurationen tatsächlich als Übergangsgebilde dar zu den Kármánstraßen, in welche sich die verallgemeinerten Straßen sozusagen 'stabilisieren'. Andererseits werden sich auch diese allgemeinen Wirbelstraßen als solche erweisen, die 'am wenigsten instabil sind', sobald sie eine neue ...Stabilitätsbedingung erfüllen ..." [*ibid.*]

Both papers [23] and [25] are reviewed by J. Kravtchenko in *Mathematical Reviews* (vols. **14**, 1953, and **13**, 1952, respectively). Apropos of [23] Kravtchenko writes: "Considérons une rue de tourbillons formée de deux files rectilignes et parallèles de tourbillons ponctuels, régulièrement espacés sur chaque file, mais présentant un décalage quelconque d'une file à l'autre. Les conditions nécéssaires de stabilité de cette configuration ont été etudiées antérierement (entre autre par Kotchin et par l'A.) en particularisations d'un file plus générale. Il conclut à l'insuffisance du critère de stabilité classique: $\operatorname{sh}\kappa\pi = \sin\lambda\pi$ ($\kappa = \frac{h}{l}$, $\lambda = \frac{d}{l}$, ou h est la distance des files, l l'especement des tourbillons sur celles-ci, d le d'écalage des files); mais les configurations correspondantes seraient les 'moins instables' au sense attachée à ce mot par l'A."

Kravtchenko's review apropos of [25] reads: "L'auteur étudie la stabilité d'un rue de tourbillons plan (formée de deux files parallèles). Si les perturbations initiales sont quelconques, la configuration précédente est instable in sens absolu, même si la condition nécéssaire de Kármán est remplie: car, on peut toujours particulariser les conditions initiales de manière que l'axe du système tourbillonaire s'écarte indéfiniment de sa position primitive en se déplaçant parallèlement a lui même. L'auteur cherche alors les conditions de stabilité au sens restreint de Kotchine (*C. R. (Doklady) Acad. Sci. USSR*, (N. S.), **24**, 1939, 18–22; ces *Rev.*, **2**, 26); il est conduit ainsi à caractériser les configurations les moindre instabiles."

Article [25] is reviewed in greater detail by Karl Maruhn: "Verf. untersucht zwei parallele 'Straßen' mit äquidistanten, gegeneinander versetzten Wirbeln mittels der Methode der kleinen Schwingungen (Linearisierung der Bewegungsgleichungen) auf Stabilität. Es zeigt sich, daß unter der Bedingung $\operatorname{sh}(\kappa\pi) = \sin(\lambda\pi)$ (κ, λ für die Wirbelanordnung charakteristische Parameter), die für $\lambda = \frac{1}{2}$ die v. Kármánsche stabile Anordnung ergibt, die Bewegung (sie liefert bei Störung eine Schrägverschiebung der Straßen) bei genügend allgemein gehaltener Definition noch gewissermaßen als stabil angesehen werden kann. Solche Bewegungen ordnen sich folgender, von N. E. Kočin [*Doklady Akad. Nauk SSSR* (n. Ser.), **24**, Nr. 1, 1939] formulierten Definition ein: ein Wirbelsystem heißt dann stabil, wenn nach Vorgabe einer beliebig kleinen positiven Zahl ε eine Zahl $\delta > 0$ so bestimmt werden kann, daß bei einer Anfangsverschiebung der Wirbel um weniger als δ der Abstand zweier beliebiger Wirbel während der ganzen Zeit der Bewegung sich von dem entsprechenden Abstand bei der ungestörten Bewegung um weniger als ε unterscheidet."

In 1950 Dolaptchiev published his article *A generalizing method for the establishment of the stability of arbitrarily ordered vortex streets* [24] and its Russian version [22]. By his own words, "es wird in vorliegenden Aufsatz eine einfache und allgemeine Ausführung der notwendigen Gleichgewichtsbedingungen einer Wirbelstraße mit beliebiger Anordnung ihrer zwei parallelen Wirbelreihen gegeben, aus welcher Bedingung man Schlüße auf die Stabilität oder Instabilität jeder der drei möglichen Anordnungen: a) der symmetrischen Straßen, b) schachmaten (Kármánschen) und c) asymmetrischen Straßen ziehen kann. Zum Zwecke wird einerseits die Methode der unendlich kleinen Störungen von N. E. Kotchin benutzt; andererseits wird dasjenige Verfahren zur Lösung der Differentialgleichungen der gestörten Wirbelbewegungen verwendet, welches in einer früheren Arbeit von uns befolgt wurde" [24, S. 375].

Article [22] is reviewed by J. Kravtchenko in *Zentralblatt fur Mathematik* (Bd. **42**, Hf. 1–5, 1952, S. 188) in the following manner: "L'auteur forme une condition nécéssaire de la stabilité d'une rue de tourbillons formée de deux files parallèles, distantés de $2h$ et decolées de $2l$ l'une par rapport à l'autre. Si $2l$ est la distance de deux tourbillons consecutifs d'une file, cette condition prend la forme $\operatorname{sh}\kappa\pi = \sin\lambda\pi$ en posant $\kappa = \frac{h}{l}$, $\lambda = \frac{d}{l}$."

According to the review of E. Leimanis of the same paper, "Author derives in a simple and very general manner the necessary conditions for the equilibrium of a vortex street composed of equally spaced point vortexes in two parallel and arbitrarily situated rows. These conditions permit one to solve the problem of stability or instability in each of the three possible cases: (1) symmetric, (2) alternating, and (3) asymmetric vortex streets. The method used is that of Kochin [*Dokladi Akad. Nauk SSSR*, **24**, 1939, 18–22; *Sobranie sochinenii*, **2**, 479–485, Izd. Akad. Nauk SSSR, Moscow–Leningrad, 1949; or Kochin, Kibel' and Roze, *AMR*, **3**, Rev. 510] for small finite displacements of vortexes as applied in cases (1) and (2), and, on the other hand, that of solution of the equations for the perturbed motion of the vortexes as outlined by the author in an earlier paper [*Z. angew. Math. Mech.*, **17**, 1937, p. 313]. Author shows that the most general necessary condition for stability of a vortex street is the fulfillment of the relation (*) $\text{sh}\kappa\pi = \sin\lambda\pi$, obtained first by a different method by the author [*Godishnik Sof. Univ.*, **39**, 1942, p. 287], where $\kappa = \frac{h}{l}$, $\lambda = \frac{d}{l}$, h the distance between the rows, l spacing of the vortexed in each row, and d the shift of one row with respect to the other from the symmetrical configuration of the vortex street. From this it follows that an asymmetric vortex street is stable, provided the parameters κ and λ of the configuration satisfy the relation (*). For $\lambda = \frac{1}{2}$, the well-known necessary condition of stability for von Kármán alternating vortex street is obtained. Symmetric vortex streets turn out to be unstable. It should be noted, that the theory of stability of generalized (two parameter) vortex streets for small displacements is only an approximative one, and a revision of this theory will be given in a future paper in the same journal."

