ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 101

ANNUAL OF SOFIA UNIVERSITY „ST. KLIMENT OHRIDSKI“

FACULTY OF MATHEMATICS AND INFORMATICS
Volume 101

ON THE EXTENDABILITY OF GRIESMER ARCS

IVAN N. LANDJEV, ASSIA P. ROUSSEVA

We prove a new sufficient condition for the extendability of Griesmer arcs with certain parameters.

Keywords: Finite projecive geometries, arcs, blocking sets, divisible arcs, quasi-divisible arcs, the griesmer bound, Griesmer codes, Griesmer arcs, extendable arcs, minihypers

2000 Math. Subject Classification: Primary 51A20, 51A21, 51A22; Secondary 94B65

1. INTRODUCTION

The geometric nature of certain problems in coding theory has been long known. In this paper we present a new result on the extendability of arcs in finite projective spaces which translates in a natural way into a result about the extendability of linear codes.

It is a well-known fact that adding a parity check to a binary $[n, k, d]$-code of odd minimum distance d increases the minimum distance of the codes, i.e. the resulting codes have parameters $[n + 1, k, d + 1]$. This result has been generalized by Hill and Lizak in [4,5]. They showed that if all weights in an $[n, k, d]_q$ code are congruent to 0 or $d \pmod q$, with $(d, q) = 1$, then it can be extended to an $[n+1, k, d+1]_q$-code. This fact has a natural explanation in terms of blocking sets containing a hyperplane. It was proved independently in [6] and [9] that the theorem of Hill and Lizak can be obtained from the well-known Bose-Burton theorem for blocking sets in $\mathrm{PG}(k-1, q)$. This result was further generalized in [7] by using a result of Beutelspacher and Heim on the size of the minimal non-trivial (i.e. not containing a hyperplane) blocking set in a finite projective geometry.

In a series of papers, Maruta obtained further results [9,10,11,12,13] on the extendability of linear codes. He introduced the notion of diversity of a linear code with spectrum (A_i) as the pair (Φ_0, Φ_1), where

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i\neq 0} A_i, \quad \Phi_1 = \frac{1}{q-1} \sum_{i\not\equiv 0, d(q)} A_i.$$

Maruta proved that for various values of the diversity the investigated codes are indeed extendable. In particular, he showed that a linear $[n, k, d]$-code over $\mathbb{F}_q$, with $q \geq 5$, $d \equiv -2 \pmod q$, having all non-zero weights congruent to $-2, -1$, and 0 modulo q is extendable.

Dodunekov and Simonis proved in [3] that linear $[n,k,d]_q$-codes of full length and $(n,n-d)$-arcs in $\mathrm{PG}(k-1,q)$ are in some sense equivalent objects. With each linear code one can associate an arc (possibly in an non-unique way) so that semilinearly isomorphic codes give rise to equivalent arcs and vice versa. Arcs associated with codes meeting the Griesmer bound are called Griesmer arcs.

This paper deals with the question of the extendability of arcs associated with codes meeting the Griesmer bound. The results translate in an obvious way for linear codes over finite fields. In section 2, we give some basic definitions and introduce the important notion of t-quasidivisibility modulo q. In section 3, we define a special arc $\widetilde{\mathcal{K}}$ in the dual geometry and relate the extendability property for $\mathcal{K}$ with the existence of a hyperplane in the support of $\widetilde{\mathcal{K}}$. Section 4 contains the main theorem stating that a t-quasidivisible Griesmer arc with divisor q, $t<\sqrt{q}$, which has an additional numerical condition on the parameters, is t-times extendable.

2. BASIC DEFINITIONS

Let $\mathcal{P}$ be the set of points of the projective geometry $\mathrm{PG}(k-1,q)$. Every mapping $\mathcal{K}:\mathcal{P}\to\mathbb{N}_0$ from the pointset of the geometry to the non-negative integers is called a multiset in $\mathrm{PG}(k-1,q)$. This mapping is extended additively to the subsets of $\mathcal{P}$: for every $\mathcal{Q}\subseteq\mathcal{P}$, $\mathcal{K}(\mathcal{Q})=\sum_{P\in\mathcal{Q}}\mathcal{K}(P)$. The integer $n:=\mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$. For every set of points $\mathcal{Q}\subset\mathcal{P}$ we define its characteristic (multi)set $\chi_{\mathcal{Q}}$ by

$$\chi_{\mathcal{Q}}(P)=\begin{cases}1 & \text{if } P\in\mathcal{Q},\\ 0 & \text{otherwise.}\end{cases}$$

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-multiarc (or simply (n,w)-arc) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\le w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-blocking set with respect to the hyperplanes (or (n,w)-minihyper) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\ge w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$.

