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A POLYNOMIAL PROBLEM

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We show that the roots of the equation (5) with respect to z are among the roots of the equation (6). Therefore the roots of the given equation (5) are determined by means of a check of the roots of the resolvent equation (6). Some examples and applications are given.

Keywords: two polynomial equations in two variables, common roots, Sylvester method of elimination, determinants, a check of the roots of the resolvent equation in the given equation

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EXPOSITION OF THE PROBLEM

First we shall prove the following

Theorem 1. *Let*

$$p \equiv P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \quad a_n \neq 0,\ n \geq 1, \tag{1}$$

and

$$Q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0, \quad b_m \neq 0,\ m \geq 1, \tag{2}$$

and let

$$q \equiv Q(\bar{z}) = b_m \bar{z}^m + b_{m-1} \bar{z}^{m-1} + \cdots + b_1 \bar{z} + b_0, \tag{3}$$

and

$$\bar{q} \equiv \overline{Q(\bar{z})} = \bar{b}_m z^m + \bar{b}_{m-1} z^{m-1} + \cdots + \bar{b}_1 z + \bar{b}_0. \tag{4}$$

Then all roots of the equation

$$Q(\bar{z}) = P(z) \tag{5}$$

with respect to z are roots of the determinant (resolvent) equation

$$\begin{array}{r} n \\ \text{rows} \end{array}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \begin{array}{r} m \\ \text{rows} \end{array}\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \begin{vmatrix} b_m & b_{m-1} & \dots & b_1 & b_0 - p & 0 & 0 & \dots & 0 \\ 0 & b_m & \dots & b_2 & b_1 & b_0 - p & 0 & \dots & 0 \\ \multicolumn{9}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 & 0 & \dots & b_m & b_{m-1} & \dots & \dots & b_1 & b_0 - p \\ \bar{a}_n & \bar{a}_{n-1} & \dots & \bar{a}_1 & \bar{a}_0 - \bar{q} & 0 & 0 & \dots & 0 \\ 0 & \bar{a}_n & \dots & \bar{a}_2 & \bar{a}_1 & \bar{a}_0 - \bar{q} & 0 & \dots & 0 \\ \multicolumn{9}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ 0 & 0 & \dots & \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_1 & \bar{a}_0 - \bar{q} \end{vmatrix} \tag{6}$$

as well, but, conversely, not always all roots of the equation (6) *are roots of the equation* (5) *as well, where the determinant is of order $n+m$.*

The determinant equation (6) *has:*

(i) *exactly n^2 roots if $n > m$,*

(ii) *exactly $n^2 = m^2$ roots if $n = m$ and $|a_m| \neq |b_m|$, and less than $n^2 = m^2$ roots if $n = m$ and $|a_m| = |b_m|$, both under the condition that all the equations*

$$\bar{a}_s = b_s e^{-i\varphi},\ 1 \le s \le m, \qquad a_0 = b_0 \pm i r_0 e^{i\frac{\varphi}{2}}, \tag{7}$$

where $\varphi \equiv \operatorname{Arg} a_m + \operatorname{Arg} b_m \pmod{2\pi}$, $r_0 \ge 0$ and the signs $\pm$ are taken singly, cannot exist simultaneously, and

(iii) *exactly m^2 roots if $n < m$.*

Proof. Let us examine the equations

$$b_m\zeta^m + b_{m-1}\zeta^{m-1} + \dots + b_1\zeta + b_0 - p = 0 \tag{8}$$

and

$$\bar{a}_n\zeta^n + \bar{a}_{n-1}\zeta^{n-1} + \dots + \bar{a}_1\zeta + \bar{a}_0 - \bar{q} = 0. \tag{9}$$

According to the classical Sylvester method of elimination, the two equations (8) and (9) have a common root ζ only if z is a root of the eliminating equation (6), and conversely (see the Sylvester method, for example, in Dickson's book [1, p. 164]). Hence, if a common root ζ of the two equations (8) and (9) is equal to $\bar{z}$, where z is a root of the resolvent (determinant) equation (6), then z is a root of the given equation (5) as well, taking into account the same multiplicity of z as a root of the determinant (resolvent) equation (6). If a common root ζ of the two equations (8) and (9) is not equal to $\bar{z}$, where z is a root of the determinant equation (6), then z is not a root of the given equation (5) as well.