Article [24] is reviewed by J. Kravtchenko in *Mathematical Reviews* (vol. **14**, 1953, p. 422) in the following manner: "En utilisant la méthode de Kotchine [cf. Kočin, Kibel', Roze, Hydrodynamique théorique (en russe), t. I, p. 207, Gostechizdat, Moscou–Leningrad, 1948] l'A. forme la condition nécéssaire de stabilité d'une rue de tourbillons, constituée par deux files rectilignes et parallèles, les tourbillons sur chaque file étant équidistants, mais le décalage des files étant quelconque. Dans le cas particulier des configurations: 1) symetriques, 2) en quinconce ou alternés (v. Kármán), l'A. retrouve les conclusions de son travail antérieur [*Spisanie Bulgar. Akad. Nauk*, **57**, 1938, 149–218].

Dolaptchiev's papers [14, 20, 23, 24] are mentioned in R. Wille's article *Kármánsche Wirbelstraßen* in *Z. Flugwiss.* (**9**, No. 4–5, 1961, S. 151) with the comment: "Mit linearisierten Bewegungsgleichungen formulierten Maue und Dolaptschieff unabhängig voneinander die Stabilitätsbedingung $\sin\pi\mu = \text{sh}\,\pi\kappa$. Domm beweist, daß alle Wirbelstraßen, die die Maue–Dolaptschieffsche–Bedingung befriedigen ..." (misprints in the titles of articles and journal). The same condition is quoted in U. Domm's article *Über die Wirbelstraßen von geringster Instabilität* (the latter term being introduced by Dolaptchiev) in *Zeitschr. angew. Math. Mech.* (Bd. **36**, Nos. 9–10, 1956, S. 370): "Die weitere Rechnung kann durch die Verwendung der Maue–Dolaptschiew–Bedingung der linearisierten Theorie ... vereinfacht werden."

Dolaptchiev's paper [26] of 1951 is described by Kravtchenko [*ibidem*] in the following way: "On sait qu'une rue de tourbillons a deux files parallèles est, en général, instable. L'expèrience montre que la configuration initiale se déforme et tend vers une configuration stable quinconque. Moyenant des hypothèses supplémentaires concernant la nature des perturbations, l'auteur donne une explication théorique du phénomène en explicant les lois de passage de l'état instable à l'état stable."

In 1953 the recapitulative article *Vortex streets in incompressible media* of M. Z. v. Krzywoblocki (*Applied Mechanics Reviews*, **6**, No. 9, 393–397) was published, where Dolaptchiev's papers [5, 7, 14, 19, 20, 22–26] are cited. Apropos of some of them the author writes: "Dolapchiev presented analytical construction of stable and unstable paths described by a single point vortex" and "Dolapchiev discussed the stability of an arbitrarily situated vortex street and conditions of stability in the sense of Kochin" (p. 393). In his article *On the stability of Bénard-Kármán vortex street in compressible fluids*. I (*Acta physica Austriaca*, **7**, No. 3, 1953, 283) the same author, apropos of Dolaptchiev's papers [7, 25] and especially [26], writes: "Dolaptchiev published several papers on the problem in question. His most recent paper (1951) contains the stability considerations in the sense of Kochin" (p. 284).

Dolaptchiev's papers [5, 7, 22, 26] have been noticed in G. Birkhoff's article *Formation of vortex streets* in *Journal of Applied Physics* (**24**, 1953, 98–103, p. 100): "For the stability of vortex streets, see also ... B. Dolaptschiev, *Z. angew. Math. Mech.*, **17**, 313–323 (1937) and **18**, 263–271 (1938) ..., also *Doklady Akad. Nauk S.S.S.R.*, **77**, 985–988 (1951) and **78**, 29–32 (1951) (*Math. Revs.*, **13**, 81 (1952)."

In 1954 Dolaptchiev published two recent notes [27] and [28] connected with vortex configurations. The first of them deals with the stability and the oblique flow of the two-parametric vortex streets, while the second is dedicated to the approximative determination of the vortex resistance. Both were reviewed in *Applied Mechanics Reviews* (vol. **9**, No 11, 1956).

As regards article [27], it is described by M. D. Friedmann in the following manner: "Author gives a simple derivation of the stability condition $\mathrm{sh}\,\kappa\pi = \sin\lambda\pi$, where $\kappa = \frac{h}{l}$, $\lambda = \frac{d}{l}$ for two-parametric (asymmetric) vortex streets, which leads directly to the instability of symmetric streets and to the condition for stability of staggered streets for $\lambda = 0$ and $\lambda = \frac{1}{2}$. Author claims the interpretation of the flow of a two-parametric vortex street given in Sommerfeld's 'Lectures on theoretical physics' is incorrect. The correct interpretation should be that the axis of symmetry of an infinite street maintains a constant direction — the direction of the free stream flow — but is displaced parallel to itself rather than at an angle to the free stream direction as Maue [*ZAMM*, **20**, 1940] predicts. Attempts to verify latter statement experimentally at the Budapest Aerodynamic Institute failed."

Article [28] is reviewed by J. Beranek as follows: "Applications of the well-known von Kármán formula for vortex resistance meet with some difficulties due to the fact that two of the six quantities determining the vortex resistance should be found experimentally. Kosmodemianski introduced further assumptions allowing one to determine the circulation theoretically, at least with some approximation. The present paper deals with the question: For what kind of rising vortices and following vortex configuration is the Kosmodemianski hypothesis sufficiently adequate?"

The same papers are reviewed also by J. Kravtchenko in *Mathematical Reviews* (vol. **16**, No 11, 1955, p. 169) as follows. Apropos of [27] he writes: "L'auteur considére la configuration C, formee de deux files parallèles de tourbillons ponctuels; C est caracterisée par les paramètres classiques: $\alpha = \frac{h}{l}$ et $\lambda = \frac{d}{l}$ ($d \neq 0$, $d \neq \frac{1}{2}$). Au premier ordre, la condition de stabilité de C s'écrit

(1) $$\operatorname{sh}(\alpha\pi) = \sin(\lambda\pi)$$

L'auteur montre que, moyennant (1), la stabilité de C subsiste pour les perturbations les plus générales du second ordre; globalement, la file ne peut subir qu'un déplacement de translation. L'A. insiste sur cette conclusion qui contredit les prévisions de Maue, reprises par A. Sommerfeld, mais qui confirment, dans une certaine mesure, les essais effectues à la demande de l'Auteur."

Apropos of [28] Kravtchenko states: "Une formule classique de Kármán donne la resistance frontale, éprouvée par un contour circulaire place dans un courant plan avec des tourbillons regulièrement distribuées suivant deux files parallèles. La relation en cause, valable aux faibles vitesses, contient six paramètres dont deux ne sont susceptibles que d'une détermination experiméntale. A. Kosmodemiamski a émis des hypothèses complémentaires permetant le calcul théorique des coefficients en cause. L'auteur approfundit la question et precise la nature des configurations pour lesqueles ces hypothèses sont valables."

