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Π_1^0-POSITIVE INDUCTIVE DEFINABILITY ON ABSTRACT STRUCTURES*

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Inductive definability by means of Π_1^0-positive formulas is studied in the paper. An explicit characterization of the Π_1^0-positive inductive sets on an arbitrary abstract structure with equality is presented. A relationship between these sets and the sets of all points of $\forall$-definedness of non-deterministic programs is established.

Keywords: inductive definability, non-deterministic computability, points of $\forall$-definedness, prime computability

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1. INTRODUCTION

It is well-known that on the first order structure of arithmetic every inductive set can be defined inductively also by means of very simple formula — Π_1^0, and positive with respect to its set variable ([2, 6]). When study induction on abstract structures, it is reasonable to consider the so-called acceptable structures. Even in this case, however, the above mentioned result of Kleene and Spector is no longer valid. In other words, there are acceptable structures for which the class of the Π_1^0-positive inductive sets is strictly included in the class of all inductive sets (cf. for example [1]). So, a question arises to find a characterization for this type of inductive definability.

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Let us mention that a similar problem is considered in [1], where the sets, which have Σ_2^0 and simpler (excepting Π_1^0 positive) inductive definitions, are characterized by means of prime and search computability in Kleene's quantifier $\mathbb{E}$. Here we describe Π_1^0 positive inductive definability in terms of some particular characteristic of the non-deterministic programs — the so-called sets of points of $\forall$-definedness ([3, 5]). It turns out that a set is Π_1^0-positive inductive iff it is the set of all points of $\forall$-definedness of some non-deterministic program.

From here it is easily obtained that every Π_1^0-positive inductive set on acceptable structure $\mathfrak{A}$ can be represented as $\{\bar{s} \mid \forall\alpha\exists n R(\alpha(\bar{n}), n, \bar{s})\}$ and vice versa. Here the predicate R is prime computable over $\mathfrak{A}$ and the second order variable α ranges over the set of all infinite sequences with elements in $|\mathfrak{A}|$.

In view of further applications, we shall consider here some special acceptable structures, namely least acceptable extensions [4]. It will be transparent, however, how to modify the proofs for an arbitrary acceptable structure with a sufficiently simple coding scheme.

2. PRELIMINARIES

Given an arbitrary total structure $\mathfrak{A}_0 = (B; f_1, \ldots, f_a; R_1, \ldots, R_b)$ (the case f_i is 0-ary is admitted as well), we define its least acceptable extension $\mathfrak{A}$ in the following way. Take an object $\mathbb{O} \notin B$ and fix some pairing operation $\Pi : C \times C \to C$ ($C \supseteq B \cup \{\mathbb{O}\}$) such that no element of $B \cup \{\mathbb{O}\}$ is an ordered pair. Let B^* be the smallest set containing $B \cup \{\mathbb{O}\}$ and closed under Π. Denote by $\langle\ \rangle$ the restriction of Π on B^*. We extend the initial functions and predicates of $\mathfrak{A}$ on B^*, setting $f_i(s_1, \ldots, s_n) = \mathbb{O}$ if $(s_1, \ldots, s_n) \notin B^n$, and $R_j(s_1, \ldots, s_m) =$ "falsity" if $(s_1, \ldots, s_m) \notin B^m$.

Now put $\mathfrak{A} = (B^*; \mathbb{O}, \langle\ \rangle, f_1, \ldots, f_a; B, R_1, \ldots, R_b)$. From now on we shall suppose that the equality relation is among the basic predicates of $\mathfrak{A}$. Throughout the paper we shall assume this structure fixed.

Let $\varphi(x_1, \ldots, x_k, X)$ be a formula in the first order language $\mathfrak{L}_{\mathfrak{A}}$ of $\mathfrak{A}$ with k object variables $x_1, \ldots, x_k$ and one k-ary relational variable X which occurs in φ only positively. Then φ determines the following mapping $\Gamma_\varphi : (B^*)^k \to (B^*)^k$:

$$\Gamma_\varphi(A) = \{(s_1, \ldots, s_k) \mid \varphi(s_1, \ldots, s_k, A)\}.$$

Define I_φ^ξ by transfinite induction on ξ as follows:

$$I_\varphi^\xi = \Gamma_\varphi\Big(\bigcup_{\eta<\xi} I_\varphi^\eta\Big).$$

Then the set $I_\varphi = \bigcup_\xi I_\varphi^\xi$ is the least fixed point of Γ_φ (see for example [4, Ch. 1A]).

For every $\bar{s} \in I_\varphi$ set

$$|\bar{s}| = \min\{\xi \mid \bar{s} \in I_\varphi^\xi\}.$$

It will be convenient to consider that $|\bar{s}| = |B^*|^+$ for $\bar{s} \notin I_\varphi$, where $|B^*|^+$ is the least cardinal, greater than the cardinal number of B^*.

A set $A \subseteq (B^*)^k$ is called *inductively definable (by φ on $\mathfrak{A}$)* if $A = \{(s_1, \ldots, s_k) \mid (s_1, \ldots, s_k, t_1, \ldots, t_n) \in I_\varphi)\}$ for some fixed $t_1, \ldots, t_n$, $n \geq 0$, built up from the basic functions and constants of $\mathfrak{A}$.

Remark. The last requirement imposed on $t_1, \ldots, t_n$ is a slight deviation from the usual definition in [4, Ch. 1D], where these parameters are supposed arbitrary. In our case perhaps more appropriate would be to say that A is absolutely inductively definable.

We shall say that the set A is Σ_k^0 (Π_k^0) *positive inductive (on $\mathfrak{A}$)* iff it is inductively definable by some Σ_k^0 (Π_k^0) X-positive formula $\varphi(\bar{x}, X)$.

In this paper we shall consider non-deterministic programs, in which the non-determinism is understood as possibility of choosing arbitrary elements of $|\mathfrak{A}|$. These programs are built up from the following three types of (eventually labeled) operators: assignment operator $x_i := \tau(x_{j_1}, \ldots, x_{j_m})$, conditional operator **if** $R(x_{j_1}, \ldots, x_{j_n})$ **then go to** q (τ and R being a term and a quantifier-free formula in $\mathfrak{L}_{\mathfrak{A}}$, respectively), and choice operator $x_i :=$ **arbitrary**(B^*).

Semantics of the assignment and conditional operators is the usual one. The execution of the choice operator assigns to the variable x_i an arbitrary element of B^*. The choice of this element is arbitrary: it does not depend on the input, on the current configuration, etc.

Now let P be such non-deterministic program. Along with the usual input-output relation R_P in this case we can speak also about the so-called *set of points of $\forall$-definedness* of P, to be denoted by D_P. An input $\bar{s}$ belongs to D_P iff all possible executions of P, starting from this input, are finite.

The main part of the exposition is based on a certain syntactical description of these sets of points of $\forall$-definedness. It is easily obtained from a more general uniform characterization of all possible pairs (R_P, D_P) which we are going to formulate below.

Let us call χ elementary if it is atomic or a negation of an atomic formula. A clause is an expression of the form $\Pi \Rightarrow \tau$, where τ is a term and Π is a finite conjunction of elementary formulas in the language $\mathfrak{L}_{\mathfrak{A}}$. A sequence of clauses $\left\{\Pi^{(n)} \Rightarrow \tau^{(n)}\right\}_n$ is regarded primitive recursive if the function, which assigns to each n the code of $\Pi^{(n)} \Rightarrow \tau^{(n)}$, is primitive recursive.

Throughout the paper $\alpha = \{\alpha(n)\}_{n=1}^{\infty}$ will denote an infinite sequence with elements from B^*. As usual, $\bar{\alpha}(n)$ will stand for $\langle\!\langle \alpha(1), \ldots, \alpha(n) \rangle\!\rangle$, where $\langle\!\langle\ \rangle\!\rangle$ is some effective coding of all finite sequences from B^*.

