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ON THE SUBPLANES OF THE HUGHES PLANES OF ODD SQUARE PRIME ORDER

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In this paper we give a construction of a family of Baer subplanes of the Hughes plane $\mathbf{H}$ of odd square prime order q^2, $q \geq 5$, which are not isomorphic to its well-known desarguesian Baer subplane $\mathbf{H}_0$ [1, 5.4].

Keywords: finite geometries, Hughes planes, Baer subplanes

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1. DEFINITION AND MAIN PROPERTIES OF THE HUGHES PLANES

We recall the well-known properties of the Hughes plane H over an arbitrary regular nearfield R of odd square prime power order q^2 [1]:

a. H is a projective plane of odd square prime power order q^2, and it is of Lenz–Barlotti type I.1 [1, 5.4].

b. Let $P := \{\mathrm{P} = (x_1, x_2, x_3)R,\ x_i \in Z(R),\ i = 1,2,3\}$; $T := \{$all the rest points$\}$; $L(1) := \{L_1^{\gamma^i},\ i = 0,1,2,\dots,r-1\}$; $L(t) := \{L_t^{\gamma^i},\ t \in R \setminus Z(R),\ i = 0,1,2,\dots,r-1\}$.

Then the points in the set P together with the lines joining them (that are the lines of the set $L(1)$) form a desarguesian Baer subplane H_0 of H.

c. The projective group of H_0 is faightfully induced by the collineation group $\Gamma = \mathrm{GL}_3(q)$ ($\mathrm{GL}_3(q)$ is the group of nonsingular (3,3)-matrices with elements in

$Z(R) = \mathrm{GF}(q)$). Every central collineation of H_0 extends to a central collineation of H in Γ.

d. The full collineation group G of H has two points — P and T, two lines — $L(1)$ and $L(t)$, and two flag orbits. Also, G is a semidirect product $\Gamma.\mathrm{Aut}\,R$; this product is direct iff q is prime [1, 5.4].

Further we consider the case when q is an odd prime ≥ 5 and we use the notation $\mathrm{F} = \mathrm{GF}(q)$. Let $f = z^2 - \alpha$ be an irreducible polynomial over F. Then one can describe the quadratic extension $\Phi = \mathrm{GF}(q^2)$ of F as follows: $\Phi = \{az+b,\ a, b \in \mathrm{F},\ z$ — a root of $f\}$.

Let θ be a primitive element of Φ. Then $\Phi^* = \Phi \setminus \{0\} = \Phi_S \cup \Phi_N$, where Φ_S (Φ_N) is the set of squares (nonsquares) in Φ, i.e. $\Phi_S = \{u : u \in \Phi^*,\ u = \theta^{2k},\ k \in \mathbb{N}\}$ and $\Phi_N = \{u : u \in \Phi^*,\ u = \theta^{2l+1},\ l \in \mathbb{N}\}$.

Let K be the regular nearfield of order q^2 with the same elements as Φ, in which the addition is the same as in Φ, while the multiplication, denoted by $\circ$, is defined as follows: $w \circ u = uw$ if $u \in \Phi_S$ and $w \circ u = uw^q$ if $u \in \Phi_N$ [2].

Since $\alpha^q = \alpha$ for each $\alpha \in F$, we have $\alpha \circ u = \alpha u$ when $\alpha \in F$ and $u \in \mathrm{K}$. It is known that the centre $\mathrm{Z(K)}$ is just the field F.

Let $\mathrm{H} = \mathrm{H(K)}$ be the Hughes plane over K. The points of H are all ordered triples $\mathrm{P} = (x_1, x_2, x_3) = (x_1, x_2, x_3) \circ k = (x_1 \circ k, x_2 \circ k, x_3 \circ k)$, $k \in \mathrm{K}^* = \mathrm{K} \setminus (0)$, $x_i \in \mathrm{K}$, $i = 1, 2, 3$, and $(x_1, x_2, x_3) \neq (0, 0, 0)$.

