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FIELD FLUCTUATIONS AND THE EFFECTIVE BEHAVIOUR OF MICRO-INHOMOGENEOUS SOLIDS

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The problem of predicting the effective mechanical properties and response of micro-inhomogeneous solids is revisited. The aim is to highlight the influence of field fluctuations which, as a rule, is neglected by the numerous existing theories that interconnect the micro- and macro-behaviour of such solids. The key observation is that in a heterogeneous solid of random constitution when, say, macroscopic quantities like the mean stress tensor are prescribed, fluctuations always create regions in which considerably higher stresses appear. In these regions either plastic flow or a certain kind of deterioration takes place, which affects the macroscopic behaviour of the solid. The result is that the latter should start exhibiting deviations from linear response from the very beginning of straining, despite the assumed linearity of its constituents. A quantitative approach that takes into account the field fluctuations is proposed and outlined in the lecture. For the simplicity sake, the scalar conductivity problem for a dilute dispersion of spheres, possessing properties different from those of the matrix, is employed in order to illustrate better the basic ideas. The progressive deviation from linearity, when the macroscopic "straining" increases, indeed shows up clearly in the performed analysis.

Keywords: random media, effective properties, fluctuations, excursion sets

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1. INTRODUCTION

The problem of predicting the macroscopic mechanical behaviour of a solid with a given internal structure is the central problem of micromechanics, see, e.g.

[1], where the well-known approximate theories, like self-consistent one, differential scheme, effective fields approach, etc., are considered in detail and compared to experimental findings. If the microstructure is periodic, there exist rigorous mathematical theories that yield numerical algorithms for evaluating the macroproperties [2]. For a random solid, the problem is considerably more complicated. In the simplest linear scalar case it is posed in the following typical manner [3].

Assume that $\kappa(\mathbf{x})$ is the known random field of conductivity coefficient for the medium. (For a two-phase one, $\kappa(\mathbf{x})$ takes the values κ_m or κ_f depending on whether $\mathbf{x}$ lies in the matrix or in a particle, respectively.) The temperature field, $\theta(\mathbf{x})$, in such a medium is governed by the equations

$$\nabla \cdot \mathbf{q}(\mathbf{x}) = 0, \quad \mathbf{q}(\mathbf{x}) = \kappa(\mathbf{x})\nabla\theta(\mathbf{x}), \quad \langle \nabla\theta(\mathbf{x}) \rangle = \mathbf{G}, \tag{1.1}$$

where $\mathbf{q}(\mathbf{x})$ is the (opposite) heat flux, $\mathbf{G}$ denotes the prescribed macroscopic temperature gradient. Hereafter $\langle \cdot \rangle$ signifies ensemble averaging. The problem (1.1) should be solved in statistical sense — for a given (infinite) hierarchy of multipoint moments $\langle \kappa(\mathbf{x}_1) \ldots \kappa(\mathbf{x}_q) \rangle$, $q = 1, 2, \ldots$, one should find the similar hierarchy of *all* multipoint moments of $\theta(\mathbf{x})$ and the joint moments of $\theta(\mathbf{x})$ and $\kappa(\mathbf{x})$. In particular among the latter, one of the simplest is of special interest, namely,

$$\mathbf{Q} = \langle \kappa(\mathbf{x})\nabla\theta(\mathbf{x}) \rangle = \kappa^* \mathbf{G}, \tag{1.2}$$

since it defines the well-known effective conductivity, κ^*, of the medium. The definition (1.2) of κ^* reflects the familiar "homogenization" of the problem under study, in the sense that from a macroscopic point of view, when only the macroscopic values of the flux $\mathbf{Q}$ and of the temperature gradient $\mathbf{G}$ are of interest, the medium behaves as if it were homogeneous with a certain macroscopic conductivity κ^*. This interpretation explains why the value κ^* and its counterparts, say, the effective elastic moduli, have been extensively studied in the literature on micro-inhomogeneous and composite materials, see [1] and the references therein.