Proposition 2.1 (Normal form theorem). *Let P be a non-deterministic program with k input and one output variables. Then there exists a primitive recursive sequence of clauses $\{\Pi^{(n)} \Rightarrow \tau^{(n)}\}_n$, each with variables among $x_0, x_1, \ldots, x_k$, such that*

$$(s_1, \ldots, s_k) \in D_P \Leftrightarrow \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s_1, \ldots, s_k),$$

$$(s_1, \ldots, s_k, t) \in R_P \Leftrightarrow \exists\alpha\exists n\big(\Pi^{(n)}(\bar{\alpha}(n), s_1, \ldots, s_k) \,\&\, \tau^{(n)}(\bar{\alpha}(n), s_1, \ldots, s_k) = t \,\&\, \forall m_{m<n} \neg\Pi^{(m)}(\bar{\alpha}(m), s_1, \ldots, s_k)\big)$$

for every $s_1, \ldots, s_k, t$ in B^*.

Conversely, for every such sequence $\{\Pi^{(n)} \Rightarrow \tau^{(n)}\}_n$ *there exists a program* P *such that* D_P *and* R_P *satisfy the above equivalence.*

The proof of this proposition is rather technical to be presented here. We are going to make some comments instead. Every $\alpha : N \to B^*$ can be thought of as being a sequence of successive values of the choice operator (which may be assumed unique). So, every particular (finite or infinite) execution of P is uniquely determined by the input $\bar{s}$ and some choice sequence α. In addition, this execution is carried out in elementary steps in some canonical way. More precisely, given an input $\bar{s}$ and a sequence α, the values $\Pi^{(1)}(\bar{\alpha}(1), \bar{s})$, $\Pi^{(2)}(\bar{\alpha}(2), \bar{s})$, ... are computed in turn until the first n with $\Pi^{(n)}(\bar{\alpha}(n), \bar{s}) =$ true is reached. Then an output $\tau^{(n)}(\bar{\alpha}(n), \bar{s})$ is returned.

From this point of view it is clear that an input $\bar{s}$ belongs to D_P iff for every α there exists n such that $\Pi^{(n)}(\bar{\alpha}(n), s)$ holds. Further we shall be interested in sets of the type D_P rather than of D_P and R_P as a pair. For this purpose it will be enough (and even more appropriate) to consider non-deterministic programs without output variable. In this case a particular execution is regarded finite if the output operator **stop** is reached during the computation.

So, as a consequence of Proposition 2.1, for any non-deterministic program P (with or without output variable) we have

Proposition 2.2. *Let* P *has* k *input variables. Then there exists a primitive recursive sequence* $\{\Pi^{(n)}\}_n$ *with variables among* $x_0, x_1, \ldots, x_k$ *such that*

$$(s_1, \ldots, s_k) \in D_P \Leftrightarrow \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s_1, \ldots, s_k),$$

whenever $s_1, \ldots, s_k \in |\mathfrak{A}|$.

3. INDUCTIVE DEFINABILITY OF THE SETS OF ALL POINTS OF ∀-DEFINEDNESS

We begin with some preliminary definitions. Set

$$\mathrm{Nat} = \{\mathbb{O}, \langle\mathbb{O},\mathbb{O}\rangle, \langle\langle\mathbb{O},\mathbb{O}\rangle,\mathbb{O}\rangle, \langle\langle\langle\mathbb{O},\mathbb{O}\rangle,\mathbb{O}\rangle,\mathbb{O}\rangle, \ldots\}.$$

We shall identify the natural numbers $0, 1, 2, \ldots$ with elements of Nat (as listed above). For every $n \in \mathrm{Nat}$, $n+1$ will stand for $\langle n, 0\rangle$. Let L and R be the left and right decoding functions for the mapping $\langle\,\rangle$. We shall assume that $L(0) = R(0) = 0$ and $L(s) = R(s) = 1$ for $s \in B$.

A coding $\langle\!\langle\,\rangle\!\rangle$ of all finite sequences from B^* is defined inductively by the equalities

$$\langle\!\langle\,\rangle\!\rangle = 0, \ \langle\!\langle s_1\rangle\!\rangle = \langle 0, s_1\rangle, \ \langle\!\langle s_1, \ldots, s_{n+1}\rangle\!\rangle = \langle\langle\!\langle s_1, \ldots, s_n\rangle\!\rangle, s_{n+1}\rangle.$$

Set also $\mathrm{Seq} = \{s \mid s = \langle\!\langle s_1, \ldots, s_n\rangle\!\rangle$ for some $s_1, \ldots, s_n, \ n \geq 0\}$. The function lh (length) is defined in the usual way:

$$\mathrm{lh}(s) = \begin{cases} n & \text{if } s = \langle\!\langle s_1, \ldots, s_n\rangle\!\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, denote by $(s)_i$ the decoding function corresponding to $\langle\!\langle\ \rangle\!\rangle$:

$$(s)_i = \begin{cases} s_i & \text{if } s = \langle\!\langle s_1, \dots, s_n \rangle\!\rangle \text{ and } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $t = \langle\!\langle t_1, \dots, t_n \rangle\!\rangle$. As customary, $t * q$ will stand for $\langle t, q \rangle = \langle\!\langle t_1, \dots, t_n, q \rangle\!\rangle$. For any sequence α we shall write $\alpha \succ t$ to denote that $\alpha(i) = t_i$ for every $i = 1, \dots, \mathrm{lh}(t)$.

Now let P be an arbitrary non-deterministic program over $\mathfrak{A}$. We shall assume for simplicity that P has one input variable. According to Proposition 2.2, there is a primitive recursive sequence $\{\Pi^{(n)}\}_{n=0}^{\infty}$ with variables x_0 and x_1 such that for every $s \in B^*$

$$s \in D_P \;\Leftrightarrow\; \forall\alpha\exists n \Pi^{(n)}(\bar{\alpha}(n), s).$$

Let us fix some effective coding $\ulcorner \dots \urcorner$ of all finite conjunctions of elementary formulas with variables among x_0 and x_1. Let Φ be the universal relation for this class of formulas, defined by the equivalence

$$\Phi(n, t, s) \;\Leftrightarrow\; n \text{ is a code of some elementary formula } \chi \text{ and } \chi(t, s) \text{ holds.}$$

Denote by g the primitive recursive function $\lambda n. \ulcorner \Pi^{(n)} \urcorner$. Then we have

$$s \in D_P \;\Leftrightarrow\; \forall\alpha\exists n \Phi(g(n), \bar{\alpha}(n), s). \tag{3.1}$$

Now set

$$\varphi^*(n, t, s, X) \;\Leftrightarrow\; \mathrm{Nat}(n) \,\&\, \mathrm{Seq}(t) \,\&\, (\Phi(g(n), t, s) \vee \forall q((n+1, t * q, s) \in X))).$$

Let us first check that D_P is a section of I_{φ^*}.

Lemma 3.1. $D_P = \{s \mid (0, 0, s) \in I_{\varphi^*}\}$.

Proof. For the inclusion $D_P \supseteq \{s \mid (0, 0, s) \in I_{\varphi^*}\}$ we need the following more general assertion:

$$\mathrm{Seq}(t) \,\&\, \mathrm{lh}(t) = n \,\&\, (n, t, s) \in I_{\varphi^*} \;\Rightarrow\; \forall\alpha_{\succ t}\exists m \Phi(g(m), \bar{\alpha}(m), s). \tag{3.2}$$

We are going to prove (3.2) by transfinite induction on $|n, t, s|$. It can be easily seen that

$$|n, t, s| = \begin{cases} 0 & \text{if } \Phi(g(n), t, s), \\ \sup\{|n+1, t * q, s| + 1 \mid q \in B^*\} & \text{otherwise.} \end{cases}$$

If $|n, t, s| = 0$, take $m = n$. We have by assumption

$$\bar{\alpha}(m) = \bar{\alpha}(n) = \langle\!\langle \alpha(1), \dots, \alpha(n) \rangle\!\rangle = t$$

and therefore $\Phi(g(m), \bar{\alpha}(m), s)$.

Now choose some $(n, t, s) \in I_{\varphi^*}$ with $|n, t, s| > 0$ (and, of course, $\mathrm{Seq}(t)$ and $\mathrm{lh}(t) = n$). Take some $\alpha \succ t$ and set $\alpha(n+1) = q$. We have $|n+1, t*q, s| < |n, t, s|$, as well as $\mathrm{Seq}(t*q)$ and $\mathrm{lh}(t*q) = n+1$. So by induction hypothesis $\Phi(g(m), \bar{\alpha}(m), s)$ for some m, which completes the verification of (3.2).

Now put $t = \langle\!\langle\ \rangle\!\rangle$ ($= 0$) and $n = 0$ in (3.2) and use (3.1) to conclude that $D_P \supseteq \{s \mid (0, 0, s) \in I_{\varphi^*}\}$.