The theorem of Singer [3] gives us the existence of a transformation

$$(x_1, x_2, x_3) \mapsto \Big(\sum_{j=1}^{3} a_{1j}x_j, \sum_{j=1}^{3} a_{2j}x_j, \sum_{j=1}^{3} a_{3j}x_j\Big), \quad a_{ij} \in F,$$

such that the mapping $(x_1, x_2, x_3) = \mathrm{P} \mapsto \mathrm{AP} = (a_{11}x_1, \ldots, a_{33}x_3)$ is a collineation γ of order $r = q^2 + q + 1$ of the desarguesian plane $\pi(q)$ of order q over the field F.

The basic lines L of **H** are defined by the equations

$$\mathrm{L}_t : x_1 + t \circ x_2 + x_3 = 0, \quad t \in \{\Phi \setminus F\} \cup \{1\}. \tag{1}$$

The point $\mathrm{P} = (x_1, x_2, x_3) \circ k$ is incident with the line L_t iff the triple (x_1, x_2, x_3) is a solution of (1). The remaining lines of **H** are $\mathrm{L}_t^{\gamma^i}$, $i = 0, 1, 2, \ldots, r-1$, and $\mathrm{A}^i\mathrm{P}$ is incident with $\mathrm{L}_t^{\gamma^i}$, $i = 0, 1, 2, \ldots, r-1$, iff P is incident with L_t [2].

2. AUTOMORPHISMS OF THE NEARFIELD $\mathrm{K} = \mathrm{K}(q^2)$

It is quite evident that the automorphism group $\mathrm{Aut}\,\mathrm{K}$ is isomorphic to $\mathbb{Z}_2$. Indeed, on the one hand, $|\mathrm{Aut}\,\mathrm{K}| \leq 2$ [1, 5.2.2]. On the other hand, it is easy to check that the mapping $\sigma : \mathrm{K} \to \mathrm{K}$, defined by the correspondance $az + b \mapsto (az+b)^\sigma = -az + b$, is a nontrivial automorphism of K, usually called conjugation in K.

3. THE MAIN RESULT

Let $\chi = \chi(\mathrm{S}, \mathrm{L})$ be an arbitrary homology of order two of $\mathbf{H}$ with centre S and axis L (since $o(\chi) = 2$ and q is odd, S is not incident with L). Actually, $\mathrm{S} \in P$ and $\mathrm{L} \in L(1)$.

From **1.c**, **d** and **2** it follows that there exists a collineation $\psi = \sigma\chi$, where $\sigma \in \mathrm{Aut}\,\mathrm{K}$, $\sigma \neq \mathrm{id}$. That gives us a reason to investigate the geometry of the ψ-invariant points and lines of $\mathbf{H}$. The main result in this paper is the following statement:

Theorem. *The ψ-invariant points and lines of* $\mathbf{H}$ *form a Baer subplane* $\pi = \pi(\mathrm{S}, \mathrm{L})$ *of order q of* $\mathbf{H}$*, non-isomorphic to the subplane* $\mathbf{H}_0$.

Since the group Aut K is flag-transitive, it is sufficient to prove the theorem in the case when the homology χ has as an axis the line $\mathrm{L}_1 : x_1 + x_2 + x_3 = 0$ and as a centre an arbitrary point $\mathrm{S} \in P$ which is not incident with L_1.