An (n,w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called t-extendable, if there exists an $(n+t,w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\ge\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. An arc is called simply extendable if it is 1-extendable. Similarly, an (n,w)-blocking set $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called reducible, if there exists an $(n-1,w)$-blocking set $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\le\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. A blocking set is called irreducible if it is not reducible.

Given a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$, we denote by a_i the number of hyperplanes H with $\mathcal{K}(H)=i$. The sequence (a_i) is called the spectrum of $\mathcal{K}$. An (n,w)-arc $\mathcal{K}$ with spectrum (a_i) is said to be divisible with divisor $\Delta>1$ if $a_i=0$ for all $i\not\equiv n \pmod{\Delta}$. The (n,w)-arc $\mathcal{K}$ with $w\equiv n+t \pmod{q}$ is called t-quasidivisible with divisor $\Delta>1$ (or t-quasidivisible modulo Δ) if $a_i=0$ for all $i\not\equiv n,n+1,\dots,n+t \pmod{\Delta}$, $1\le t\le q-1$. The result of Hill and Lizak says that every 1-quasidivisible arc with divisor q is extendable; Maruta's theorem from [11] claims that for q odd every 2-quasidivisible arc with divisor q is extendable.

3. THE CONNECTION BETWEEN QUASIDIVISIBILITY AND EXTENDABILITY OF GRIESMER ARCS

As already noted, there exists a one-to-one correspondence between the classes of isomorphic $[n,k,d]_q$-codes and the classes of projectively equivalent $(n,n-d)$-arcs in $\mathrm{PG}(k-1,q)$ [3]. With every multiset $\mathcal{K}$ we can associate many isomorphic linear codes. Fix arbitrarily one of these codes and denote it by $C_{\mathcal{K}}$. If $C_{\mathcal{K}}$ is a Griesmer code then we call $\mathcal{K}$ a Griesmer arc.

Let $\mathcal{K}$ be a t-quasidivisible (n, w)-arc with divisor q in $\Sigma = \mathrm{PG}(k-1, q)$, $t < q$. Set $d = n - w$. This is a typical situation when one investigates the existence of Griesmer arcs with given parameters.

Define a new multiset $\widetilde{\mathcal{K}}$ in the dual geometry $\widetilde{\Sigma}$ by

$$\widetilde{\mathcal{K}} : \begin{cases} \mathcal{H} & \to \quad \{0, 1, \ldots, t\} \\ H & \to \quad \widetilde{\mathcal{K}}(H) \equiv n + t - \mathcal{K}(H) \pmod{q}, \end{cases} \tag{3.1}$$

where $\mathcal{H}$ is the set of all hyperplanes in Σ, i.e. the set of all ponts in $\widetilde{\Sigma}$. In other words, hyperplanes of multiplicity congruent to $n+a \pmod{q}$ become $(t-a)$-points in the dual geometry. The following result is straightforward.

Theorem 1. *Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(k-1, q)$, which is t-quasidivisible modulo q with $t < q$. Let $\widetilde{\mathcal{K}}$ be defined by (3.1). If*

$$\widetilde{\mathcal{K}} = \sum_{i=1}^{c} \chi_{\widetilde{H}_i} + \widetilde{\mathcal{K}}'$$

for some multiset $\mathcal{K}'$ and c not necessarily different hyperplanes $\widetilde{H}_1, \ldots, \widetilde{H}_c$, then $\mathcal{K}$ is c-extendable. In particular, if $\widetilde{\mathcal{K}}$ contains a hyperplane in its support, then $\mathcal{K}$ is extendable.

Proof. Since maximal hyperplanes correspond to 0-points in the dual geometry, the condition of the theorem is that there exist points in Σ of total multiplicity c that are not incident with maximal hyperplanes. □

By Theorem 1, the extendability of t-quasidivisible arcs is linked with the structure of the multiset $\widetilde{\mathcal{K}}$ defined in the dual geometry. It turns out that this multiset is highly divisible.

Theorem 2. *Let $\mathcal{K}$ be an (n, w)-arc in $\Sigma = \mathrm{PG}(k-1, q)$ which is t-quasidivisible modulo q with $t < q$. For every subspace $\widetilde{S}$ of $\widetilde{\Sigma}$ with $\dim \widetilde{S} \geq 1$,*

$$\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod{q}.$$

Proof. Let $\widetilde{S}$ be a line in the dual geometry $\widetilde{\Sigma}$. It corresponds to a subspace S of codimension 2 in Σ. Denote by H_i, $i = 0, \ldots, q$, the set of all hyperpalnes through S. We have

$$n = \sum_{i=0}^{q} \mathcal{K}(H_i) - q\mathcal{K}(S).$$

Reducing both sides modulo q and using the fact that $\mathcal{K}(H_i) + \widetilde{\mathcal{K}}(H_i) \equiv n + t \pmod{q}$, one gets

$$(q+1)(n+t) - \sum_{i=0}^{q} \widetilde{\mathcal{K}}(H_i) \equiv n \pmod{q},$$

whence

$$\widetilde{\mathcal{K}}(\widetilde{S}) = \sum_{i=0}^{q} \widetilde{\mathcal{K}}(H_i) \equiv t \pmod{q}.$$