To see that the converse inclusion also holds, take some s such that $(0,0,s) \notin I_{\varphi^*}$. By the definition of φ^* we have $\neg\Phi(g(0),0,s)$ and $(1,\langle\langle q_1\rangle\rangle,s) \notin I_{\varphi^*}$ for at least one q_1. Assume that for some $n \geq 1$ we have found $q_1,\ldots,q_n$ with $\neg\Phi(g(i),\langle\langle q_1,\ldots,q_i\rangle\rangle,s)$ for $i=1,\ldots,n-1$ and $(n,\langle\langle q_1,\ldots,q_n\rangle\rangle,s) \notin I_{\varphi^*}$. From the latter it follows that $\neg\Phi(g(n),\langle\langle q_1,\ldots,q_n\rangle\rangle,s)$ and $\langle n+1,\langle\langle q_1,\ldots,q_{n+1}\rangle\rangle,s\rangle \notin I_{\varphi^*}$ for some q_{n+1}.

In this way we construct a sequence $\alpha = q_1,q_2,\ldots$ for which the right-hand side of (3.1) fails. Therefore $s \notin D_P$.

For any list $g_1,\ldots,g_k;Q_1,\ldots,Q_l$ of functions and predicates in B^* denote by $(\mathfrak{A};g_1,\ldots,g_k;Q_1,\ldots,Q_l)$ the extended structure

$$(B^*;\mathbb{O},\langle\,\rangle,f_1,\ldots,f_a,g_1,\ldots,g_k;B,R_1,\ldots,R_b,Q_1,\ldots,Q_l).$$

We have established so far that the set D_P is inductive in the structure $(\mathfrak{A};g;\mathrm{Nat},\mathrm{Seq},\Phi)$. To eliminate the additional function and predicates, we shall need the following refinement of the Transitivity Theorem [4, Th. 1C.3], which follows immediately from the corresponding proof in [4]:

If $\varphi(\bar{x},X)$ is Σ^0_1 (Π^0_1) formula in $(\mathfrak{A};Q)$, positive with respect to X and Q, and Q is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$, then I_φ is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$. (3.3)

Further we shall apply this fact in the next modified form.

Lemma 3.2 (Transitivity Lemma). *Let $\varphi(\bar{x},X)$ be Σ^0_1 (Π^0_1) formula in $(\mathfrak{A};g_1,\ldots,g_k;Q_1,\ldots,Q_l)$ in which $Q_1,\ldots,Q_l$ and X occur only positively. Suppose also that the graphs $G_{g_1},\ldots,G_{g_k}$ are Σ^0_1 ($\bar{G}_{g_1},\ldots,\bar{G}_{g_k}$ are Π^0_1) positive inductive on $\mathfrak{A}$ and $Q_1,\ldots,Q_l$ are Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$. Then I_φ is Σ^0_1 (Π^0_1) positive inductive on $\mathfrak{A}$.*

Proof. Assume first that g_1 has a unique occurrence in φ and let $\tau_1,\ldots,\tau_m$ be the arguments of g_1 in this occurrence. Let ψ be the formula which is obtained from φ when we replace $g_1(\tau_1,\ldots,\tau_m)$ by y. If φ is Σ^0_1, then ψ is Σ^0_1 in $(\mathfrak{A};g_2,\ldots,g_k;Q_1,\ldots,Q_l)$. Set

$$\chi(\bar{x},X) \;\Leftrightarrow\; \exists y\,(G_{g_1}(\tau_1,\ldots,\tau_m,y)\,\&\,\psi)\,.$$

Obviously, $\varphi \Leftrightarrow \chi$. By assumption G_{g_1} is Σ^0_1 positive inductive on $\mathfrak{A}$ and according to (3.3) I_χ (and hence I_φ) is Σ^0_1 positive inductive on $(\mathfrak{A};g_2,\ldots,g_k;Q_1,\ldots,Q_l)$.

When φ is Π^0_1, consider the Π^0_1 formula

$$\chi(\bar{x},X) \;\Leftrightarrow\; \forall y\,(\neg G_{g_1}(\tau_1,\ldots,\tau_m,y)\vee\psi)\,.$$

If g_1 has $i>1$ occurrences in φ, proceed by induction on i, applying the above argument to the innermost g_1. Iterating this procedure, exclude successively $g_2,\ldots,g_k$. For the elimination of $Q_1,\ldots,Q_l$ apply directly (3.3).

Lemma 3.3. *Let $\varphi(\bar{x},X)$ is Σ^0_k (Π^0_k) positive formula and I_φ is the unique fixed point of Γ_φ. Then $\bar{I}_\varphi$ (the complement of I_φ) is Π^0_k (Σ^0_k) positive inductive.*

Proof. We have $\bar{x} \in I_\varphi \Leftrightarrow \varphi(\bar{x}, I_\varphi)$ and hence

$$\bar{x} \in \bar{I}_\varphi \Leftrightarrow \neg\varphi(\bar{x}, I_\varphi) \Leftrightarrow \varphi'(\bar{x}, \bar{I}_\varphi),$$

where $\varphi'(\bar{x}, X)$ is obtained from $\neg\varphi$ after the replacement of every occurrence of X by $\neg X$. Obviously, φ' is equivalent to some $\varphi''(\bar{x}, X)$, which is X-positive and Π^0_k (Σ^0_k) if φ is Σ^0_k (Π^0_k) respectively. From the above equivalence it follows that

$$\bar{x} \in \bar{I}_\varphi \Leftrightarrow \varphi''(\bar{x}, \bar{I}_\varphi),$$

i.e. $\bar{I}_\varphi$ is a fixed point of $\Gamma_{\varphi''}$. Assuming that A is another fixed point of $\Gamma_{\varphi''}$ we obtain successively

$$\bar{x} \in \bar{A} \Leftrightarrow \neg(\bar{x} \in A) \Leftrightarrow \neg\varphi''(\bar{x}, A) \Leftrightarrow \varphi(\bar{x}, \bar{A}).$$

So $\bar{A}$ is a fixed point of Γ_φ and hence $\bar{A} = I_\varphi$, $A = \bar{I}_\varphi$, i.e. $\bar{I}_\varphi$ is the unique fixed point of $\Gamma_{\varphi''}$.

Say that a set is Δ^0_1 *positive inductive on* $\mathfrak{A}$ if it is both Σ^0_1 and Π^0_1 positive inductive on $\mathfrak{A}$.

For any $s \in B^*$ define $\|s\|$ (norm of s) as follows: $\|s\| = 0$ for $s \in B \cup \{\mathbb{O}\}$, $\|\langle s_1, s_2\rangle\| = \max(\|s_1\|, \|s_2\|) + 1$. We shall use systematically an induction on this norm when proving the next lemma.

Lemma 3.4. (i) *Let* $f \in \{L, R, \mathrm{lh}, \lambda x, i.(x)_i\}$. *Then* G_f *(the graph of* f*) is* Σ^0_1 *positive inductive on* $\mathfrak{A}$ *and* $\bar{G}_f$ *is* Π^0_1 *positive inductive on* $\mathfrak{A}$.

(ii) Nat, $\overline{\mathrm{Nat}}$, Seq *and* $\overline{\mathrm{Seq}}$ *are* Δ^0_1 *positive inductive on* $\mathfrak{A}$.

Proof. By definition we have

$$L(s) = t \Leftrightarrow s = t = 0 \vee s \in B \,\&\, t = 1 \vee \exists q(s = \langle t, q\rangle).$$

So the set G_L is explicitly definable by Σ^0_1 formula and, in particular, it is Σ^0_1 positive inductive on $\mathfrak{A}$. Similarly, $\bar{G}_L$ is Π^0_1 explicitly (and hence inductively) definable on $\mathfrak{A}$. The case $f = R$ is analogous.