Let χ^* be an arbitrary homology of $\mathbf{H}$ with centre $\mathrm{S} = (a, b, c)$ $(a, b, c \in \mathrm{F})$ and axis L_1. Then the action of χ^* over the points of $\mathbf{H}$ can be presented as follows:

$$\chi^* : \bar{x}' \circ k^* = \mathbf{A}^* \bar{x}, \quad k^* \in K \setminus \{0\},$$

where the column vector $\bar{x} = (x_1, x_2, x_3)^t$ is an arbitrary point of $\mathbf{H}$, $\bar{x}' = (x'_1, x'_2, x'_3)^t$ is its image under χ^*, and the matrix $\mathbf{A}^* \in \mathrm{GL}_3(q)$ has the form

$$\mathbf{A}^* = \begin{pmatrix} a+\rho & a & a \\ b & b+\rho & b \\ c & c & c+\rho \end{pmatrix},$$

where $\rho \in \mathrm{F}$. Let us point out that $a + b + c \neq 0$ since S is not incident with L_1.

We denote by 2 the element $1 + 1$, where 1 is the unit element of the multiplication in F (and, respectively, in K). Since the characteristic of F is odd, $2 \neq 0$. Then the homology $\chi = (\mathrm{S}, \mathrm{L}_1)$ of order two with centre $\mathrm{S} = (a, b, c)$ and axis the line L_1 is defined by the matrix

$$\mathbf{A} = \begin{pmatrix} 2^{q-2}(a-b-c) & a & a \\ b & 2^{q-2}(b-a-c) & b \\ c & c & 2^{q-2}(c-a-b) \end{pmatrix}.$$

If the collineation ψ fixes some quadrangle (four points, no three of which are incident with one and the same line) pointwisely, then ψ maps to itself some proper subplane of $\mathbf{H}$. We will show that there exist four points, no three of which are incident with one and the same line, and which are invariant with respect to ψ.

The ψ-invariant points and lines are exactly these, which the homology χ maps onto their conjugated ones, respectively. With the line L_1 are incident precisely $q+1$ different points of the orbit P. Obviously, these points together with the centre of ψ — the point $\mathrm{S} = (a, b, c)$ — are invariant with respect to the collineation ψ. If the ψ-invariant points and lines of $\mathbf{H}$ form a Baer subplane, then every point of the supposed subplane will be incident with exactly $q + 1$ different ψ-invariant lines. Since all basic lines $\mathrm{L}_t(1)$ are incident with the point $\mathrm{P}_1 = (-1, 0, 1)$, one can expect that there exist $q + 1$ ψ-invariant ones among them. That is why we have to find the number of lines $L_t(1)$, which are mapped under the homology of order two $\chi = \chi(S, L_1)$ in their conjugated lines $L_t\sigma = (L_t)^\sigma$, respectively.

Let $L_t \neq L_1$, $L_t : x_1 + t \circ x_2 + x_3 = 0$, $t \in K \setminus F$. Since the line L_t is different from the line $y = 0$, an arbitrary point T, not incident with L_t, $T \neq P_1$, has coordinates $(x_1, 1, -(x_1 + t))$, $x_1 \in K$. Then $\chi(T) = T'$, where $T' = (2^{q-2}(a - b - c)x_1 + a - a(x_1+t), 2^{q-2}(b-a-c)-bt, cx_1+c-2^{q-2}(c-a-b)(x_1+t))$. Therefore $\chi(L_t) = L'_t$, $L'_t = P_1T'$ and the element $t' \in K \setminus F$ is uniquely determined by the equation

$$t' \circ (m + bt) = 2m + b - mt, \tag{2}$$

where $m = 2^{q-2}(a - b + c)$. Hence $m \in F$ and m depends on the coordinates of the centre $S = (a, b, c)$ only.

Now we will find the number of distinct solutions of the equation (2) when $t' = t^\sigma$. It is conveniant to consider two cases with respect to b: $b = 0$ and $b \neq 0$.

