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DEFINABILITY VIA PARTIAL ENUMERATIONS WITH SEMICOMPUTABLE CODOMAINS*

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Let $\mathfrak{A}$ be a total abstract structure. We prove that if a set $A \subseteq |\mathfrak{A}|^n$ is admissible in every partial enumeration of $\mathfrak{A}$ with semicomputable codomain, then A is semicomputable in $\mathfrak{A}$ in the sense of Friedman – Shepherdson.

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1. INTRODUCTION

There are two major ways to introduce a notion of computable function on an arbitrary abstract structure $\mathfrak{A}$. Using the first one, which we may call explicit, computable functions are defined by means of relativized programs, generalized algorithms, formulas, etc. The second approach, known as implicit, reduces the problem to computability on natural numbers making use of various types of enumerations of the structure.

It turns out that most of the well-known explicit notions of abstract computability can be characterized via enumerations. As a rule, when considering computability without "search" over the domain, we need a suitable notion of partial enumeration. A typical result of this type is the next theorem from [3], which

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characterizes the semicomputable sets, i.e. the sets, whose semicharacteristic functions are computable in the sense of Friedman – Shepherdson ([1, 2]), using finitely many constants from $|\mathfrak{A}|$.

1.1. Theorem. *A set $A \subseteq |\mathfrak{A}|^n$ is semicomputable in $\mathfrak{A}$ if and only if A is admissible in every partial enumeration of $\mathfrak{A}$.*

Here an interesting question is whether there exists a subclass $\mathfrak{K}$ of the class of all partial enumerations such that admissibility in every enumeration in $\mathfrak{K}$ guarantees semicomputability, and further, whether there exists minimal such class. This question is answered partially in [4], where it is proved that admissibility in all partial enumerations with Σ_2^0 domains yields semicomputability. Here we show that the same is true if we confine ourselves to the class of all enumerations with Π_1^0 domains and this is the minimal class with this property.

2. PRELIMINARIES

Let an abstract structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k, \Sigma_1, \ldots, \Sigma_m)$ be given, where the set B is finite or denumerable, θ_i is a total function of a_i arguments in B and Σ_j is a total predicate of b_j arguments in B. The equality relation is not supposed to be among the initial predicates of $\mathfrak{A}$. We shall write $\Sigma_j(\bar{s}) = 0$ (1) when $\Sigma_j(\bar{s})$ is *true* (resp. *false*).

2.1. Definition. *Partial enumeration* of $\mathfrak{A}$ is an ordered pair $(f, \mathfrak{B})$, where f is a partial function from N (the set of all natural numbers) onto B, $\mathfrak{B} = (N; \varphi_1, \ldots, \varphi_k, Q_1, \ldots, Q_m)$ is a total structure in the signature of $\mathfrak{A}$, and the following conditions hold for $1 \leq i \leq k$ and $1 \leq j \leq m$:

(1) if $x_1, \ldots, x_{a_i}$ are in $Dom(f)$, then $\varphi_i(x_1, \ldots, x_{a_i}) \in Dom(f)$;

(2) $f(\varphi_i(x_1, \ldots, x_{a_i})) = \theta_i(f(x_1), \ldots, f(x_{a_i}))$ for $x_1, \ldots, x_{a_i}$ in $Dom(f)$;

(3) $Q_j(x_1, \ldots, x_{b_j}) \iff \Sigma_j(f(x_1), \ldots, f(x_{b_j}))$ for $x_1, \ldots, x_{b_j}$ in $Dom(f)$.

In other words, the pair $(f, \mathfrak{B})$ is a partial enumeration of $\mathfrak{A}$ if the mapping $f \restriction Dom(f)$ is a strong homomorphism from $\mathfrak{B} \restriction Dom(f)$ onto $\mathfrak{A}$.

The set $Dom(f)$ is called *domain* of the enumeration $(f, \mathfrak{B})$.

A set $W \subseteq N^n$ is *semicomputable* in $\mathfrak{B}$ iff the semicharacteristic function of W is Turing computable relative to $\varphi_1, \ldots, \varphi_k, Q_1, \ldots, Q_m$.

2.2. Definition. A set $A \subseteq B^n$ is *admissible* in the enumeration $(f, \mathfrak{B})$ iff there exists a semicomputable in $\mathfrak{B}$ set $W \subseteq N^n$ such that for all $x_1, \ldots, x_n$ in $Dom(f)$

$$(x_1, \ldots, x_n) \in W \iff (f(x_1), \ldots, f(x_n)) \in A.$$

The set W is called an *associate* of A (in the enumeration $(f, \mathfrak{B})$).

Next we introduce the notion of semicomputable set in the sense of Friedman – Shepherdson [1, 2]. Say that the n-ary predicate Π in B is *elementary* iff it is a finite conjunction of atomic predicates or their negations. Suppose that some effective coding of all elementary predicates (of arbitrary number of arguments) is fixed and denote by Π^v the predicate with code v.

A set $A \subseteq B^n$ is *semicomputable* in $\mathfrak{A}$ iff there exist constants $t_1, \ldots, t_l$ in B, $l \geq 0$, and an unary recursive function γ such that for every v, $\gamma(v)$ is a code of an elementary predicate with variables among $X_1, \ldots, X_n, Y_1, \ldots, Y_l$ and the equivalence

$$(s_1, \ldots, s_n) \in A \iff \exists v \left(\Pi^{\gamma(v)}(X_1/s_1, \ldots, X_n/s_n, Y_1/t_1, \ldots, Y_l/t_l) = 0\right)$$

is satisfied for every $(s_1, \ldots, s_n) \in B^n$.

3. STANDARD ENUMERATIONS

In order to save space, from now on we shall suppose that the initial functions and predicates of the structure $\mathfrak{A} = (B; \theta_1, \ldots, \theta_k, \Sigma_1, \ldots, \Sigma_m)$ are unary. We shall consider also subsets of B (instead of B^n).

For our goal it is sufficient to confine ourselves to some special type of enumerations, called standard enumerations [3]. To introduce the precise version of this notion that we will need here, let us first fix some recursive coding $\langle\, , \rangle$ of ordered pairs of natural numbers, chosen in such a way that the decoding functions L and R satisfy the condition

$$L(x) < x \;\&\; R(x) < x$$

for all $x \in N$. (Take, for example, $\langle x, y\rangle = 2^x(2y+1)$.) We shall write sometimes $(x)_0$ and $(x)_1$ instead of $L(x)$ and $R(x)$, respectively.