However, κ^* is only a *tiny* part of the full statistical solution of the random problem (1.1). Moreover, its evaluation *cannot* be torn away from the full statistical solution of (1.1), i.e., of specifying the *whole* infinite hierarchy of multipoint moments, as argued, e.g., in [3 – 5] et al. (The latter fact explains the failure of all schemes that try to determine solely the effective property κ^* without trying to solve the whole stochastic problem (1.1).) Besides, there are plenty of reasons why one should pay much more attention to other statistical characteristics of random fields like $\theta(\mathbf{x})$ in (1.1), that appear in problems in random heterogeneous media. For instance, one of the most important quantities is often the variance of local fields, connected with the square of its fluctuation, see, e.g., [6 – 7] for more details and references. For transport-like problems of the type of (1.1), the fluctuations of the local fields are of primary importance when there exists a transition to nonlinearity (or a deterioration starts) after a certain threshold. Their effect will then consists in a deviation from a linear response, however small is the macroscopic "loading" $\mathbf{G}$, compared to the respective threshold value.

The aim of the present work is to quantify this statement to a certain extent, sketching very briefly a theory that describes such an effect in a highly idealized

situation and thus will stimulate, hopefully, further interest and research. The core of the approach is a combination between the functional (Volterra-Wiener) series method, proposed and used in the last years by one of the authors in the study of micro-inhomogeneous solids [4, 5], and Vanmarcke's theory [8] of excursion sets for random fields.

2. STATISTICAL SOLUTION OF EQ. (1.1) FOR A DILUTE DISPERSION

To illustrate the basic ideas, assume that the medium is a random dispersion of spheres — a typical representative of the wide and important class of particulate micro-inhomogeneous media, extensively studied in the literature.

Let $\eta_f = nV_a$, $V_a = \frac{4}{3}\pi a^3$, be the volume fraction of the spheres, n be their number density, a be the spheres' radius. In this case the random temperature field, $\theta(\mathbf{x})$, that solves the problem (1.1), can be conveniently constructed by means of the functional series approach, see [5, 6]. In particular, in the dilute limit $\eta_f \ll 1$, $\theta(\mathbf{x})$ has the form of the truncated functional series

$$\theta(\mathbf{x}) = \mathbf{G}\cdot\mathbf{x} + \int T_1(\mathbf{x}-\mathbf{y})\,\psi'(\mathbf{y})\,d\mathbf{y} + o(\eta_f), \tag{2.1}$$

where $\psi'(\mathbf{x}) = \psi(\mathbf{x}) - n$ is the fluctuating part of Stratonovich's random density field $\psi(\mathbf{x}) = \sum_\alpha \delta(\mathbf{x}-\mathbf{x}_\alpha)$, see [9]. The integrals hereafter are over the entire space $\mathbb{R}^3$ if the integration domain is not explicitly indicated. In Eq. (2.1)

$$T_1(\mathbf{x}) = 3\beta\mathbf{G}\cdot\nabla\varphi(\mathbf{x}), \quad \beta = \frac{[\kappa]}{\kappa_f + 2\kappa_m}, \tag{2.2}$$

$[\kappa] = \kappa_f - \kappa_m$, is the "single-sphere" field, i.e. the disturbance to the temperature field $\mathbf{G}\cdot\mathbf{x}$ in the homogeneous matrix of conductivity κ_m, introduced by a single spherical inclusion of conductivity κ_f; $\varphi(x)$ is the Newtonian potential for the latter inclusion. Recall that

$$\nabla\nabla\varphi(\mathbf{x}) = -\frac{1}{3}\begin{cases}\mathbf{I}, & \text{if } |\mathbf{x}| < a,\\ \dfrac{1}{\rho^3}(\mathbf{I} - 3\mathbf{e}_r\mathbf{e}_r), & \text{if } |\mathbf{x}| > a,\end{cases} \tag{2.3}$$

$\rho = r/a$, $\mathbf{e}_r = \mathbf{r}/r$, $r = |\mathbf{x}|$ and $\mathbf{I}$ is the unit 2nd-rank tensor.

The representation (2.1) allows us to obtain all statistical characteristics of the field $\theta(\mathbf{x})$, asymptotically correctly to the order $O(\eta_f)$. In particular, we shall need in what follows the full statistical information about the random variable

$$\tau = \tau(0) = |\nabla\theta(0)|^2, \tag{2.4}$$

i.e. its probability distribution function. Using Eq. (2.1) and the formula

$$\langle\psi'(\mathbf{y}_1)\psi'(\mathbf{y}_2)\rangle = n\delta(\mathbf{y}_1 - \mathbf{y}_2) + o(n), \tag{2.5}$$

see [9], yields after some algebra:

