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100 YEARS FROM THE BIRTH OF YAROSLAV TAGAMLITZKI *

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The life and the deed of the eminent Bulgarian mathematician Yaroslav Tagamlitzki (1917–1983) are considered.

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About a century ago, less than two months before the October Revolution, a Russian family acquired a son whom fate has ordered to become after several decades one of the distinguished Bulgarian mathematicians (when mentioning fate, the consequences of the revolution are also taken into account). The child was born in the southern Russian city of Armavir on September 11, 1917.[1] The father Eng. Alexander Mihailovich Tagamlitzki (1881–1945) and the mother Vera Leonidovna (1894–1976) chose the name Yaroslav for their son. He was, however, named Yaroslav-Roman, because the priest who performed the baptism said the name Yaroslav is not a Christian one (this has not prevented later actually using only the first part of the official double name).

There was another child in the family - the daughter Galina, born in 1916,

*This is a revised and extended version of the talk [19].

[1] Perhaps it is difficult to determine whether this date is Old or New Style. Such a problem does not arise for people born in 1917 in Bulgaria, where the transition from Old to New Style happened in 1916. In Russia, however, it was in 1918.

a future Bulgarian specialist in Russian philology[2]. After a few hard years that followed the revolution, the Tagamlitzki family moved to Bulgaria in 1921 and settled in Sofia. Here, however, as seen by Galina's recollections [21], their lives were not at all easy. A few years after the immigration, the father became seriously ill and, in addition, the company where he found a job went bankrupt. In the sequel, the father's health continued to deteriorate, and finding a permanent job became impossible. The family care services lay mainly on the shoulders of the mother, who started work as a seamstress and ironer, and later as an embroidress. The penury became an everyday occurrence in the life of the four for decades. Many changes of abode occurred aiming at reducing the burden of the rent expenses.[3]

Absorbed in their troubles, the parents missed sending children in time to school. Fortunately, the American elementary school in Sofia which was opened shortly before allowed them to go straight into classes corresponding to their ages – Yaroslav into the second grade, and Galina into the third one (thanks to the fact that the two children already had become literate and had sufficiently educated themselves, no essential difficulties arose for them at school classes).

Concerning the school years of Yaroslav Tagamlitzki, let us quote [4] (with a footnote added here): "During the education of the young Tagamlitzki in the primary school no indication about the great talents hidden in him could be observed, but when he entered the famous Second Boys' High School in Sofia the things changed radically. Obviously an exceptionally favorable combination has arisen of, on the one hand, the great innate ability of the already mature pupil and his irresistible pursuit of science and, on the other hand, the high professional level of teachers and their genuine love towards their profession and care for the trainees. Already in this period the mathematical interests of Tagamlitzki far exceeded the matter studied in the secondary school, and he had also serious manifestations of his own scientific work, although for his disappointment the results turned out to be already known. Again in this period Tagamlitzki was a regular listener of the guest lectures in 1935 in Sofia of the prominent German mathematician Otto Blumenthal.[4] Not only, however, in mathematics and not only in science has shown his abilities the gifted and studious young man. To that time, for example, goes back his great attraction to music, the interest in which from aesthetic and scientific point of view does not leave him for the rest of his life."

In 1936, Yaroslav Tagamlitzki graduated from secondary education and became a mathematics student at the then Faculty of Physics and Mathematics of the Sofia University. After his second year of study there, his name appeared in

[2]Brief information about her can be found for instance in the calendar accessible from the web page [53] (the information is in the July 2016 section of the calendar; the link to that section is currently on the second page of the calendar).

[3]The dwelling places mentioned in [53] are indicated there by the corresponding street names. More complete information about the location of the fourth of them is present in the return address 46 Milin Kamak Str. of a letter sent in 1939 by the student Yaroslav Tagamlitzki to Professor Lyubomir Chakalov.