Case 1. Let $b = 0$, i.e. $S = (a, 0, c)$. In this case the equation (2) is of the form

$$mt' = m(2 - t). \tag{3}$$

Here $m = 2^{q-2}(a+c)$ and since S is not incident with L_1, $a+c \neq 0$, i.e. $m \neq 0$. Therefore $m^{-1} \in F^*$ and (3) yields that

$$t' = 2 - t. \tag{3'}$$

In general, each $t \in K \setminus F$ is of the form $t = dz + e$, $d \in F^*$, $e \in F$. As it was mentioned above, we are looking for the solutions of (3′) when $t' = t^\sigma$, i.e. when $t' = -dz + e$. Then from (3′) we obtain that

$$-dz + e = -dz + 2 - e. \tag{3''}$$

Hence d is an arbitrary element of F^* and $e = 1$. These solutions give us exactly $q - 1$ distinct L_t-lines ($t = dz + 1$, $d \in F^*$), which are mapped under the homology χ in their conjugated ones, respectively.

Case 2. Let $b \neq 0$, i.e. $S = (a, 1, c)$. Now from (2) we have the following equation when $t^\sigma = t'$, namely

$$t^\sigma \circ (m + t) = 2m + 1 - mt. \tag{4}$$

Here $m = 2^{q-2}(a + c - 1)$ and since $a + 1 + c \neq 0$, $m \neq -1$. As t has the representation $t = dz + e$, so that $t^\sigma = -dz + e$, $d \in F^*$, $e \in F$, the relation (4) immediately gives

$$(-dz + e) \circ (dz + m + e) = -mdz + 2m - me + 1. \tag{4'}$$

For the result of multiplication in the left-hand side of (4′) we have two possibilities, namely,

$$(-dz + e) \circ (dz + m + e) = \begin{cases} (-dz + e)(dz + m + e) & \text{if } dz + m + e \in \Phi_S, \\ (dz + e)(dz + m + e) & \text{if } dz + m + e \in \Phi_N. \end{cases}$$

If we assume that $dz + m + e \in \Phi_N$, then for d and e we have from (4′) that $(m + 2e)dz + d^2z^2 + e(m + e) = -mdz + 2m + 1 - me$. Since $z^2 = \alpha$, we obtain

$$(m + 2e)dz + \alpha d^2 + e(m + e) = -mdz + 2m + 1 - me. \tag{4''}$$

Hence $e = -m$, i.e. $e \neq 1$, and $d^2\alpha = (e-1)^2$, i.e. $d \neq 0$. Therefore $\alpha = [d^{-1}(e-1)]^2$, i.e. α is a square in the field F, but this contradicts the choice of α. Therefore, if there exists a solution of the equation (4′), then $dz + m + e \in \Phi_S$.

Suppose that $dz + m + e \in \Phi_S$. Now (4′) is reduced to

$$\alpha d^2 = e^2 + 2me - 2m - 1. \tag{4'''}$$

We transform the right-hand side of (4‴) as follows: $e^2 + 2me - 2m - 1 = e^2 + 2me + m^2 - m^2 - 2m - 1 = (e+m)^2 - (m+1)^2$, and then (4‴) becomes

$$(-\alpha)d^2 + (e+m)^2 = (m+1)^2, \tag{5}$$

where $m \neq -1$ is a fixed element of F and $\alpha = z^2$.

Let η be a quadratic character of GF(q) (q — odd), i.e. $\eta(c) = 1$ if c is a square in GF(q) and $\eta(c) = -1$ if c is a nonsquare in GF(q). Define the function v on GF(q) by $v(b) = -1$ if $b \in \mathrm{GF}^*(q)$ and $v(0) = q-1$. Let $N(\alpha_1 y_1^2 + \alpha_2 y_2^2 = b)$ ($b \in \mathrm{GF}(q)$, $\alpha_1, \alpha_2 \in \mathrm{GF}^*(q)$) be the number of the solutions of the equation $\alpha_1 y_1^2 + \alpha_2 y_2^2 = b$ in the field GF(q). Then [4, 6.24]

$$N(\alpha_1 y_1^2 + \alpha_2 y_2^2 = b) = q + v(b)\eta(-\alpha_1\alpha_2).$$

In the case of the equation (5) in the variables $y_1 = d$ and $y_2 = e + m$ we have $\alpha_1 = -\alpha$, $\alpha_2 = 1$ and $b = (m+1)^2$. Since $m \neq -1$ and α is a nonsquare in GF(q), $v((m+1)^2) = -1$ and $\eta(-\alpha_1\alpha_2) = \eta(\alpha) = -1$. Therefore $N((-\alpha)d^2 + (e+m)^2 = (m+1)^2) = q+1$.