If $n = m$, the condition in (ii) (see (7)) ensures that the equation (6) is not an identity with respect to z. Indeed, for $n = m$, the determinant in (6) is identically equal to zero with respect to z only if the two equations (8) and (9) are reduced to one equation, i.e. keeping in mind (1)–(4), if we have the identity

$$\bar{p} - \bar{q} \equiv \lambda(q - p) \quad (\zeta = \bar{z}) \tag{10}$$

for some complex (or real) number $\lambda \neq 0$ which does not depend on z and ζ. Thus from (10) we obtain the equations

$$\bar{a}_s = \lambda b_s, \quad 1 \le s \le m, \tag{11}$$

and the identity

$$\bar{a}_0 - \bar{q} \equiv \lambda(b_0 - p). \tag{12}$$

Further, from (12) it follows that

$$\bar{b}_s = \lambda a_s, \quad 1 \le s \le m, \tag{13}$$

and

$$\bar{b}_0 - \bar{a}_0 = \lambda(a_0 - b_0). \tag{14}$$

Now, from (11) and (13) for $s = m$, we obtain $|a_m| = |b_m|$ and hence $|\lambda| = 1$, i.e.

$$\lambda = e^{-i\varphi}, \tag{15}$$

where $\varphi \equiv \operatorname{Arg} a_m + \operatorname{Arg} b_m \pmod{2\pi}$. Therefore from (11) and (15) we get the first equations in (7). Finally, if we set $a_0 - b_0 = r_0 e^{i\alpha}$, $r_0 \ge 0$, α being real (α is arbitrary if $r_0 = 0$), from (14) and (15) we find $2\alpha = \pi + \varphi + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, if $r_0 > 0$, i.e. we obtain the second equations in (7).

Now we shall determine the degree of the resolvent equation (6) with respect to z. For the solution of this problem we shall use the fact that each summand in (6) consists of a product of elements of different columns and rows.

(i) Let $n > m$. Let k be a non-negative integer such that $0 \le k \le m$. If we take k times the binomial $\bar{a}_0 - \bar{q}$ and $n - k$ times the binomial $b_0 - p$, then we obtain the expression $(b_0 - p)^{n-k}(\bar{a}_0 - \bar{q})^k$ in which the highest degree of z is $n(n-k) + mk$. But $n(n-k) + mk \le n^2$ for the considered n, m and k with equality sign only for $k = 0$. Thus we proved that the determinant development of (6) includes only one summand of the form $(-1)^{nm}\bar{a}_n^m(b_0 - p)^n$ (the sign is $(-1)^{nm}$ since the number of the inversions of the permutation of the columns in order $m+1$, $m+2$, $\ldots$, $m+n$, 1, 2, $\ldots$, m is nm), i.e. the resolvent equation (6) is exactly of degree n^2 with respect to z.

(ii) Let $n = m$ and the equations (7) not exist simultaneously. Then if we take k times ($0 \le k \le m$) the binomial $\bar{a}_0 - \bar{q}$ and $m - k$ times the binomial $b_0 - p$, we obtain the expression $(b_0 - p)^{m-k}(\bar{a}_0 - \bar{q})^k$ in which the highest degree of z is $m(m-k) + mk = m^2$. Hence the determinant development of (6) contains the sum

$$\sum_{k=0}^{m} \sum_{k} (-1)^{\nu_{mk}} \bar{b}_m^k (\bar{a}_0 - \bar{q})^k \bar{a}_m^{m-k} (b_0 - p)^{m-k}, \tag{16}$$

where the number of the summands in the inner sum is equal to the number of the combinations of m elements of the class k, and ν_{mk} is equal to the number of the inversions of the columns to which the considered non-zero elements of the determinant in (6) belong. Now we shall determine the coefficient of z^{m^2} and the exponent ν_{mk} in (16) with the help of the following method: From (1)–(4) we obtain the limit equations

$$\lim_{z \to \infty} \frac{\bar{a}_0 - \bar{q}}{z^m} = -\bar{b}_m \tag{17}$$

and

$$\lim_{z \to \infty} \frac{b_0 - p}{z^m} = -a_m. \tag{18}$$

From (16)–(18) it follows that the coefficient of z^{m^2} is $(-1)^m \Delta_{2m}(b_m, a_m)$, where

$$\Delta_{2m}(b_m, a_m) = \sum_{k=0}^{m} \sum_{k} (-1)^{\nu_{mk}} |b_m|^{2k} |a_m|^{2m-2k}. \tag{19}$$

