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AN ALGORITHMIC APPROACH TO SOME PROBLEMS ON THE REPRESENTATION OF NATURAL NUMBERS AS SUMS WITHOUT REPETITIONS[1]

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Given any strictly increasing computable function in the set of natural numbers, certain algorithmic problems arise on the representation of numbers as sums of distinct values of the function. The problem whether a given natural number is representable in this form is obviously algorithmically solvable, but we propose some methods for the solution of the problem that seem to be better than the straightforward ones.

It is easy to see the algorithmic unsolvability of the problem whether all natural numbers are representable (under the usual assumption that an index of the given computable function is used as input data). However, under an appropriate restriction concerning, roughly speaking, the speed of the growth of the function, we present an algorithm for solving this problem and the more general one whether all natural numbers greater than a given one are representable (the restriction is satisfied, for example, when the given function is a polynomial).

We make applications of the presented positive results to concrete problems concerning, for instance, the representation as sums of distinct squares or as sums of distinct positive cubes.

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1. INTRODUCTION

Let $\mathbf{N}_+$ be the set of the positive integers. Suppose f is a strictly increasing function in $\mathbf{N}_+$. An integer n will be called *additively f–representable without*

[1] Lecture presented at the Session, dedicated to the centenary of the birth of Nikola Obreshkoff.

repetitions (*f–representable*, for short) iff

$$n = \sum_{i \in A} f(i)$$

for some finite subset A of $\mathbf{N}_+$; any such A will be called an *f–representation of n*. Of course, all f–representable integers are non-negative, and the number 0 is f–representable (with an empty f–representation).

There is a case when any non-negative integer is f–representable and has a unique f–representation. This is the case when $f(i) = 2^{i-1}$ for $i = 1, 2, 3, \dots$ To have a more complicated example concerning f–representability, let us consider the case when $f(i) = i^2$ for any i in $\mathbf{N}_+$. Then there exist positive integers that are not f–representable, as well as ones having more than one f–representation. Some results connected to f–representability in this case have been presented in [2–5], but without giving a complete description of the set of the representable integers. Such a description can be derived from certain results given in [1] that show the f-representability of all integers greater than 128 as well as of the most of the smaller positive integers. By checking individually the few remaining positive integers, one gets the following conclusion: there are exactly 31 positive integers that are not f-representable, namely the integers 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112, 128.

The mentioned results from [1] are proved by using tools from Number Theory (such as, e.g., divisibility considerations). Those results give in fact considerably more precise information about the f–representations in question. For example, it is seen that each f–representable integer in the considered case has an f–representation consisting of not more than six elements. However, it could be possibly interesting to know that the less precise statement, formulated at the end of the previous paragraph, can be proved in an algorithmic way without using any specific tools from Number Theory. This can be done as an application of a certain method that will be exposed in the present paper.

2. A USEFUEXTENSION OF THE INVERSE FUNCTION f^{-1}

We turn back to the general case described in the first paragraph of the introduction. Given the function f, we define three other functions REPR_f, L_f and $f^\dagger$ with domain $\mathbf{N}_+$, the first two of them being set-valued and the third one being integer-valued. We define them as follows. Let n be an arbitrary element of $\mathbf{N}_+$. We adopt $\mathsf{REPR}_f(n)$ to be the set of all f–representations of n; clearly, this set is finite (possibly empty) and its elements (if any) are non-empty finite subsets of $\mathbf{N}_+$. Then we set

$$L_f(n) = \{\min A \mid A \in \mathsf{REPR}_f(n)\}.$$

Of course, $L_f(n)$ is a finite subset of $\mathbf{N}_+$, and $L_f(n)$ is empty iff $\mathsf{REPR}_f(n)$ is empty, i.e. iff n is not f–representable. Finally, if $L_f(n) \neq \emptyset$, then we set $f^\dagger(n)$ to be the maximal element of $L_f(n)$, otherwise we set $f^\dagger(n) = 0$. Thus $f^\dagger(n)$ is a non-negative integer that is equal to 0 iff n is not f–representable.