The solution $(0, 1+m)$ of the equation (5) gives the line L_1 and to the solution $(0, -2m-1) = (0, -(a+c))$ corresponds the line $\mathrm{SP}_1 : x_1 - (a+c)x_2 + x_3 = 0$, $\mathrm{SP}_1 \in L(1)$.

For each of the remaining $q-1$ solutions $(d, e+m)$ of the equation (5) we will prove that $dz + (m+e) \in \Phi_S$. Suppose that $(d_1, e_1 + m)$ is a solution of (5) such that $u = d_1 z + (m + e_1) \in \Phi_N$, i.e. $u = \theta^{2l+1}$. Then $u \circ u = uu^q = (-\alpha)d_1^2 + (e_1+m)^2 = (m+1)^2$, i.e. $\theta^{(q+1)(2l+1)} = (m+1)^2$, and therefore $\theta^{[(q^2-1)/2](2l+1)} = (m+1)^{q-1} = 1$. But, on the other hand, $\theta^{[(q^2-1)/2](2l+1)} = (-1)^{2l+1} = -1$. It turns out that $1 = -1$, which contradicts the oddness of the characteristic of the field F = GF(q). Therefore every solution $(d, e+m)$ of the equation (5) with $d \neq 0$ gives an element $w = dz + (m+e) \in \Phi_S$. That means there exist exactly $q-1$ distinct basic lines L_t ($\neq \mathrm{L}_1$) invariant with respect to the collineation ψ.

Now it is easy to find ψ-invariant points, no three of which are incident with one and the same line. Let $\mathrm{P}_i \in P$, $i = 1,2,3$, be three different points which are incident with the axis L_1 of the homology of order two $\chi = \chi(\mathrm{S}, \mathrm{L}_1)$ and differ from the point $(-1, 0, 1)$. Then the points S, P_1, P_2, $\mathrm{T} = \mathrm{SP}_3 \cap \mathrm{L}_t$ have the desired property, L_t is an arbitrary invariant with respect to the ψ basic line different from L_1 and $\mathrm{S} = (a, b, c)$ is the centre of χ.

Hence for any point $\mathrm{S} \in P$ and any line $\mathrm{L} \in L(1)$, S non-incident with L, the points and lines of **H** invariant with respect to the collineation $\psi = \sigma\chi$ form a Baer subplane $\pi = \pi(\chi)$ of **H** of order q (χ is the homology of order two with centre S and axis L, $\sigma \in \mathrm{Aut\,K}$, $\sigma \neq \mathrm{id}$). It is clear that this subplane $\pi = \pi(\chi)$ is not isomorphic to the well-known Baer subplane $\mathbf{H}_0$ of **H** with respect to the group

Aut **H**. Otherwise there exists an element $\phi \in \mathrm{Aut}\,\mathbf{H}$ such that $\phi(\mathrm{P}_0) = \mathrm{T}_0$, where P_0 and T_0 are arbitrary points from the orbits P and T, respectively. Hence Aut **H** acts transitively over the set of all points of **H**, which is inadmissible in any Hughes plane.

It is naturaly to ask whether the subplanes of the kind $\pi = \pi(\chi)$ are isomorphic to each other with respect to the automorphism group of the plane **H**. We claim that the answer of this question is positive.