The article [29] of 1955 is an extension of the theory developed in [28] concerning a method of approximative calculation of the resistance experienced by a circular cylinder in an ideal fluid in the presence of a vortex street behind it. Dolaptchiev considers a vortex pair and seeks those curve lines in the plane of the fluid flow for which the vertical velocities of the vortices are zero and the horizontal velocities are maximal. These conditions define a family (*verallgemeinerte Föpplsche Kurven*) with parameter λ equal to $\kappa\pi\coth(\kappa\pi)$. The curve $\lambda = \frac{5}{4}$ corresponds to the condition $\kappa = 0{,}280$ of stability of Kármán vortex streets. Now Dolaptchiev supposes that the locus of tearing off of the first vortex pair is where this particular curve intersects the straight lines $y = \pm a$, accepting Kosmodemianski's hypothesis that the breadth of Kármán street equals the diameter $2a$ of the cylinder. He juxtaposes his theoretical results with photographs of vortices in the wake of a cylinder and finds a satisfactory concurrence. In any case, as he himself notes, the question of the mechanism by means of which the symmetric disposition of the vortices turns into a staggered one remains open.

Dolaptchiev's papers [14] and [29] are mentioned in R. Wille's article *Kármán vortex streets* in *Advances of Applied Mechanics* (**6**, 1960, 273–294): "With linearized equations A. Maue [10] and B. Dolaptschieff [11] indepently formulated the stability condition $\sin\pi\mu = \operatorname{sh}\pi\kappa$. Domm proves that all vortex streets satisfying Maue–Dolaptschieff condition ..." and "B. Dolaptschieff [21, misprint in the title of the journal] formulated the laws of motion for any single fluid particle in a given vortex street. The evaluation of the integrals and the physical conclusions are yet to be published". As a matter of fact, the latter is done in the paper [39].

Apropos of Dolaptchiev's article [29], in his paper *К вопросу обтекания эллиптического цилиндра потоком несжимаемой жидкости с парой присоединенных вихрей* (*Известия Сибирского отделения Акад. Наук СССР*, **9**, 1958) I. G. Legtchenko writes: "Анализируя их, Б. Долапчиев [2] доказал справедливость введенных А. А. Космодемьянским гипотез и показал, что постановка задачи, при которой рассматривается одна-единственная вихревая пара, связанная с цилиндром, приводит к удовлетворительному результату, когда не требуется большой точности", i.e. "analyzing A. A. Kosmodemjanski's hypotheses B. Dolaptchiev [2] proved their truthworthyness and displayed that the formulation of the problem, in which one and only vortex pair connected with the zylinder is regarded, leads to a satisfactory results in those cases when no great accuracy is required".

The paper *Notes on the investigations of the stability of vortex streets* [30] of 1955 has a surveying character and reflects Dolaptchiev's twenty-year-old experience in the domain. By his own words, "im Vorliegenden berichte ich über einige zusammenfassende und kritische Bemerkungen, welche die Fragen der Stabilität oder der Instabilität der Wirbelstraßen betreffen. Diese Bemerkungen beziehen sich konkret auf die Themata: Stabilität und Instabilität der Wirbelstraßen bei verschiedene Wirbelanordnungen und unter verschiedenen Stabilitätskriterien; Eigenschaften der Wirbelstraßen; schräge Fortbewegung der Wirbelstraßen; Stabilisierung der Wirbelstraßen. Parallel mit diesem Hauptziel berühre ich noch gewisse Fragen, die eng mit der Theorie der Wirbelstraßen verbunden sind und zwar beziehen sie sich auf den Widerstand der Körper, hinter welchen die genannten Wirbelkonfigurationen entstehen" (S. 110). After an introduction and a review of the content, Dolaptchiev treats the following items: finite or infinite number of degrees of freedom of perturbances, stability criterion in the narrow sense, general periodic law of perturbance, special "group" perturbances, group perturbances of first order, group perturbances of higher order, Prandtl's assumption, generalized condition of stability, "oblique" flow of vortex streets, properties of vortex streets, stabilization of vortex streets, a criticism of the stability criteria, the most general criterion of stability, instability and destruction of a vortex street, other investigations concerning vortex streets, vortex streets or vortex pairs in the presence of a body, investigation of the stability of vortex streets in compressible fluid, cylindrical vortex streets, viscous vortex streets and their stability.

The article *On an approximate scheme for the calculation of the vortex resistence* [31] of 1955 represents, as the author himself notes, a detailed exposition of his investigations on vortex resistance of a circular cylinder and the related "generalized Föppl's curves", briefly announced in the notes [28] and [29]. It consists of the following paragraphs: Kármán's formulation of the problem, Kosmodemianski's hypotheses, an additional supposition to hypothesis II, velocities of the vortex pair, a geometrical picture of the loci of vortex centers with horizontal relative velocities, determination of the breadth of the vortex street, determination of the circulation, interpretation of the diagrams.

In 1956 the article *On the integrals of motion of ideal fluid in the presence of Kármán vortex streets* [33] was published along with its abbreviated German version [38]; the note [36] is a summary of the report on the same subject delivered by Dolaptchiev before the *Ninth International Congress of Applied Mechanics* in Brussels. The paper [33] consists of the following paragraphs: connection between the complex potentials with respect to various systems of reference, superposition of complex potentials, complex potentials at uniform translations along the real axis, construction of complex potentials in the presence of a vortex street, differential equations of motion of an arbitrary non-vortex fluid particle, Riccati differential equations connected with the system of motion, equipotential and current lines, integrals of the system of motion, a direct integration of the system of motion, a direct induction of the equipotential lines.

The paper [38] is reflected in *Schnittbeleg aus der physikalischen Verhandlungen* (Bd. **8**, Folge 3, 1957, S. 66) in the following manner: "Die bisherigen Untersuchungen über Kármánsche Wirbelstraßen behandeln entweder das Stabilitätsproblem oder das Widerstandsproblem. Demgegenüber untersuchen die Verf. die Flüssigkeitsbewegungen, die sich entweder in großer Entfernung oder nahe bei einer Wirbelstraße einstellen. Zunächst werden nur die Integrale der Bewegungsgleichungen eines Flüssigkeitsteilchen erhalten, das nicht mit einem Wirbelzentrum zusammenfällt. Die Auswertung dieser Integrale und die physikalischen Schlußfolgerungen sind noch nicht abgeschlossen. Wenn auch ideale Ffüssigkeit vorausgesetzt wird und man von der Physik der Wirbelentstehung absieht, kann man durchaus gewisse Vorgänge in den beiden Zonen deuten und die entsprechenden Mikro- oder Makroströmungen studieren. Mathematisch führt man die Gleichungssysteme auf Riccatische Differentialgleichungen vom gleichen Typ zurück, sowohl bei schachbrettartigen als auch bei symmetrischen Wirbelanordnungen hinsichtlich aller drei möglichen Bezugssysteme. Diese Riccatischen Gleichungen werden dann teils hydrodynamisch durch das komplexe Potential bei stationärer Strömung gelöst, teils durch die Christl-Halmsche Seichegleichung (Gleichung für die Eigenschwingungen eines Sees)."