[4]Ludwig Otto Blumenthal (1876–1944). Some information about his lectures in Sofia is given on page 10 of the comprehensive biographical paper [2].

the papers [11, 12], where some results obtained by the student Tagamlitzki were included by his mathematical analysis professor, the prominent mathematician Professor Kiril Popov. A little later, still as a student, Tagamlitzki himself wrote three papers which appeared in editions of the Bulgarian Association for Physics and Mathematics, namely the articles [22, 23, 24]. The third of them, quite different in spirit from the other Bulgarian publications of that time, shows a profound knowledge of Lebesgue's integration theory.[5] It is worth noting that Tagamlitzki's letter to Chakalov mentioned in a preceding footnote concerns the subject matter of [24].

Tagamlitzki graduated from the University in 1940, and was seconded by the Ministry of Education to the then existing Mathematical Institute of the Sofia University. In 1942 and 1943 he had an academic specialization at Leipzig University and completed it with the defense of a doctoral dissertation in the field of complex analysis, namely [25].[6]

Here are the autobiographical data given by Yaroslav Tagamlitzki at the end of his dissertation:[7]

"I, Yaroslav Alexandrov Tagamlitzki, am a Bulgarian citizen, born on 11 September 1917 in Armavir, and I am the son of the engineer Alexander Tagamlitzki and his wife Vera. In Sofia I attended primary school and the semi-classical department of Second Boys' High School, which I graduated in 1936. Eight semesters (academic years from 1936/37 to 1939/40) I studied mathematics at Sofia University (Bulgaria). I attended there the lectures on mathematics of Messrs. Popov, Chakalov, Obreshkov, Tabakov, Tsenov and Stoyanov, the lectures on physics of Messrs. Nadzhakov, Penchev, Manev and Raynov and those of astronomy of Mr. Bonev.[8] During the academic year 1940/41 I was assisting at the Sofia Univer-

[5]However, a statement in [19] connected with this turns out to be not correct. It is claimed there that lectures on the theory in question started to be read in Sofia University much later. Actually such ones were read by Professor Kiril Popov already in the academic year 1939/1940, and a corresponding book by him appeared in 1941.

[6]Information about the fact of the defense and some other data can be found for instance by means of a search in the website [54] by the word *Tagamlitski* (this is the used there transcription of his family name). A scanned copy of the dissertation itself (as well as of almost all other works of him) is accessible from the page "A bibliography of Yaroslav Tagamlitzkis works" of the website [52] (in the dissertation, its author's name is transcribed as *Jaroslaw Tagamlizki*, and several other transcriptions of that name are indicated on the page "Web resources about Tagamlitzki" of [52]).

[7]The translation from German and the footnotes are mine, the Bulgarian names in the fourth sentence and in the footnote to it being transliterated according to the present-day official Bulgarian transliteration system (most of these names had other transcriptions in publications of the time – for instance the transcriptions Kyrille Popoff and Ljubomir Tschakaloff of the names Kiril Popov and Lyubomir Chakalov, as well as the transcription Obrechkoff of the family name Obreshkov). Let us note that we depart in this paper from the above-mentioned transliteration system in the case of Tagamlitzki's name by adhering to its transcription which is most frequently used in his Western language publications.

[8]These are Kiril Popov (1880–1966), Lyubomir Chakalov (1886–1963), Nikola Obreshkov (1896-1963), Dimitar Tabakov (1879–1973), Ivan Tsenov (1883–1967), Arkadi Stoyanov (1896–1963), Georgi Nadzhakov (1896–1981), Petar Penchev (1873–1956), Georgi Manev (1884–1965), Ruscho Raynov (1886–1965) and Nikola Bonev (1898–1979). A lot of information about the first six of them can be found in [5].

sity. During three semesters in the academic years 1941/42 and 1942/43 I attended as a regular student in Leipzig the lectures by Messrs. Koebe, van der Waerden, Schnee, Hopf, Heisenberg, Hund and Hopmann.[9] Then I started developing this treatise."

The thesis defended by Tagamlitzki in Leipzig is on a topic suggested to him by Paul Koebe, one of the world's leading experts in complex analysis at that time. The results presented in the dissertation generalize some important Koebe's results in the conformal mapping theory.