Let S_1 and S_2 be arbitrary points of P and L^1, L^2 be lines of $L(1)$, S_i be non-incident with L^i $(i = 1, 2)$. Denote by χ_i the homology of order two with centre S_i and axis L^i $(i = 1, 2)$. Let $\pi_i = \pi_i(\chi_i)$ be the subplane generated by the collineation $\psi_i = \sigma\chi_i$ $(i = 1, 2)$. If there exists an element $\phi \in \mathrm{Aut}\,\mathbf{H}$ such that $\phi(\mathrm{S}_1) = \mathrm{S}_2$ and $\phi(\mathrm{L}^1) = L^2$, then $\phi(\pi_1) = \pi_2$.

Due to the fact that the group Aut **H** is flag-transitive, it is sufficient to consider only the case when the axis of χ_1 and χ_2 is the line $\mathrm{L}_1 : x_1 + x_2 + x_3 = 0$, the centre S_1 of χ_1 has coordinates $(1, 0, 0)$, and the centre S_2 of χ_2 is an arbitrary point from the orbit P non-incident with L_1. Then S_2 has coordinates (a, b, c), $a, b, c \in \mathrm{F}$ and $a + b + c \neq 0$.

In order to prove that the subplanes $\pi_1 = \pi(\chi_1)$ and $\pi_2 = \pi(\chi_2)$ are isomorphic, it is sufficient to show that there exists an automorphism $\phi \in \mathrm{Aut}\,\mathbf{H}$ which maps certain quadrangle of π_1 into a quadrangle of π_2.

In the case when $\mathrm{S}_2 = (a, 0, b)$, the points S_1, S_2 and $\mathrm{P}_1 = (-1, 0, 1)$ are incident with the line $x_2 = 0$. Then the isomorphism between the subplanes π_1 and π_2 is realized by the elation $\varepsilon_1 = \varepsilon_1(\mathrm{P}_1, \mathrm{L}_1)$ with centre $\mathrm{P}_1 = (-1, 0, 1)$, axis L_1 and $\varepsilon(\mathrm{S}_1) = \mathrm{S}_2$. It is obvious that in this case both subplanes π_1 and π_2 contain the lines of the form $\mathrm{L}_t : x_1 + t \circ x_2 + x_3 = 0$, where $t = dz + 1$, $d \in \mathrm{F}^*$.

Let the points $\mathrm{P}_2 \neq \mathrm{P}_3$ be in the orbit P, P_2 and P_3 be incident with L_1, and suppose that $\mathrm{P}_2 \neq \mathrm{P}_1$ and $\mathrm{P}_3 \neq \mathrm{P}_1$. The line $\mathrm{S}_1\mathrm{P}_2$ intersects an arbitrary line $\mathrm{L}_{t_1} \in \pi_1$, $t_1 = d_1 z + 1$, $d_1 \in \mathrm{F}^*$ at a point $\mathrm{T}_1 \in \pi_1$, and the line $\mathrm{S}_2\mathrm{P}_2$ intersects the same line $\mathrm{L}_{t_1} \in \pi_2$ at a point $\mathrm{T}_2 \in \pi_2$. Then the points S_1, P_2, P_3, T_1 form a quadrangle in π_1 and the points S_2, P_2, P_3, T_2 — quadrangle in π_2. Since the point P_1 is incident with all the lines L_t, $\varepsilon_1(\mathrm{L}_t) = \mathrm{L}_t$, hence $\varepsilon_1(\mathrm{S}_1, \mathrm{P}_2, \mathrm{P}_3, \mathrm{T}_1) = (\mathrm{S}_2, \mathrm{P}_2, \mathrm{P}_3, \mathrm{T}_2)$, which gives us that $\varepsilon_(\pi_1) = \pi_2$.