Set

$$N_0 = N \setminus \{\langle i, x\rangle \mid 1 \leq i \leq k,\ x \in N\}.$$

Let f_0 be an arbitrary partial mapping from N_0 onto B. Using a course-by-value-recursion, define f as:

$$f(x) \simeq \begin{cases} f_0(x), & x \in N_0, \\ \theta_i(f(x_0)), & x = \langle i, x_0\rangle \text{ for some } 1 \leq i \leq k. \end{cases}$$

Now for $1 \leq i \leq k$ and $1 \leq j \leq m$ set

$$\varphi_i(x) = \langle i, x\rangle$$

and

$$Q_j(x) = \begin{cases} \Sigma_j(f(x)), & x \in Dom(f), \\ \text{arbitrary}, & \text{otherwise}. \end{cases}$$

It is an easy exercise to check that the pair $(f, \mathfrak{B} = (N; \varphi_1, \ldots, \varphi_k, Q_1, \ldots, Q_m))$ is an enumeration of $\mathfrak{A}$. Every enumeration, obtained in the way just described, we shall call a *standard* enumeration.

Let $W \subseteq N$ is semicomputable in $\mathfrak{B}$. Since $\mathfrak{B}$ is total, it is equivalent to the fact that $W = \Gamma_e(\langle\mathfrak{B}\rangle)$ for some enumeration operator with index e, more precisely,

$$W = \{x \mid \exists v(\langle v, x\rangle \in W_e \;\&\; D_v \subseteq \langle\mathfrak{B}\rangle)\},$$

where W_e is the e-th r. e. subset of N, $D_v \subseteq N$ is the finite set with canonical index v and $\langle\mathfrak{B}\rangle$ is "the code" of $\mathfrak{B}$, i.e. the set, which consists of the codes $\langle i, x, y\rangle$ of the triples (i, x, y), such that

$$(1 \leq i \leq k \ \&\ y = \langle i, x\rangle) \ \vee\ (k+1 \leq i \leq k+m \ \&\ Q_i(x) = y).$$

Let $root(x)$ be the recursive function such that

$$root(x) = \begin{cases} x, & \text{if } x \in N_0, \\ root(x_0), & \text{if } x = \langle i, x_0\rangle \text{ for some } 1 \leq i \leq k. \end{cases}$$

Clearly, $root(x) \in N_0$ for every $x \in N$. Define $F : N \times B \longrightarrow B$ as

$$F(x, s) = \begin{cases} s, & \text{if } x \in N_0, \\ \theta_i(F(x_0, s)), & \text{if } x = \langle i, x_0\rangle \text{ for some } 1 \leq i \leq k. \end{cases}$$

So, thinking of x as a code of an one-variable term in the signature of $\mathfrak{A}$, $F(x, s)$ is the value (in $\mathfrak{A}$) of this term, when its variable is evaluated to s.

Below we introduce appropriate notions of a finite part and a forcing relation. Let $z_0, z_1, \ldots$ be an enumeration of the elements of N_0 in an ascending order.

3.1. Definition. *Finite part* (of a standard enumeration) is an $(m+2)$-tuple

$$\tau = (f_\tau; H_\tau; q_1, \ldots, q_m),$$

where f_τ is a finite function from N into B, $H_\tau \subseteq N_0$ is a finite set, $Dom(f_\tau) \cap H_\tau = \varnothing$, $Dom(f_\tau) \cup H_\tau = \{z_0, \ldots, z_l\}$ for some $l \geq 0$, and $q_1, \ldots, q_m$ are unary finite predicates satisfying the additional condition

$$x \in Dom(q_i) \implies root(x) \in H_\tau.$$

The set $Dom(f_\tau) \cup H_\tau$ we shall call *domain* of τ (to be denoted by $Dom(\tau)$). If $Dom(\tau) = \{z_0, \ldots, z_l\}$, then l is the *length* of τ (in symbols $|\tau|$).

Whenever $\tau = (f_\tau; H_\tau; q_1, \ldots, q_m)$ is a finite part, $x \in N_0$ is the first not in $Dom(\tau)$ and $s \in B$, by $\tau * s$ we shall denote the tuple $(g; H_\tau; q_1, \ldots, q_m)$, where g is the function with graph $G_{f_\tau} \cup \{(x, s)\}$. Clearly, $\tau * s$ is a finite part.

Let $\tau = (f_\tau; H_\tau; q_1, \ldots, q_m)$ and $\delta = (f_\delta; H_\delta; r_1, \ldots, r_m)$ be arbitrary finite parts. We introduce three types of partial relations between finite parts:

$$\tau \subseteq \delta \iff f_\tau \subseteq f_\delta \ \&\ H_\tau \subseteq H_\delta \ \&\ q_1 \subseteq r_1 \ \&\ \ldots \ \&\ q_m \subseteq r_m;$$

$$\tau \leqslant \delta \iff \tau \subseteq \delta \ \&\ f_\tau = f_\delta;$$

$$\tau \preccurlyeq \delta \iff \tau \leqslant \delta \ \&\ H_\tau = H_\delta.$$

As usual, we will write $\delta \supseteq \tau$, $\delta \geqslant \tau$, $\ldots$ for $\tau \subseteq \delta$, $\tau \leqslant \delta$, etc.

The enumeration $(f, \mathfrak{B} = (N; \varphi_1, \ldots, \varphi_k, Q_1, \ldots, Q_m))$ *extends* τ ($\tau \subseteq (f, \mathfrak{B})$) iff $f_\tau \subseteq f$, $H_\tau \subseteq N \setminus Dom(f)$ and $q_i \subseteq Q_i$ for $i = 1, \ldots, m$.

Now set

$$\begin{aligned} \tau \Vdash u \iff & \exists x \exists y \exists i (u = \langle i, x, y\rangle \ \&\ 1 \leq i \leq k \ \&\ y = \langle i, x\rangle \ \vee \\ & u = \langle k+i, x, y\rangle \ \&\ 1 \leq i \leq m \ \&\ (q_i(x) = y \ \vee \\ & root(x) \in Dom(f_\tau) \ \&\ \Sigma_i(F(x, f_\tau(root(x)))) = y)); \end{aligned}$$

$$\tau \Vdash D_v \iff \forall u (u \in D_v \implies \tau \Vdash u);$$

$$\tau \Vdash R_e(x) \iff \exists v (\langle v, x\rangle \in W_e \ \&\ \tau \Vdash D_v).$$

The next simple observation will be of use in the sequel.