We will continue with a brief listing of further facts from the biography of Yaroslav Tagamlitzki, following closely [20, p. 231–232].

After his military service in the turbulent 1943 and 1944, Tagamlitzki was appointed in 1945 as Assistant Professor in the Faculty of Physics and Mathematics of the Sofia University – at the Department of Differential and Integral Calculus which was then led by Professor Kiril Popov (academician from 1947 on). In 1947 and 1949, Tagamlitzki was consecutively elected as a private and a regular associate professor at the same department. Since 1954 he has been Professor, Head of the Department of Differential and Integral Calculus. In 1958, Tagamlitzki was awarded a second doctor grade – this time according to the new rules for the scientific degrees in Bulgaria.[10] In 1961 he was elected as a corresponding member of the Bulgarian Academy of Sciences. Besides the Department on Differential and Integral Calculus, he also leads the Section of Functional Analysis at the Mathematical Institute of the Bulgarian Academy of Sciences, and after the unification of the two units in the early 1970s – the resulting sector of Real and Functional Analysis in the then established United Center of Mathematics and Mechanics at the Sofia University and the Bulgarian Academy of Sciences.

For Tagamlitzki's research on Dirichlet series and on the Laplace integral equation, he received in 1947 the Award for Science from the Committee for Science, Art and Culture[11]. In 1952 he was awarded the Dimitrov Prize[12] for the work [32]. For his scientific and teaching activities he was awarded the first grade "Cyrille and Methodius" order in 1953 and 1967, as well as a jubilee medal in 1969. In 1982 he was awarded the title Merited Scientist.

The active and versatile activity of Professor Yaroslav Tagamlitzki was cut short by his sudden death in Sofia on 28 November 1983.

Without any attempt at completeness, we will further consider the major scientific achievements of Tagamlitzki. But before going to them we will say, essentially

[9]Envisaged are Paul Koebe (1882-1945), Bartel Leendert van der Waerden (1903-1996), Walter Schnee (1885–1958), Eberhard Friedrich Ferdinand Hopf (1902–1983), Werner Heisenberg (1901–1976), Friedrich Hund (1896–1997) and Josef Hopmann (1890–1975). Biographical information about each of them can be found on the website [55].

[10]The degree was awarded for the work [32]; the reviewers were the academicians Lyubomir Chakalov, Nikola Obreshkov and Kiril Popov.

[11]This was a state institution which existed in the period from 1947 to 1954 and had the rank of a ministry.

[12]A high Bulgarian state award at that time.

following [20, p. 246], a few words about his teaching activity and his care for improvement of mathematical education. Their place in his life was very essential. From the beginning of his academic career to his last days Tagamlitzki was engaged most actively in the process of students education. Only a few years after this beginning, he developed the first modern calculus course in Bulgaria, and he taught it for the rest of his life, steadily bringing improvements and refinements. Typical of this course is the skillful combining of logical rigor with accessibility. The corresponding textbook which appeared first in a cyclostyle edition and then in six regular print editions stands out with its high qualities. Tagamlitzki delivered his lectures with remarkable pedagogical ability and took great care to ensure a thorough mastering of the taught material. Throughout several decades, he also read advanced lecture courses in various other fields of mathematics as integral equations, combinatorial topology, Fourier series, interpolation series, real functions theory, generalized functions. Tagamlitzki's functional analysis lectures which were read for more than a quarter of a century had a special place in his lecturing. They were quite different from the traditional functional analysis courses because aimed at displaying the results of the research of the lecturer himself.

In the talk [43] delivered in 1978 at a conference of the Union of Mathematicians in Bulgaria, Tagamlitzki presents his views on the teaching of mathematics at the university, supporting them by instructive examples.