In the process of composing [33] an interesting mathematical phenomenon has been observed. As it is well-known, a plane fluid flow is called *potential* if there exists such an analytical function $f(z)$ of the complex variable z that its derivative with respect to z equals the conjugate of the complex velocity of the fluid particle z, i.e. if $z = x + iy$, then

$$\frac{dx}{dt} = \Re\frac{df(z)}{dz}, \quad \frac{dy}{dt} = -\Im\frac{df(z)}{dz} \tag{28}$$

or, for the sake of brevity,

$$\frac{d\bar{z}}{dt} = \frac{df(z)}{dz}. \tag{29}$$

If now the right-hand sides of (28) do not depend explicitly on the time, t, then (28) imply the differential equation

$$\frac{dx}{dy} = -\frac{\Re\dfrac{df(z)}{dz}}{\Im\dfrac{df(z)}{dz}} \tag{30}$$

of the trajectory of the fluid particle z or, just the same, of the point (x, y). As a rule, the appearance of the right-hand side of (30) is such that all hopes for a direct integration of this differential equation seem illusionary. On the other hand, if

$$f(z) = g(x, y) + ih(x, y), \tag{31}$$

then

$$\frac{df(z)}{dz} = \frac{\partial g(x, y)}{\partial x} + i\frac{\partial h(x, y)}{\partial x}, \tag{32}$$

whence the equation (30) may be rewritten in the form

$$\frac{dx}{dy} = -\frac{\partial g(x,y)}{\partial x} : \frac{\partial h(x,y)}{\partial x}. \tag{33}$$

Now (33) and

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial h(x,y)}{\partial y}, \quad \frac{\partial h(x,y)}{\partial x} = -\frac{\partial g(x,y)}{\partial y} \tag{34}$$

imply

$$\frac{\partial h(x,y)}{\partial x}dx + \frac{\partial h(x,y)}{\partial y}dy = 0, \tag{35}$$

consequently the current lines

$$h(x,y) = \text{const.} \tag{36}$$

propose a solution of (30). In such a way, if conversely one aims at the solution of a particular differential equation of the kind

$$\frac{dx}{dy} = F(x,y) \tag{37}$$

and if one succeeds to represent the right-hand side of (37) in the form (33), $g(x,y)$ and $h(x,y)$ satisfying (31) for some analytical function $f(z)$, then one arrives at the solution (36) of (37). At first sight the premises of these explanations seem somewhat implausibly, but there are cases when the scheme works, as we shall immediately see.

In the particular case considered in [33] the equation (30) is a Riccati equation. As it is well known, to any Riccati equation there corresponds an equivalent linear homogeneous equation of second order, so that the solution of the first one may be reduced to the solution of the second one and vice versa. The linear homogeneous equation of second order corresponding to the Riccati equation of the paper [33] is of the kind

$$4(x^2+1)^2\, y'' + (ax^2 + a - 3)\, y = 0, \tag{38}$$

a denoting an arbitrary parameter. This is a particular case of the two-parametric set of differential equations

$$4(x^2+1)^2\, y'' + (ax^2 + b)\, y = 0, \tag{39}$$

namely that with

$$b = a - 3. \tag{40}$$

It is a freak of chance that the equation (39) is solved in the well-known reference book *Differentialgleichungen. Lösungsmethoden und Lösungen* of E. Kamke namely in the particular case (40). Therein one may read that the equation (38) has been solved by J. Halm (*Transactions of the Royal Society of Edinburgh*, **41**, 1906). For this reason the equation (39) has been called the *Halm's equation* in the paper [33] and afterwards.

Halm's equation (39) or, just the same, the equation

$$(1+x^2)y'' + (ax^2 + b)y = 0, \tag{41}$$

is the subject of the papers [32, 34, 37], where various relations between the parameters a and b are found when this equation may be integrated by quadratures, making use of complex potentials as explained above. The same topic is treated in the paper [35], too, by a different approach, however: in this last case the hypergeometric differential equation has been used.

Chronologically, the next paper *Trajectories of a symmetrical vortex pair behind a circular cylinder and their connection with the resistence of the cylinder* [39] of 1956 and its Russian version [41] contain "eine exakte Lösung des Problems 'Zylinder–Wirbelpaar', das eng mit dem Problem des Zylinderwiderstandes im Zusammenhang steht und zum ersten mal von L. Föppl (1913) und H. Rubach (1916) behandelt worden ist" [39, p. 240]. The exposition includes the following items: the model of L. Föppl, stationary state of the vortex pair, integration of the system of differential equations of motion of the vortex pair, analytical investigation of the family of trajectories, construction and kinematical interpretation of the vortex paths, notes on the inference of Föppl's resistance formula, interpretation of the latter, another interpretation.

The content of the survey [40] of 1957 becomes clear from the synopsis of the paper by the editor of the *Schrifrenreihe des Forschungsinstituts für Mathematik bei der Deutschen Akademie der Wissenschaften zu Berlin*: "In dieser Schrift gibt der bekannte Forscher einen Überblick über Methoden und Ergebnisse der ausgedehnten theoretische Forschung über das Mathematiker wie Physiker interessierende Problem. Ausgehend von dem berühmten Kármánschen Kriterium wird die Stabilität von Straßen mit differenten Anordnungen der Wirbel unter dem Gesichtspunkt verschiedenartiger Stabilitätskriterien betrachtet. Eine wichtige Rolle spielen dabei die sogenannten Gruppenstörungen erster und zweiter Ordnung; für endliche Störungen liegt die Theorie von N. J. Kotschin vor. Eingehend wird das Phänomen der schrägen Fortbewegung von Wirbelstraßen diskutiert; dabei gelangt der Verfasser zu dem wichtigen Begriff der Stabilisierung von Wirbelstraßen. Es folgen die hypothetischen Ansätze zur Bestimmung der Parameter der Kármánschen Wirbelstraße. Anschließend geht der Verfasser auf die neuerdings erfolgte Erweiterung der Theorie um den Einfluß der Reibung ein."

The articles *Fluid transport inducted by Kármán vortex streets*, parts I and II [42, 43] of 1959, as well as their German summary [46] and the report [47] on the same subject delivered by Dolaptchiev before the *Tenth International Congress of Applied Mechanics* in Stresa (Italy) in 1960, concern the problem of fluid masses driven by a Kármán vortex street through a (curvilinear in the general case) wall perpendicular to the plane of the moving ideal fluid. Mathematically, this problem is reduced to the calculation of integrals of the kind

$$f(x) = \int_0^x \ln(a + \sin x)\, dx \quad (a > 1). \tag{42}$$

Fourier expansions are found in [42, 43] of the right-hand side of (42), which are analogues of the expansion

$$L(x) = x \ln 2 + \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{\sin 2nx}{n^2} \tag{43}$$

of the function of Lobatchevski

$$L(x) = -\int_0^x \ln \cos x \, dx. \tag{44}$$

The aims of the article *Some mechanical considerations on curve lines traced on surfaces* [44] of 1959 are formulated by its authors in the following manner: "Es werden in vorliegender Arbeit die folgenden mechanisch-geometrischen Probleme betrachtet: Bestimmung des Kraftpotentials, bei welchem ein materieller Punkt, der sich auf einer Fläche bewegt, eine vorgeschriebene Flächenkurve beschreibt; Auffindung der Differentialgleichung der Flächenkurven, die gewissen Bedingungen entsprechen; Gleichgewichtskurven schwerer, homogener, unausdehnbarer und biegsamer Fäden auf Flächen; Bestimmung der Rotationsflächen, deren Gleichgewichtskurven solcher Fäden Kurven gegebener Neigung sind" (p. 33). In a sense, this article is an extension of the considerations of the papers [1, 3, 8, 15, 18].