3.2. Lemma.

(1) $\tau \Vdash R_e(x)$ & $\tau \subseteq \rho \implies \rho \Vdash R_e(x)$ (*monotonisity*);

(2) $\tau \Vdash R_e(x)$ & $\tau \subseteq (f, \mathfrak{B}) \implies x \in \Gamma_e(\langle \mathfrak{B} \rangle)$;

(3) $x \in \Gamma_e(\langle \mathfrak{B} \rangle) \implies \exists \tau(\tau \subseteq (f, \mathfrak{B})$ & $\tau \Vdash R_e(x))$;

(4) $\tau \subseteq (f, \mathfrak{B})$ & $\forall \rho(\rho \geqslant \tau \implies \rho \nVdash R_e(x)) \implies x \notin \Gamma_e(\langle \mathfrak{B} \rangle)$.

Proof. The verification of (1) – (3) is straightforward.

(4) Towards contradiction assume that $x \in \Gamma_e(\langle \mathfrak{B} \rangle)$. Then by (3) there exists some $\delta = (f_\delta; H_\delta; r_1, \ldots, r_m)$ such that $\delta \subseteq (f, \mathfrak{B})$ and $\delta \Vdash R_e(x)$. Since $\tau = (f_\tau; H_\tau; q_1, \ldots, q_m) \subseteq (f, \mathfrak{B})$ as well, the sets $G = Dom(f_\tau) \cup Dom(f_\delta)$ and $H = H_\tau \cup H_\delta$ are disjoint and $G \cup H$ is an initial segment of N_0. It is clear also that the predicates $r'_i = q_i \cup r_i$, $1 \le i \le m$, are single-valued and the ordered tuple $\delta' = (f_\tau \cup f_\delta; H_\tau \cup H_\delta; r'_1, \ldots, r'_m)$ is a finite part. We have $\delta' \supseteq \delta$, hence $\delta' \Vdash R_e(x)$. It means that $\delta' \Vdash D_v$ for some v with $\langle v, x \rangle \in W_e$. Now consider the tuple $\rho = (f_\tau; H \cup (G \setminus Dom(\tau)); p_1, \ldots, p_m)$, where

$$p_i = r'_i \cup \{\langle x, y \rangle \mid x \in G \setminus Dom(\tau) \& \exists u(u \in D_v \ \& \ u = \langle k+i, x, y \rangle\ )\}.$$

Clearly, $\rho \Vdash D_v$ and $\rho \geqslant \tau$ — a contradiction. □

4. THE MAIN RESULT

In order to establish our result, we introduce a suitable notion of normal form of a subset of B.

4.1. Definition. A set $A \subseteq B$ has a *normal form*, if there exist a finite part δ and a natural number e such that if $x \in N_0$ is the first not in $Dom(\delta)$, then

$$s \in A \iff \exists \rho(\rho \geqslant \delta * s \ \& \ \rho \Vdash R_e(x))$$

for every $s \in B$.

Now the rest of the paper is devoted to the proof of the next theorem.

4.2. Theorem. *Let $A \subseteq B$. The following conditions are equivalent:*

(1) *A is semicomputable in $\mathfrak{A}$;*

(2) *A is admissible in every enumeration $(f, \mathfrak{B})$ such that $N \setminus Dom(f)$ is semicomputable in $\mathfrak{B}$;*

(3) *A has a normal form.*

Proof. The implication (1) $\implies$ (2) follows immediately from the definitions. To see that (3) $\implies$ (1) holds, take into account the next two observations:

(a) The set $R = \{(e, x) \mid \rho \Vdash R_e(x)\}$ is semicomputable in $\mathfrak{A}$.

(b) In order to find (if it exists) a finite part ρ such that $\rho \geqslant \tau$ (with τ fixed), it is sufficient to search over natural numbers (notice that for the more common inclusion "$\supseteq$" this is not true). More precisely:

$$\rho \geqslant \tau \iff \exists H \exists r_1, \ldots, r_m \big(H \supseteq H_\tau \ \& \ r_1 \supseteq q_1^\tau \ \& \ \ldots r_m \supseteq q_m^\tau \ \& \\ (f_\tau; H; r_1, \ldots, r_m) \text{ is a finite part}\big).$$

The interesting part of the theorem is the direction (2) $\Longrightarrow$ (3), which now we prepare to prove using some auxiliary lemmas.

Indeed, assume that (2) holds, but the set A does not have a normal form. We are going to construct a standard enumeration $(f, \mathfrak{B})$ in which A is not admissible and such that $N \setminus Dom(f)$ is semicomputable in $\mathfrak{B}$.

Clearly, $A \neq \varnothing$ — check that the empty set has a normal form. Therefore, if $\Gamma_e(\langle\mathfrak{B}\rangle)$ is an associate of A, then $W_e \neq \varnothing$. So it will be sufficient to consider only the indexes of non-empty r. e. subsets of N. As it is well-known, every set of this type can be enumerated by some unary primitive recursive function. We will need in fact some uniform procedure that enumerates the elements of nonempty r. e. sets. So consider $U(n, x)$ — the universal for all unary primitive recursive functions. By the S_n^m-theorem, there exists recursive function σ:

$$W_{\sigma(e)} = Range(\lambda x.U(e.x)).$$

Our aim is to construct successively a sequence of finite parts $\tau^{(0)} \subseteq \tau^{(1)} \subseteq \ldots$ such that for every enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$ and every n it is true that

$$\text{if } \tau^{(2n+1)} \subseteq (f, \mathfrak{B}), \text{ then } \Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle) \text{ is not an associate of } A. \qquad (*)$$

It is clear from here that if $(f, \mathfrak{B})$ is an enumeration, such that $\tau^{(n)} \subseteq (f, \mathfrak{B})$ for every n, then A is not admissible in $\mathfrak{A}$. Indeed, assuming the contrary, we will have an index e such that $\Gamma_e(\langle\mathfrak{B}\rangle)$ is an associate of A. Since $W_e \neq \varnothing$, there exists n with $W_{\sigma(n)} = W_e$. Therefore $\Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle)$ is an associate of A, which contradicts $(*)$.

Let us fix some enumeration $s_0, s_1, \ldots$ of the elements of B. Now we are going to define the sequence $\{\tau^{(n)}\}_n$ satisfying $(*)$. The definition is by induction on n.