On the other hand, we can determine directly the coefficient of z^{m^2} from (6). If we take out a factor z^m of each one of the last m columns of the determinant in (6) and set $z \to \infty$, then by means of (17)–(18) we obtain that the coefficient of z^{m^2} is $(-1)^m \Delta_{2m}(b_m, a_m)$, where

$$\Delta_{2m}(b_m, a_m) \equiv \Delta_{2m} \begin{pmatrix} b_m, & \dots, & b_1 \\ \bar{a}_m, & \dots, & \bar{a}_1 \end{pmatrix}$$

$$\equiv \begin{array}{r} m \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ m \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \end{array} \begin{vmatrix} b_m & b_{m-1} & \dots & b_1 & a_m & 0 & 0 & \dots & 0 \\ 0 & b_m & \dots & b_2 & 0 & a_m & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & \dots & b_m & 0 & 0 & 0 & \dots & a_m \\ \bar{a}_m & \bar{a}_{m-1} & \dots & \bar{a}_1 & \bar{b}_m & 0 & 0 & \dots & 0 \\ 0 & \bar{a}_m & \dots & \bar{a}_2 & 0 & \bar{b}_m & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & \dots & \bar{a}_m & 0 & 0 & 0 & \dots & \bar{b}_m \end{vmatrix}, \tag{20}$$

and the determinant is of order $2m$. Now we develop the determinant (20) with respect to the first column and again we develop the obtained two subdeterminants with respect to the m-th columns, respectively. Thus we obtain the recurrence relation

$$\Delta_{2m} \begin{pmatrix} b_m, & \dots, & b_1 \\ \bar{a}_m, & \dots, & \bar{a}_1 \end{pmatrix} = \left(|b_m|^2 - |a_m|^2\right) \Delta_{2m-2} \begin{pmatrix} b_m, & \dots, & b_2 \\ \bar{a}_m, & \dots, & \bar{a}_2 \end{pmatrix} \tag{21}$$

for $m \geq 2$, where

$$\Delta_2 \begin{pmatrix} b_m \\ \bar{a}_m \end{pmatrix} = \begin{vmatrix} b_m & a_m \\ \bar{a}_m & \bar{b}_m \end{vmatrix} = |b_m|^2 - |a_m|^2. \tag{22}$$

From (21)–(22), by induction on m, we get the formula

$$\Delta_{2m}(b_m, a_m) = \left(|b_m|^2 - |a_m|^2\right)^m \tag{23}$$

for $m \geq 1$, keeping in mind the notations (20). Hence the resolvent equation (6) for $n = m$ is exactly of degree m^2 if $|a_m| \neq |b_m|$, and of degree less than m^2 if $|a_m| = |b_m|$. Further we compare (19) with the binomial expansion of (23). This yields the formula

$$\nu_{mk} = m - k. \tag{24}$$

By means of the formula (24) we find that the part (16) of the development of the determinant (6) for $n = m$ has the form

$$[b_m(\bar{a}_0 - \bar{q}) - \bar{a}_m(b_0 - p)]^m = \left[b_m \bar{a}_0 - \bar{a}_m b_0 + \sum_{s=0}^{m} (\bar{a}_m a_s - b_m \bar{b}_s) z^s \right]^m, \tag{25}$$

keeping in mind (1)–(4). Finally, from (25) for $s = m$ again our assertion for the degree becomes evident.

(iii) Let $n < m$. Then we interchange the roles of n and m, i.e. we examine the case $m > n$ as in point (i). Hence the resolvent equation (6) is exactly of degree m^2 with respect to z.

This completes the proof of Theorem 1.

Now we shall examine the equation (5) for $n = m$ under the conditions (7). In this case from (5) and (7), keeping in mind (1)–(4), we obtain the corresponding two equations

$$\sum_{s=1}^{m} b_s \bar{z}^s = e^{i\varphi} \sum_{s=1}^{m} \bar{b}_s z^s \pm i r_0 e^{i\frac{\varphi}{2}}, \tag{26}$$

which coincide with their conjugate equations

$$\sum_{s=1}^{m} \bar{b}_s z^s = e^{-i\varphi} \sum_{s=1}^{m} b_s \bar{z}^s \mp i r_0 e^{-i\frac{\varphi}{2}},$$

respectively, if the last equations are multiplied by $e^{i\varphi}$.