Throughout the decades of his teaching at the Sofia University Yaroslav Tagamlitzki maintained close connections with secondary school, reading repeatedly there lectures on appropriate mathematical questions.[13] In the years 1963–1965 he conducted systematic work with bright secondary school students. The main subjects were the method of mathematical induction and a new method developed by him for building a certain essential part of calculus without using limit transition. Later he examined the applicability of the latter method for teaching the basics of calculus in secondary school, and during the school year 1973/1974 personally participated in the experimental application of this method in a school in Sofia. The method is described briefly in [39, 40], and a detailed description of it is given in [44] (the article [40] is a paper presented at the 1974 Spring Conference of the Bulgarian Mathematical Society[14]; besides the method in question also other important issues are considered in this paper).

The book [47] gives a fairly complete picture of the principles of Tagamlitzki's pedagogical activity in the mass education in mathematics, the depth of its implementation and his innovative approach to it. However, the devotional work of Tagamlitzki with capable mathematics students was especially fruitful. He did it with an indisputable talent. By attracting such students to research, he forwarded the growth as highly qualified mathematicians of many people from the next generations. The road to this for most of them went through Tagamlitzki's seminar.

[13] In the early fifties, being a secondary school student, the present author had the chance to be among the listeners of one of these lectures. Its title was "Solvable and unsolvable problems in mathematics".

[14] So then it was called the Union of Mathematicians in Bulgaria.

Despite of being modestly named ‘Calculus Study Group’, this seminar operated at a very high scientific level. In the paper [3], after a description of the scarcity of opportunities for contacts of the Bulgarian students with scientific life during the first post-war decade, the following is written about the seminar: “... we can imagine what an impression the work did in a study group where the participants were proposed to be full and equal associates of their enthusiastic teacher, and how easily that enthusiasm could find its way to their hearts. The unique atmosphere created determined the further path of many of the participants in the group”. Eight PhD theses and many master theses created by Tagamlitzki’s students contain essential scientific contributions on a variety of subjects in the stream of his investigations.

An exciting picture of the most essential of the above things and of Tagamlitzki’s personality can be taken from the recollections [51, 14, 6, 7, 15] (all in the first chapter of the book [10]).

As has already been said, three publications of Tagamlitzki are from the time when he was a student. The next one is his PhD dissertation. Several years later, during the period from 1946 to 1951, a series of works of Tagamlitzki appeared, which make him the pioneer of functional analysis in Bulgaria. These works are internally united by the idea of indecomposability – in part of them it is present implicitly, and in others it occurs in an explicit form by the notion of prime vector. We will briefly mention some of the first ones and some of the others.

The above-mentioned series of works begins with the papers [26, 27, 28] inspired by certain investigations of N. Obreshkov. In the first of these papers, the following result is established: *if the function $f(x)$ is defined and has derivatives of any order for every x to the left of a fixed point a of the real axis, and A is a constant such that $|f^{(k)}(x)| \leq Ae^x$ for all $x < a$ and any nonnegative integer k, then the function $f(x)$ has the form Be^x, where B is a constant satisfying the inequality $|B| \leq A$* (the opposite is, of course, trivial). Of course, the result can be reformulated for the case of functions defined for $x > a$ – with e^{-x} instead of e^x; the considered property of the function e^{-x} is indicated and other proofs of it are given in [27, 30]. An analogous result, but for infinite sequences of numbers is established in [27, 28], wherein finite differences and infinite geometric progressions with terms between 0 and 1 are considered instead of derivatives and the exponential function.

In the paper [29], which is Tagamlitzki’s habilitation work for the position of a regular associate professor[15], Tagamlitzki considers for each real number a a linear space $K(a)$ of functions defined in the interval $(a, +\infty)$,[16] and proves that among them are the functions $f(x)$, defined for $x > a$, tending to 0 as $x \to +\infty$, having derivatives of any order and satisfying the inequalities

$$(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, 2, \dots ,$$

for each $x > a$ (obviously the functions $e^{-\lambda x}$, where $\lambda > 0$, have the above properties; other functions with these properties are, for example, the functions of the

[15] For the position of private associate professor, his habilitation work is [28].