Set $\tau^{(0)} = \left(f^{(0)}; H^{(0)}; q_1^{(0)}, \ldots, q_m^{(0)}\right)$, where $f^{(0)}$ is the function (with graph) $\{(z_0, s_0)\}$, $H^{(0)} = \varnothing$ and all $q_1^{(0)}, \ldots, q_m^{(0)}$ are unary predicates with empty domains. Assuming that $\tau^{(2n)} = \left(f^{(2n)}; H^{(2n)}; q_1^{(2n)}, \ldots, q_m^{(2n)}\right)$, $n \geq 0$, is already determined, we define $\tau^{(2n+1)}$ and $\tau^{(2n+2)}$ as follows.

Let $x \in N_0$ be the first which does not belong to $Dom\left(\tau^{(2n)}\right)$. By assumption A does not have a normal form, so there exists $s \in B$ such that exactly one of the next two conditions holds:

- $s \in A \ \& \ \forall \rho(\rho \geqslant \tau^{(2n)} * s \implies \rho \nVdash R_{\sigma(n)}(x))$;

•• $s \notin A \ \& \ \exists \rho(\rho \geqslant \tau^{(2n)} * s \ \& \ \rho \Vdash R_{\sigma(n)}(x))$.

In the first case put

$$\tau^{(2n+1)} = \tau^{(2n)} * s.$$

If the second case holds, take some finite part ρ such that

$$\rho \geqslant \tau^{(2n)} * s \ \&\ \rho \Vdash R_{\sigma(n)}(x)$$

and set $\tau^{(2n+1)} = \rho$.

Let us notice that this rather arbitrary choice of the above ρ is sufficient only to establish (*). To claim that the codomain of $(f, \mathfrak{B})$ is semicomputable in $\mathfrak{B}$, we will further have to choose ρ more carefully.

Let s be the first in the list $s_0, s_1, \ldots$, which is not in $Range\left(\tau^{(2n+1)}\right)$. Set

$$\tau^{(2n+2)} = \tau^{(2n+1)} * s.$$

Now let $(f, \mathfrak{B})$ be an enumeration which extends $\tau^{(2n+1)}$. In order to establish (*), assume that $\Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle)$ is an associate of A in $(f, \mathfrak{B})$. Then for every $x \in Dom(f)$

$$x \in \Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle) \iff f(x) \in A. \tag{4.1}$$

Now let $x \in N_0$ be the first not in $Dom\left(\tau^{(2n)}\right)$.

By definition $f^{(2n+1)}(x) = s$, hence $x \in Dom(f)$. According to the choice of s we have that either • or •• is true. Suppose first that $s \in A \ \&\ \forall\rho(\rho \geqslant \tau^{(2n)} * s \implies \rho \nVdash R_{\sigma(n)}(x))$. By Lemma 3.2 (4) $x \notin \Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle)$, so using 4.1, we obtain $f(x) = s \notin A$ — a contradiction. Therefore it is the case $s \notin A \ \&\ \exists\rho(\rho \geqslant \tau^{(2n)} * s \ \&\ \rho \Vdash R_{\sigma(n)}(x))$. According to Lemma 3.2 (2) $x \in \Gamma_{\sigma(n)}(\langle\mathfrak{B}\rangle)$ and again by (4.1) $f(x) = s \in A$, which is also impossible.

Now set

$$f_0 = \bigcup_n f^{(n)},\ H = \bigcup_n H^{(n)},\ q_i = \bigcup_n q_i^{(n)} \quad \text{for } 1 \leq i \leq m.$$

Obviously, $Dom(f_0) \cup H = \varnothing$ (otherwise there will be some n such that $x \in Dom\left(f^{(n)}\right) \ \&\ x \in H^{(n)}$) and $Dom(f_0) \cup H = N_0$. Notice also that with the even steps of the definition of $\left\{\tau^{(n)}\right\}_n$ it is ensured that f_0 is a partial mapping onto B.

Define the predicates Q_j, $1 \leq j \leq m$, as

$$Q_j(x) = \begin{cases} \Sigma_j(f(x)), & \text{if } x \in Dom(f), \\ q_j(x), & \text{if } x \in Dom(q_j), \\ 0, & \text{otherwise.} \end{cases}$$

This definition is correct since for any $1 \leq j \leq m$:

$$x \in Dom(f) \iff root(x) \in Dom(f_0) \iff root(x) \notin \bigcup_n H^{(n)} \implies$$

$$\forall n \left(x \notin Dom\left(q_j^{(n)}\right)\right) \iff x \notin Dom(q_j).$$

Now putting

$$f(x) \simeq \begin{cases} f_0(x), & \text{if } x \in Dom(f_0), \\ \theta_i(f(x_0)), & \text{if } x = \langle i, x_0\rangle \text{ for some } 1 \leq i \leq k, \end{cases}$$

we obtain a standard enumeration $(f, \mathfrak{B} = (N; \lambda x.\langle 1, x\rangle, \ldots, \lambda x.\langle k, x\rangle, Q_1, \ldots, Q_m))$ of $\mathfrak{A}$. It is clear that for arbitrary n:

$$f^{(n)} \subseteq f_0 \subseteq f \ \&\ q_j^{(n)} \subseteq q_j \subseteq Q_j,\ 1 \leq j \leq m,\ \&\ H^{(n)} \subseteq H \subseteq N \setminus Dom(f).$$

Therefore $\tau^{(n)} \subseteq (f, \mathfrak{B})$ for every n, which immediately brings us to the conclusion that A is not admissible in $\mathfrak{B}$.

The work done so far repeats in essence the respective proof in [3]. Our aim is to show that if we choose more precisely the finite parts ρ (when it is the case $\bullet\bullet$), then we may claim that $N \setminus Dom(f)$ is semicomputable in $\mathfrak{B}$.

Set for brevity

$$\tau \Vdash P_e(x) \iff \tau \Vdash R_{\sigma(e)}(x).$$

We have by definition

$$\tau \Vdash P_e(x) \iff \exists v(\langle v, x\rangle \in W_{\sigma(e)} \ \&\ \tau \Vdash D_v) \iff$$

$$\exists t \exists v(U(e,t) = \langle v, x\rangle \ \&\ \tau \Vdash D_v)) \iff \exists t((U(e,t))_1 = x \ \&\ \tau \Vdash D_{(U(e,t))_0}).$$

Put

$$\tau \Vdash_t P_e(x) \iff \exists t_0 \left(t_0 \leq t \ \&\ (U(e,t_0))_1 = x \ \&\ \tau \Vdash D_{(U(e,t_0))_0}\right).$$

Obviously,

$$\tau \Vdash P_e(x) \iff \exists t(\tau \Vdash_t P_e(x)).$$

The first t with $\tau \Vdash_t P_e(x)$ may be thought of as the first step at which the validity of the fact that $\tau \Vdash P_e(x)$ is established.