Example 1. If $f(i) = i^2$ for $i = 1, 2, 3, \dots$, then

$$\mathsf{REPR}_f(50) = \{\{1,7\}, \{1,2,3,6\}, \{3,4,5\}\},$$

hence $L_f(50) = \{1,3\}$, $f^\dagger(50) = 3$.

For any positive integer i the singleton $\{i\}$ is an f–representation of the number $f(i)$, and any f–representation of this number contains some element not greater than i, hence the equality $f^\dagger(f(i)) = i$ holds. Thus the function $f^\dagger$ is an extension of the inverse function f^{-1}.

We also note that for any f–representable positive integer n the number $f^\dagger(n)$ belongs to some f–representation of n, hence the inequality $f(f^\dagger(n)) \leq n$ holds.

The consecutive values of the function $f^\dagger$ can be recursively computed on the base of the next proposition.

Theorem 1. *For any positive integer n we have the equality*

$$L_f(n) = \{k \in \mathbf{N}_+ \mid f(k) = n \ \text{ or } \ (f(k) < n \ \text{ and } \ f^\dagger(n - f(k)) > k)\}.$$

Proof. Let n be a positive integer. Consider first any k belonging to $L_f(n)$. Then $k = \min A$ for some f–representation A of n, hence $k \in \mathbf{N}_+$. If k is the only element of A, then $f(k) = n$. Otherwise $n - f(k)$ is a positive integer, and $A \setminus \{k\}$ is an f-representation of $n - f(k)$. Therefore

$$f^\dagger(n - f(k)) \geq \min(A \setminus \{k\}) > k.$$

Thus in both cases k belongs to the right-hand side of the equality. For the reasoning in the opposite direction, suppose now that k belongs to the right-hand side of this equality. Then $k \in \mathbf{N}_+$, and either $f(k) = n$ or $f(k) < n$ and $f^\dagger(n - f(k)) > k$. If $f(k) = n$, then we set $A = \{k\}$. Otherwise we consider an f-representation B of $n - f(k)$ such that $f^\dagger(n - f(k)) = \min B$, and we set $A = \{k\} \cup B$. In both cases A is an f–representation of n and $k = \min A$, hence $k \in L_f(n)$.

Example 2. Let f enumerate the set of the prime numbers, i.e. $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 7$, $f(5) = 11$ and so on. Then, making use of Theorem 1 and of the definition of the function $f^\dagger$, we get consecutively:

$$\begin{aligned}
L_f(1) &= \emptyset, & f^\dagger(1) &= 0,\\
L_f(2) &= \{1\}, & f^\dagger(2) &= 1,\\
L_f(3) &= \{2\}, & f^\dagger(3) &= 2,\\
L_f(4) &= \emptyset, & f^\dagger(4) &= 0,\\
L_f(5) &= \{1,3\}, & f^\dagger(5) &= 3,\\
L_f(6) &= \emptyset, & f^\dagger(6) &= 0,\\
L_f(7) &= \{1,4\}, & f^\dagger(7) &= 4,\\
L_f(8) &= \{2\}, & f^\dagger(8) &= 2,\\
L_f(9) &= \{1\}, & f^\dagger(9) &= 1,\\
L_f(10) &= \{1,2\}, & f^\dagger(10) &= 2.
\end{aligned}$$

x	$f^{\dagger}(10x+y)$									
	$y=0$	$y=1$	$y=2$	$y=3$	$y=4$	$y=5$	$y=6$	$y=7$	$y=8$	$y=9$
0		1	0	0	2	1	0	0	0	3
1	1	0	0	2	1	0	4	1	0	0
2	2	1	0	0	0	5	1	0	0	2
3	1	0	0	0	3	1	6	1	2	1
4	2	4	1	0	0	3	1	0	0	7
5	3	1	4	2	2	1	2	1	3	1
6	0	5	2	1	8	4	1	0	2	2
7	3	1	0	3	5	1	0	4	2	1
8	4	9	1	3	2	6	3	2	1	5
9	4	1	0	2	3	1	0	4	3	3
10	10	4	2	2	2	4	5	1	0	3
11	5	1	0	7	3	3	4	6	2	3
12	2	11	4	2	1	5	4	1	0	4

