ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 100

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 100

NON-EXISTENCE OF FLAT PARACONTACT METRIC STRUCTURES IN DIMENSION GREATER THAN OR EQUAL TO FIVE [1]

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An example of a three dimensional flat paracontact metric manifold with respect to Levi-Civita connection is constructed. It is shown that no such manifold exists for odd dimensions greater than or equal to five.

Keywords: paracontact metric manifold, integral submanifold, maximal integral submanifold

2000 MSC: 53C15, 5350, 53C25, 53C26, 53B30

1. INTRODUCTION

The almost paracontact structure on pseudo-Riemannian manifold M of dimension $(2n+1)$ is defined in [7]. An almost paracomplex structure on $M^{(2n+1)} \times \mathbb{R}$ is constructed in [5]. Some properties of an almost paracontact metric manifold and the gauge (conformal) transformations of a paracontact metric manifold, i.e., transformations preserving the paracontact structure, are studied in [8]. Furthermore, in this paper a canonical paracontact connection on a paracontact metric manifold is defined. This connection is the paracontact analogue of the (generalized) Tanaka-Webster connection. It is shown that the torsion of the canonical

[1]Simeon Zamkovoy acknowledges support from the European Operational programm HRD through contract BGO051PO001/07/3.3-02/53 with the Bulgarian Ministry of Education. He also was partially supported by Contract 082/2009 with the University of Sofia "St. Kl. Ohridski".

paracontact connection vanishes exactly when the structure is para-Sasakian and the gauge transformation of its scalar curvature is computed.

An example of a paracontact structure of flat canonical connection is the hyperbolic Heisenberg group [3]. The paraconformal tensor gives a necessary and sufficient condition for a $(2n+1)$-dimensional paracontact manifold to be locally paracontact conformal to the hyperbolic Heisenberg group [3].

In this paper, we show that there is no flat, with respect to Levi-Civita connection, paracontact metric structures in dimension greater than or equal to five, whereas in dimension equal to three there is.

2. PRELIMINARIES

A (2n+1)-dimensional smooth manifold $M^{(2n+1)}$ has an *almost paracontact structure* (φ, ξ, η) if it admits a tensor field φ of type $(1,1)$, a vector field ξ, and a 1-form η satisfying the following compatibility conditions

$$\begin{aligned} &(i) \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \\ &(ii) \quad \eta(\xi) = 1 \quad \varphi^2 = id - \eta \otimes \xi, \\ &(iii) \quad \text{the tensor field } \varphi \text{ induces an almost paracomplex structure (see [4])} \\ &\qquad \text{on each fibre on the horizontal distribution } \mathbb{D} = Ker\ \eta. \end{aligned} \tag{2.1}$$

Recall that an almost paracomplex structure on a $2n$-dimensional manifold is a (1,1)-tensor J such that $J^2 = 1$ and the eigensubbundles T^+, T^- corresponding to the eigenvalues $1, -1$ of J respectively, have dimensions equal to n. The Nijenhuis tensor N of J, given by $N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y]$, is the obstruction for the integrability of the eigensubbundles T^+, T^-. If $N = 0$ then the almost paracomplex structure is called paracomplex or integrable.

An immediate consequence of the definition of the almost paracontact structure is that the endomorphism φ has rank $2n$, $\varphi\xi = 0$ and $\eta \circ \varphi = 0$, (see [1, 2] for the almost contact case).

If a manifold $M^{(2n+1)}$ with (φ, ξ, η)-structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{2.2}$$

then we say that $M^{(2n+1)}$ has an almost paracontact metric structure and g is called *compatible* metric. Any compatible metric g of a given almost paracontact structure is necessarily of signature $(n+1, n)$.

Setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$. From here and (2.2) follows

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

The fundamental 2-form

$$F(X,Y) = g(X, \varphi Y) \tag{2.3}$$

is non-degenerate on the horizontal distribution $\mathbb{D}$ and $\eta \wedge F^n \neq 0$.

Definition 2.1. *If $g(X, \varphi Y) = d\eta(X,Y)$ (where $d\eta(X,Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X,Y])$), then η is a paracontact form and the almost paracontact metric manifold (M, φ, η, g) is said to be* paracontact metric manifold.