For a finite part $\tau = (f_\tau; H_\tau; q_1, \ldots, q_m)$ with $|\tau| = w$ put

$$\tau_l = (f_\tau; H_\tau \cup \{z_{w+1}, \ldots, z_{w+l}\}; q_1, \ldots, q_m).$$

Clearly, for each $l \geq 1$, τ_l is a finite part, too.

The next lemma will be of use when constructing the modified sequence $\{\tau^{(n)}\}_n$.

4.3. Lemma. *Suppose that $\exists \rho(\rho \geqslant \tau \ \&\ \rho \Vdash P_e(x))$. Then there exist $l \geq 1$ and $\rho^* \succcurlyeq \tau_l$ such that $\rho^* \Vdash_{l-1} P_e(x)$.*

Proof. Let $\rho \Vdash_t P_e(x)$, where $\rho = (f_\rho; H_\rho; q_1, \ldots, q_m)$. Set $l = \max(l_0, t+1)$, where $l_0 = |\rho| - |\tau|$. We claim that the finite part

$$\rho^* = (f_\tau; H_\tau \cup \{z_{|\tau|+1}, \ldots, z_{|\tau|+l}\}; q_1, \ldots, q_m)$$

fulfills the requirements of the lemma.

Indeed, we have $f_{\rho^*} = f_{\tau_l}$, $H_{\rho^*} = H_{\tau_l}$ and q_i extends the i-th initial predicate of τ_l for $1 \leq i \leq m$, so $\rho^* \succcurlyeq \tau_l$. Besides, $\rho^* \supseteq \rho$ and $l - 1 \geq t$, hence $\rho^* \Vdash_{l-1} P_e(x)$. □

Now we make the following refinement in the definition of the odd members of $\{\tau^{(n)}\}_n$. Again assuming that $\tau^{(2n)}$ is already defined and denoting by x_n the first number in the list $z_0, z_1, \ldots$, which is not in $Dom\left(\tau^{(2n)}\right)$, we will have that there exists $p_n \in B$ such that

$$s \in A \ \&\ \forall \rho \left(\rho \geqslant \tau^{(2n)} * p_n \implies \rho \nVdash P_n(x_n)\right) \quad \text{or}$$

$$s \notin A \ \& \ \exists \rho \left(\rho \geqslant \tau^{(2n)} * p_n \ \& \ \rho \Vdash P_n(x_n) \right) .$$

Set for brevity $\tau = \tau^{(2n)} * p_n$. If the first of the above two cases holds, set $\tau^{(2n+1)} = \tau$, otherwise start to look for the least $l \geq 1$, for which there exists $\rho^* \succcurlyeq \tau_l$ with $\rho^* \Vdash_{l-1} P_n(x)$ (the definition of τ_l is given immediately before Lemma 4.3). It follows from that lemma that such l exists. Now put $\tau^{(2n+1)} = \rho^*$, where $\rho^* \succcurlyeq \tau_l$ and $\rho^* \Vdash_{l-1} P_e(x)$.

Now let $(f, \mathfrak{B} = (N; \varphi_1, \dots, \varphi_k, Q_1, \dots, Q_m))$ be a standard enumeration, obtained from the sequence $\left\{\tau^{(n)}\right\}_n$ in the way described before. We have that A is not admissible in $(f, \mathfrak{B})$, so to complete the proof of the theorem, it remains to see that $N \setminus Dom(f)$ is semicomputable in $\mathfrak{B}$.

Let us notice that

$$x \in N \setminus Dom(f) \iff x \notin Dom(f) \iff root(x) \notin Dom(f_0) \iff root(x) \in H.$$

Since $root(x)$ is a recursive function, it is sufficient to see that the set $H = \bigcup_n H^{(n)}$ is semicomputable in $\mathfrak{B}$.

For $t, l \in N$ set

$$\begin{aligned} \mathfrak{B}^{(t)} \models_l P_e(x) \iff & \exists l_0 \exists v (l_0 \leq l \ \& \ U(e, l_0) = \langle v, x \rangle \ \& \ \forall u (u \in D_v \implies \\ & \exists i \exists y (u = \langle i, x, y \rangle \ \& \ ((1 \leq i \leq k \ \& \ y = \langle i, x \rangle) \vee \\ & (k + 1 \leq i \leq k + m \ \& \ Q_i(x) = y \ \& \ root(x) \leq z_t)))). \end{aligned}$$

Below we describe a procedure P that generates effectively the elements of the set H, asking questions of the type "$\mathfrak{B}^{(t)} \models_l P_e(x)$?".

Obviously, the set $R = \{(t, l, e, x) \mid \mathfrak{B}^{(t)} \models_l P_e(x)\}$ is decidable in $\mathfrak{B}$, so the set generated by P is semicomputable in $\mathfrak{B}$.

Let $l_n = lh\left(\tau^{(2n)}\right) + 1$. Then $x_n = z_{l_n}$, in particular $x_0 = z_1$. Informally, the procedure for generating $N \setminus Dom(f)$ is the following.

We should begin with asking questions

$$\mathfrak{B}^{(l_0+1)} \models_0 P_0(x_0)?, \quad \mathfrak{B}^{(l_0+2)} \models_1 P_0(x_0)?, \ \dots$$

in order to find (if it exists) the first t such that $\mathfrak{B}^{(l_0+t+1)} \models_t P_0(x_0)$. If such t does exist, then according to the construction of $\tau^{(1)}$ we put $H^{(1)} = \{z_{l_0+1}, \dots, z_{l_0+t+1}\}$. Since z_{l_0+t+2} and z_{l_0+t+3} are added to $Dom\left(f^{(2)}\right)$ and $Dom\left(f^{(3)}\right)$, resp., they are not in H. So we should set $l_1 = l_0 + t + 3$, $x_1 = z_{l_1}$ and then start searching for t with $\mathfrak{B}^{(l_1+t+1)} \models_t P_1(x_1)$.