Set

$$\varphi(x, X) \Leftrightarrow x = 0 \vee R(x) = 0 \,\&\, L(x) \in X.$$

Evidently, Nat is a fixed point of Γ_φ. Towards establishing that it is the unique fixed point of Γ_φ, assume that A and A' are some fixed points of Γ_φ. Then

$$\begin{aligned} x \in A &\Leftrightarrow x = 0 \vee R(x) = 0 \,\&\, L(x) \in A \quad \text{and} \\ x \in A' &\Leftrightarrow x = 0 \vee R(x) = 0 \,\&\, L(x) \in A'. \end{aligned} \tag{3.4}$$

Let $s \in$A. We shall use induction on $\|s\|$ to see that $s \in A'$. Suppose, first, that $\|s\| = 0$. The case $s = 0$ is obvious; the other case $s \in B$ is impossible, since then we would have $R(s) = 0$. If $s = \langle s_1, s_2\rangle$, then by (3.4) $R(s) = 0$ and $L(s) = s_1 \in A$. By induction hypothesis $s_1 \in A'$ and applying again (3.4) we conclude that $s \in A'$. So $A \subseteq A'$ and symmetrically, $A' \subseteq A$. Thus Nat is the only fixed point of Γ_φ. Consequently, Nat and (by Lemma 3.3) $\overline{\mathrm{Nat}}$ are Δ^0_0 inductive on $(\mathfrak{A}; L, R)$. Now, having in mind the facts about L and R just established and applying Lemma 3.2, we obtain that Nat and $\overline{\mathrm{Nat}}$ are Δ^0_1 positive inductive on $\mathfrak{A}$.

For the predicate Seq apply the same argument to the formula

$$\psi(x, X) \Leftrightarrow x \notin B \,\&\, (x = 0 \vee L(x) \in X).$$

By virtue of its definition lh satisfies the equivalence

$$\begin{aligned}\mathrm{lh}(s) = n \Leftrightarrow\ & (\neg\mathrm{Seq}(s) \vee s = 0) \,\&\, n = 0 \\ & \vee \mathrm{Seq}(s) \,\&\, s \neq 0 \,\&\, \mathrm{Nat}(n) \,\&\, n \neq 0 \,\&\, \mathrm{lh}(L(s)) = L(n).\end{aligned}$$

An easy induction on $||s||$ convinces us that lh is the only function with this property. In other words, G_{lh} is the unique fixed point of Γ_χ, where

$$\begin{aligned}\chi(x, y, X) \Leftrightarrow\ & (\neg\mathrm{Seq}(x) \vee x = 0) \,\&\, y = 0 \\ & \vee \mathrm{Seq}(x) \,\&\, x \neq 0 \,\&\, \mathrm{Nat}(y) \,\&\, y \neq 0 \,\&\, (L(x), L(y)) \in X.\end{aligned}$$

So, by Lemma 3.3 G_{lh} and $\bar{G}_{\mathrm{lh}}$ are Δ_0^0 positive inductive on $(\mathfrak{A}; L, R, \mathrm{Nat}, \mathrm{Seq}, \overline{\mathrm{Seq}})$. To see that G_{lh} is Σ_1^0 positive inductive on $\mathfrak{A}$, apply again the Transitivity Lemma and the previous results; similarly for $\bar{G}_{\mathrm{lh}}$.

Finally, for the function $(x)_i$ it is also immediate that $G_{(x)_i}$ is the unique fixed point of Γ_θ, where

$$\begin{aligned}\theta(x, i, y, X) \Leftrightarrow\ & (\neg\mathrm{Nat}(i) \vee i = 0 \vee i > \mathrm{lh}(x)) \,\&\, y = 0 \\ & \vee \mathrm{Seq}(x) \,\&\, x \neq 0 \,\&\, (\mathrm{lh}(x) = i \,\&\, y = R(x) \vee (L(x), i, y) \in X).\end{aligned}$$

Here the predicate "$>$" (greater than) over Nat is defined inductively as

$$n > k \Leftrightarrow \mathrm{Nat}(n) \,\&\, \mathrm{Nat}(k) \,\&\, n \neq 0 \,\&\, (k = 0 \vee L(n) > L(k))$$

and therefore is Δ_1^0 positive inductive on $\mathfrak{A}$. To complete the proof, repeat the arguments used above.

Let f is a k-ary function in B^*. Say that f is primitive recursive if the restriction of f over Nat^k is primitive recursive (considered as a function over the natural numbers) and $f(\bar{s}) = 0$ for $\bar{s} \notin \mathrm{Nat}^k$.

Lemma 3.5. *Let f be primitive recursive. Then G_f is Σ_1^0 positive inductive on $\mathfrak{A}$ and $\bar{G}_f$ is Π_1^0 positive inductive on $\mathfrak{A}$.*

Proof. By induction on the definition of f. If f is initial primitive recursive, then it has a Δ_0^0 explicit definition on $\mathfrak{A}$. If f is a superposition, say $f = f_0(f_1, \ldots, f_n)$, then we have the representation

$$\begin{aligned}f(s_1, \ldots, s_k) = t \Leftrightarrow\ & \mathrm{Nat}(s_1) \,\&\, \ldots \,\&\, \mathrm{Nat}(s_k) \,\&\, \exists q_1 \ldots \exists q_k (f_1(s_1, \ldots, s_k) = q_1 \,\&\, \ldots \\ & \&\, f_n(s_1, .., s_k) = q_n \,\&\, f_0(q_1, .., q_n) = t) \\ & \vee \neg(\mathrm{Nat}(s_1) \,\&\, \ldots \,\&\, \mathrm{Nat}(s_k)) \,\&\, t = 0.\end{aligned}$$

Now the result follows easily from the induction hypothesis, Lemma 3.4 and the Transitivity Lemma.

Finally, assume that f is obtained by primitive recursion from some g and h. Then

$$\begin{aligned}f(s_1, .., s_k, q) = t \Leftrightarrow\ & \mathrm{Nat}(s_1) \,\&\, \ldots \,\&\, \mathrm{Nat}(s_k) \,\&\, \mathrm{Nat}(q) \,\&\, (q = 0 \,\&\, t = g(s_1, .., s_k) \\ & \vee q \neq 0 \,\&\, \exists r (f(s_1, \ldots, s_k, L(q)) = r \,\&\, h(s_1, \ldots, s_k, L(q), r) = t) \\ & \vee \neg(\mathrm{Nat}(s_1) \,\&\, \ldots \,\&\, \mathrm{Nat}(s_k) \,\&\, \mathrm{Nat}(q)) \,\&\, t = 0.\end{aligned}$$

A trivial induction on $q \in \text{Nat}$ convinces us that f is the unique function, satisfying this equivalence. To see that f has the desired properties, proceed as in the proof of the previous lemma.

It remains to check that the universal relation Φ from the definition of φ^* is inductive on $\mathfrak{A}$. Of course, Φ depends on the particular coding of the syntactical objects that we have fixed. Below we specify some primitive recursive coding, which allows us to assert that Φ has Δ^0_1 inductive definition on $\mathfrak{A}$.

In order to save space, here we shall assume that the basic functions and predicates of $\mathfrak{A}_0 = (B; f_1, \ldots, f_a; R_1, \ldots, R_b)$ are unary. We shall use also the same letters for the corresponding symbols in $\mathfrak{L}_{\mathfrak{A}}$.

Let p_i be the i-th prime number (starting from $p_0 = 2$). Set also $\langle 0\rangle_i = 0$; $\langle n\rangle_i = \max\{j \mid p_i^j \text{ divides } n\}$ for $n > 0$.

Now put

$$\ulcorner \mathbb{O} \urcorner = 0, \quad \ulcorner x_0 \urcorner = 1, \quad \ulcorner x_1 \urcorner = 2, \quad \ulcorner \langle \tau_1, \tau_2\rangle \urcorner = 2^2.3^{\ulcorner \tau_1 \urcorner}5^{\ulcorner \tau_2 \urcorner},$$

$$\ulcorner f_i(\tau) \urcorner = 2^{2+i}.3^{\ulcorner \tau \urcorner} \text{ for } 1 \le i \le a,$$

$$\ulcorner R_i(\tau) \urcorner = 3^i.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \neg R_i(\tau) \urcorner = 3^{b+i}.5^{\ulcorner \tau \urcorner} \text{ for } 1 \le i \le b,$$

$$\ulcorner B(\tau) \urcorner = 3^{2b+1}.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \neg B(\tau) \urcorner = 3^{2b+2}.5^{\ulcorner \tau \urcorner}, \quad \ulcorner \psi_1 \,\&\, \psi_2 \urcorner = 3^{2b+3}.5^{\ulcorner \psi_1 \urcorner}.7^{\ulcorner \psi_2 \urcorner}.$$

Obviously, the predicates

$$K(n) \Leftrightarrow n \text{ is a code of a term with variables among } x_0,\, x_1,$$
$$M(n) \Leftrightarrow n \text{ is a code of some finite conjunction of elementary formulas with variables among } x_0,\, x_1$$

are primitive recursive.