Definition 2.2. *An r-dimensional submanifold N of $M^{(2n+1)}$ is said to be an* integral submanifold *(of the horizontal distribution $\mathbb{D}$) if and only if every tangent vector of N at every point p of N belongs to $\mathbb{D}$.*

Definition 2.3. *An integral submanifold of dimension r in $M^{(2n+1)}$ is said to be a* maximal integral submanifold *if it is not a proper subset of any other integral submanifold of dimension r.*

Similarly to the contact metric case [6], we may obtain the following

Proposition 2.4. *Let $(M^{2n+1}, \varphi, \eta, g)$ be a paracontact metric manifold. Then the highest dimension of integral submanifolds of the horizontal distribution $\mathbb{D}$ is equal to n.*

The tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ are defined [8] by

$$N^{(1)}(X,Y) = N_\varphi(X,Y) - 2d\eta(X,Y)\xi,$$

$$N^{(2)}(X,Y) = (\pounds_{\varphi X}\eta)Y - (\pounds_{\varphi Y}\eta)X,$$

$$N^{(3)}(X) = (\pounds_\xi \varphi)X,$$

$$N^{(4)}(X) = (\pounds_\xi \eta)X,$$

where $N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y]$.

The tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ are analogs of the tensors denoted in the same way in the almost contact case [1, 2].

They have the following propositions [8].

Proposition 2.5. *For an almost paracontact structure (φ, ξ, η) the vanishing of $N^{(1)}$ implies the vanishing $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$;*

For a paracontact structure (φ, ξ, η, g), $N^{(2)}$ and $N^{(4)}$ vanish. Moreover $N^{(3)}$ vanishes if and only if ξ is a Killing vector field.

Proposition 2.6. *For an almost paracontact metric structure (φ, ξ, η, g), the covariant derivative $\nabla\varphi$ of φ with respect to the Levi-Civita connection ∇ is given by*

$$\begin{aligned} 2g((\nabla_X\varphi)Y, Z) = &-dF(X,Y,Z) - dF(X, \varphi Y, \varphi Z) - g(N^{(1)}(Y,Z), \varphi X) \\ &+N^{(2)}(Y,Z)\eta(X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z). \end{aligned} \tag{2.4}$$

For a paracontact metric structure (φ, ξ, η, g)*, the formula* (2.4) *simplifies to*

$$2g((\nabla_X\varphi)Y, Z) = -g(N^{(1)}(Y, Z), \varphi X) - 2d\eta(\varphi Z, X)\eta(Y) + 2d\eta(\varphi Y, X)\eta(Z) \quad (2.5)$$

Lemma 2.7. *On a paracontact metric manifold,* $h = \frac{1}{2}N^{(3)}$ *is a symmetric operator,*

$$\nabla_X\xi = -\varphi X + \varphi hX, \quad (2.6)$$

h *anti-commutes with* φ *,and* $trh = h\xi = 0$.

3. NON-EXISTENCE OF FLAT PARACONTACT METRIC STRUCTURES IN DIMENSION GREATER THAN OR EQUAL TO FIVE

In this section we shall show that every paracontact metric manifold of dimension greater than or equal to five must have some curvature, though not necessarily in the plane sections containing ξ.

Theorem 3.1. *Let* M^{2n+1} *be a manifold of dimension greater than or equal to five. Then* M^{2n+1} *cannot admit a paracontact structure of vanishing curvature.*

Proof. The proof will be by contradiction. We let (φ, ξ, η, g) denote the structure tensors of a paracontact metric structure and assume that g is flat. From [8] we have, for a paracontact metric structure,

$$\frac{1}{2}(R(\xi, X)\xi + \varphi R(\xi, \varphi X)\xi) = \varphi^2 X - h^2 X$$

where $h = \frac{1}{2}\mathcal{L}_\xi\varphi$. Thus if g is flat, $h^2 = \varphi^2$, and hence $h\xi = 0$ and $rank(h) = 2n$. The eigenvectors corresponding to the non-zero eigenvalues of h are orthogonal to ξ and the non-zero eigenvalues are ± 1. Recall that $d\eta(X, Y) = \frac{1}{2}(g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X))$ and that for a paracontact metric structure

$$\nabla_X\xi = -\varphi X + \varphi hX. \quad (3.1)$$

From Lemma 2.7 follows that whenever X is an eigenvector of eigenvalue $+1$, φX is an eigenvector of -1 and vice-versa. Thus the paracontact distribution $\mathbb{D}$ is decomposed into the orthogonal eigenspaces of ± 1 which we denote by $[+1]$ and $[-1]$.