$$\tau = \widetilde{G}^2 + 3\beta \mathbf{G} \cdot \int \widetilde{\mathbf{T}}(\mathbf{y}) \cdot \mathbf{G}\, \psi'(\mathbf{y})\, d\mathbf{y} + o(\eta_f), \tag{2.6}$$

$$\widetilde{G}^2 = G^2(1 + 3\beta^2 \eta_f), \quad \widetilde{\mathbf{T}}(\mathbf{y}) = \nabla\nabla\varphi(\mathbf{y}) \cdot \big(2\mathbf{I} + 3\beta \nabla\nabla\varphi(\mathbf{y})\big), \tag{2.7}$$

having employed some results of [5]. In a similar way, other statistical characteristics of the field $\theta(\mathbf{x})$ can be obtained in a closed form. Details can be found, e.g., in [5] and [10] (in the last paper the effective conductivity κ^* of the dispersion is rigorously evaluated to the order $O(\eta_f^2)$, using the truncated functional series approach).

3. THE IDEALIZED MODEL

Hereafter we shall consider a highly idealized situation in which there exists a threshold G_0 of the temperature gradient with the following properties: if $|\nabla\theta(\mathbf{x})| < G_0$ locally (at the point $\mathbf{x}$ that is), then both the matrix and the particles behave linearly, obeying the Fourier law, see Eq. (1.1).

If, however, $|\nabla\theta(\mathbf{x})| \geq G_0$, the constituents become nonconducting, i.e. both κ_f and κ_m vanish. In other words, the following constitutive equation is adopted:

$$\mathbf{q}(\mathbf{x}) = \widetilde{\kappa}(\mathbf{x})\nabla\theta(\mathbf{x}), \quad \widetilde{\kappa}(\mathbf{x}) = \begin{cases} \kappa(\mathbf{x}), & \text{if } |\nabla\theta(\mathbf{x})| < G_0, \\ 0, & \text{if } |\nabla\theta(\mathbf{x})| \geq G_0, \end{cases} \tag{3.1}$$

$$\kappa(\mathbf{x}) = \begin{cases} \kappa_m, & \text{if } \mathbf{x} \in \text{matrix}, \\ \kappa_f, & \text{if } \mathbf{x} \in \text{spheres}. \end{cases}$$

The model is not claimed to have any specific physical meaning — its sole role here is to illustrate the basic ideas and techniques as simply as possible. The generalizations to more realistic situations when, say, plastic flow and/or damaging take place, according to certain well-known criteria, can be performed along a similar line of reasoning (provided the volume fraction η_f of the inhomogeneities is small enough).

As it follows from Eqs. (2.2) and (2.3), the "stress-concentration factor" for the single-spherical inhomogeneity, in the scalar conductivity context under discussion, does not exceed 2, whatever the values of κ_f and κ_m. In other words, if $G = |\mathbf{G}|$ is the magnitude of the temperature gradient at infinity, the magnitude of this gradient within or around the single inhomogeneity does not exceed $2G$. This means that if $G < 2G_0$, the linear Fourier law in (1.1) is applicable throughout the whole infinite space, comprising the matrix with the single spherical inhomogeneity. In turn, for the considered dilute dispersion the spheres are, as a rule, far one from

another[1] and hence each one can be considered as single, immersed into the infinite matrix constituent. This means that under the condition

$$G < G_0/2, \tag{3.2}$$

the linear equation (1.1) for the temperature field $\theta(\mathbf{x})$ is still applicable, despite the obvious strong nonlinearity of the model (3.1). The applicability of this linear equation does not mean, though, that the effective behaviour of the solid will be linear even in the region (3.2). The reason is that however small is the mean gradient's magnitude G, the solution $\theta(\mathbf{x})$ of Eq. (1.1), being *random*, will always exhibit fluctuations, some of which will be big enough to generate regions in which the local gradient $|\nabla\theta(\mathbf{x})| \geq G_0$. These are just the so-called *excursion* sets to be discussed in the next section.

4. THE EXCURSION SETS FOR THE RANDOM FIELD $|\nabla\theta(\mathbf{x})|^2$

Let $f(\mathbf{x})$ be a random field, whose realizations are defined over the region $\Omega \subset \mathbb{R}^3$. The sets $\Omega_A = \{\mathbf{x} \in \Omega \mid f(\mathbf{x}) \geq A\}$ are called excursion for the field $f(\mathbf{x})$, [8, 11]. A problem of central importance for many applications concerns a more detailed description of these sets and, in particular, estimates of their mean volume Ω_A/Ω.