Let U be the universal for the class of all terms with variables among x_0 and x_1, in other words,

$$U(n,t,s) = q \Leftrightarrow n \text{ is a code of a term } \tau \text{ with variables } x_1,\, x_2 \text{ and } \tau(t,s) = q.$$

Lemma 3.6. (i) *G_U is Σ^0_1 positive inductive on $\mathfrak{A}$ and $\bar{G}_U$ is Π^0_1 positive inductive on $\mathfrak{A}$.*

(ii) *Φ is Δ^0_1 positive inductive on $\mathfrak{A}$.*

Proof. (i) By definition we have

$$\begin{aligned} U(n,t,s) = q \Leftrightarrow\ & \neg K(n) \,\&\, q = 0 \\ & \vee M(n) \,\&\, (n = 0 \,\&\, q = \mathbb{O} \vee n = 1 \,\&\, q = t \vee n = 2 \,\&\, q = s \\ & \vee \langle n\rangle_0 = 2 \,\&\, \exists q_1 \exists q_2 (U(\langle n\rangle_1, t, s) = q_1 \,\&\, U(\langle n\rangle_2, t, s) = q_2 \,\&\, q = \langle q_1, q_2\rangle) \\ & \vee \langle n\rangle_0 = 3 \,\&\, \exists q_1 (U(\langle n\rangle_1, t, s) = q_1 \,\&\, f_1(q_1) = q) \vee \ldots \\ & \vee \langle n\rangle_0 = a + 2 \,\&\, \exists q_1 (U(\langle n\rangle_1, t, s) = q_1 \,\&\, f_a(q_1) = q)). \end{aligned}$$

Moreover, U is the unique function which satisfies this equivalence (a simple induction on n). All arithmetic functions which appear in the above formula are primitive recursive. Now proceed again as in the proof of Lemma 3.4, using also the previous lemma.

(ii) To see that Φ is both Σ^0_1 and Π^0_1 positive inductive on $\mathfrak{A}$, use the same argumentation, noticing first that Φ is the unique relation satisfying the equivalence

$$\begin{aligned}\Phi(n,t,s) \Leftrightarrow M(n) \,\&\, (\langle n\rangle_1 = 1 \,\&\, R_1(U(\langle n\rangle_2,t,s)) \vee \dots \\ \vee \langle n\rangle_1 = b \,\&\, R_b(U(\langle n\rangle_2,t,s)) \vee \langle n\rangle_1 = b+1 \,\&\, \neg R_1(U(\langle n\rangle_2,t,s)) \vee \dots \\ \vee \langle n\rangle_1 = 2b \,\&\, \neg R_b(U(\langle n\rangle_2,t,s)) \vee \langle n\rangle_1 = 2b+1 \,\&\, B(U(\langle n\rangle_2,t,s)) \\ \vee \langle n\rangle_1 = 2b+2 \,\&\, \neg B(U(\langle n\rangle_2,t,s)) \\ \vee \langle n\rangle_1 = 2b+3 \,\&\, \Phi(\langle n\rangle_2,t,s) \,\&\, \Phi(\langle n\rangle_3,t,s)).\end{aligned}$$

Now we are in a position to claim

Proposition 3.7. *Every set of points of* $\forall$*-definedness* D_P *is* Π_1^0 *positive inductive on* $\mathfrak{A}$.

Proof. By Lemma 3.1 D_P is a section of I_{φ^*}, where φ^* is Π_1^0 positive formula in some extended structure $(\mathfrak{A}; g; \text{Nat}, \text{Seq}, \Phi)$. Here g is primitive recursive, so by Lemma 3.5 $\bar{G}_g$ is Π_1^0 positive inductive on $\mathfrak{A}$. According to Lemma 3.4 and Lemma 3.6 the predicates Nat, Seq and Φ are Π_1^0 positive inductive on $\mathfrak{A}$. Now apply the Transitivity Lemma to conclude that I_{φ^*} is Π_1^0 positive inductive on $\mathfrak{A}$.

4. PROGRAM CHARACTERIZATION OF THE Π_1^0 POSITIVE INDUCTIVE DEFINITIONS

Let $\varphi(x_1, \dots, x_k, X)$ be an arbitrary Π_1^0 formula in which the k-ary relational variable X occurs positively. After contracting the quantifiers and converting the matrix into the disjunctive normal form, φ becomes equivalent to a formula of the following type:

$$\forall y\Big(\psi(\bar{x},y) \vee \psi_1(\bar{x},y) \,\&\, \Big(\tau_{1,1}^{(1)},\dots,\tau_{1,1}^{(k)}\Big) \in X \,\&\, \dots \,\&\, \Big(\tau_{1,n_1}^{(1)},\dots,\tau_{1,n_1}^{(k)}\Big) \in X \vee \dots$$
$$\vee \psi_m(\bar{x},y) \,\&\, \Big(\tau_{m,1}^{(1)},\dots,\tau_{m,1}^{(k)}\Big) \in X \,\&\, \dots \,\&\, \Big(\tau_{m,n_m}^{(1)},\dots,\tau_{m,n_m}^{(k)}\Big) \in X\Big),$$

where $\psi, \psi_1, \dots, \psi_m$ are quantifier-free formulas in which X does not occur.

Further we shall consider the case $k = 1$, $m = 2$ and $n_1 = n_2 = 1$, since it is sufficiently representative. Without essential loss of generality we may omit also ψ_1 and ψ_2 (dropping the formula ψ, however, trivializes the problem). So φ takes the form

$$\forall y(\psi(x,y) \vee \tau(x,y) \in X \vee \mu(x,y) \in X).$$

Now consider the following simple non-deterministic program P, for which we are going to establish that D_P coincides with the fixed point I_φ:

```
P:   input(x); x := ⟪x⟫;
  1: y := arbitrary(B*); z := head(x);
     if ψ(z, y) then stop;
     x := append(tail(x), ⟪τ(z, y), μ(z, y)⟫);
     if x = x then go to 1.
```

Head, *tail* and *append* act as the usual string-transforming operations, here applied to codes of sequences. It is an easy exercise to show that these functions can be computed by means of programs of the type considered here. Of course, in the above program P the operators, involving this functions, should be considered as macros rather than as assignments.

The proof of the equality $D_P = I_\varphi$ will be carried out by two lemmas. For the first one let us denote by D the set of all $t \in B^*$ such that $\mathrm{Seq}(t)$ and every computation of P, starting from the choice operator y := **arbitrary**(B^*) with current value of the variable x equal to t, is terminating.

Obviously, $s \in D_P$ iff $\langle\!\langle s \rangle\!\rangle \in D$. Further, if $\mathrm{Seq}(t)$, then

$$t \in D \Leftrightarrow \forall q(\psi(head(t), q) \vee append(tail(t), \langle\!\langle \tau(head(t), q), \mu(head(t), q)\rangle\!\rangle) \in D). \quad (4.1)$$

Lemma 4.1. *Let $s \in I_\varphi$. Then for every $t = \langle\!\langle t_1, \ldots, s, \ldots, t_n \rangle\!\rangle$ is true that $t \in D$.*

Proof. Transfinite induction on $|s|$. Let us notice (having in mind the agreement $|s| = |B^*|^+$ for $s \notin I_\varphi$) that whenever $s \in I_\varphi$

$$|s| = \begin{cases} 0 & \text{if } \forall q\psi(s,q), \\ \sup\limits_{q:\neg\psi(s,q)} (\min(|\tau(s,q)|, |\mu(s,q)|) + 1) & \text{otherwise.} \end{cases} \quad (4.2)$$

For every $t = \langle\!\langle t_1, \ldots, s, \ldots, t_n \rangle\!\rangle$ set $\mathrm{pos}(s,t) = \min\{i \mid s = t_i\}$.

Now let $s \in I_\varphi$ and suppose first that $|s| = 0$. Using induction on $\mathrm{pos}(s,t)$, we are going to prove that $t \in D$ for every $t = \langle\!\langle t_1, \ldots, s, \ldots, t_n \rangle\!\rangle$.