We now show that the distribution $[+1]$ is integrable. If X and Y are vector fields belonging to $[+1]$, equation (3.1) gives $\nabla_X\xi = 0$ and $\nabla_Y\xi = 0$. Thus since M^{2n+1} is flat:

$$0 = R(X, Y)\xi = -\nabla_{[X,Y]}\xi = \varphi[X, Y] - \varphi h[X, Y];$$

but $\eta([X, Y]) = -2d\eta(X, Y) = -2g(X, \varphi Y) = 0$, so that $h[X, Y] = [X, Y]$. Applying the same argument to ξ and $X \in [+1]$ we see that the distribution $[+1] \oplus [\xi]$ spanned by $[+1]$ and ξ is also integrable.

Since $[+1] \oplus [\xi]$ is integrable, we can choose local coordinates $(u^0, u^1, \ldots, u^{2n})$ such that $\frac{\partial}{\partial u^0}, \frac{\partial}{\partial u^1}, \ldots, \frac{\partial}{\partial u^n} \in [+1] \oplus [\xi]$. For $i = 1 \ldots n$ the vector $\frac{\partial}{\partial u^{n+i}}$ can be uniquely presented as $\frac{\partial}{\partial u^{n+i}} = v^{+1}_{n+i} + v^{\xi}_{n+i} + v^{-1}_{n+i}$ where $v^{+1}_{n+i} \in [+1]$, $v^{\xi}_{n+i} \in [\xi]$, $v^{-1}_{n+i} \in [-1]$, and $v^{-1}_{n+i} \neq \overrightarrow{0}$. Let $v^{+1}_{n+i} + v^{\xi}_{n+i} = -\sum_{j=0}^{n} f^j_i \frac{\partial}{\partial u^j}$. We define local vector fields X_i, $i = 1, \ldots, n$ by $X_i = \frac{\partial}{\partial u^{n+i}} + \sum_{j=0}^{n} f^j_i \frac{\partial}{\partial u^j}$, i.e. $X_i = v^{-1}_{n+i}$ so that $X_i \in [-1]$. Note $X_1, \ldots, X_n$ are n linearly independent vector fields spanning $[-1]$. Clearly $[\frac{\partial}{\partial u^k}, X_i] \in [+1] \oplus [\xi]$ for $k = 0, \ldots, n$ and hence ξ is parallel along $[\frac{\partial}{\partial u^k}, X_i]$. Therefore using (3.1) and the vanishing curvature

$$0 = \nabla_{[\frac{\partial}{\partial u^k}, X_i]}\xi = \nabla_{\frac{\partial}{\partial u^k}} \nabla_{X_i}\xi - \nabla_{X_i}\nabla_{\frac{\partial}{\partial u^k}}\xi = -2\nabla_{\frac{\partial}{\partial u^k}}\varphi X_i$$

from which we have

$$\nabla_{\varphi X_j}\varphi X_i = 0. \tag{3.2}$$

In particular $\nabla_\xi \varphi X_i = 0$. Furthermore, from equation (3.1) we obtain $\nabla_{\varphi X_i}\xi = 0$ and hence $[\varphi X_i, \xi] = 0$.

Similarly, noting that $[X_i, X_j] \in [+1]$,

$$0 = R(X_i, X_j)\xi = \nabla_{X_i}\nabla_{X_j}\xi - \nabla_{X_j}\nabla_{X_i}\xi - \nabla_{[X_i,X_j]}\xi = -2\nabla_{X_i}\varphi X_j + 2\nabla_{X_j}\varphi X_i$$

giving

$$\nabla_{X_i}\varphi X_j = \nabla_{X_j}\varphi X_i, \tag{3.3}$$

or equivalently

$$\varphi[X_i, X_j] = -(\nabla_{X_i}\varphi)X_j + (\nabla_{X_j}\varphi)X_i. \tag{3.4}$$