Fig. 1. The first 129 values of the function $f^{\dagger}$ in the case of $f(i) = i^2$

Clearly, it is not always necessary to find all elements of the set $L_f(n)$ in order to see that it is not empty and to find its maximal element. We have $f(k) \leq n$ for any k in $L_f(n)$. Therefore, to calculate $f^{\dagger}(n)$, one could simply find the least positive integer k such that $f(k) > n$ and then execute the operator

$$\textbf{repeat } k := k - 1 \textbf{ until } k = 0 \text{ or } k \in L_f(n)$$

(interpreted in a Pascal-like way).

Example 3. Fig. 1 contains a table of the values of $f^{\dagger}(n)$ for $n = 1, 2, \ldots, 129$, calculated by computer in the above way in the case of $f(i) = i^2$. The table shows that among the positive integers not greater than 129, exactly the 31 ones listed in the introduction are not f–representable.

The amount of operations can be somewhat reduced by noticing that for positive integers n, not belonging to the range of f, one could start executing the above operator from the least positive integer k such that $f(k) \geq n/2$ (if $n \in \mathbf{N}_+ \setminus \mathrm{range}(f)$, then $f(k) < n/2$ for any k in $L_f(n)$, since any k in $L_f(n)$ belongs to some f–representation of n together with at least one greater number). Working in this way, one could manually verify the correctness of the values in Fig. 1 in the course of, let us say, one and a half hour.

Let $\mathbf{N}$ be the set of all non-negative integers. The indicated method for computing values of the function $f^{\dagger}$ can be modified by introducing a binary relation H_f in $\mathbf{N}$ as follows: $n\ H_f\ i$ iff n has an f–representation A such that all elements of A are greater than i. We have $0\ H_f\ i$ for any i in $\mathbf{N}$ by trivial reasons. On the other hand, the following equivalence holds for any n in $\mathbf{N}_+$ and any i in $\mathbf{N}$: $n\ H_f\ i$ iff $f^{\dagger}(n) > i$. Making use of these properties of H_f and of Theorem 1, we get the following result.

Theorem 2. *Let $n \in \mathbf{N}_+$. Then*

$$L_f(n) = \{k \in \mathbf{N}_+ \mid f(k) \le n \text{ and } n - f(k)\ H_f\ k)\}$$

and for any i in $\mathbf{N}$

$$n\ H_f\ i \Leftrightarrow \exists k \in \mathbf{N}_+\ (k > i \text{ and } f(k) \le n \text{ and } n - f(k)\ H_f\ k).$$

To illustrate the application of the relation H_f to the computation of values of $f^\dagger$, we shall consider one more example.

Example 4. Let, as in Examples 1 and 3, $f(i) = i^2$ for $i = 1, 2, 3, \ldots$ We shall compute $f^\dagger(50)$ by using the properties of the relation H_f. Since 50 is not a value of the function f and the least positive integer k satisfying the inequality $k^2 \ge 50/2$ is 5, the value of $f^\dagger(50)$ can be obtained from $k = 5$ by applying the operator

$$\textbf{repeat } k := k - 1 \textbf{ until } k = 0 \text{ or } k \in L_f(50).$$

By Theorem 2 we have

$$4 \in L_f(50) \Leftrightarrow 50 - 4^2\ H_f\ 4 \Leftrightarrow 34\ H_f\ 4 \Leftrightarrow$$

$$\exists k \in \mathbf{N}_+\ (k > 4 \text{ and } k^2 \le 34 \text{ and } 34 - k^2\ H_f\ k) \Leftrightarrow 34 - 5^2\ H_f\ 5 \Leftrightarrow$$

$$9\ H_f\ 5 \Leftrightarrow \exists k \in \mathbf{N}_+\ (k > 5 \text{ and } k^2 \le 9 \text{ and } 9 - k^2\ H_f\ k),$$