Case 1: $t = \langle\!\langle s, t_2, \ldots, t_n \rangle\!\rangle$. From the assumption $|s| = 0$ it follows that $\forall q\psi(s,q)$, in other words, $\forall q\psi(head(t), q)$ and therefore by (4.1) $t \in D$.

Case 2: $t = \langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ with $s = t_i$ for some $i > 1$. Pick any $q \in B^*$ and set $t' = \langle\!\langle t_2, \ldots, t_n, \tau(t_1, q), \mu(t_1, q) \rangle\!\rangle$. Obviously, $\mathrm{pos}(t', s) = i - 1$ and hence $t' \in D$ by induction supposition. Since q is arbitrary, applying again (4.1) we obtain $t \in D$.

Now let $s \in I_\varphi$, $|s| > 0$ and assume that for all s' with $|s'| < |s|$ the lemma is true. We shall use second induction on $\mathrm{pos}(s,t)$ again. Suppose first that $t = \langle\!\langle s, t_2, \ldots, t_n \rangle\!\rangle$. In order to establish that the right-hand side of (4.1) holds, take an arbitrary $q \in B^*$. If $\psi(s,q)$, there is nothing to prove; if not, by (4.2) $|s| > \min\{|\tau(s,q)|, |\mu(s,q)|\}$. Suppose for definiteness that $|s| > |\tau(s,q)|$. Then, in particular, $\tau(s,q) \in I_\varphi$ and by induction hypothesis for $s' = \tau(s,q)$, applied to $t' = \langle\!\langle t_2, \ldots, t_n, \tau(s,q), \mu(s,q) \rangle\!\rangle$, we obtain $t' \in D$. So by (4.1) $t \in D$. Finally, consider $t = \langle\!\langle t_1, \ldots, t_n \rangle\!\rangle$ with $\mathrm{pos}(s,t) = i > 1$. For $t' = \langle\!\langle t_2, \ldots, t_n, \tau(t_1, q), \mu(t_1, q) \rangle\!\rangle$, where q is any element of B^*, we have $\mathrm{pos}(s, t') = i - 1$. Therefore $t' \in D$ and hence $t \in D$.

Applying this result to $t = \langle\!\langle s \rangle\!\rangle$, we obtain $I_\varphi \subseteq D_P$. The opposite inclusion is given by the next lemma.

Lemma 4.2. $D_P \subseteq I_\varphi$.

Proof. Consider particular execution (finite or infinite) of P with some input s. Let x_n and y_n be the current values of the variables x and y immediately after the n-th running of the choice operator (if the execution has stopped after the m-th run of this operator, we assume that $x_n = x_m$ and $y_n = y_m$ for $n > m$). Set $\alpha(n) = y_n$, $n = 1, 2, \ldots$ Obviously, every x_n is uniquely determined by the input s and α. Let F be such that $F(s, \alpha, n) = x_n$.

Now choose some $s \notin I_\varphi$. In order to prove that $s \notin D_P$, it is enough to find a sequence α such that

$$\neg\psi(head(F(s, \alpha, n)), \alpha(n)) \text{ for every } n = 1, 2, \ldots \tag{4.3}$$

To this end we are going to define recursively two sequences $\{s_n\}_n$ and $\{q_n\}_n$ satisfying the condition

$$\begin{aligned} &s_1 = s; \quad s_{2k} = \tau(s_k, q_k); \quad s_{2k+1} = \mu(s_k, q_k); \\ &\neg\psi(s_k, q_k) \text{ and } s_k \notin I_\varphi \text{ for } k = 1, 2, \ldots \end{aligned} \tag{$*$}$$

We shall see later that (4.3) is true for $\alpha = \{q_n\}_n$.

Indeed, set $s_1 = s$. By assumption $s \notin I_\varphi$ and therefore $\neg\varphi(s, I_\varphi)$. Consequently, there is $q_1 \in B^*$ such that $\neg\psi(s_1, q_1)$, $s_2 = \tau(s_1, q_1) \notin I_\varphi$ and $s_3 = \mu(s_1, q_1) \notin I_\varphi$. Let us assume that for some $n > 1$ we have found $s_1, \ldots, s_{2^n-1}$ and $q_1, \ldots, q_{2^{n-1}-1}$ with the property

$$\begin{aligned} &s_1 = s; \quad s_{2k} = \tau(s_k, q_k); \quad s_{2k+1} = \mu(s_k, q_k); \\ &\neg\psi(s_k, q_k) \text{ for every } k = 1, 2, \ldots, 2^{n-1} - 1; \quad s_1 \notin I_\varphi, \ldots, s_{2^n-1} \notin I_\varphi. \end{aligned} \tag{$*$}_n$$

We shall construct elements $q_{2^{n-1}}, \ldots, q_{2^n-1}$ and $s_{2^n}, \ldots, s_{2^{n+1}-1}$ such that $(*)_{n+1}$ holds for $s_1, \ldots, s_{2^{n+1}-1}$ and $q_1, \ldots, q_{2^n-1}$. Let k be an arbitrary number between 2^{n-1} and $2^n - 1$. From the fact that $s_k \notin I_\varphi$ it follows that for some $q_k \in B^*$, $\neg\psi(s_k, q_k)$, $\tau(s_k, q_k) \notin I_\varphi$ and $\mu(s_k, q_k) \notin I_\varphi$. Set $s_{2k} = \tau(s_k, q_k)$ and $s_{2k+1} = \mu(s_k, q_k)$. Obviously, all the requirements of $(*)_{n+1}$ are satisfied. Therefore $(*)$ is true for the sequences $\{s_n\}_n$ and $\{q_n\}_n$ constructed in this way.

Set finally $\alpha(n) = q_n$, $n = 1, 2, \ldots$ We are going to check that (4.3) holds for this sequence α. Let us first notice that for every n, $F(s, \alpha, n) = \langle\langle s_n, \ldots, s_{2n-1}\rangle\rangle$. Indeed, the case $n = 1$ is obvious. Assuming that this is true for some $n \geq 1$, we shall have $head(F(s, \alpha, n)) = s_n$. By $(*)$ $\neg\psi(s_n, q_n)$ and therefore

$$\begin{aligned} F(s, \alpha, n+1) &= append(tail(F(s, \alpha, n)), \langle\langle \tau(s_n, q_n), \mu(s_n, q_n)\rangle\rangle) \\ &= append(\langle\langle s_{n+1}, \ldots, s_{2n-1}\rangle\rangle, \langle\langle s_{2n}, s_{2n+1}\rangle\rangle) = \langle\langle s_{n+1}, \ldots, s_{2n+1}\rangle\rangle. \end{aligned}$$

So, in particular, $head(F(s, \alpha, n)) = s_n$ for every $n = 1, 2, \ldots$ and using again $(*)$ we conclude that (4.3) is true.

Proposition 4.3. *Let A be a Π^0_1 positive inductive set. Then there exists a non-deterministic program P such that $A = D_P$.*

Proof. If A is I_φ, the result follows directly from the lemmas just verified. Otherwise A is a section of some I_φ, i.e.

$$A = \{(s_1, \ldots, s_k) \mid (s_1, \ldots, s_k, t_1, \ldots, t_m) \in I_\varphi)\}$$

for some fixed $t_1, \ldots, t_m$, built up from the basic constants and functions of $\mathfrak{A}$. Choose some $\tau_1, \ldots, \tau_m$ such that $\tau_{1\mathfrak{A}} = t_1, \ldots, \tau_{m\mathfrak{A}} = t_m$. Let P_0 be such that

$I_\varphi = D_P$. Denote by $x_1, \ldots, x_{k+m}$ the input variables of P_0. Now consider the following program P with input variables $x_1, \ldots, x_k$:

$$P: \; x_{k+1} := \tau_1; \; \ldots; \; x_{k+m} := \tau_m; \; P_0.$$

Obviously, $D_{P_0} = A$.

5. INDUCTIVE DEFINABILITY BY EXISTENTIALLY RESTRICTED FORMULAS

Let us call φ *existentially restricted* (*e.r.*) iff all existential quantifiers of φ range over the set Nat $\subseteq B^*$. In this section we establish that these quantifiers do not increase the inductive expressive power, i.e. the least fixed point I_φ of every e.r. X-positive $\varphi(\bar{x}, X)$ is Π^0_1 positive inductive. This result is based on the next lemma, which ascertains the same for the Π^0_2 positive formulas.