The paper *A critical analysis of an attempt at an axiomatical consolidation of analytical mechanics* [45] of 1960 stands in a sense outside the main trends of Dolaptchiev's scientific interests. As it is well-known, in his famous address *Mathematische Probleme* before the *International Mathematical Congress* in Paris 1900 Hilbert formulated 23 mathematical problems set for this century to solve, the sixth of which, in Hilbert's own formulation, reads: "Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, *nach diesem Vorbilde diejenigen physikalischen Disziplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt: dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik.*" As Truesdell quite justly noted, however, "this problem, like all those he proposed concerning the relation between mathematics and physical experience, has been neglected by mathematicians" (see his paper *Recent advances in rational mechanics* in *Science*, **127**, 1958, 729–739). One of the few attempts to solve Hilbert's Sixth Problem as far as rational mechanics is concerned has been undertaken by G. Hamel in his article *Die Axiome der Mechanik* (*Handbuch der Physik*, Bd. V. *Grundlagen der Mechanik. Mechanik der Punkte und starren Korper.* Berlin, 1927). The paper [45] under consideration represents an analytical criticism of this article of Hamel's, the final conclusion of the authors being that this try is a hopeless failure.

The article *Velocity regime of a fluid in the presence of bilaterally infinite Kármán vortex streets* [48] of 1962 is written apropos of H. Wehner's note *Betrachtungen über Strömungsverhältnisse einer wandernden Kármánschen Wirbelstraße* (*Zeitschr. für Meteorologie*, **4**, Hf. 7/8, 1950, 248–250), where an attempt is made to solve the following problem: if a bilaterally infinite staggered Kármán vortex street is at hand in the plane of an ideal fluid, then examine the behaviour of the direction of the velocity in a fixed point of that plane. The author claims to have proved that if the point in question is invariably connected with the obstacle generating the vortex street and presupposed located at minus infinity (in order that the street is *bilaterally* infinite and situated *behind* the obstacle, if the undisturbed by its presence fluid is moving from left to right) and if it is sufficiently far from the axis of the street, then its local velocity undergoes a periodic rotation, the period of which does not depend on the distance of that point from the axis of the street. It is proved in the article [48] that the reason for this improbable conclusion of

Wehner is rooted in the erroneous complex potential of the fluid motion he uses as a starting point for his calculation; moreover, the authentic complex potential is given in [48] along with the true solution of Wehner's problem.

The main aim of the article *Motions of parallel vortex rows in an ideal fluid* [49] of 1963 is to propose a most complete investigation of the circumstances concomitant and conditioning the flow of a certain number of vortex rows parallel to the motion of the indisturbed by their presence ideal fluid. The cause for this study has been the recent publication *Stable vortex waves near rigid boundaries* (*Journ. of Math. and Phys.*, **40**, 1961, 33–40) of J. Hunt treating the problem of the conditions under which the existence of a bilaterally infinite staggered Kármán vortex street is possible parallel to an infinite wall perpendicular to the plane of the fluid motion. There are two other publications that have a formal or a factual attitude to this problem: the articles *Über zweidimensionale Flüssigkeitsströmung zwischen parallelen ebenen Wänden* (*Ann. der Phys.*, **61** [**366**], 1920, 173–194) of G. Jaffe and *The stability of infinite differential systems associated with vortex streets* (*Journ. of Math. and Phys.*, **30**, 1952, 171–199) of E. A. Coddington. The first one contains a study of fluid flow in a rectilinear canal in the presence of infinite vortex rows parallel to its walls; the second one treats the phenomenon of simultaneous motion of two parallel infinite staggered Kármán vortex streets without a wall in the fluid, with such mutual disposition, however, and with such a kinematical symmetry, which correspond to the conditions realized namely in Hunt's case. An error commited by Hunt in his article is discussed and corrected in [49]; it is due to incorrect use of complex potentials and makes the interpretation of some of his results illusionary.

The note [50] represents the report delivered by Dolaptchiev before the III. *Konferenz über nichtlineare Schwingungen* (Berlin, 25–30 Mai, 1964). It contains critical remarks apropos of the papers *Mathematical investigation if the thrust experienced by a cylinder in a current, the motion being periodic* (*Proc. Royal Irish Acad.*, **37**, 1927, 95–109) of J. L. Synge and *О вихревой теории лобового сопротивления* (*Сб. научн. труд. Казаньск. авиац. инст.*, **2**, 1934, 33–43) of G. V. Kamenkov. As regards Synge's paper, Dolaptchiev firstly refreshes the reader's memory by the remark that although Kármán investigated bilaterally infinite antisymmetric vortex streets and Synge only unilateraly ones, both authors arrive at the same expressions for the vortex resistance. Afterwards he promulgates this coincidence illusionary on the grounds of the fact that the velocity of translation of the vortex configuration along the fluid flow simply does not exist in Synge's case. On the other hand, Kamenkov proposed for the ratio $\frac{h}{l}$ the value 0,245 rather than Kármán's 0,281 by virtue of his finding that 0,245 minimizes the resistance; Dolaptchiev's objection is that the thrust experienced by the cylinder neither is an integral of some system of perturbed motions nor has an extremum, contrary to Kamenkov's claims.

The article *On Synge's formula for the vortex resistance and on Kamenkov's criterion for the stability of Kármán vortex streets* [56] of 1965 is a Bulgarian version of the report [50], while the note *Über die Deutung (von Synge) des Wirbelwiderstandes und (von Kamenkow) der Wirbelstabilität bei einer Kármánstraße* [59] announces that "eine ausführliche Darstellung erscheint in den Veröffentlichungen über 'Nichtlineare Schwingungen', Berlin".

The article *On the vortex concept and the motions of vortex configurations* [51] of 1964 treats a problem which, for a long period of time, has remained unnoticed by the investigators of Kármán vortex streets. As it is well known, a point V in the plane of motion of an ideal fluid is called a *vortex center* or simply a *vortex* if, and only if, P being any fluid particle in the same plane, V induces on P velocity perpendicular to the segment VP, the magnitude of which is inversely proportional to the length of VP; the proportionality factor being $\frac{\Gamma}{2\pi}$, Γ is called the *circulation* of V and its sign determines the direction of rotation of the vortex or, just the same, of the velocity in question. This definition becomes senseless in the case when P coincides with V and is, therefore, supplemented by the acceptance that in this case the induced velocity is zero by definition; this circumstance is described by the phrase that a vortex does not induce velocity on itself. Obviously, from a purely logical point of view (physical considerations being cast aside) the latter definition is a quite arbitrary one.