In general, such questions are very hard since the answers should involve the multipoint statistics of $f(\mathbf{x})$. Comparatively simple results are achieved for infinite regions $\Omega = \mathbb{R}^3$ under the assumption that the field $f(\mathbf{x})$ is Gaussian, see again [8, 11]. The latter assumption unfortunately is not appropriate for the fields that, like $\theta(\mathbf{x})$, emerge as solution of the random equations of the type of (1.1) in media of particulate microstructure, see [4] for a more detailed discussion. For arbitrary (statistically homogeneous) random fields convenient, though approximate, results are given by Vanmarcke [8, Ch. 4], and they will turn out very useful for our study, as we shall see in a moment.

Namely, Vanmarcke observed that if the excursion value A is considerably higher than the mean value of the field (say, two or three time at least, which as a matter of fact is just our case, as it follows from Eq. (3.2)), the excursion sets have a simple structure — they represent well separated areas in $\mathbb{R}^3$, whose volume fraction, η_A, is just the complementary cumulative distribution function $F_f^c(A)$ of the random variable $f = f(0)$. More precisely,

$$\eta_A = \lim_{\Omega \to \mathbb{R}^3} \Pr\{f \geq A\} = F_f^c(A) = 1 - F_f(A). \tag{4.1}$$

The result (4.1), though not mentioned explicitly in [8], immediately follows from the formulae (4.6.4) of the same book.

[1] More precisely, this is true if the inclusions are "well-separated"; however, there can exist realizations of the arrays of spheres when they form clusters. The latter may result in a considerable increase of the local temperature gradient, as pointed out in the final section.

Note that due to the assumed statistical homogeneity, all random variables $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, possess one and the same probability distribution function $F_f(A)$ and hence it suffices to take $\mathbf{x} = 0$, i.e. to consider the random variable $f = f(0)$ only.

It is to be also noted that the formula (4.1) has a simple and clear interpretation: it states that for high enough excursion levels the multipoint statistics of the random field $f(\mathbf{x})$ *does not* influence the volume fraction η_A of the respective excursion sets. This volume fraction is thus specified by the "one-point" statistics only, i.e. by the cumulative distribution function (c.d.f.) $F_f(A) = \Pr\{f < A\}$ of the random field $f(\mathbf{x})$ in a fixed point $\mathbf{x}$ (which can always be chosen at the origin due to the assumed statistical homogeneity). With this interpretation, the formula (3.2) becomes natural enough.

For the dispersion under study which obeys the constitutive law (3.1), the quantity of central interest is just the field $|\nabla\theta(\mathbf{x})|^2$ since its excursion sets above the level G_0^2 will represent, so to say, the "plastified" region of the volume fraction η_{G_0} of the composite. According to (4.1),

$$\eta_A = F_\tau^c(G_0) = 1 - F_\tau(G_0), \tag{4.2}$$

where τ is the random variable, introduced in Eq. (2.4), and $F_\tau(A)$ is its cumulative distribution function. These regions will cover both matrix and inhomogeneities with probabilities η_m and η_f, respectively. Hence the dispersion under study will become a three-phase material, comprising:

- phase '1' — matrix of conductivity κ_m and volume fraction $\eta_m(1 - \eta_{G_0})$;
- phase '2' — inhomogeneities of conductivity κ_f and volume fraction $\eta_f(1 - \eta_{G_0})$;
- phase '3' — nonconducting excursion sets ("plastified" regions) of volume fraction η_{G_0}.

Since the dispersion is dilute, $\eta_f \ll 1$, we can imagine that the foregoing three-phase material can be adequately homogenized in the following simple and obvious way. First, the matrix with the inhomogeneities is replaced by a homogeneous medium of effective conductivity

$$\kappa^* = \kappa_m(1 + 3\beta\eta_f) + o(\eta_f), \tag{4.3}$$

without paying attention to the excursion regions. In the next step we introduce the latter (whose conductivity is zero) into the already homogenized medium of conductivity κ^* and get a material of conductivity

$$\widetilde{\kappa}^* = \kappa^* \frac{1 - \eta_{G_0}}{1 + \eta_{G_0}/2} = \kappa_m(1 + 3\beta\eta_f)(1 - \frac{3}{2}\eta_{G_0}), \tag{4.4}$$

so that, eventually,

$$\frac{\widetilde{\kappa}^*}{\kappa_m} = 1 + 3\beta\eta_f - \frac{3}{2}\eta_{G_0} + o(\eta_f). \tag{4.5}$$