Lemma 5.1. *Let $\varphi(\bar{x}, X)$ be existentially restricted Π^0_2 positive formula. Then I_φ is Π^0_1 positive inductive.*

Proof. Our aim is to build a non-deterministic program Q with $D_Q = I_\varphi$ and, applying Proposition 3.7, to conclude that I_φ is Π^0_1 positive inductive.

We shall assume for simplicity that φ has one object variable x, so it is in the following general form:

$$\forall y(\exists z \in \text{Nat})(\psi(x,y,z) \vee \psi_1(x,y,z) \,\&\, \tau_{1,1} \in X \,\&\, \ldots \,\&\, \tau_{1,n_1} \in X \vee \ldots$$
$$\vee\, \psi_m(x,y,z) \,\&\, \tau_{m,1} \in X \,\&\, \ldots \,\&\, \tau_{m,n_m} \in X), \quad (5.1)$$

where ψ, ψ_1, $\ldots$, ψ_m are quantifier-free.

For the sake of clarity we shall confine ourselves to the case $m = 1$. When $m = 2$ (a case which is typical of the general case), combine the idea used in the construction of the program Q below with the in-width search in an appropriate binary tree, as carried out in the previous section.

Dropping also ψ_1 in (5.1) (since it is unessential here), we come to the following formula φ:

$$\forall y(\exists z \in \text{Nat})(\psi(x,y,z) \vee \tau(x,y,z) \in X).$$

Now define the program Q as follows:

```
Q :  input(x); t := ⟨⟨⟩⟩; u := 0;
  1: z := x; y := arbitrary(B*);
     t := append(t, ⟨⟨y⟩⟩); u := u + 1; v := 1;
  2: if ψ(z, (t)_v, [u]_v) then stop;
     if u = v then go to 1;
     z := τ(z, (t)_v, [u]_v); v := v + 1;
     if x = x then go to 2.
```

By $\lambda n, i.[n]_i$ we have denoted the decoding function for a fixed effective coding $\varkappa$ of all finite sequences of natural numbers (assuming, as customary, that $[n]_i = 0$ if

$i > \mathrm{lh}(n)$). Obviously, $\lambda n, i.[n]_i$, being recursive, can be computed with a program of our type.

Let us mention that the program Q that we propose here is far from being the most efficient one with the property $D_Q = I_\varphi$. Its advantage is the easy way to prove this fact.

Let us consider a particular execution of Q with input s. Suppose that during the computation we have arrived at the operator 2: **if** $\psi(z, (t)_v, [u]_v)$ **then stop** with current values of variables z, t, u and v, respectively q, r, n and i. Obviously, q is uniquely determined by s, r, n, i, i.e. there is a function g_0 such that $q = g_0(s, r, n, i)$. Set $g(s, \alpha, n, i) = g_0(s, \alpha(\bar{n}), n, i)$. Clearly, g satisfies the following equalities:

$$g(s, \alpha, n, 1) = s,$$
$$g(s, \alpha, n, i+1) = \tau(g(s, \alpha, n, i), \alpha(i), [n]_i) \text{ for } 1 < i \leq n.$$

Using this observation, one can easily check that

$$s \in D_Q \quad \text{iff} \quad \forall \alpha \exists n_{>0} (\exists i_{1 \leq i \leq n} \psi(g(s, \alpha, n, i), \alpha(i), [n]_i). \tag{5.2}$$

We shall use this equivalence in proving that the program Q has the desired property $I_\varphi = D_Q$.

Now let us agree until the end of the proof that n (eventually indexed) denotes an element of Nat.

For the first inclusion $I_\varphi \subseteq D_Q$ we shall use induction on $|s|$. A straightforward verification convinces us that for every $s \in I_\varphi$

$$|s| = \begin{cases} 0 & \text{if } \forall q \exists n \psi(s, q, n), \\ \sup\limits_{q : \forall n \neg \psi(s,q,n)} (\min\{|\tau(s, q, n)| + 1 \mid n \in \mathrm{Nat}\}) & \text{otherwise.} \end{cases}$$

Now take some $s \in I_\varphi$. In order to show that $s \in D_Q$, it suffices to see that the right-hand side of (5.2) holds. To do this, pick a sequence α and denote its first element by q. If $|s| = 0$, then there exists $n_q : \psi(s, q, n_q)$. Set $i = 1$ and $n = \varkappa(n_q)$ (or, for example, $n = \varkappa(n_q, 0)$ if $\varkappa(n_q)$ happens to be 0). Then, obviously, $\psi(g(s, \alpha, n, i), \alpha(i), [n]_i)$.

Now suppose that $|s| > 0$. If there is n_q with $\psi(s, q, n_q)$, proceed as above. Otherwise there should exist n_q such that $\tau(s, q, n_q) \in I_\varphi$ and $|\tau(s, q, n_q)| < |s|$. Set $\beta(n) = \alpha(n+1)$, $n = 1, 2, \ldots$ By induction hypothesis $\tau(s, q, n_q) \in D_Q$ and according to (5.2) exist m and $j_{1 \leq j \leq m}$:

$$\psi(g(\tau(s, q, n_q), \beta, m, j), \beta(j), [m]_j). \tag{5.3}$$

Now take $n \geq j + 1$ such that

$$[n]_1 = n_q \quad \text{and} \quad [n]_l = [m]_{l-1} \text{ for } l = 2, \ldots, j+1.$$

An easy induction on $i = 1, \ldots, j$ convinces us that

$$g(s, \alpha, n, i+1) = g(\tau(s, q, n_q), \beta, m, i).$$

In particular, $g(s, \alpha, n, j+1) = g(\tau(s, q, n_q), \beta, m, j)$. From here, using (5.3) and taking $i = j + 1$, we get the desired

$$\psi(g(s, \alpha, n, i), \alpha(i), [n]_i).$$

Towards establishing the converse inclusion $D_Q \subseteq I_\varphi$, suppose that A is an arbitrary fixed point of Γ_φ and take some $s \notin A$. Then $\neg\varphi(s, A)$ and therefore for at least one $q_1 \in B^*$

$$\forall n_1 \neg\psi(s, q_1, n_1) \quad \text{and} \quad \forall n_1 \tau(s, q_1, n_1) \notin A.$$

Analogously, from the latter there exists some $q_2 \in B^*$:

$$\forall n_1 \forall n_2 \neg\psi(\tau(s, q_1, n_1), q_2, n_2) \quad \text{and} \quad \forall n_1 \forall n_2 \tau(\tau(s, q_1, n_1), q_2, n_2) \notin A.$$

Iterating this procedure, we build a sequence $\alpha = q_1, q_2, \ldots$ satisfying for each n and $i \in \{1, \ldots, n\}$

$$\neg\psi(g(s, \alpha, n, i), \alpha(i), [n]_i).$$

From here, applying (5.2), we get $s \notin D_Q$.

Proposition 5.2. *Let $\varphi(\bar{x}, X)$ be an arbitrary existentially restricted positive formula. Then I_φ is Π^0_1 positive inductive.*

Proof. Our idea is to reduce φ to a Π^0_2 positive formula φ^* such that I_φ is a section of I_{φ^*} and to apply the result just obtained.

With no loss of generality we may assume that φ has one object variable x. Now consider first the case when φ is Π^0_4 formula, i.e. it is equivalent to

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} \psi(x, z, t, z', t')$$

with ψ — a quantifier-free. Let $\varphi^*(x, y, X)$ be the formula

$$\forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x, z, t\rangle\!\rangle, 1) \in X \vee y = 1 \,\&\, \hat{\psi}(x, z, t, X)),$$

where $\hat{\psi}$ is constructed from ψ in the following way: first replace simultaneously the variables x, z, t, z', t' by $(x)_1$, $(x)_2$, $(x)_3$, z and t, respectively. Then in the formula thus obtained replace each formula $\tau \in X$ by $(\tau, 0) \in X$. We claim that for every $s \in B^*$

$$s \in I_\varphi \;\Leftrightarrow\; (s, 0) \in I_{\varphi^*}. \tag{5.4}$$

We are going to prove (5.4) for the case when the matrix ψ is in the following simple form:

$$\chi(x, z, t, z', t') \vee \alpha(x, z, t, z', t') \in X,$$

since the verification of the general case is much similar to it.