hence $4 \notin L_f(50)$. Again by Theorem 2

$$3 \in L_f(50) \Leftrightarrow 50 - 3^2\ H_f\ 3 \Leftrightarrow 41\ H_f\ 3 \Leftrightarrow$$

$$\exists k \in \mathbf{N}_+\ (k > 3 \text{ and } k^2 \le 41 \text{ and } 41 - k^2\ H_f\ k) \Leftrightarrow$$

$$41 - 4^2\ H_f\ 4 \text{ or } 41 - 5^2\ H_f\ 5 \text{ or } 41 - 6^2\ H_f\ 6,$$

$$41 - 4^2\ H_f\ 4 \Leftrightarrow 25\ H_f\ 4 \Leftrightarrow$$

$$\exists k \in \mathbf{N}_+\ (k > 4 \text{ and } k^2 \le 25 \text{ and } 25 - k^2\ H_f\ k) \Leftrightarrow$$

$$25 - 5^2\ H_f\ 5 \Leftrightarrow 0\ H_f\ 5,$$

hence $41 - 4^2\ H_f\ 4$, and therefore $3 \in L_f(50)$. Thus $f^\dagger(50) = 3$.

Remark. The method used in the above example is convenient when some value of the function $f^\dagger$ has to be computed without necessarily computing the preceding ones (an additional reduction of the count of the operations could be achieved by noticing that the statements of Theorem 2, in particular the second one, hold also with "$f(k) < n/2$" instead of "$f(k) \le n$" in the case of $n \in \mathbf{N}_+ \setminus \text{range}(f)$). However, if one has to make a table of the values of $f^\dagger(n)$ for $n = 1, 2, \ldots, m$, where m is a given positive integer, then it seems more reasonable to proceed by consecutive straightforward applications of Theorem 1 as in Example 3.

The function $f^\dagger$ can be used not only for checking whether a given positive integer is f-representable, but also for finding one of the f-representations of a given f-representable natural number. This way of using $f^\dagger$ is possible on the basis of the next proposition.

Theorem 3. *Let n be an f-representable non-negative integer. Let the integers $n_0, n_1, \ldots$ be defined as follows, taken for granted that n_{j+1} is defined iff the right-hand side of the second equality makes sense:*

$$n_0 = n, \quad n_{j+1} = n_j - f(f^\dagger(n_j)).$$

Then there is a non-negative integer r such that $n_r = 0$, and if r is such an integer, then the set $\{f^\dagger(n_j) \mid 0 \le j < r\}$ is an f-representation of n.

Proof. It is clear that n_{j+1} is defined iff n_j is positive and f-representable. Hence, if n_r is defined for a certain r, then n_j is defined, positive and f-representable for any $j < r$, and if $n_r = 0$, then n_j is undefined for any $j > r$. Applying the last statement in the case of $r = 0$, we see that the theorem is trivial if $n = 0$. Suppose now that $n > 0$. Then n_0 is positive and f-representable. On the other hand, if for a certain j the number n_j is defined, positive and f-representable, then, by the definition of the function $f^\dagger$, the number n_{j+1} is not only defined, but it has an f-representation whose elements are all greater than $f^\dagger(n_j)$, and this implies the inequality $f^\dagger(n_j) < f^\dagger(n_{j+1})$ in the case of $n_{j+1} > 0$. Since the values of the function f are positive, we thus see that the defined numbers n_j form a strictly decreasing sequence of f-representable and hence non-negative integers, and the defined numbers $f^\dagger(n_j)$ form a strictly increasing sequence. The sequence $n_0, n_1, \ldots$ should be necessarily finite, and it is clear that its last member should be 0. Consider now an r such that $n_r = 0$, and set $A = \{f^\dagger(n_j) \mid 0 \le j < r\}$. Then

$$n = n_0 - n_r = \sum_{j=0}^{r-1}(n_j - n_{j+1}) = \sum_{j=0}^{r-1} f(f^\dagger(n_j)) = \sum_{i \in A} f(i).$$

Hence A is an f-representation of n.