In the last formulae (4.3) to (4.5) we have applied the well-known Maxwell (or Clausius-Mossotti) result for predicting the effective conductivity of a dispersion, which is exact in the dilute limit. Also, we have tacitly assumed that the applied gradient G is considerably smaller than the limit one G_0, so that the volume fraction η_{G_0} of the "plastified" regions is small as well; the parameter β in the foregoing relations is defined in Eq. (2.2). The latter assumption is in full agreement with the one that assures the applicability of Vanmarcke's formula (4.1), so that

$$\eta_{G_0} = F_\tau(G_0). \tag{4.6}$$

Hence, as it follows from Eqs. (4.5) and (4.6), to predict the effective behaviour of the dispersion under study, with fluctuations of the appropriate random fields taken into account, it is necessary that the c.d.f. $F_\tau(A)$ of the random variable τ, defined by Eq. (2.4), be evaluated.

5. EVALUATION OF THE DISTRIBUTION FUNCTION $F_\tau(A)$

Let

$$f_\tau(u) = \frac{\mathrm{d}F_\tau(u)}{\mathrm{d}u} \tag{5.1}$$

be the probability density function (p.d.f.) of the random variable $\tau = |\nabla\theta(0)|^2$. Then the moments of τ read

$$t_p = \langle \tau^p \rangle = \int_0^\infty u^p f_\tau(u)\,\mathrm{d}u. \tag{5.2}$$

The integration is over $(0, \infty)$ since, obviously, the random variable τ is nonnegative and hence both $f_\tau(u)$ and $F_\tau(u)$ vanish if $u < 0$.

To find the moments t_p, the representation (2.6) of τ is to be used together with the formulae

$$\langle \psi'(\mathbf{y}_1)\ldots\psi'(\mathbf{y}_k)\rangle = n\delta(\mathbf{y}_1-\mathbf{y}_2)\ldots\delta(\mathbf{y}_1-\mathbf{y}_2) + o(n), \tag{5.3}$$

$k = 2, 3, \ldots$, which generalize Eq. (2.5) in an obvious manner, see [9]. The final result reads

$$\begin{aligned}
&t_0^{(1)} = t_1^{(1)} = 0, \quad t_p = \tilde{G}^{2p} + \eta_f t_p^{(1)} G^{2p},\\
&\tilde{G}^{2p} = G^{2p}(1 + 6p\beta^2\eta_f),\\
&t_p^{(1)} = \sum_{k=0}^{p} \frac{1}{3^k} C_k^p \left[(\beta-2)^k + 3I_k\right]^k, \quad p \geq 2,\\
&I_k = \frac{1}{3}\int_0^1 \mathrm{d}z \int_1^\infty \frac{\mathrm{d}\rho}{\rho^{6k-2}} \left[\beta + 2\rho^3 + 3(\beta - 2\rho^3)z^2\right]^k\\
&\quad = \frac{1}{3}\sum_{l=0}^{k} \frac{2^l\beta^{k-1}}{2k-l-1} C_l^k \int_0^1 (1+3z^2)^{k-l}(1-\beta z^2)^l\,\mathrm{d}z.
\end{aligned} \tag{5.4}$$

Knowledge of the moments t_p allows us to evaluate the Laplace transform of $f_\tau(u)$:

$$\overline{f}_\tau(s) = \mathcal{L}[f_\tau](s) = \int_0^\infty \mathrm{e}^{-su} f_\tau(u)\,\mathrm{d}u = \overline{f}_\tau^{(0)}(s) + \eta_f \overline{f}_\tau^{(1)}(s),$$

$$\overline{f}_\tau^{(0)}(s) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!}(\widetilde{G}^2 s)^p = \mathrm{e}^{-\widetilde{G}^2 s}, \tag{5.5}$$

$$\overline{f}_\tau^{(1)}(s) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!}(\widetilde{G}^2 s)^p t_p^{(1)}.$$

Hence

$$f_\tau(u) = \delta(u - \widetilde{G}^2) + \eta_f f_\tau^{(1)}(u), \tag{5.6}$$

where

$$f_\tau^{(1)}(u) = \mathcal{L}^{-1}[\overline{f}_\tau^{(1)}](u) \tag{5.7}$$

is the inverse Laplace transform of the function $\overline{f}_\tau^{(1)}(s)$, defined in the last line of Eq. (5.5). Let us recall that all the foregoing formulae hold in the dilute limit, i.e. they are correct to the order $O(\eta_f)$ only.