Here the problem is that we do not know in advance whether there exists t with $\mathfrak{B}^{(l_0+t+1)} \models_t P_0(x_0)$. So if two unsuccessful steps in this search are done (i.e. when $\mathfrak{B}^{(l_0+1)} \not\models_0 P_0(x_0)$ and $\mathfrak{B}^{(l_0+2)} \not\models_1 P_0(x_0)$), we decide temporarily that such t does not exist and start simultaneously a similar procedure for seeking the first $t \leq 1$: $\mathfrak{B}^{(l'_1+t+1)} \models_t P_1(x'_1)$ for $l'_1 = l_0 + 2$ and $x'_1 = z'_{l_1}$, i.e. $l'_1 = 3$ and $x'_1 = z_3$. If such $t \leq 1$ again does not exist, we repeat the same for $n = 2$ and so on. Meanwhile, if we have found (for example for $n = 0$) some t_0 such that $\mathfrak{B}^{(l_0+t_0+1)} \models_{t_0} P_0(x_0)$, we interrupt all started procedures for finding out t with $\mathfrak{B}^{(l'_i+t+1)} \models_t P_i(x'_i)$ for

$i > 0$. Then we print numbers $z_{l_0+1}, \dots, z_{l_0+t_0+1}$, set $l_1 = l_0 + t + 3$, $x_1 = z_{l_1}$, start searching for t: $\mathfrak{B}^{(l'_1+t+1)} \models_t P_1(x'_1)$ etc.

However, this least t_0 with $\mathfrak{B}^{(l_0+t_0+1)} \models_t P_0(x_0)$ could be found after we have come across some (let say) t_1: $\mathfrak{B}^{(l'_1+t_1+1)} \models_{t_1} P_1(x'_1)$, where l'_1 and x'_1 are calculated under the wrong supposition that $\forall t \mathfrak{B}^{(l_0+t+1)} \not\models_t P_0(x_0)$. So, on the one hand, our algorithm requires the respective set $\{z_{l'_1+1}, \dots, z_{l'_1+t_1+1}\}$ to be printed right after such a t_1 has been found (since it is supposed to be the set $H^{(3)}$). On the other hand, the "real" $H^{(3)}$ may be different (and in fact is different). However, thanks to the special choice of $\tau^{(1)}$ (to be long enough), it turns out that the printed numbers $z_{l'_1+1}, \dots, z_{l'_1+t_1+1}$ actually belong to $H^{(1)}$ and hence to H.

Below we describe formally the procedure P that generates H. There the function $g(n,t)$ is intended to be such that $x_n = z_{g(t,n)}$ for sufficiently large t, namely $t \geq |\tau^{(2n-1)}|$. The function $G(t,n)$ from the program P is used to code the information about questions of the type "$\mathfrak{B}^{(t)} \models_v P_n(y)$?" for every $n \leq y$ (v, y depending on answers of similar questions for the numbers less than n).

Set

$$\langle y_1 \rangle = y_1; \quad \langle y_1, \dots, y_{n+1} \rangle = \langle \langle y_1, \dots, y_n \rangle, y_{n+1} \rangle \quad \text{for } n > 1.$$

Let $\lambda z.\langle z \rangle_i$ be the recursive function such that if $z = \langle y_1, \dots, y_n \rangle$ and $i \in \{1, \dots, n\}$, then $\langle z \rangle_i = y_i$, and $\langle z \rangle_i = 1$ — otherwise. We shall obtain $G(t,n)$ in the format $\langle y_0, \dots, y_t \rangle$, where each y_i will indicate (if t is large enough) whether $z_i \in H^{(n)}$ or not (writing $y_i = 0$ if $z_i \in H^{(n)}$, and $y_i = 1$ if $z_i \in Dom(f_0)$). The value $G(t+1,n)$ will depend on the last member y_n of $G(t,n)$, which is in fact $R(G(t,n))$. Since certainly $z_0 \in Dom(f)$ and $x_0 = z_1$, we put $G(t,0) = \langle 1 \rangle$ and $g(t,0) = 1$ for $t = 0$.

Here follows the exact description of the procedure P.

$t := 0$; $G(t,0) := \langle 1 \rangle$; $g(t,0) := 1$; $1 : n := 0$;

2: **if** $R(G(t,n)) = 0$ **then**
  if $g(t,n) = t+2$
    then $G(t+1,n) := \langle G(t,n), 1 \rangle$; $g(t+1,n) := g(t,n)$; $t := t+1$; **go to** 1
    else $G(t+1,n) := \langle G(t,n), 3 \rangle$; $g(t+1,n) := g(t,n)$; $n := n+1$; **go to** 2 **fi**
        else
if $R(G(t,n)) = 1$ **then**
  if $g(t,n) = t+2$
    then $G(t+1,n) := \langle G(t,n), 1 \rangle$; $g(t+1,n) := g(t,n)$; $t := t+1$; **go to** 1
    else if $g(t,n) = t+1$
      then $G(t+1,n) := \langle G(t,n), 1 \rangle$; $G(t+1,n+1) := G(t+1,n)$;
        $g(t+1,n) := g(t,n)$; $g(t+1,n+1) := t+3$; $t := t+1$; **go to** 1
      else if $\mathfrak{B}^{(t+1)} \models_0 P_n(z_{g(t,n)})$
        then $G(t+1,n) := \langle G(t,n), 0 \rangle$; **print**$(z_{t+1})$; $G(t+1,n+1) := G(t+1,n)$;
          $g(t+1,n) := g(t,n)$; $g(t+1,n+1) := t+3$; $t := t+1$; **go to** 1