[16] The specific definition of this space will be not important here.

form $(x-b)^n$, where $b \le a$ and $n < 0$). The functions with the listed properties are called *positively definite*. A partial ordering is defined in the space $K(a)$ by the convention that a function f majorizes a function g if and only if the difference $f-g$ is positively definite. The functions of the form $e^{-\lambda x}$, where $\lambda > 0$, turn out to be simple vectors of the so obtained partially ordered linear space in the sense that these positively definite functions are different from the zero element of the space, but any positively definite function majorized by such a function is a product of it with a constant. This fact is used to prove a theorem which, for an arbitrary sequence $\{\lambda_n\}$ of distinct positive real numbers gives a necessary and sufficient condition for the expandability in a series of the form $\sum A_n e^{-\lambda_n x}$, absolutely convergent in the interval $(a, +\infty)$ (the expandability in such a series could be intuitively regarded as a generalized representability as linear combination of prime vectors). This and other results in [29] are generalized in the article [31] for the case of arbitrary partially ordered linear spaces satisfying certain natural requirements (as is close to mind, the definition of a prime vector in this case is the following: *an element p of the space is called its prime vector if p majorizes the zero element, p is different from it, and p is collinear with each of its minorants which majorize the zero element*).

The above-mentioned intuitive idea plays an important heuristic role in obtaining the main results in the extremely deep and contentful work [32]. Let the infinite sequence of real numbers $x_0, x_1, x_2, \ldots$ be an arithmetic progression with positive difference τ. Interesting is the question about expandability of functions in series of the form $\sum_{n=0}^{\infty} a_n P_n(x)$, where the coefficients a_n do not depend on x, and $P_0(x), P_1(x), P_2(x), \ldots$ are the Abel interpolation polynomials defined as follows: $P_n(x) = (x_0 - x)(x_n - x)^{n-1}/n!, \quad n = 0, 1, 2, \ldots$ The answer to this question is obviously positive for a function $f(x)$ which is a polynomial (in this case we have $a_n = (-1)^n f^{(n)}(x_n)$ for $n = 0, 1, 2, \ldots$). Generally, however, the situation is much more complicated. The results proven in [32] reveal a substantial reason for this. Let a be a number less than x_0, and A be the linear space whose elements are the functions defined and infinitely many times differentiable in the interval $(a, +\infty)$. A function $f(x)$ from A is called *positively definite*, if, for each nonnegative integer k, we have $(-1)^k f^{(k)}(x) \ge 0$ when $a < x \le x_k$ (the positively definite functions of the space $K(a)$ from the paper [29] obviously satisfy this condition, but some other functions in A also meet it – it can be shown, for example, that it is satisfied also by the Abel interpolation polynomials). Let a partial order in the space A be defined by the convention that a function f majorizes a function g when the difference $f-g$ is positively definite. It turns out that the Abel interpolation polynomials are prime vectors of the so obtained partially ordered linear space, but besides them and their products with positive constants there are many other prime vectors – it is proven in the paper that the set of prime vectors of that space consists of the products with positive constants of the Abel interpolation polynomials and of the functions of x of the form $R(x,t)$, $0 < t \leq 1$, wherein

$$R(x,t)=\begin{cases} \dfrac{e^{\lambda(x_0-x)}-e^{\mu(x_0-x)}}{\lambda-\mu} & \text{for } 0<t<1,\ \tau\lambda e^{1-\tau\lambda}=\tau\mu e^{1-\tau\mu}=t,\ \tau\mu<1<\tau\lambda, \\ (x_0-x)e^{(x_0-x)/\tau} & \text{for } t=1. \end{cases}$$

Due to this, the realization of the intuitive idea about generalized representability as a linear combination of prime vectors becomes more complicated – it naturally uses not only series but also integrals. One of the main results in [32] is exactly in this spirit. It concerns the representability of functions $f(x)$ in the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) + \int_0^1 R(x,t)d\theta(t), \tag{1}$$

where the coefficients a_n are nonnegative numbers not depending on x, and $\theta(t)$ is a monotonically increasing function which also does not depend on x. The result is that a function $f(x)$ of the space A is representable in this form if and only if $f(x)$ is positively definite. In that case a uniqueness theorem holds according to which the coefficients a_n are uniquely determined, and the function $\theta(t)$ is substantially uniquely determined. These results can be interpreted as an indication that the expandability in a series on the Abel polynomials is a rare and unusual case, and the natural task is that of representation in the form (1) (possibly without requirements about nonnegativity of the coefficients a_n and monotony of the function $\theta(t)$).