What now if P is another vortex center itself, say W? One is faced with the alternative:

1. The above definition makes no exception for W.
2. W is excepted in the above definition.

At that, both postulates are enjoying equal logical rights. This means that both are equally acceptable from a purely *logical* point of view. Quite on the contrary, only one of them may be accepted from a *physical* point of view.

There is a Latin proverb: *De principiis non est disputandum.* It may be paraphrased: *De definitionibus non est disputandum.* In other words, no mathematical objections against the first or the second of the above potentialities can be tolerated. This fact accentuates on the responsibility of the physical factor.

As far as our knowledge goes, all researchers in the field of vortex configurations, without a single exception, are unanimous in their conviction that there is no problem at all: none of them feels any doubt that the first alternative is a question of proof. According to them, consequently, the second hypothesis is impossible on principle.

On the contrary, in the opinion of the authors of [51], the initial definition of the vortex concept quoted above must be accepted only in the case when P is a non-vortex fluid particle; if P coincides with another vortex W, then another definition for the velocity induced by V upon W must be accepted (possibly identical with the one for the non-vortex case). In any case, the law of vortex interaction is a self-dependent hypothesis of fluid mechanics completely detached from the other fundamental laws of this theory: it is quite possible to reconstruct the whole doctrine of vortex configurations on another hypothesis concerning the vortex interaction (the magnitude of the velocity, induced on W by V, being, *exempli causa*, inversely proportional to the square or some other degree of the distance VW; or being a function of the circulation of W too, rather than of the circulation of V only, etcetera). The authors of [51] do not actually propose any particular hypothesis whichever to this end: they only emphasize on the mathematical possibility of such a procedure, in the same time giving utterance of their doubts in the physical advisability and trustworthiness of the traditional assumptions concerning the vortex interaction. And that is that.

The paper [77] of 1971 represents a German version of the reviewed article [51] published by the *Danish Center for Applied Mathematics and Mechanics.* As a matter of fact, [77] is a report delivered by Dolaptchiev in the capacity of a "Gastprofessor in Danemarks Technischer Hochschule".

In the light of these explications the last work of Dolaptchiev in the field of the vortex configurations *On an unknown theorem (of Synge) in an old hydrodynamical problem (of Kármán)* [81] of 1973 (printed posthumously) seems somewhat oddly. As a matter of fact, in this paper he renounces the previous articles [51, 77] and takes in the standpoint of J. L. Synge expressed glibly by the latter in a letter to Dolaptchiev. In it Synge sustains that the law of vortex interaction is a question of proof rather than of a hypothesis; in the same letter he is pretending having produced such a proof. In actual fact, disregarding some verbal explanations, Dolaptchiev's paper [81] contains a reproduction of Synge's "proof". The inverted commas are used here by no means accidentally: Synge's considerations are entiredly void of sense, the proof of the traditional hypothesis of vortex interaction being as unattainable as the proof of the Fifth Postulate; worst of all, there are counter-examples. At that, Dolaptchiev's co-author of the article [51] has come to know about the paper [81] as late as it has been printed.

Chronologically, the next paper after [51] is *Vortex resistance of fluid flows in the presence of two-parametric vortex streets* [52] of 1964. The main motive for this investigation is an erroneous formula of the vortex resistance in the presence of an asymmetric Kármán vortex street in the ideal fluid, reproduced in the renowned work *Vorlesungen über theoretische Physik* of A. Sommerfeld (Bd. II. *Mechanik der deformierbaren Medien.* Leipzig, 1957) and attributed by him to A. W. Maue. The authors of [51] point at the origin of Maue's mistake and give a correct version of the formula in question.

The next work of Dolaptchiev in the mathematical theory of vortex configuration is the article *Velocity regime of an ideal fluid in the presence of bilaterally infinite Kármán vortex streets* [66] of 1968; it represents an extension of the considerations exposed in the reviewed paper [48].

The article *On the 'least instability' of the two-parametric vortex streets* [70] of 1968 is dealing with the same problem as the paper [14], namely with the conditions (26) or (27), characterizing the corresponding two-parametric vortex street as possessing "geringste Instabiliutät". The latter is established in [70] by means of considerations different from those exposed in [14], namely by a detailed algebraic analysis of the characteristic equation of the respective system of differential equations of the disturbed vortex configuration.

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Hitherto we have retraced Dolaptchiev's mathematical papers in pure mathematics (geometry and analysis) and in the theory of vortex configurations. We proceed now to the last domain of his scientific activity, namely analytical mechanics, in general, and dynamical equations of motion, in particular. If we should characterize in a few words the tendencies of his occupations in this field, we would say that here he is entirely in the swim with Lagrangean dynamical tradition.

The first paper [53] of Dolaptchiev in this domain is dated 1965, when he was sixty, lesser than ten years before his death and two years before the decease of Ivan Tzenov, his predecessor in the Chair of Analytical Mechanics in the Mathematical Faculty of the University of Sofia. We mention the name of Tzenov since, as already underlined, he has worked all his life exclusively in the field of the equations of analytical dynamics, in general, and in non-holonomic dynamics, in particular. The authors of these notes cannot account for the fact that — having worked in the Chair of Analytical Mechanics more than three decades, in the encirclement of Tzenov's scientific tradition at that — Dolaptchiev did not write a single line apropos of any dynamical problem whichever. But one cannot get away from facts and, having marked out this one, we proceed farther, coming down to brass tacks.

One can only guess which kind of professional or personal motives Dolaptchiev might have had to start his work in the new field, but we have our conjectures. About 1965 he was working upon the second, wholly revised edition of his lecture course *Аналитична механика*. In the process of this revision he came across a most interesting in a pedagogical respect booklet, the *Vorlesungen über elementare Mechanik* of J. Nielsen (Berlin, 1935). Therein he bump up against "eine zweite, weniger bekannte Form" [53, S. 351]

$$\frac{\partial \dot{T}}{\partial \dot{q}_\kappa} - 2\frac{\partial T}{\partial q_\kappa} = Q_\kappa \quad (\kappa = 1, 2, 3, \ldots, k) \tag{45}$$

of Lagrange's dynamical equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\kappa} - \frac{\partial T}{\partial q_\kappa} = Q_\kappa \quad (\kappa = 1, 2, 3, \ldots, k). \tag{46}$$

The aims of his first article [53] in Lagrangean subject matter Dolaptchiev formulated in the following manner: "Das Ziel vorliegender Betrachtungen ... es wird zuerst gezeigt, daß wir die Gleichungen ... von Nielsen an ein Variationsprinzip anknüpfen konnen; andererseits, daß sie auf die Appellsche Form zurückgeführt und damit auch auf nichtholonome mechanische Systeme übertragen werden können" [*ibid.*, S. 351–352]. The variational principle in question is "das ... ebenfalls weniger bekannte Differentialprincip von Jourdain ..., nämlich

$$\sum_{\nu=1}^{N}(m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu)\delta\dot{\boldsymbol{r}}_\nu = 0 \tag{47}$$

— "genau so wie die Gleichungen ... von Lagrange dem Prinzip von D'Alembert"

$$\sum_{\nu=1}^{N}(m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu)\delta\boldsymbol{r}_\nu = 0 \tag{48}$$

"entsprechen" [*ibid.*, S. 352]. As regards the reduction of the equations (45) to the equations of Appell

$$\frac{\partial S}{\ddot{q}_\kappa} = Q_\kappa \quad (\kappa = 1, 2, 3, \ldots, k), \tag{49}$$

where

$$S = \sum_{\nu=1}^{N} m_\nu {w_\nu}^2, \tag{50}$$

Dolaptchiev underlines that "wir folgen dazu den Weg, den Ivan Tzenoff ... in einigen Arbeiten vorgeschlagen hat" [*ibid.*, S. 353].