Theorem 2. *The two equations* (26) *are indeterminate, i.e. they have infinitely many roots* z.

Proof. We set

$$b_s = r_s e^{i\beta_s}, \quad 1 \le s \le m, \tag{27}$$

where $r_s \ge 0$ ($r_m > 0$), β_s are real (β_s is arbitrary if the corresponding $r_s = 0$), and

$$z = \rho e^{i\psi}, \tag{28}$$

where $\rho \ge 0$, ψ is real (ψ is arbitrary if $\rho = 0$). Then by means of (27) and (28) the two equations (26) become

$$\sum_{s=1}^{m} \rho^s r_s e^{i(\beta_s - s\psi)} = \sum_{s=1}^{m} \rho^s r_s e^{i(\varphi - \beta_s + s\psi)} \pm i r_0 e^{i\frac{\varphi}{2}},$$

which, after multiplication by $e^{-i\frac{\varphi}{2}}$, takes the form

$$2\sum_{s=1}^{m} \rho^s r_s \sin\left(s\psi - \beta_s + \frac{\varphi}{2}\right) \pm r_0 = 0. \tag{29}$$

The equations (29) are indeterminate with respect to ρ and ψ, depending on r_0, φ, r_s and β_s ($1 \le s \le m$).

This completes the proof of Theorem 2.

EXAMPLES AND APPLICATIONS

1. In particular, if $m = 1$, $b_1 = 1$, $b_0 = 0$ ($\bar{q} = Q(z) = z$) and $n \ge 1$ ($p = P(z)$), the equation (5) is reduced to the equation

$$\bar{z} = P(z), \tag{30}$$

keeping in mind (1). According to (6), the resolvent equation of (30) is the equation

$$D_{n+1}(\bar{a}_n, \bar{a}_{n-1}, \dots, \bar{a}_1, \bar{a}_0 - z) \equiv$$

$$\equiv \begin{array}{r} n \\ \text{rows} \\ \\ 1 \\ \text{row} \end{array} \left\{ \begin{vmatrix} 1 & -p & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & -p & 0 & \dots & 0 & 0 \\ & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \dots & 1 & -p \\ \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \dots & \bar{a}_1 & \bar{a}_0 - z \end{vmatrix} \right. = 0, \tag{31}$$

where the determinant is of order $n+1$. If we develop the determinant in (31) by the first column, then we obtain the recurrence relation

$$D_{n+1}(\bar{a}_n, \bar{a}_{n-1}, \dots, \bar{a}_1, \bar{a}_0 - z) = D_n(\bar{a}_{n-1}, \bar{a}_{n-2}, \dots, \bar{a}_1, \bar{a}_0 - z) + \bar{a}_n p^n \tag{32}$$

for $n \geq 2$, where

$$D_2(\bar{a}_1, \bar{a}_0 - z) = \begin{vmatrix} 1 & -p \\ \bar{a}_1 & \bar{a}_0 - z \end{vmatrix} = \bar{a}_0 - z + \bar{a}_1 p. \tag{33}$$

From (32) and (33) by induction on n we get the resolvent equation (31) of the equation (30) in the form

$$\bar{a}_n p^n + \bar{a}_{n-1} p^{n-1} + \dots + \bar{a}_1 p + \bar{a}_0 - z = 0, \tag{34}$$

keeping in mind (1). We shall note that the equation (34) follows directly from (30) with the help of the conjugate equation $z = \overline{P(z)}$ as well. If $n > 1$, the resolvent equation (34) is of degree n^2 and hence the given equation (30) has at most n^2 roots determined by (34). If $n = 1$, the resolvent equation (34) ($p = a_1 z + a_0$) is

$$\left(|a_1|^2 - 1\right) z + \bar{a}_1 a_0 + \bar{a}_0 = 0, \quad a_1 \neq 0. \tag{35}$$

Thus:

(I) If $|a_1| \neq 1$, from (35) it follows that the equation (30) ($n = 1$) has only one root which is

$$z = -\frac{\bar{a}_1 a_0 + \bar{a}_0}{|a_1|^2 - 1};$$

(II) If $|a_1| = 1$, i.e. $a_1 = e^{i\varphi}$, φ is real, the resolvent equation (35) is reduced to

$$0.z + e^{-i\varphi} a_0 + \bar{a}_0 = 0. \tag{36}$$

Now:

(II$_1$) If $e^{-i\varphi} a_0 + \bar{a}_0 \neq 0$, i.e. $a_0 \neq \pm i r_0 e^{i\frac{\varphi}{2}}$, $r_0 \geq 0$, the resolvent equation (36), and hence the given equation (30) for $n = 1$ and $a_1 = e^{i\varphi}$, i.e. the equation $\bar{z} = e^{i\varphi} z + a_0$, has not a root;

(II$_2$) If $e^{-i\varphi} a_0 + \bar{a}_0 = 0$, i.e. $a_0 = \pm i r_0 e^{i\frac{\varphi}{2}}$, $r_0 \geq 0$, the resolvent equation (36) is the identity

$$0.z + 0 = 0.$$

This is so, since for $m = 1$ the two equations (7) exist simultaneously ($b_1 = 1$, $b_0 = 0$, $\bar{a}_1 = e^{-i\varphi}$, $a_0 = \pm i r_0 e^{i\frac{\varphi}{2}}$). In this case the given equation (30) ($n = 1$) yields the two equations

$$\bar{z} = e^{i\varphi} z \pm i r_0 e^{i\frac{\varphi}{2}}, \tag{37}$$

which coincide with their conjugate equations

$$z = e^{-i\varphi}\bar{z} \mp i r_0 e^{-i\frac{\varphi}{2}},$$

respectively, if the last equations are multiplied by $e^{i\varphi}$. If we set $z = \rho e^{i\psi}$, $\rho \geq 0$, ψ is real (ψ is arbitrary if $\rho = 0$), then from (37), after multiplication by $e^{-i\frac{\varphi}{2}}$, we obtain the corresponding two indeterminate equations

$$2\rho \sin\left(\psi + \frac{\varphi}{2}\right) \pm r_0 = 0,$$

which yield the unknown values ρ and ψ, depending on r_0 and φ.

2. In particular, if $m = 2$, $b_2 = 1$, b_1 is arbitrary, $b_0 = 0$ ($\bar{q} = \overline{Q(\bar{z})} = z^2 + \bar{b}_1 z$) and $n = 2$ ($p = P(z) = a_2 z^2 + a_1 z + a_0$, $a_2 \neq 0$), the equation (5) is reduced to the equation

$$\bar{z}^2 + b_1\bar{z} = a_2 z^2 + a_1 z + a_0. \tag{38}$$

For this case the equations (7) ($m = 2$) are

$$\bar{a}_2 = e^{-i\varphi}, \quad \bar{a}_1 = b_1 e^{-i\varphi}, \quad a_0 = \pm i r_0 e^{i\frac{\varphi}{2}} \tag{39}$$

with $r_0 \geq 0$ and an arbitrary real φ. From (38) and (39) we obtain the two equations

$$\bar{z}^2 + b_1\bar{z} = e^{i\varphi} z^2 + \bar{b}_1 e^{i\varphi} z \pm i r_0 e^{i\frac{\varphi}{2}}, \tag{40}$$

which coincide with their conjugate equations

$$z^2 + \bar{b}_1 z = e^{-i\varphi}\bar{z}^2 + b_1 e^{-i\varphi}\bar{z} \mp i r_0 e^{-i\frac{\varphi}{2}},$$

respectively, if the last equations are multiplied by $e^{i\varphi}$. If we set $z = \rho e^{i\psi}$, $\rho \geq 0$, ψ is real (ψ is arbitrary if $\rho = 0$), then from (40), after multiplication by $e^{-i\frac{\varphi}{2}}$, we obtain the corresponding two indeterminate equations

$$2\rho^2 \sin\left(2\psi + \frac{\varphi}{2}\right) + 2\rho r_1 \sin\left(\psi - \beta_1 + \frac{\varphi}{2}\right) \pm r_0 = 0,$$

which yield the unknown values ρ and ψ, depending on r_0, φ, $r_1 = |b_1|$ and $\beta_1 = \operatorname{Arg} b_1$ (β_1 is arbitrary if $r_1 = 0$).