The representability in the form (1) with nonnegative a_k and monotone increasing $\theta(t)$ is obvious for the Abel interpolation polynomials and the functions of x of the form $R(x,t)$, where $0 < t \leq 1$, so it is present for each prime vector of the space A. If we denote by K the set of the elements of a partially ordered linear space which majorize the zero element of the space, then the prime vectors of this space are actually those nonzero elements of K which cannot be represented as the sum of two noncollinear elements of K. The set K is a cone, i.e. belonging to K is preserved under multiplying with nonnegative numbers. In the work [33], which appeared a few years later, Tagamlitzki calls *indecomposable* elements of a cone those of its nonzero elements which cannot be represented as a sum of two noncollinear its elements, and proves under certain assumptions that *if all indecomposable elements of a cone belong to a convex cone*[17] *then all elements of the first cone belong to the second one.* The most restrictive of the assumptions is maybe that the considered cone is located in a linear space with scalar product and there exists a fixed nonzero element of the space forming acute angles with all nonzero elements of the cone (such a cone cannot for example have nonzero mutually opposite elements). Nevertheless, the said result provides a general method for proving a number of statements that are in the spirit of the statement about the representability of the positively definite elements of the space A in the form (1). Several examples are

[17]A cone is called convex if belonging to it is preserved under addition (it is easy to show that this is equivalent to the requirement the cone in question to be a convex set).

given in the paper which illustrate this method, namely proofs of the positive case of the Hausdorff moment theorem, of Bernstein's theorem about the integral representation of completely monotone functions in an infinite interval, of a statement equivalent to the theorem of Bernstein about the analyticity of functions completely monotone in a finite interval, and of a theorem about representability by integral, similar to the one in the representation (1). As indicated in [9], another application made then by Tagamlitzki remained unpublished, namely an elegant proof of the classical Bochner's theorem on the positive definite functions (one of the first applications of Tagamlitzki's theorem that also remained unpublished is its application to the positive case of F. Riesz's representation theorem for linear functionals on $C[a, b]$).

The above-mentioned restrictive assumption is removed in the published in [34] two years later another version of the above theorem. Let K be a convex cone, and P be a nonnegative real-valued function such that $P(\lambda a) = \lambda P(a)$ for each nonnegative number λ and each $a \in K$, $P(a + b) \leq P(a) + P(b)$ for any $a, b \in K$, and the value of P can be 0 only for the zero element of the cone. Deviating slightly from the terminology of [34], we will call such a function a *norm of* K (the definition of norm in [34] imposes also a certain semicontinuity requirement). By definition, an *indecomposable element of* K *with respect to* P is such a nonzero element a of K that no noncollinear elements b and c of K exist satisfying the equations $a = b + c$ and $P(a) = P(b) + P(c)$. In the new version of the theorem, it is assumed to be given a convex cone K with a norm P and a contained therein convex cone L with a norm Q, and it is proved under some additional assumptions that *if the conditions* $x \in L$ *and* $P(x) \geq Q(x)$ *are satisfied whenever* x *is an indecomposable element of* K *with respect to* P*, then these conditions are met for each* x *of* K. It is shown in the article that the method derived from this theorem (now known as *Cones Theorem*), allows proving with its help the Hausdorff moment theorem (first in the general case and then in the positive one), as well as Widder's theorem about the integral representation of the functions $f(x)$, defined and infinitely many times differentiable for $x > 0$, which satisfy the condition $\dfrac{1}{n!}\displaystyle\int_0^\infty x^n \left|f^{(n+1)}(x)\right| dx \leq A$, $n = 0, 1, 2, \ldots$, with constant A, independent of n (it is indicated how then one can get also Bernstein's theorem about integral representation of the absolutely monotone functions). Numerous other applications of the method are listed in the summary [35] of a talk presented by Tagamlitzki in 1956 at an international mathematical conference in Sofia (in the following years appeared also many other applications). The application of the theorem to the general case of Riesz's theorem on the representation of the bounded linear functionals in $C[a, b]$ was the first among the applications listed in the summary. However, a detailed presentation of this application was published only thirty years later in [8], and such presentations of some other ones remained unpublished at all, although all these results were considered in detail at Tagamlitzki's seminar or on his functional analysis lectures.