So the corresponding formula $\varphi^*(x, y, X)$ is the following:

$$\begin{aligned} &\forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x, z, t\rangle\!\rangle, 1) \in X \\ &\qquad \vee y = 1 \,\&\, (\chi((x)_1, (x)_2, (x)_3, z, t) \vee (\alpha((x)_1, (x)_2, (x)_3, z, t), 0) \in X). \end{aligned}$$

The set I_{φ^*} is a fixed point of Γ_{φ^*}, therefore for any x

$$\begin{aligned} (x, 0) \in I_{\varphi^*} &\Leftrightarrow \forall z \exists t_{\in \mathrm{Nat}} ((\langle\!\langle x, z, t\rangle\!\rangle, 1) \in I_{\varphi^*}) \\ &\Leftrightarrow \forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(x, z, t, z', t') \vee (\alpha(x, z, t, z', t'), 0) \in I_{\varphi^*}). \end{aligned} \tag{5.5}$$

Towards establishing the equivalence (5.4) suppose that $s \in I_\varphi$. Then $s \in I^\xi_\varphi$ for some ordinal ξ. Using transfinite induction on ξ, we are going to check that $(s, 0) \in I_{\varphi^*}$. Indeed, under definition, $s \in I^\xi_\varphi$ if and only if

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} \Big(\chi(s, z, t, z', t') \vee \alpha(s, z, t, z', t') \in \bigcup_{\eta < \xi} I^\eta_\varphi\Big). \tag{5.6}$$

Taking an ordinal $\eta < \xi$, we get by inductive supposition that

$$\alpha(s,z,t,z',t') \in I^\eta_\varphi \Rightarrow (\alpha(s,z,t,z',t'),0) \in I_{\varphi^*}.$$

So, using (5.6), we obtain

$$\forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s,z,t,z',t') \vee (\alpha(s,z,t,z',t'),0) \in I_{\varphi^*}),$$

which according to (5.5) means that $(s,0) \in I_{\varphi^*}$.

Now, conversely, assuming that $(s,0) \in I^\xi_{\varphi^*}$ for some ξ, by induction on ξ we prove that $s \in I_\varphi$. We have

$$(s,0) \in I^\xi_{\varphi^*} \Leftrightarrow (s,0) \in \Gamma_{\varphi^*}\Big(\bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big) \Leftrightarrow \forall z \exists t_{\in \mathrm{Nat}} \Big((\langle\!\langle s,z,t\rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big).$$

Now suppose that $(\langle\!\langle s,z,t\rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}$ for some $\eta < \xi$. Then

$$\forall z' \exists t'_{\in \mathrm{Nat}} \Big(\chi(s,z,t,z',t') \vee \Big(\alpha(s,z,t,z',t'),0) \in \bigcup_{\mu<\eta} I^\mu_{\varphi^*}\Big)\Big)$$

and by the induction hypothesis for μ

$$\forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s,z,t,z',t') \vee \alpha(s,z,t,z',t') \in I_\varphi).$$

So we obtained

$$(\langle\!\langle s,z,t\rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*} \Rightarrow \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s,z,t,z',t') \vee \alpha(s,z,t,z',t') \in I_\varphi).$$

From here

$$\forall z \exists t_{\in \mathrm{Nat}} \Big((\langle\!\langle s,z,t\rangle\!\rangle, 1) \in \bigcup_{\eta<\xi} I^\eta_{\varphi^*}\Big)$$
$$\Rightarrow \forall z \exists t_{\in \mathrm{Nat}} \forall z' \exists t'_{\in \mathrm{Nat}} (\chi(s,z,t,z',t') \vee \alpha(s,z,t,z',t') \in I_\varphi),$$

in other words,

$$(s,0) \in I^\xi_{\varphi^*} \Rightarrow s \in I_\varphi$$

and hence $s \in I_\varphi$.

Thereby, the verification of equivalence (5.4) is completed. So I_φ is a section of I_{φ^*}, which is Π^0_2 positive inductive on $(\mathfrak{A}, \lambda x, i.(x)_i)$. From here, I_{φ^*} is Π^0_2 positive inductive on $\mathfrak{A}$ (under Lemma 3.4 and Transitivity Lemma). By Lemma 5.1 I_{φ^*} is Π^0_1 positive inductive on $\mathfrak{A}$ and hence I_φ is Π^0_1 positive inductive on $\mathfrak{A}$.

Now let $\varphi(x,X)$ be an arbitrary e.r. positive formula. We may assume that φ is Π^0_{2k}, $k \geq 2$, i.e. that φ is equivalent to

$$\forall z^1 \exists t^1_{\in \mathrm{Nat}} \ldots \forall z^k \exists t^k_{\in \mathrm{Nat}} \psi(x, z^1, t^1, \ldots, z^k, t^k, X)$$

with ψ — a quantifier-free. Set φ^* to be the following formula:

$$\forall z \exists t_{\in \mathrm{Nat}} (y = 0 \,\&\, (\langle\!\langle x,z,t\rangle\!\rangle, 1) \in X \vee \ldots \vee y = k-2 \,\&\, (\langle\!\langle x,z,t\rangle\!\rangle, k-1) \in X$$
$$\vee\, y = k-1 \,\&\, \hat{\psi}(x,z,t)).$$

Here the formula $\hat{\psi}(x,z,t)$ is constructed from ψ as it follows: first replace in ψ variables x, z^1 and t^1 by $(x)_1$, $(x)_2$ and $(x)_3$, respectively, and denote the formula thus obtained by $\psi^{(1)}$. To define $\psi^{(2)}$, replace in $\psi^{(1)}$ each occurrence of $(x)_i$ by $((x)_1)_i$ for $i = 1,2,3$ and z^2 and t^2 by $(x)_2$ and $(x)_3$, respectively. Repeat this procedure $k-1$ times. Finally, in the formula $\psi^{(k-1)}$ replace z^k and t^k by z and t, respectively, and then replace all formulas of the type $\tau \in X$ by $(\tau, 0) \in$X. The formula, constructed in this way, is $\hat{\psi}$.

Now the equality $I_\varphi = \{s \mid (s,0) \in I_{\varphi^*}\}$ is verified as above. As we have already seen, it immediately implies that I_φ is Π_1^0 positive inductive on $\mathfrak{A}$.

Theorem 5.3. *Let $A \subseteq (B^*)^k$. The following conditions are equivalent:*

(i) *A is the set of all points of $\forall$-definedness for some non-deterministic program P;*

(ii) *A is inductively definable by some Π_1^0 positive formula;*

(iii) *A is inductively definable by some existentially restricted positive formula;*

(iv) *$A = \{\bar{s} \mid \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s})\}$, where the predicate R is prime computable on $\mathfrak{A}$.*

Proof. The equivalence between the first three conditions follows from Proposition 3.7, Proposition 4.3 and Proposition 5.2. The easiest way to complete the proof of the theorem, is to show that (i) and (iv) are equivalent. We shall use the observation that prime computability (PC) is equivalent to computability by means of deterministic programs (see, for example, [5, Ch. 1.3]).

Now, assuming that the non-deterministic program P is determined by the recursive sequence $\{\Pi^{(n)}\}_n$ (in the sense of Proposition 2.2), let us set

$$R(t,n,\bar{s}) \Leftrightarrow \mathrm{Seq}(t)\,\&\,\mathrm{Nat}(n)\,\&\,\mathrm{lh}(t) = n\,\&\,\Pi^{(n)}(t,\bar{s}).$$

Having in mind some basic facts about prime computability, we may assert that R is prime computable on $\mathfrak{A}$. It is clear that $D_P = (\bar{s} \mid \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s})\}$.

Conversely, let $R(t,n,\bar{s})$ be a prime computable predicate and P_0 be some deterministic program that computes it. Denote by y, z, x_1, ..., x_k the input variables of P_0. Now set

P : **input**$(x_1,\ldots,x_k)$; $y := \langle\!\langle\ \rangle\!\rangle$; $z := 0$;
  1: $t :=$ **arbitrary**(B^*); $y := y * t$; $z := z+1$;
  P_0; **if** $x_1 = x_1$ **then go to** 1.

It is immediate by the construction of P that

$$\bar{s} \in D_P \Leftrightarrow \forall\alpha\exists n(P_0 \text{ stops at input } (\bar{\alpha}(n), n, \bar{s})) \Leftrightarrow \forall\alpha\exists n R(\bar{\alpha}(n), n, \bar{s}).$$

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