The article *The principle of Jourdain and the equations of Nielsen* [55] of 1965 is closely connected with the reviewed paper [53], the latter being actually an abbreviated version. It includes the following items: the principle of Philip Jourdain, the equations of Jakob Nielsen, deduction of Nielsen's equations from Jourdain's principle, Appell's form of Nielsen's equations for holonomic material systems, Tzenov's form of Nielsen's equations for non-holonomic material systems. As a matter of fact, the principle of Jourdain (proposed by him in the article *Note on an analogue of Gauss' principle*, printed in 1909 in the *Quart. Journ. of Pure and Appl. Math.*, p. 251) has been hitherto neglected by mechanicians, as well as the equations of Nielsen (though so far it is not wholly clear whether Nielsen namely is the genuine author of these equations, as well as where have they been published for the first time), and one of Dolaptchiev's special merits in this field is that he wormed them out of oblivion.

The chronologically next Dolaptchiev's papers connected with the dynamical equations of motion are *Generalization of Nielsen–Tzenov equations* [57] and its abbreviated German version [54]. The first of them involves the following items: deduction of a kinematical identity, derivation of "generalized Lagrange equations", formulation of a "generalized D'Alembert principle", reduction of the generalized equations to Appell's and Tzenov's forms, application of the latters to non-holonomic mechanical systems. As a prelude the identities (3) of the paper are derived, by the aid of which the "most general kinematical identities" (according to the author's terminology)

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\lambda} = \frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q}_\lambda} - \frac{\partial T}{\partial q_\lambda}\right) \qquad (\lambda = 1, \ldots, l) \tag{51}$$

are deduced for any natural n under the traditional notations for the mechanical entities involved, the holonomity of the constraints being implied. The ordinary Lagrange equations (46) being presupposed, the "generalized Lagrangean equations"

$$\frac{1}{n}\left(\frac{\partial \overset{(n)}{T}}{\partial \overset{(n)}{q}_\lambda} - (n+1)\frac{\partial T}{\partial q_\lambda}\right) = Q_\lambda \qquad (\lambda = 1, \ldots, l) \tag{52}$$

for any natural n are in such a manner obtained. Nielsen's equations (45) and Tzenov's equations

$$\frac{1}{2}\left(\frac{\partial \ddot{T}}{\partial \ddot{q}_\lambda} - 3\,\frac{\partial T}{q_\lambda}\right) = Q_\lambda \qquad (\lambda = 1, \ldots, l) \tag{53}$$

are at once deduced from (52) for $n = 1$ and $n = 2$, respectively. As the author notes apropos of the equations (52), they are "auch von Mangeron–Deleanu abgeleitet" [54, S. 446] (see *Sur une classe d'équations de la mécanique analytique au sens de I. Tzenoff*, C. R. Acad. bulg. Sci. **15**, No. 1, 1962, 9–12); the following comment is, however, not uninteresting: "A similar kind of generalization has been already alluded by Wassmuth ...and realized by Mangeron-Deleanu ...; in both cases, however, it has not been brought to an end. In the first case it remained at the level analytical mechanics has been carried to by Appell, by virtue of the unavoidable introduction of accelerating energy; in the second case the liaisons have not been conformed between those equations and the corresponding variational principles they may be derived from, as well as the structure of the non-holonomic constraints and the nature of the forces" [57, p. 41]. Besides, Dolaptchiev notes that he obtains (52) by means of two particular approaches different from that of MangeronDeleanu [*ibid*]. Afterwards he formulates the "general D'Alembert's principle"

$$\sum_{\nu=1}^{N}(m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu)\delta \overset{(n)}{\boldsymbol{r}}_\nu = 0 \quad (n = 0, 1, 2, \dots) \tag{54}$$

provided

$$\delta t = 0, \quad \delta \overset{(m)}{\boldsymbol{r}}_\nu = \boldsymbol{0} \ (m = 1, \dots, n-1), \quad \delta \overset{(n)}{\boldsymbol{r}}_\nu \neq \boldsymbol{0} \tag{55}$$

($\nu = 1, \dots, N$); it is obvious that the common principle of D'Alembert (48), the principle of Jourdain (47), and the principle of Gauss

$$\sum_{\nu=1}^{N}(m_\nu \boldsymbol{w}_\nu - \boldsymbol{F}_\nu)\delta \ddot{\boldsymbol{r}}_\nu = 0 \tag{56}$$

are formally obtained from (55) for $n = 0$, $n = 1$ and $n = 2$, respectively.

АЗБУЧНА БИБЛИОГРАФИЯ НА БЛАГОВЕСТ ДОЛАПЧИЕВ

А. КИРИЛИЦА

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ХРОНОЛОГИЧНА БИБЛИОГРАФИЯ НА БЛАГОВЕСТ ДОЛАПЧИЕВ*

1. Върху единъ родъ криви линии, трасирани върху дадена повърхнина (1931).
2. Върху единъ начинъ за раздѣляне равнината на области отъ n прави линии (24.3.1932).
3. Sur certaines courbes tracees sur une surface donnee (15.3.1935).
4. Приносъ къмъ стабилитета на Kármán'овитѣ вихрени улици и траекториитѣ на отделнитѣ вихри (18.6.1937).
5. Über die Stabilität der Kármánschen Wirbelstraße (1937).
6. Erwiderung (1938).
7. Störungsbewegungen (Bahnen) der einzelnen Wirbel der Kármánschen Wirbelstraße (1938).
8. Върху единъ родъ цилиндрични криви (30.5.1939).
9. Една допирателна трансформация въ геометрията; приложение. (Възлови параболи) (17.4.1940).
10. Новъ начинъ за изследване ортогоналната проекция на кривата на пресичане на две ротационни повърхнини отъ втора степень върху равнината на оситѣ имъ (5.6.1941).
11. Нѣкои екстремални случаи при образуването на евклидово-конвекснитѣ полигони и трирълбести полиедри (29.3.1942).
12. Едно дескриптивно-геометрично приложение на проективнитѣ системи конични сѣчения (28.5.1942).
13. Eine einfache geometrisch-mechanische Analogie (1942).

* За подробни библиографски данни читателят се препраща към азбучната библиография. В скоби след заглавията на трудовете са посочени датите на приемането им за печат.