Example 5. We shall apply the above theorem to $f(i) = i^2$ and $n = 124$. In this case we get (using the table from Fig. 1)

$$n_0 = 124,\ f^\dagger(n_0) = 1,\ n_1 = 123,\ f^\dagger(n_1) = 2,\ n_2 = 119,\ f^\dagger(n_2) = 3,$$

$$n_3 = 110,\ f^\dagger(n_3) = 5,\ n_4 = 85,\ f^\dagger(n_4) = 6,\ n_5 = 49,\ f^\dagger(n_5) = 7,\ n_6 = 0.$$

Hence, by Theorem 3, the set $\{1, 2, 3, 5, 6, 7\}$ is an f-representation of the number 124.

3. CHECKING IF ALL NATURAL NUMBERS GREATER THAN A GIVEN ONE ARE f-REPRESENTABLE

As until now, a strictly increasing function f from $\mathbf{N}_+$ to $\mathbf{N}_+$ is supposed to be given. If this function is computable (in the precise sense given by Recursive Function Theory), then there are obvious algorithms solving the problem whether a given natural number is f-representable, and the considerations from the previous section yield certain better algorithms for the same purpose. A more difficult problem is to decide whether all natural numbers are f-representable. This problem is algorithmically unsolvable in the following natural sense: there is no computable function h defined on the indices of all strictly increasing computable functions f in $\mathbf{N}_+$ and transforming such an index into 0 exactly when all natural numbers are f-representable with respect to the corresponding function f. To prove this, let us consider a two-argument primitive recursive function g such that the set $P = \{x \mid \exists y\,(g(x,y) = 0)\}$ is not recursive. For each x in $\mathbf{N}$ we define a strictly increasing function f_x from $\mathbf{N}_+$ into $\mathbf{N}_+$ as follows: for any i in $\mathbf{N}_+$, if $g(x,y) > 0$ for all y less than i, then $f_x(i) = 2^{i-1}$, otherwise $f_x(i) = 2^i$. If $x \in P$, then the range of the corresponding function f_x is the set $\{1, 2, 2^2, 2^3, \ldots\}$ with one of its elements missing, otherwise the range of f_x is the whole set $\{1, 2, 2^2, 2^3, \ldots\}$. Hence, if $x \in P$, then there are infinitely many natural numbers that are not f_x-representable, otherwise all natural numbers are f_x-representable. If we suppose that a computable function h exists telling apart indices as said above, then we get a contradiction with the non-recursiveness of P.

Of course, the established algorithmic unsolvability directly implies the unsolvability of the more general problem to decide whether all natural numbers greater than a given one are f-representable. However, we cannot exclude the possibility of an algorithmic solution of the last problem under some reasonable restrictions imposed on the function f. A realization of this possibility will be demonstrated in the present section.

For any two integers a and b let $[a\,..\,b)$ denote the set of all integers n satisfying the inequalities $a \le n < b$ (of course, this set is non-empty iff $a < b$). Let $[a\,..\,\infty)$ denote the set of all integers n satisfying the inequality $a \le n$.

Theorem 4. *Suppose $i_0 \in \mathbf{N}_+$, $n_0 \in \mathbf{N}$, and the following two conditions are satisfied:*

1. *For any i in $[i_0\,..\,\infty)$ the inequality $2f(i) - f(i+1) \ge n_0$ holds.*
2. *All elements of $[n_0\,..\,n_0 + f(i_0))$ are f-representable.*

Then all elements of $[n_0\,..\,\infty)$ are f-representable.