For subspaces of larger dimension, we can use the fact that the multiplicity of each line in $\widetilde{S}$ is t modulo q. Then we sum the multiplicities of all lines through a fixed 0-point in $\widetilde{S}$. □

By the above theorem, the multiset $\widetilde{\mathcal{K}}$ has the following properties:

- the multiplicity of each point is at most t;

- the multiplicity of each subspace of dimension r, $1 \le r \le k-1$, is at least tv_r.

Here we use the conventional notation $v_r = (q^r - 1)/(q-1)$. Let us note that in the general case the cardinality of $\widetilde{\mathcal{K}}$ is not known.

For $t = 1$, the arc $\mathcal{K}$ is always extendable. In fact, this is another formulation of the theorem by Hill and Lizak. A plane arc with the above properties for $t = 1$ turns out to be projective. Then every line is 1- or $(q+1)$-line, the arc is either a line or the complete plane. More generally, in higher dimensions such an arc is either a hyperplane or the complete space. The second case does not occur since a maximal hyperplane maps to a 0-point. Therefore every 1-quasidivisible arc $\mathcal{K}$ is extendable by Theorem 1.

For $t = 2$ and odd $q \ge 5$, the arcs $\widetilde{\mathcal{K}}$ were characterized by Maruta [11]. He proved that in this case, the arc $\widetilde{\mathcal{K}}$ contains a hyperplane without 0-points, which implies that the arc $\mathcal{K}$ is again extendable.

The next theorem relates the extendability of $\mathcal{K}$ with the spectrum of a maximal hyperplane of Σ with respect to $\mathcal{K}$.

Theorem 3. *Let $\mathcal{K}$ be a Griesmer t-quasidivisible modulo q arc with parameters (n, w) in* $\mathrm{PG}(k-1, q)$*, where $w = n - d$. For a fixed hyperplane H_0 of multiplicity w, denote by (a_i) the spectrum of the arc $\mathcal{K}|_{H_0}$, the restriction of $\mathcal{K}$ to the hyperplane H_0. Let A be the largest integer such that a $(tv_{k-1} + A, tv_{k-2})$-minihyper contains a hyperplane in its support. If*

$$qa_{w-\lceil d/q\rceil-1} + 2qa_{w-\lceil d/q\rceil-2} + \ldots + (t-2)qa_{w-\lceil d/q\rceil-t+2}(t-1)q \sum_{u \le w-\lceil d/q\rceil-t+1} a_u \le A,$$

then $\mathcal{K}$ is extendable.

Proof. By the fact that $\mathcal{K}$ is a Griesmer arc, we have that

$$n = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil, \qquad w = \sum_{i=1}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

By straightforward counting, one gets that the maximal multiplicity of a subspace of codimension 2 contained in H_0 is

$$w' = w - \lceil \frac{d}{q} \rceil = \sum_{i=2}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

Let $\widetilde{\mathcal{K}}$ be the arc in $\widetilde{\Sigma}$ defined earlier in this section. The point $\widetilde{P} = H_0$ is a 0-point in $\widetilde{\Sigma}$. Denote by $\widetilde{L}_i$ all lines in $\widetilde{\Sigma}$ through $\widetilde{P}$. They correspond to the hyperlines δ_i in H_0, i.e. the subspaces of codimension 2 that are contained in H_0.

Consider a fixed line $\widetilde{L} = \delta$, where $\mathcal{K}(\delta) = w' - \lambda$, $\lambda \in \{0, \ldots, t-1\}$. Denote by $H_0, H_1, \ldots, H_q$ all hyperplanes through δ. Set

$$\mathcal{K}(H_i) = w - \alpha_i q - \beta_i, \ \ \beta_i \in \{0, \ldots, t\}.$$