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            else G(t+1,n) := ⟨G(t,n),2⟩; g(t+1,n) := g(t,n); n := n+1; go to 2
                 fi fi fi
                    else
if R(G(t,n)) = 2
  then l := t − g(t,n); if 𝔅^(t+1) ⊨_l P_n(z_g(t,n))
    then G(t+1,n) := ⟨L^l(G(t,n)), 0,...,0 (l+1 times)⟩; print (z_g(t,n)+1, z_g(t,n)+2, ..., z_t+1);
            G(t+1,n+1) := G(t+1,n); g(t+1,n) := g(t,n); g(t+1,n+1) := t+3;
            t := t+1; go to 1
    else G(t+1,n) := ⟨G(t,n),2⟩; g(t+1,n) := g(t,n); n := n+1; go to 2 fi
  else G(t+1,n) := ⟨G(t,n),3⟩; g(t+1,n) := g(t,n); n := n+1; go to 2
       fi fi fi.
```

Let us mention that some of the assignments in the above program are redundant. They are put there only to facilitate the verification of the algorithm.

Denote by $Output(P)$ the collection of all numbers, printed by P. We have to prove that

$$x \in H \iff x \in Output(P). \tag{4.2}$$

By an immediate inspection of P one can notice that for every t, n

$$g(t,n) \leq t+2 \;\&\; g(t,n) \leq g(t,n+1).$$

Set

$$S(n) = \begin{cases} 0, & \text{if } \exists \rho \left(\rho \geqslant \tau^{(2n)} * p_n \;\&\; \rho \Vdash P_n(x_n)\right), \\ 1, & \text{otherwise.} \end{cases}$$

To establish the first direction of (4.2), we will make use of the fact that $G\left(|\tau^{(2n+1)}|, n\right)$ is the code of the characteristic function C_H of H, restricted to the first $|\tau^{(2n+1)}|+1$ members of N_0. In order to prove this, the next more common observation will be needed.

4.4. Lemma.

(1) *For each* $t \geq |\tau_{(2n-1)}|$, $g(t,n) = |\tau^{(2n)}|+1$ $(|\tau^{(-1)}| = 0)$;

(2) *For each* $t \geq |\tau^{(2n+1)}|$, $G(t,n) = \langle y_0, \ldots, y_t\rangle$, *where*

$$y_i = \begin{cases} 0, & \text{if } i \leq |\tau^{(2n+1)}| \;\&\; z_i \in H^{(2n+1)}, \\ 1, & \text{if } i \leq |\tau^{(2n+1)}| \;\&\; z_i \notin H^{(2n+1)}, \\ 2, & \text{if } i > |\tau^{(2n+1)}| \;\&\; S(n) = 1, \\ 3, & \text{if } i > |\tau^{(2n+1)}| \;\&\; S(n) = 0. \end{cases}$$

Proof. Induction on n. (1) The case $n=0$ is obvious. In order to check (2) for $n=0$, we shall separately consider the cases $S(0)=0$ and $S(0)=1$. If the latter is true, i.e.

$$\forall \rho \left(\rho \geqslant \tau^{(0)} * p_0 \implies \rho \nVdash P_0(z_0)\right),$$

then $\mathfrak{B}^{(u)} \not\models_v P_0(z_0)$ for no $u \geq 1$ and v. So for every t, $G(t,0) = \langle 1,1,\underbrace{2,\ldots,2}_{t-1}\rangle$.

Now suppose that $S(0) = 0$ and denote the finite part $\tau^{(0)} * p_0$ by τ. According to Lemma 4.3 there exists least $l \geq 1$ for which there are finite parts ρ^* such that $\rho^* \geqslant \tau_l$ and $\rho^* \Vdash_{l-1} P_0(x_0)$. Let us remind that by construction $\tau^{(1)} = \rho^*$, where ρ^* satisfies the last conditions. We claim that

$$G(t,0) = \langle 1,1,\underbrace{2,\ldots,2}_{t-1}\rangle \quad \text{for each } 1 \leq t < l+1 \tag{4.3}$$

and

$$G(l+1,0) = \langle 1,1,\underbrace{0,\ldots,0}_{l}\rangle. \tag{4.4}$$

Indeed, assuming that (4.3) does not hold and looking at the program P, we may claim that there is a step $t < l$ such that $1 \leq t < l$ with $\mathfrak{B}^{(t+1)} \models_{t-1} P_0(x_0)$. From here, there exists $\delta \subseteq (f, \mathfrak{B})$ with $|\delta| = t+1$ such that $\delta \Vdash_{t-1} P_0(x_0)$. Since $\tau \subseteq (f, \mathfrak{B})$, we may assume that $\delta \geqslant \tau$. We have $|\tau_t| = t+1$ and $\delta \geqslant \tau_t$. In addition, $\delta \Vdash_{t-1} P_0(x_0)$ and $t < l$ — a contradiction with the choice of l.

To see that (4.4) also holds, recall that $\tau^{(1)} \Vdash_{l-1} P_0(x_0)$, $|\tau^{(1)}| = l+1$ and $\tau^{(1)} \subseteq (f, \mathfrak{B})$. So $\mathfrak{B}^{(l+1)} \models_{l-1} P_0(x_0)$, and as we have just seen, l is the first one with this property. Hence at step $t = l$ we shall have for the first time that $\mathfrak{B}^{(t+1)} \models_{t-1} P_0(x_0)$, so $G(t+1,0) = \langle \underbrace{L(\ldots,L}_{t-1}(G(t,0))\ldots), \underbrace{0,\ldots,0}_{t}\rangle =$

$\langle 1,1,\underbrace{0,\ldots,0}_{t}\rangle$, or $G(l+1,0) = \langle 1,1,\underbrace{0,\ldots,0}_{l}\rangle$.

Clearly, $l \geq 1$, hence $R(G(l+1,0)) = 0$. Then for every $t > l$ we shall have $G(t+1,0) = \langle G(t,0), 3\rangle$, in other words, $G(t+1,0) = \langle 1,1,\underbrace{0,\ldots,0}_{l}\underbrace{3,\ldots,3}_{t-l}\rangle$.

Now suppose that for each $j \leq n$ (1) and (2) are true. In particular, for $l = |\tau^{(2n+1)}|$ we have that $G(l,n) = \langle y_0,\ldots,y_l\rangle$, where $y_i = C_H(z_i)$, $0 \leq i \leq l$. Suppose first that $S(n) = 0$. According to the construction of $\tau^{(2n+1)}$ and the program P, at step t with $t+1 = |\tau^{(2n+1)}|$ we have $G(t+1,n+1) = G(t,n)$ and $g(t+1,n+1) = t+3$, in other words, $G(l,n+1) = \langle y_0,\ldots,y_l\rangle$ and $g(l,n+1) = |\tau^{(2n+1)}| + 2$. From the latter, $g(l,n+1) = |\tau^{(2n+2)}| + 1$, and since $g(t,n+1) = g(l,n+1)$ for every $t \geq l$, (1) is established for $n+1$. From here $x_{n+1} = z_{|\tau^{(2n+2)}|+1} = z_{g(t,n+1)}$ for $t \geq l$.