Proof (making use of an idea from [5]). For any positive integer i we set $S_i = [n_0 + f(i)..2f(i))$. We shall first show that any element of the set $[n_0 + f(i_0)..\infty)$ belongs to some S_i (with $i \ge i_0$). In fact, given an element n of $[n_0 + f(i_0)\,..\,\infty)$, let us consider the greatest i in $\mathbf{N}_+$ satisfying the inequality $n_0 + f(i) \le n$. For that i we have the inequalities $i \ge i_0$, $n_0 + f(i+1) > n$. From them and Condition 1, the inequality $n < 2f(i)$ follows, hence $n \in S_i$. Now we shall prove the conclusion of the theorem by means of an induction of the following kind: we shall show that whenever an integer n belongs to the set $[n_0 \, .. \, \infty)$ and all smaller integers belonging to this set are f–representable, then n is also f–representable. Suppose n is an integer satisfying the above assumptions; we shall prove that n is f–representable. By Condition 2, we have to examine only the case when $n \geq n_0 + f(i_0)$. Then we consider a positive integer i such that $n \in S_i$. The last condition is equivalent to the inequalities $n_0 \leq n - f(i) < f(i)$. The first of them, together with the inequality $n - f(i) < n$ and the induction hypothesis, shows that $n - f(i)$ is f–representable. Let A be an f–representation of $n - f(i)$. The inequality $n - f(i) < f(i)$ implies that $i \notin A$. This fact, combined with the equality $n = (n - f(i)) + f(i)$, shows that $A \cup \{i\}$ is an f–representation of n, hence n is f–representable.

Remark. An inspection of the proof shows that Condition 2 may be weakened by requiring f–representability only of the elements of $[n_0 \, .. \, n_0 + f(i_0))$ that belong to none of the sets S_i, $i = 1, 2, 3, \ldots$

Suppose now some $n_0 \in \mathbf{N}$ is given. Theorem 4 immediately implies the following statement: whenever $i_0 \in \mathbf{N}_+$ and Condition 1 is satisfied, then the f–representability of all elements of $[n_0 \, .. \, \infty)$ is equivalent to the representability of the elements of $[n_0 \, .. \, n_0 + f(i_0))$. If the function f is computable, then the last condition can be checked in an algorithmic way, and this will be an algorithmic way to check whether all elements of $[n_0 \, .. \, \infty)$ are f–representable. Of course, we may use this way only if we succeed to find some $i_0 \in \mathbf{N}_+$ satisfying Condition 1. We shall show now some examples when such an i_0 really can be found.

Example 6. Let $f(i) = 2^{i-1}$ for $i = 1, 2, 3, \ldots$ Then $2f(i) - f(i+1) = 0$ for any such i, hence Condition 1 is satisfied with $n_0 = 0$, $i_0 = 1$. Therefore the well-known f–representability of all non-negative integers in this case can be proved by checking the f–representability of the elements of the set $[0 \, .. \, f(1))$. Thus the f–representability of all non-negative integers is reduced to the trivial fact that 0 is f–representable.

Example 7 (generalization of the previous example). If $2f(i) - f(i+1) \geq 0$ for any i, then the f–representability of all non-negative integers is equivalent to the equality $f(1) = 1$ (since no f–representable positive integer can be less than $f(1)$). As a particular instance of this we could consider the case when f enumerates the Fibonacci numbers $1, 2, 3, 5, 8, 13, \ldots$, i.e. $f(1) = 1$, $f(2) = 2$ and $f(i) = f(i-1) + f(i-2)$ for $i = 3, 4, 5, \ldots$ In this case, if $i = 1$, then $2f(i) - f(i+1) = 0$, otherwise $2f(i) - f(i+1) = f(i) - f(i-1) > 0$. Thus all non-negative integers are f–representable with respect to this particular function f.

Example 8. Let the function f be a polynomial, i.e.

$$f(i) = a_0 i^r + a_1 i^{r-1} + a_2 i^{r-2} \ldots + a_{r-1} i + a_r,$$

where $r \in \mathbf{N}$, $r, a_0, a_1, \ldots, a_{r-1}, a_r$ do not depend on i, and $a_0 \neq 0$. Obviously, we should have $r > 0$, $a_0 > 0$, and all coefficients $a_0, a_1, \ldots, a_{r-1}, a_r$ must be rational numbers. The function $2f(i) - f(i+1)$ is also a polynomial, namely