Let $\mathrm{S}_2 = (a, 1, c)$. Since S_2 is non-incident with L_1, $a + c + 1 \neq 0$ and the line $\mathrm{S}_1\mathrm{S}_2$ intersects the line L_1 at the point $\mathrm{P}_12 = (-1 - c, 1, c)$. Similarly, the isomorphism between the subplanes π_1 and π_2 is realized by the elation $\varepsilon_2 = \varepsilon_2(\mathrm{P}_12, \mathrm{L}_1)$ with centre P_12, axis L_1 and $\varepsilon_2(\mathrm{S}_1) = \mathrm{S}_2$. This elation is given by the matrix

$$\mathbf{B} = \begin{pmatrix} a & -1-c & -1-c \\ 1 & a+c+2 & 1 \\ c & c & a+2c+1 \end{pmatrix}.$$

Actually, we have that each point $\mathrm{G}_1 \in \pi_1$ which is incident with the line $\mathrm{S}_1\mathrm{P}_1$ is mapped under the elation ε_2 into a point $\mathrm{G}_2 \in \pi_2$ which is incident with the line $\mathrm{S}_2\mathrm{P}_1$.

The homology of order two $\chi_1 = \chi_1(S_1, L_1)$ with centre $S_1 = (1,0,0)$ is given by the matrix

$$\mathbf{C} = \begin{pmatrix} 2^{q-2} & 1 & 1 \\ 0 & -2^{q-2} & 0 \\ 0 & 0 & -2^{q-2} \end{pmatrix}.$$

The line $S_1P_1 \in L(1)$ has an equation $x_2 = 0$ and therefore it differs from the line $x_3 = 0$. Then each point $G \neq S_1$ which is incident with the line S_1P_2 has coordinates $(dz + e, 0, 1)$, $d, e \in F$. The point G belongs to the subplane π_1 iff $\chi_1(G) = \sigma(G)$. Hence the points G_1 of π_1, incident with the line S_1P_1, have coordinates $(dz - 1, 0, 1)$, $d \in F$ (when $d = 0$, $G_1 = P_1$).

The homology of order two $\chi_2 = \chi_2(S_2, L_1)$ with centre $S_2 = (a, 1, c)$ and axis L_1 is given by the matrix

$$\mathbf{D} = \begin{pmatrix} 2^{q-2}(a-c-1) & a & a \\ 1 & 2^{q-2}(-1-a-c) & 1 \\ c & c & 2^{q-2}(c-a-1) \end{pmatrix}.$$

The equation of the line $S_2P_2 \in L(1)$ is $x_1 - (a+c)x_2 - x_3 = 0$ and it is easy to see that the points $G_2 \in \pi_2$, which are incident with S_2P_2, have coordinates $(dz + a, 1, -dz + c)$, $d \in F$.

We have $\varepsilon(G_1) = G'$, where the coordinates of the point G' are $(adz - (a+c+1), dz, cdz + (a+c+1))$, $a+c+1 \neq 0$. If $d \neq 0$, then the point G' has coordinates $(adz - (a+c+1), dz, cdz + (a+c+1)) \circ (dz)^{-1}$, that is $G' = (-(a+c+1)(dz)^{-1} + a, 1, (a+c+1)(dz)^{-1} + c)$. Since $(dz)^{-1} = \bar{d}z$, $G' = (d^* + a, 1, -d^* + c)$, where $d^* = -(a+c+1)\bar{d}$, $d^* \in F^*$.

If $d = 0$, then $G_1 = P_1 = G' = (-(a+c+1), 0, (a+c+1))$. Hence G' is incident with S_2P_1 and $G' \in \pi_2$.

Let P be an arbitrary point in the orbit P, let P be incident with the line L_1 and $P_1 \neq P \neq P_{12}$. That is why the point P belongs to the subplane π_1 as well as to the subplane π_2. Then the elation $\varepsilon_2 = \varepsilon_2(P_12, L_1)$ maps the quadrangle (P_12, P, S_1, G_1) of π_1 onto the quadrangle $(P_12, P, S_2, G_2 = G')$ of π_2 ($S_1 \neq G_1 \neq P_12$, $S_2 \neq G_2 \neq P_12$).

In this way we have proved that the subplanes π_1 and π_2 are isomorphic with respect to the automorphism group of **H**.

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