By the induction hypothesis, for every $t \geq l$ and $i < n$ we have that $R(G(t,i)) = 2$ or 3 if $S(i)$ is 1, resp. 0. Further, $R(G(l,n)) = 0$ and hence for every $t > l$, $R(G(t,n)) = 3$. From here, for every $t \geq l$ there will not be situations that may cause changes in $G(t,n+1)$, due to the extension of an assignment of the type $G(t+1,i+1) := g(t+1,i)$ for some $i \leq n$. In other words, the value of $G(t+1,n+1)$ depends uniquely on the answers of the questions of the type

"$\mathfrak{B}^{(t+1)} \models_{t-(|\tau^{(2n+2)}|+1)} P_{n+1}(x_{n+1})$?" (recall that $g(t, n+1) = |\tau^{(2n+2)}| + 1$, as we have already noticed).

Now to complete the verification of (2) for $n+1$, we proceed in essence as in the case $n = 0$. The second case $S(n) = 1$ is treated similarly. □

Our last lemma, which asserts that the program P is correct, completes the proof of the theorem.

4.5. Lemma. $H = Output(P)$.

Proof. For the first inclusion, recall that $H = \bigcup_n H^{(n)}$, where $H^{(0)} = \varnothing$, $H^{(2n)} = H^{(2n-1)}$ and $H^{(0)} \subseteq H^{(1)} \subseteq \ldots$ So if $x \in H$, then there exists n such that $x \in H^{(2n+1)}$ and $x \notin H^{(2n)}$. Further, $x = z_j$ for some $j \leq |\tau^{(2n+1)}|$. According to Lemma 4.4, $G\left(|\tau^{(2n+1)}|, n\right) = \langle y_0, \ldots, y_{|\tau^{(2n+1)}|}\rangle$ with $y_j = 0$. Since $x \notin H^{(2n)}$, $j > |\tau^{(2n)}|$, then at step t with $t+1 = |\tau^{2n+1}|$ the number z_j will be among the numbers, printed by P.

Towards proving the inclusion $Output(P) \subseteq H$, let us notice that

$$z_j \in Output(P) \iff \exists n \exists t \, (\langle G(t+1, n)\rangle_j = 0 \ \& \ g(t,n) \leq j \leq t+1).$$

Then it is sufficient to show that

$$(\exists n \exists t \, \langle G(t,n)\rangle_j = 0) \implies z_j \in H. \tag{4.5}$$

Define the predicate T as follows:

$$T(n) \iff \forall j ((\exists t \langle G(t,n)\rangle_j = 0) \implies z_j \in H).$$

We are going to establish $\forall t T(n)$ using induction on n. From here it follows (4.5) for arbitrary j.

To facilitate the inductive step, we suppose that when an assignment of the type $G(t+1, n+1) := G(t+1, n)$ is executed, the value $G(t+1, n)$ is assigned also to $G(t+1, k)$ for every $k > n+1$ for which there exists step $l \leq t$, at which $G(l, k)$ is determined. In other words, instead of single assignment $G(t+1, n+1) := G(t+1, n)$ we perform the finite list of assignments

$$\begin{aligned} &G(t+1, n+1) := G(t+1, n), \\ &G(t+1, n+2) := G(t+1, n), \\ &\ldots \\ &G(t+1, n') := G(t+1, n), \end{aligned}$$

where $n' > n$ can be found effectively.

Let $In(l, \langle y_0, \ldots, y_t\rangle) = \langle y_0, \ldots, y_l\rangle$ for $l \leq t$. The validity of $T(0)$ follows from the proof of Lemma 4.4. We obtained there that for $t < |\tau^{(1)}|$ we have $G(t, 0) = \langle 1, 1, \underbrace{2, \ldots, 2}_{t-1}\rangle$, so if for some j there exists t such that $\langle G(t,0)\rangle_j = 0$, then $t \geq |\tau^{(1)}|$, hence $\langle G(t,0)\rangle_j = C_H(z_j)$, i.e. $C_H(z_j) = 0$ and $z_j \in H$.

Now let for some $n > 0$ and some j there exists t_0 such that $\langle G(t_0, n)\rangle_j = 0$. Clearly $j \leq t_0$. We may assume that for every $k < n$ and every t, $\langle G(t, k)\rangle_j \neq 0$ since otherwise we can apply the induction hypothesis for that k.

If for every $t > t_0$ $In(t_0, G(t, n)) = G(t_0, n)$, then for $t > |\tau^{(2n+1)}|$ we will have according to Lemma 4.4 that

$$\langle G(t_0, n)\rangle_j = \langle In(t_0, G(t, n))\rangle_j = \langle G(t, n)\rangle_j = C_H(z_j) = 0,$$

hence $z_j \in H$.

Now assume that there exists $t' \geq t_0$ with $In(t_0, G(t'+1, n)) \neq G(t_0, n)$ and suppose that t' is the first one with that property. Clearly, $j > g(t_0, n)$ — otherwise we will have that $\langle G(g(t_0, n-1), n-1)\rangle_j = 0$, which contradicts the choice of n. So the fact that $G(t_0, n) \neq In(t_0, G(t'+1, n))$ is not due to an assignment of the type $G(t+1, n) := \langle L^l(G(t, n)), \underbrace{0, \ldots, 0}_{l+1}\rangle$ at step $t = t'$, since at the preceding step $t' - 1$ we would have $\langle G(t', n)\rangle_j = 2$ (if $|\tau^{(2n+1)}| - |\tau^{(2n)}| > 2$) or $\langle G(t', n)\rangle_j = 1$ (if $|\tau^{(2n+1)}| - |\tau^{(2n)}| = 2$).

Therefore the change of $G(t'+1, n)$ is caused by an assignment of the type $G(t+1, n) := G(t+1, n_0)$ for some $n_0 < n$. It is easy to see that this is preceded by an operator of the type $G(t+1, n_0) := \langle L^l(G(t, n_0)), \underbrace{0, \ldots, 0}_{l+1}\rangle$ at the same step $t = t'$, where $l = t - g(t, n_0)$. In other words,

$$G(t'+1, n_0) = \langle L^l(G(t', n_0)), \underbrace{0, \ldots, 0}_{l+1}\rangle$$

for $l = t' - g(t', n_0)$. From here, for $g(t', n_0) < i \leq t'+1$ we have $\langle G(t'+1, n_0)\rangle_i = 0$. We may claim that $g(t', n_0) = g(t_0, n_0)$ — convince yourselves that any change of $g(t, n_0)$ for $t_0 \leq t < t'$ will produce changes in $In(t_0, G(t, n))$ and take into consideration that t' is the first one with that property.

Further, we have $g(t_0, n_0) < g(t_0, n)$, since $n_0 < n$. So

$$g(t', n_0) = g(t_0, n_0) < g(t_0, n) < j$$

and, obviously, $j \leq t_0 \leq t'$. Hence $\langle G(t'+1, n_0)\rangle_j = 0$ and using the induction hypothesis $T(n_0)$ we conclude that $z_j \in H$. □

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