In the general case, if the equations (39) do not exist simultaneously, then according to (6) ($m = n = 2$) the resolvent equation of (38) is the equation

$$(\bar{a}_0 - \bar{q} + \bar{a}_2 p)^2 + (\bar{a}_2 b_1 - \bar{a}_1)\left[b_1(\bar{a}_0 - \bar{q}) + \bar{a}_1 p\right] = 0, \tag{41}$$

where

$$\bar{a}_0 - \bar{q} + \bar{a}_2 p = \left(|a_2|^2 - 1\right) z^2 + \left(a_1\bar{a}_2 - \bar{b}_1\right) z + a_0\bar{a}_2 + \bar{a}_0 \tag{42}$$

and

$$b_1(\bar{a}_0 - \bar{q}) + \bar{a}_1 p = (\bar{a}_1 a_2 - b_1) z^2 + \left(|a_1|^2 - |b_1|^2\right) z + a_0\bar{a}_1 + \bar{a}_0 b_1. \tag{43}$$

If $|a_2| \neq 1$, then from (41)–(43) it follows that the resolvent equation (41) is of degree 4 and hence the given equation (38) has at most four roots z. If $|a_2| = 1$, then from (41)–(43) it follows that the resolvent equation (41) is of degree at most 2 and hence the given equation (38) has at most two roots z.

3. In particular, for $a_s = 0$, $0 \leq s \leq n-1$, $a_n \neq 0$, $b_s = 0$, $0 \leq s \leq m-1$, $b_m \neq 0$, $n \geq m \geq 1$ and $|a_m| \neq |b_m|$, if $n = m$ ($p = P(z) = a_n z^n$, $\bar{q} = \overline{Q(\bar{z})} = \bar{b}_m z^m$), from (5) and (6) we obtain the equation

$$b_m \zeta^m = a_n z^n \qquad (\zeta = \bar{z}) \tag{44}$$

and its resolvent equation

$$E_{nm}(a_n, b_m, z) \equiv$$

$$\equiv \begin{array}{r} n\ \text{rows} \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ m\ \text{rows} \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \end{array} \begin{vmatrix} b_m & 0 & \dots & 0 & -a_n z^n & 0 & 0 & \dots & 0 \\ 0 & b_m & \dots & 0 & 0 & -a_n z^n & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & \dots & b_m & 0 & 0 & 0 & \dots & -a_n z^n \\ \bar{a}_n & 0 & \dots & 0 & -\bar{b}_m z^m & 0 & 0 & \dots & 0 \\ 0 & \bar{a}_n & \dots & 0 & 0 & -\bar{b}_m z^m & 0 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & \dots & \bar{a}_n & 0 & 0 & 0 & \dots & -\bar{b}_m z^m \end{vmatrix} = 0, \tag{45}$$

where the determinant $E_{nm}(a_n, b_m, z)$ is of order $n+m$. The equation (45) is a result of the elimination of ζ from the equation (44) and the conjugate equation

$$\bar{a}_n \zeta^n = \bar{b}_m z^m \qquad (\zeta = \bar{z}). \tag{46}$$

Now we can eliminate ζ by means of another method. Namely, let

$$d \equiv (n, m) \qquad (1 \leq d \leq m) \tag{47}$$

denote the greatest common divisor of the numbers n and m, i.e.

$$n = n_1 d \quad \text{and} \quad m = m_1 d, \tag{48}$$

where n_1 ($1 \leq n_1 \leq n$) and m_1 ($1 \leq m_1 \leq m$) are the corresponding quotients which are relatively prime positive integers, i.e. their greatest common divisor $(n_1, m_1) = 1$. Since the product $n_1 m_1 d$ is the least common multiple of the numbers n and m, from (44), (46) and (48) we obtain the equations

$$\zeta^{n_1 m_1 d} = \left(\frac{a_n}{b_m}\right)^{n_1} z^{n_1^2 d} \tag{49}$$

and

$$\zeta^{n_1 m_1 d} = \left(\frac{\bar{b}_m}{\bar{a}_n}\right)^{m_1} z^{m_1^2 d}. \tag{50}$$