$$2f(i) - f(i+1) = a_0 i^r + b_1 i^{r-1} + b_2 i^{r-2} \ldots + b_{r-1} i + b_r$$

with the same a_0 and new coefficients $b_1, b_2, \ldots, b_{r-1}, b_r$ that are again rational numbers. Clearly, these new coefficients can be effectively found (assuming, of course, that the degree r and the coefficients $a_0, a_1, \ldots, a_{r-1}, a_r$ are explicitly given or can be effectively found). Therefore, given any non-negative integer n_0, one can effectively find a positive integer i_0 satisfying Condition 1. This allows us to check algorithmically whether all elements of the set $[n_0 \,..\, \infty)$ are f-representable (the result can be obviously generalized to computable functions f such that $2f(i) - f(i+1)$ effectively diverges to $+\infty$ together with i, i.e. such that there is a computable function transforming any non-negative integer n_0 into some positive integer i_0 satisfying Condition 1).

Example 9 (a particular instance of Example 8). Let $f(i) = i^2$ for any i in $\mathbf{N}_+$. Then

$$2f(i) - f(i+1) = i^2 - 2i - 1 = i(i-2) - 1,$$

and therefore $2f(i) - f(i+1) \geq 129$ for any i in $[13..\infty)$. Since $129 + f(13) = 298$, the f-representability of all elements of $[129..\infty)$ is equivalent to the f-representability of the elements of $[129 \,..\, 298)$. The f-representability of the mentioned finitely many integers can be shown by computing the corresponding values of $f^\dagger$ (using Theorem 1) and showing that they are all positive, i.e. by a certain continuation of the computations that produced the table from Fig. 1. We have done this by computer, but we do not present the corresponding continuation of the table here. We preferred to present a table of f-representations of the numbers from 129 to 297 (cf. Fig. 2), since its correctness allows an easier manual verification (the table itself is produced by computer on the basis of Theorem 3; the representations are written without the curly brackets for the sake of saving space).

Remark. Some of the considered numbers have shorter f-representations than the ones given in the table. For instance, the number 131 has also the f-representation $\{1, 3, 11\}$. Note also that one could (especially at manual verification) use the remark after the proof of Theorem 4 and somewhat reduce the count of the numbers to be checked. In the concrete situation ($f(i) = i^2$, $n_0 = 129$) we have $S_i = [129 + i^2 \,..\, 2i^2)$ for any positive integer i. We see that $S_i = \emptyset$ for $i \leq 11$, $S_{12} = [273 \,..\, 288)$, and S_i consists of numbers not less than 298 for $i \geq 13$. Hence it would be enough to check the numbers belonging to $[129 \,..\, 298) \setminus S_{12}$, i.e. one could skip the check of 15 numbers.

Example 10 (several other particular instances of Example 8). Fig. 3 contains a summary of results of applying Theorem 4 to concrete polynomials f for obtaining results of the form "All elements of $[n_0 \,..\, \infty)$ are f-representable". In any of these results the number n_0 is the least possible for the polynomial in question and has been found by means of an iterative process starting with $n_0 = 0$ as an initial value. The iteration step and the termination of the process can be described as follows. We find a positive integer i_0 satisfying Condition 1 for the current n_0 and then we consecutively check for f-representability the numbers in $[n_0 \,..\, n_0 + f(i_0))$. If all of them turn out to be f-representable, then the process terminates with the current n_0 as its result. Otherwise, if m is the least number from $[n_0 .. n_0 + f(i_0))$ that is not f-representable, then we take the number $m+1$ as a next value of n_0. Note that at the moment of the termination of the process all integers in the set $[0 \,..\, n_0 + f(i_0))$ turn out to have been already checked, hence the method can be obviously refined to compute also the total count of all positive integers that are not f-representable