Note that the formula (5.6) is fully natural — if $\eta_f = 0$, then $\widetilde{G} = G$, see Eq. (2.7), the medium is homogeneous so that $\nabla\theta(\mathbf{x}) \equiv \mathbf{G}$ and thus $\tau = |\nabla\theta(\mathbf{x})|^2 \equiv G^2$.

6. DISCUSSION

The formulae (5.4) — (5.7) specify, at least in principle, the function $f_\tau(u)$ and hence its primitive $F_\tau(u)$, see Eq. (5.1), i.e. the needed cumulative distribution function of the random variable τ, defined in Eq. (2.4). It is easily seen that the function $F_\tau(u)$ depends on the nondimensional ratio u/G^2, i.e. $F_\tau = F_\tau(u/G^2)$, and as a c.d.f. it monotonically increases, tending to 1 when $u/G^2 \to \infty$. The formulae (4.5) and (4.6) now give

$$\frac{\widetilde{\kappa}^*}{\kappa_m} = 1 + 3\beta\eta_f - \underline{\frac{3}{2}\left(1 - F_\tau(G_0^2/G^2)\right)} + o(\eta_f) \tag{6.1}$$

and the underlined term is just the result of fluctuations of the temperature gradient. When $G \to 0$, i.e. at $G \ll G_0$, κ^* tends to its classical value $\kappa_m(1 + 3\beta\eta_f)$, predicted by the Maxwell formula in the dilute limit. Therefore, the values of the effective conductivity and, more general, of the effective properties for a composite, represent but tangents to the appropriate "stress-strain" curves at the onset of loading. When G increases, the underlined term in Eq. (6.1) increases as well thus leading to progressively bigger deviation from the classical linear behaviour.

The aforesaid means that no matter how small is the macrostrain "loading" ($\mathbf{G}$ in our simplified context), imposed upon a micro-inhomogeneous medium, there will always appear zones of "plastic" yielding or deteriorated ones, due to fluctuations of the appropriate random fields. Hence such a random medium should show deviation

from linear behaviour from the *very onset* of loading. This represents the central conclusion of our study, which we have tried to quantify in the proposed scheme.

It is curious to point out immediately that the nonlinearity of stress-strain curves, even for very small strain, is an experimentally observed feature of all solids, as specially emphasized by Bell [12] as a result of his extensive and detailed review of experimental data in the last 300 years. (See, e.g., his words at the end of Ch. 2.6, p. 30: "One might dismiss nonlinearity of the response functions observed in the experiments of Hodgkinson and Dupin as merely an interesting historical development in the fields of solid mechanics, were it not for the fact that by the end of the 19th century the increasing accuracy of measurements and improved experiments demonstrated that that was indeed the precise manner in which such solids deformed.")

The proposed scheme possesses, however, certain drawbacks which should be explicitly pointed out and which make it only approximate, even in the simplest dilute case under study. The point is the following: When using ensemble averaging, one should consider a multitude of spatial realizations of the array of spheres in the dispersion. When the spheres in a given realization are "well-separated," then each one can be indeed treated as single, immersed into unbounded matrix material. The behaviour of both spheres and matrix is then linear under the condition (3.2). There will be however specific realizations of a "clustering" type, so to say, when some of the spheres are close one to another; in this case the "stress-concentration" factor may become much higher than 2. For these realizations the "plastic" or deteriorated zones will be considerable and the behaviour will not be linear already. Moreover, the number of such "clustering" realizations is not negligible, even at small G/G_0, since they influence the overall response when averaging over the set of all realizations. This means that the basic equation (1.1) should be considered as a nonlinear and random one, with $\kappa(\mathbf{x})$ replaced by $\widetilde{\kappa}(\mathbf{x})$, see Eq. (3.1), whatever the value of G. The representation (2.1), which has served as a basis of our analysis, can be viewed then as a certain approximation which allows solely to highlight the role of the fluctuations on the overall behaviour of the composite. Despite this, its adoption seems unavoidable in the proposed scheme, because it is not clear, at least to the authors, how a problem of the type (1.1) with a discontinuous coefficient $\widetilde{\kappa}(\mathbf{x})$ can be efficiently treated in the random case.

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