In 1956 and 1957, the three parts of the work [36] appeared. They present a way suggested by Tagamlitzki for building the theory of generalized functions.

Here is the abstract of the third part (with references made independent of its context):[18]

"The present work is the last part of a research, whose first two parts were published under the same title and contain the general principles of completion of cones, the definition of the space S_n^σ of pseudofunctions, and their basic properties.

Pseudofunctions, which include all summable functions and the Dirac functions on a given interval, can be differentiated arbitrary many times; moreover, the derivative of an element of S_n^σ is in $S_{n+1}^{\sigma+1}$. For $\sigma > n$ we have $S_n^\sigma \subset S_{n+1}^{\sigma+1}$. The space S_n^σ possesses a countable coordinate system. Unlike the space of the Schwartz distributions, it possesses a semicontinuous norm and it is compact with respect to it.

It follows from our earlier work *On a generalization of the notion of indecomposability* that the space S_n^σ contains indecomposable elements. The present third part is dedicated primarily to the indecomposable elements of these spaces. We establish that the indecomposable elements of S_n^σ with respect to the corresponding norm are the n-th derivatives of the Dirac functions."

The paper [50] which appeared a little later is also in such a spirit. As indicated in a footnote on its first page, it reproduces, with relevant generalizations and additions made by the second author, the main points of the investigations which Tagamlitzki expounded in his functional analysis lectures in the academic year 1955/56.

Thanks to the participation of foreign mathematicians in the 1956 conference more people abroad became aware of the method developed by Tagamlitzki. The French mathematician Choquet[19] offered Tagamlitzki to publish in France a more systematic presentation of the results obtained. A publishing house in East Germany started negotiations with Tagamlitzki to print a monograph on these results. He accepted these proposals in principle but did not hurry to implement them – both because of the continued intensive emergence of new results obtained by him and his disciples, and because of another important reason to which we will turn now.

At some point after 1956 Tagamlitzki realized that his Cones Theorem can be obtained as a consequence of the Krein-Milman Theorem. The delay in recognition of this fact is explained by the limited possibility for contacts of the Bulgarian mathematicians with the international mathematical community during the Second World War and the first postwar decade. Despite the fact in question, however, many of the applications of the Cones Theorem are scientific contributions – all its applications can be considered as applications of the Krein-Milman Theorem, and many of them turn out to be new. However, Tagamlitzki was still considering desirable a further reflection and an expansion of the achieved before proceeding to its monographic exposure. A few years later, in 1962, he was again invited to write

[18] The terminology of the earlier work mentioned in the abstract (i.e., of the work [34]) is used in it.

[19] Gustave Choquet (1915–2006).

a monograph – this time from the American publishing house Van Nostrand (the invitation was inspired by a recommendation of the famous mathematician Marshall Stone[20]). Tagamlitzki tended to accept the invitation and was considering a plan for the monograph in question, but unfortunately it remained unwritten. This is probably due to the fact that at that time he worked on a far-reaching generalization of the Krein-Milman Theorem. This generalization was reported in [37] and improved further – subsequent versions of this generalization are set forth in detail in [49] and, eight years after Tagamlitzki's death, in [46] (three years after his death, also a specific other version of the generalization was published in [13]). In the final form of the generalization in question, an arbitrary compact topological space with a collection of open sets satisfying certain conditions is considered instead of a compact subset of a linear space with a locally convex topology (the Krein-Milman Theorem is obtained by applying the general result to this subset and the collection of all its intersections with convex open sets). An application of this generalization produces a more general form of Bauer's maximum principle (cf. [49, Application 2] and [41]).