14. Двупараметрови вихрови улици (15.12.1942).
15. A felületi görbék egy fajtájáról. Hengerre írt láncgörbék (1943).
16. Über projektive Kegelschnittsysteme (1.9.1943).
17. Математически решения въ аеродинамиката на летението (8.12.1943).
18. Върху потенциала на силите, под действието на които една точка описва геодезичните линии на повърхнините (31.12.1946).
19. Върху косото протичане на вихровите улици (22.2.1947).
20. Върху стабилизирането на вихровите улици (5.3.1947).
21. Stabilisation des files de tourbillons (28.12.1949).
22. Обобщающий прием определения устойчивости произвольно расположенных вихревых дорожек (10.6.1950).
23. Приложение метода на Кочин за изучаване равновесното състояние на двупараметровите вихрови улици (14.10.1950).
24. Обобщителен метод за установяване стабилността на произволно наредени вихрови улици (14.10.1950).
25. Применение способов Н. Е. Кочина к исследованию состояния равновесия двухпараметровых вихревых дорожек (10.11.1950).
26. Устойчивость вихревых дорожек (10.11.1950).
27. Об устойчивости и косом протекании двухпараметровых вихревых дорожек (30.6.1954).
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29. Verallgemeinerte Föpplsche Kurven im Zusammenhang mit der Wirbelwiderstandsbestimmung (21.3.1955).
30. Бележки върху изследванията на стабилността на вихровите улици (30.6.1955).
31. Върху една приближена схема за пресмятане на вихровото съпротивление (19.9.1955).
32. Über eine Differentialgleichung von J. Halm (30.6.1956).
33. Върху интегралите на движение на идеален флуид при наличие на карманови вихрови улици (10.7.1956).
34. Върху едно диференциално уравнение на J. Halm (13.7.1956).
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36. Über die Integrale der Bewegungsgleichungen der Flüssigkeit um eine Kármánsche Wirbelstraße (1956).
37. Über die Halmsche Differentialgleichung (1957).
38. Integrale der Bewegungsgleichungen der Flüssigkeit um eine Wirbelstraße (1957).
39. Траектории на една симетрична вихрова двойка зад кръгов цилиндър и връзката им със съпротивлението на цилиндъра (8.11.1957).
40. Bemerkungen über die Stabilitätsuntersuchungen der Wirbelstraßen (1957).
41. Симметричное обтекание кругового цилиндра с двумя вихрями за ним. Траектории вихрей и сопротивление цилиндра (10.3.1958).
42. Флуиден транспорт, индуциран от карманови вихрови улици. I (11.6.1959).
43. Флуиден транспорт, индуциран от карманови вихрови улици. II (3.8.1959).
44. Някои механични разглеждания на криви, трасирани върху повърхнини (13.11.1959).
45. Критичен анализ на един опит за аксиоматично изграждане на аналитичната механика (25.8.1960).

46. Von Kármánstrassen induzierter Flüssigkeitstransport (20.10.1960).
47. Flüssigkeitstransporte, induziert von Kármánstraßen (1960).
48. Скоростен режим на флуид при наличие на двустранно безкрайни Kármán'ови вихрови улици (8.10.1962).
49. Движения на успоредни вихрови редици в идеален флуид (4.5.1963).
50. Zwei Bemerkungen über die Stabilität der Kármán-Strassen und den damit verbundenen Wirbelwiderstand (1964).
51. Върху понятието „вихър" и движенията на вихровите конфигурации (25.5.1964).
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53. Über die Nielsensche Form der Gleichungen von Lagrange und deren Zusammenhang mit dem Prinzip von Jourdain und mit den nichtholonomen mechanischen Systemen (20.5.1965).
54. Über die verallgemeinerte Form der Lagrangeschen Gleichungen, welche auch die Behandlung von nichtholonomen mechanischen Systemen gestattet (5.7.1965).
55. Принципът на Jourdain и уравненията на Nielsen (12.7.1965).
56. Върху формулата на Synge за вихровото съпротивление и върху критерия на Каменков за устойчивостта на кармановите вихрови улици (19.7.1965).
57. Обобщение на нилзен–ценовите уравнения (12.8.1965).
58. Über die 'verallgemeinerten' Gleichungen von Lagrange und deren Zusammenhang mit dem 'verallgemeinerten' Prinzip von d'Alembert (4.10.1965).
59. Über die Deutung (von Synge) des Wirbelwiderstandes und (von Kamenkow) der Wirbelstabilität bei einer Kármánstraße (1965).
60. Sur les systèmes mécaniques non holonomes assujettis à des liaisons arbitraires (1966).
61. Об уравнениях Нильзена — Ценова и их применении к неголономным системам с нелинейными связями (3.5.1966).
62. Critères pour l'application des équations généralisées de Lagrange aux systèmes mécaniques non holonomes (1966).
63. Върху апел-ценовите форми на уравненията на движение на холономните и нехолономните механични системи и техните обобщения и критерии за приложения (6.12.1966).
64. Über die Aufstellung und die Anwendbarkeit einigen neuen Formen der Gleichungen der analytischen Dynamik (30.12.1966).
65. Über die Appell–Tzenoffschen Formen der Gleichungen der analytischen Dynamik der holonomen und der nichtholonomen mechanischen Systeme (1967).
66. Скоростен режим на идеален флуид при наличие на двустранно безкрайни Kármán'ови вихрови улици (29.4.1968).
67. Няколко бележки върху работата „Върху уравненията на движението на холономни и нехолономни материални системи" от Иван Ценов (29.4.1968).
68. Примери за приложение на редуцираните уравнения на Нилсен върху нехолономни механични системи (8.5.1968).
69. Verwendung der einfachsten Gleichungen Tzenoffschen Typs (Nielsenschen Gleichungen) in der nichtholonomen Dynamik (29.5.1968).
70. Върху „най-малката неустойчивост" на двупараметровите вихрови улици (3.9. 1968).
71. Exemple d'application des équations de Nielsen à des systèmes mécaniques non holonomes (9.9.1968).
72. Nouvel exemple d'application des équations de Nielsen à des systèmes mécaniques non holonomes (16.9.1968).

73. Обобщенная форма уравнений Лагранжа, пригодная для исследования неголономных систем (1968).

74. Приложение на редуцираните уравнения на Нилзен за нехолономни системи върху проблема на А. Ю. Ишлински (5.7.1969).

75. Извеждане уравненията на Келдиш при „шими" на самолетното и автомобилното шаси чрез редуцираната форма на Нилсен за нехолономни системи (14.11.1969).

76. Други приложения на редуцираните уравнения на Нилсен за нехолономни механични системи (24.11.1970).

77. Über den 'Wirbelbegriff', Wirbelwiderstand und Bewegungen von Wirbelkonfigurationen (1971).

78. Няколко бележки върху Gibbs–Appell'овата форма на уравненията на динамиката на холономните и нехолономните материални системи (13.12.1971).

79. Обобщените уравнения на Лагранж като следствия от уравненията на Гибс — Апел (15.11.1972).

80. Interpretation der Gleichungsform von Gibbs — Appell (20.7.1973).

81. Върху една непозната класическа теорема (на Synge) в един стар хидродинамичен проблем (на Kármán) (23.11.1973).