Using equations (3.1) and (3.2) we obtain

$$0 = R(X_i, \varphi X_j)\xi = -\nabla_{[X_i,\varphi X_j]}\xi = \varphi[X_i, \varphi X_j] - \varphi h[X_i, \varphi X_j]$$

from which

$$g([X_i, \varphi X_j], X_k) = -g(\varphi[X_i, \varphi X_j], \varphi X_k) = g(h[X_i, \varphi X_j], X_k) =$$

$$= g([X_i, \varphi X_j], hX_k) = -g([X_i, \varphi X_j], X_k)$$

and hence

$$g([X_i, \varphi X_j], X_k) = 0. \tag{3.5}$$

Using formula (2.5) and equations (3.2), (3.4) and (3.5) we have

$$2g((\nabla_{X_i}\varphi)X_j, X_k) = -g(N^{(1)}(X_j, X_k), \varphi X_i) = -g([X_j, X_k], \varphi X_i) =$$

$$= -g((\nabla_{X_j}\varphi)X_k, X_i) + g((\nabla_{X_k}\varphi)X_j, X_i).$$

From $F = d\eta$, we obtain $dF = 0$ and hence $\sigma_{i,j,k} g((\nabla_{X_i}\varphi)X_j, X_k) = 0$. Thus our computation yields $g((\nabla_{X_i}\varphi)X_j, X_k) = 0$. Similarly

$$2g((\nabla_{X_i}\varphi)X_j, \varphi X_k) = -g(N^{(1)}(X_j, \varphi X_k), \varphi X_i) =$$

$$= -g([X_j, \varphi X_k], \varphi X_i) - g([\varphi X_j, X_k], \varphi X_i) =$$
$$= -g(\nabla_{X_j} \varphi X_k - \nabla_{\varphi X_k} X_j - \nabla_{X_k} \varphi X_j + \nabla_{\varphi X_j} X_k, \varphi X_i)$$

which vanishes by equations (3.2) and (3.3). Finally

$$2g((\nabla_{X_i}\varphi)X_j, \xi) = -g(N^{(1)}(X_j, \xi), \varphi X_i) + 2d\eta(\varphi X_j, X_i) =$$
$$= -g(\varphi^2[X_j, \xi], \varphi X_i) + 2d\eta(\varphi X_j, X_i) = -4g(X_i, X_j).$$

Thus for any vector fields X and Y in $[-1]$ on a paracontact metric manifold such that ξ is annihilated by the curvature transformation

$$(\nabla_X \varphi)Y = -2g(X, Y)\xi. \tag{3.6}$$

Note that equation (3.4) now gives $[X_i, X_j] = 0$.

Analogously, we obtain

$$2g(\nabla_{\varphi X_i} X_j, X_k) = 2g((\nabla_{\varphi X_i}\varphi)X_j, \varphi X_k) = 0. \tag{3.7}$$

Therefore by equation (3.5), we get

$$g(\nabla_{X_i} X_j, \varphi X_k) = -g(X_j, \nabla_{X_i} \varphi X_k) = -g(X_j, [X_i, \varphi X_k]) = 0.$$

It is trivial that $g(\nabla_{X_i} X_j, \xi) = 0$ and hence we obtain $\nabla_{X_i} X_j \in [-1]$.

Differentiating equation (3.6), we have

$$\nabla_{X_k} \nabla_{X_i} \varphi X_j - (\nabla_{X_k}\varphi)\nabla_{X_i} X_j - \varphi \nabla_{X_k} \nabla_{X_i} X_j =$$
$$= -2X_k(g(X_i, X_j))\xi + 4g(X_j, X_i)\varphi X_k.$$

Taking the inner product with φX_l, having in mind equation (3.6) and $\nabla_{X_i} X_j \in [-1]$, we obtain

$$g(\nabla_{X_k} \nabla_{X_i} \varphi X_j, \varphi X_l) + g(\nabla_{X_k} \nabla_{X_i} X_j, X_l) = -4g(X_j, X_i)g(X_k, X_l) \tag{3.8}$$

Interchanging i and k, $i \neq k$ and subtracting, we have

$$g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l) = 0$$

by virtue of the flatness and $[X_i, X_j] = 0$.