From (49) we obtain d equations

$$\zeta^{n_1 m_1} = \varepsilon^k z^{n_1^2} \sqrt[d]{\left(\frac{a_n}{b_m}\right)^{n_1}}, \qquad k = 1, \dots, d, \tag{51}$$

where

$$\varepsilon = e^{i\frac{2\pi}{d}} \tag{52}$$

and the radical is taken arbitrarily. Hence from (50)–(52) we obtain d equations of the form

$$\left(\frac{a_n}{b_m}\right)^{n_1} z^{n_1^2 d} - \left(\frac{\bar{b}_m}{\bar{a}_n}\right)^{m_1} z^{m_1^2 d} = 0, \tag{53}$$

which yield all roots z of the equation (44). Thus from (53) and (48) we get the resolvent equation

$$\left[\left(\frac{a_n}{b_m}\right)^{\frac{n}{d}} z^{\frac{n^2}{d}} - \left(\frac{\bar{b}_m}{\bar{a}_n}\right)^{\frac{m}{d}} z^{\frac{m^2}{d}}\right]^d = 0 \tag{54}$$

of the equation (44), keeping in mind the multiplicity of the roots z, where d is given by (47). Further, from the comparison of the equivalent equations (45) and (54) it follows that

$$E_{nm}(a_n, b_m, z) = \mu_{nm}\left(a_n^{\frac{n}{d}}\bar{a}_n^{\frac{m}{d}} z^{\frac{n^2}{d}} - b_m^{\frac{n}{d}}\bar{b}_m^{\frac{m}{d}} z^{\frac{m^2}{d}}\right)^d, \tag{55}$$

where μ_{nm} is a factor which does not depend on z. Now we shall determine μ_{nm}. From (55) we obtain

$$\left.\frac{E_{nm}(a_n, b_m, z)}{z^{m^2}}\right|_{z=0} = (-1)^d \mu_{nm} b_m^n \bar{b}_m^m \tag{56}$$

for $n > m \geq 1$, and

$$\left.\frac{E_{mm}(a_m, b_m, z)}{z^{m^2}}\right|_{z=0} = \mu_{mm}\left(|a_m|^2 - |b_m|^2\right)^m \tag{57}$$

for $n = m \geq 1$, keeping in mind that $d = (m, m) = m$. On the other hand, from (45) we obtain

$$\left.\frac{E_{nm}(a_n, b_m, z)}{z^{m^2}}\right|_{z=0} = (-1)^m b_m^n \bar{b}_m^m \tag{58}$$

for $n > m \geq 1$, and

$$\left.\frac{E_{mm}(a_m, b_m, z)}{z^{m^2}}\right|_{z=0} = (-1)^m\left(|b_m|^2 - |a_m|^2\right)^m \tag{59}$$

for $n = m \geq 1$, keeping in mind (20) (for $a_s = b_s = 0$, $1 \leq s \leq m-1$, if $m \geq 2$) and (23). If we compare (56) with (58) and (57) with (59), we obtain

$$\mu_{nm} = (-1)^{m-d}, \qquad n \geq m \geq 1. \tag{60}$$

Thus from (55) and (60) we get the formula

$$E_{nm}(a_n, b_m, z) = (-1)^{m-d}\left(a_n^{\frac{n}{d}}\bar{a}_n^{\frac{m}{d}} z^{\frac{n^2}{d}} - b_m^{\frac{n}{d}}\bar{b}_m^{\frac{m}{d}} z^{\frac{m^2}{d}}\right)^d \tag{61}$$

for the value of the determinant in (45) for $n \geq m \geq 1$ and d given by (47).

In particular, for $n = rm$, $r = 1, 2, \ldots$ $(m \geq 1)$ we have $d = (rm, m) = m$ and hence the formula (61) is reduced to the formula

$$E_{rm,m}(a_{rm}, b_m, z) = \left(a_{rm}^r \bar{a}_{rm} z^{r^2 m} - b_m^r \bar{b}_m z^m\right)^m. \tag{62}$$

In particular, from (44) for $n = m \geq 1$ and (62) for $r = 1$ it follows that all roots z of the equation

$$b_m \bar{z}^m = a_m z^m, \qquad |a_m|^2 - |b_m|^2 \neq 0, \tag{63}$$

are represented by the multiple root $z = 0$ of order m^2 of the resolvent equation

$$\left(|a_m|^2 - |b_m|^2\right)^m z^{m^2} = 0. \tag{64}$$

The resolvent equation (64) can be directly obtained if we determine $\bar{z}$ from (63), which yields

$$\bar{z} = z e^{-i\frac{2k\pi}{m}} \sqrt[m]{\frac{a_m}{b_m}}, \qquad k = 0, 1, \ldots, m-1,$$

for any value of the radical, and set these values of $\bar{z}$ in the conjugate equation of (63), namely in

$$\bar{b}_m z^m = \bar{a}_m \bar{z}^m.$$

Thus we obtain m equations of the form

$$\left(|a_m|^2 - |b_m|^2\right) z^m = 0$$

which, when multiplied, yield (64).