n	f–representation of n	n	f–representation of n	n	f–representation of n	n	f–representation of n
129	4,7,8	172	1,4,5,7,9	215	3,6,7,11	258	5,8,13
130	7,9	173	4,6,11	216	4,6,8,10	259	5,7,8,11
131	3,4,5,9	174	5,7,10	217	6,9,10	260	8,14
132	1,3,4,5,9	175	3,6,7,9	218	7,13	261	6,15
133	4,6,9	176	1,3,6,7,9	219	5,7,8,9	262	4,5,10,11
134	3,5,10	177	4,5,6,10	220	3,4,5,7,11	263	5,6,9,11
135	3,4,5,6,7	178	3,13	221	10,11	264	3,5,7,9,10
136	6,10	179	3,7,11	222	4,6,7,11	265	11,12
137	4,11	180	6,12	223	2,5,7,8,9	266	8,9,11
138	5,7,8	181	9,10	224	4,8,12	267	4,7,9,11
139	3,7,9	182	5,6,11	225	15	268	3,5,7,8,11
140	2,6,10	183	3,5,7,10	226	4,5,8,11	269	10,13
141	4,5,10	184	2,6,12	227	5,9,11	270	7,10,11
142	5,6,9	185	8,11	228	3,5,7,8,9	271	4,5,7,9,10
143	2,3,7,9	186	4,7,11	229	6,7,12	272	4,16
144	12	187	2,3,5,7,10	230	7,9,10	273	4,7,8,12
145	8,9	188	1,2,3,5,7,10	231	5,6,7,11	274	7,15
146	5,11	189	5,8,10	232	6,14	275	5,9,13
147	3,5,7,8	190	4,5,7,10	233	8,13	276	5,7,9,11
148	2,12	191	5,6,7,9	234	7,8,11	277	9,14
149	7,10	192	1,5,6,7,9	235	4,5,7,8,9	278	3,10,13
150	3,4,5,10	193	7,12	236	3,5,9,11	279	5,6,7,13
151	3,5,6,9	194	7,8,9	237	4,10,11	280	6,10,12
152	4,6,10	195	5,7,11	238	6,9,11	281	6,8,9,10
153	3,12	196	14	239	3,7,9,10	282	7,8,13
154	4,5,7,8	197	4,9,10	240	3,5,6,7,11	283	3,7,15
155	5,7,9	198	4,5,6,11	241	4,15	284	3,5,9,13
156	2,4,6,10	199	3,4,5,7,10	242	5,6,9,10	285	8,10,11
157	6,11	200	6,8,10	243	5,7,13	286	6,9,13
158	4,5,6,9	201	4,8,11	244	10,12	287	6,7,9,11
159	2,5,7,9	202	9,11	245	8,9,10	288	3,5,6,7,13
160	4,12	203	3,7,8,9	246	5,10,11	289	17
161	5,6,10	204	3,5,7,11	247	4,5,6,7,11	290	11,13
162	4,5,11	205	6,13	248	4,6,14	291	5,8,9,11
163	3,4,5,7,8	206	6,7,11	249	6,7,8,10	292	6,16
164	8,10	207	4,5,6,7,9	250	9,13	293	7,10,12
165	4,7,10	208	8,12	251	7,9,11	294	7,8,9,10
166	6,7,9	209	4,7,12	252	3,5,7,13	295	5,7,10,11
167	3,4,5,6,9	210	5,8,11	253	3,10,12	296	10,14
168	2,8,10	211	4,5,7,11	254	6,7,13	297	4,6,8,9,10
169	13	212	4,14	255	5,7,9,10		
170	7,11	213	7,8,10	256	16		
171	4,5,7,9	214	3,6,13	257	7,8,12		

Fig. 2. Some f–representations of the numbers from 129 to 297 for $f(i) = i^2$

$f(i)$	n_0	i_0	$n_0 + f(i_0)$
$i(i+1)/2$	34	9	79
i^2+1	52	9	134
$(i+1)^2-1$	157	13	352
$i(i+1)(i+2)/6$	559	16	1375
i^3	12759	25	28384

Fig. 3. Several other instances of application of Theorem 4

(we established in this way the existence of exactly 2788 positive integers that are not f–representable in the case of $f(i) = i^3$). It is easy to design the process so that the output includes also the complete list of the non-representable positive integers.

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