In the applications of the Cones Theorem, the search for the indecomposable elements is often done by means of so-called decomposing operators, and they are usually linear ones. In the paper [42], an analog of the linear decomposing operators is considered which could be useful for the search of extreme points in the applications of Krein-Milman's theorem. A characterization of the topological simplexes is given by means of the introduced notion.

Besides the results concerning indecomposability, another contribution of Tagamlitzki in functional analysis is the published in [38] generalization of certain theorems on separability of convex sets. Instead of sets in a linear space he considers such ones in mathematical structures with an appropriate notion, generalizing the notion of segment. Tagamlitzki proved two theorems of this kind, the second one concerning the case when one of the considered convex sets is open in a topology co-ordinated with the above-mentioned notion. It became known later that the first of these theorems follows from a theorem published by Ellis in 1952, but the second one could not be derived in such a way. In addition, as shown in [17], without essentially changing the proof of the first theorem, its assumptions can be weakened in a way hindering its derivation from Ellis's one. Tagamlitzki's investigations on separability of convex sets were essentially extended further in the works of Ivan Prodanov.

As already mentioned, some of Tagamlitzki's results in functional analysis remained unpublished. The results obtained in the last two decades of his life had such a fate especially often. However, some of them are mentioned by other authors who anyhow became aware of these results. For instance, the following is written in [1, p. 255]:

„Positive definite functions on semigroups have been studied by Tagamlitzki and his pupils in Bulgaria. In lectures at the University of Sofia during the years

[20] Marshal Harvey Stone (1903–1989).

1965–1969, Tagamlitzki introduced the operators $W_{\xi,a}$ for functions f on an abelian group G by

$$W_{\xi,a}f(s) = 2f(s) + \xi f(s+a) + \overline{\xi} f(s-a),$$

where $a, s \in G$ and $\xi \in \mathbf{C}$.

He used these operators for proving Bochner's theorem for discrete groups as a direct consequence of Krein-Milman's theorem."

The quoted author indicated further that a similar approach is published by Choquet in a paper from 1969, and it is mentioned that the method in question is extended to semigroups with involution by T. Tonev in his Master Thesis, written under the supervision of Tagamlitzki and defended in 1969 (Tonev's publication on this subject from 1979 in Semigroup Forum is also referred to). Some information on a paper of D. Shopova from 1970 is also given.

Among the results of Tagamlitzki in abstract areas of mathematics, other than functional analysis, we can point to a broad generalization of Tychonoff's theorem on compactness of topological products. Unfortunately the only materials in paper-like form on this subject left from Tagamlitzki are three texts in the annual scientific reports of the Sector of Real and Functional Analysis – for the years 1977, 1980 and 1982. The paper [45] was written later on the base of these texts.

Tagamlitzki developed also some new approaches to the theory of manifolds and to some mathematical questions of theoretical physics. The corresponding materials in paper-like form left by him are again texts in the annual scientific reports of the Sector of Real and Functional Analysis – for the years 1973, 1974, 1976, 1978–1981.

Besides in mathematics, Tagamlitzki carried out systematic research also on questions of archaeology, linguistics and medicine. He offered new ideas in the doctrine of tonality in music.

A lot of information waiting for its investigation can be found in Tagamlitzki's archive in the Bulgarian Academy of Sciences. An overview of the items of mathematical character in this archive is given in [16].

The person and the deed of Yaroslav Tagamlitzki left a deep and bright trace in the history of our science, in our education and in the souls of many people of several generations. The centenary of his birth is an appropriate occasion to express our admiration to the memory of this remarkable man.

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