OTHER EXAMPLES

The next simple cases illustrate the application of example 1.

(A) Consider the equation

$$\bar{z} = z. \tag{65}$$

The conjugate equation of (65) is $z = \bar{z}$ and hence the resolvent equation is the identity $z = z$, i.e. the equation

$$0.z = 0. \tag{66}$$

The solutions of (66) are all complex numbers, but the solutions of (65) are only all real numbers, because the root $\zeta = z$ of the equation $\zeta - z = 0$ is equal to $\bar{z}$ if and only if z is a real number. This result is in accordance with Theorem 2 and example 1, item (II_2), for $\varphi = 0$ and $r_0 = 0$.

(B) Consider the equation

$$\bar{z} = z^2. \tag{67}$$

The equation (67) and its conjugate equation form the two equations

$$\zeta - z^2 = 0, \qquad \zeta^2 - z = 0. \tag{68}$$

From (6), or directly from (68), we obtain the resolvent equation

$$z(z^3 - 1) = 0. \tag{69}$$

All solutions $z = 0,\ 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}}$ of (69) are roots of (67), because for these z the corresponding common root ζ of the two equations (68) is equal to the conjugate value $\bar{z}$, respectively.

(C) Consider the equation

$$\bar{z} = z^3 + z. \tag{70}$$

The equation (70) and its conjugate equation form the two equations

$$\zeta - (z^3 + z) = 0, \qquad \zeta^3 + \zeta - z = 0. \tag{71}$$

From (6), or directly from (71), we obtain the resolvent equation

$$0 = (z^3 + z)^3 + z^3 = z^3(z^2 + 2)\frac{z^6 - 1}{z^2 - 1}, \qquad z^2 \neq 1, \tag{72}$$

with the roots

$$z_{1,2,3} = 0, \quad z_{4,5} = \pm i\sqrt{2}, \quad z_6 = e^{i\frac{\pi}{3}}, \quad z_7 = e^{i\frac{2\pi}{3}}, \quad z_8 = e^{i\frac{4\pi}{3}}, \quad z_9 = e^{i\frac{5\pi}{3}}. \tag{73}$$

For the values $z = z_k$, $k = 1, 2, 3, 4, 5$, in (73), the common roots ζ of the two equations (71) are equal to $\zeta = \bar{z}_k$, $k = 1, 2, 3, 4, 5$, respectively. Hence the roots $z_{1,2,3}$ (a triple root) and $z_{4,5}$ of (72) are roots of (70) as well. For the values $z = z_k$, $k = 6, 7, 8, 9$, in (73), the two equations (71) take the forms

$$\zeta - z_{7,6,9,8} = 0, \qquad \zeta^3 + \zeta - z_{6,7,8,9} = 0, \tag{74}$$

respectively. The common roots ζ of the two equations (74) are equal to $\zeta = z_{7,6,9,8} \neq \bar{z}_{6,7,8,9}$, respectively. Hence the roots $z_{6,7,8,9}$ of the resolvent equation (72) are not roots of the given equation (70). Thus all roots of (70) are only the roots $z_{1,2,3,4,5}$ in (73).

(D) Consider the equation

$$\bar{z} = z^4. \tag{75}$$

The equation (75) and its conjugate equations form the two equations

$$\zeta - z^4 = 0, \qquad \zeta^4 - z = 0. \tag{76}$$

From (6), or directly from (76), we obtain the resolvent equation

$$z(z^{15} - 1) = 0. \tag{77}$$

But only the solutions

$$z = 0,\ 1,\ e^{i\frac{2\pi}{5}},\ e^{i\frac{4\pi}{5}},\ e^{i\frac{6\pi}{5}},\ e^{i\frac{8\pi}{5}}$$

of (77) are the unique solutions of (75), because only for these z the corresponding common root ζ of the two equations (76) is equal to the conjugate value $\bar{z}$, respectively.

The examples (B)–(D) are in accordance with Theorem 1.

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