Setting $i = j$ and $k = l$, we have

$$g(X_i, X_i)g(X_k, X_k) - g(X_i, X_k)g(X_i, X_k) = 0$$

contradicting the linear independence of X_i and X_k. $\square$

Note that in the proof of our theorem, the vanishing of $R(X, Y)\xi$ is enough to obtain the decomposition of the paracontact distribution into ± 1 eigenspaces of the operator $h = \frac{1}{2}\mathcal{L}_\xi\varphi$. Moreover, $R(X,Y)\xi = 0$ for X and Y in $[+1]$ is sufficient for the integrability of $[+1]$. Thus we have the following

Theorem 3.2. *Let M^{2n+1} be a paracontact manifold with paracontact metric structure (φ, ξ, η, g). If the sectional curvatures of all plane sections containing ξ vanish, then the operator h has rank $2n$ and the paracontact distribution is decomposed into the ± 1 eigenspaces of h. Moreover, if $R(X,Y)\xi = 0$ for $X, Y \in [+1]$, M admits a foliation by $n-$dimensional integral submanifolds of the paracontact distribution along which ξ is parallel.*

From Theorem 3.1 and Theorem 3.2 we obtain following

Theorem 3.3. *Let M^{2n+1} be a paracontact metric manifold and suppose that $R(X,Y)\xi = 0$ for all vector fields X and Y. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

Proof. We noted in Theorem 3.1 proof that $[X_i, X_j] = 0$ so that the distribution $[-1]$ is also integrable and hence we may take $X_i = \frac{\partial}{\partial u^{n+i}}$. Moreover, locally M^{2n+1} is the product of an integral submanifold M^{n+1} of $[+1] \oplus [\xi]$ and an integral submanifold M^n of $[-1]$. Since $\{\varphi X_i, \xi\}$ is a local basis of tangent vector fields on M^{n+1}, equation (3.2) and $R(X,Y)\xi = 0$ show that M^{n+1} is flat.

Now $\nabla_{\varphi X_i} X_j = 0$ since $g(\nabla_{\varphi X_i} X_j, X_k) = 0$ by equation (3.7). Moreover, $g(\nabla_{\varphi X_i} X_j, \varphi X_k) = 0$ by equation (3.2) and $g(\nabla_{\varphi X_i} X_j, \xi) = 0$ which is trivial. Interchanging i and k in equation (3.8) and subtracting, we have

$$R(X_k, X_i, \varphi X_j, \varphi X_l) + R(X_k, X_i, X_j, X_l) =$$

$$= -4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l)).$$

Using $\nabla_{\varphi X_i} X_j = 0$ and $[\varphi X_i, \varphi X_j] = 0$ we see that $R(X_k, X_i, \varphi X_j, \varphi X_l) =$ $= R(\varphi X_j, \varphi X_l, X_k, X_i) = 0$, and hence

$$R(X_k, X_i, X_j, X_l) = -4(g(X_i, X_j)g(X_k, X_l) - g(X_k, X_j)g(X_i, X_l))$$

completing the proof. □

4. FLAT ASSOCIATED METRICS ON $\mathbb{R}^3_1$

In dimension 3 it is easy to construct flat paracontact structures. For example, consider $\mathbb{R}^3_1$ with coordinates (x^1, x^2, x^3). The 1-form $\eta = \frac{1}{2}(ch(x^3)dx^1 + sh(x^3)dx^2)$ is a paracontact form. In this case $\xi = 2(ch(x^3)\frac{\partial}{\partial x^1} - sh(x^3)\frac{\partial}{\partial x^2})$ and the metric g whose components are $g_{11} = -g_{22} = g_{33} = \frac{1}{4}$ gives flat paracontact metric structure. Following the proof of the Theorem 3.1, we see that $\frac{\partial}{\partial x^3}$ spans the distribution $[-1]$, and $sh(x^3)\frac{\partial}{\partial x^1} + ch(x^3)\frac{\partial}{\partial x^2}$ spans the distribution $[+1]$.

We can now find a flat associated metric on $\mathbb{R}^3_1$ for the standard paracontact form $\eta_0 = \frac{1}{2}(dz - ydx)$. Consider the diffeomorphism $f : \mathbb{R}^3_1 \to \mathbb{R}^3_1$ given by

$$x^1 = zch(x) - ysh(x)$$

$$x^2 = zsh(x) - ych(x)$$

$$x^3 = -x$$

Then $\eta_0 = f^*\eta$, and the pseudo-Riemannian metric $g_0 = f^*g$ is a flat associated metric for the paracontact form η_0.

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Received on January 3, 2010

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