Since $\mathcal{K}(H_i) + \widetilde{\mathcal{K}}(H_i) \equiv n + t \equiv w \pmod q$, we get that $\widetilde{\mathcal{K}}(H_i) = \beta_i$. Now we have

$$\begin{aligned} n &= \sum_{i=0}^{q} \mathcal{K}(H_i) - q(w' - \lambda) \\ &= \sum_{i=0}^{q} (w - \alpha_i q - \beta_i) - q(w' - \lambda) \\ &= w - q\sum_{i=0}^{q} \alpha_i - \sum_{i=0}^{q} \beta_i + q\lceil \frac{d}{q} \rceil + q\lambda, \end{aligned}$$

whence

$$\sum_{i=0}^{q}\beta_i = q\lceil\frac{d}{q}\rceil + q\lambda - d - q\sum_{i=0}^{q}\alpha_i.$$

Since $d \equiv -t \pmod q$, we have $q\lceil\frac{d}{q}\rceil - d = t$. This gives an upper bound on the multiplicity of $\widetilde{L}$ with respect to $\widetilde{\mathcal{K}}$:

$$\widetilde{\mathcal{K}}(\widetilde{L}) = \sum_i \widetilde{\mathcal{K}}(H_i) = \sum_{i=0}^{q}\beta_i = t + q\lambda - q\sum_{i=0}^{q}\alpha_i \le t + q\lambda.$$

Now summing up the multiplicities of all lines $\widetilde{L}$ through $\widetilde{P}$ and taking into account that $\widetilde{\mathcal{K}}(\widetilde{P}) = 0$, one gets for the cardinality of $\widetilde{\mathcal{K}}$ the following estimate:

$$\begin{aligned}
|\widetilde{\mathcal{K}}| &= \sum_i \widetilde{\mathcal{K}}(\widetilde{L}_i) \\
&\le a_{w'}t + a_{w'-1}(t+q) + \ldots + a_{w'-(t-2)}(t+(t-2)q) + \sum_{u\le w'-(t-1)} a_u(t+(t-1)q) \\
&= \left(\sum_{u\le w'} a_u\right) t + a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q \\
&= v_{k-1}t + a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q.
\end{aligned}$$

Here we use the fact that for lines $\widetilde{L} = \delta$ with $\mathcal{K}(\delta) \le w' - (t-1)$, one has $\widetilde{\mathcal{K}}(\widetilde{L}) \le t + (t-1)q$. If

$$a_{w'-1}q + \ldots + a_{w'-(t-2)}(t-2)q + \sum_{u\le w'-(t-1)} a_u(t-1)q \le A$$

we have that $|\widetilde{\mathcal{K}}| \le tv_{k-1} + A$. This implies that $\widetilde{\mathcal{K}}$ contains a hyperplane without 0-points. Hence, by Theorem 1, $\mathcal{K}$ is extendable. □

The idea of Theorem 3 can be used to restrict the spectrum not only of the maximal hyperplanes, but also of hyperplanes with a smaller multiplicity. Unfortunately, the value of A is not known in general. Partial results for the plane case were proved in [1] and [2].

4. A THEOREM ON THE EXTENDABILITY OF GRIESMER ARCS

In this section we prove our main extendability result for Griesmer arcs. Consider a Griesmer t-quasidivisible arc $\mathcal{K}$, $t < q$, with parameters (n, w) in $\mathrm{PG}(k-1, q)$. Set $d = n - w$ and let $C_{\mathcal{K}}$ be a linear code associated with $\mathcal{K}$. The code $C_{\mathcal{K}}$ has parameters $[n, k, d]_q$. Write d as

$$d = sq^{k-1} - \sum_{i=0}^{k-2}\varepsilon_i q^i, \quad 0 \le \varepsilon_i < q. \tag{4.1}$$

Then we have $\lceil d/q^j\rceil = sq^{k-j-1} - \sum_{i=j}^{k-2}\varepsilon_i q^i$, which implies

$$n = sv_k - \sum_{i=0}^{k-2}\varepsilon_i v_{i+1}. \tag{4.2}$$

Let us note that with this notation $t = \varepsilon_0$, since $n + \varepsilon_0 \equiv w \pmod q$. Denote by w_j the maximal multiplicity of a subspace S of codimension j of $\mathrm{PG}(k-1,q)$: $w_j = \max_{\operatorname{codim} S = j} \mathcal{K}(S)$, $j = 1, \ldots, k-1$. We have

$$w_j = \sum_{i=j}^{k-1} \lceil \frac{d}{q^i} \rceil = s v_{k-j} - \sum_{i=j}^{k-2} \varepsilon_i v_{i-j+1}. \tag{4.3}$$

By convention, $w_0 = n$.

In the next lemmas we establish some important properties of the arc $\widetilde{\mathcal{K}}$.

Lemma 1. *Let $\mathcal{K}$ be a t-quasidivisible $(n, n-d)$-Griesmer arc with d given by (4.1). Let S be a subspace of codimension 2 contained in the hyperplane H_0 with $\mathcal{K}(H_0) = w_1 - aq$, where $a \geq 0$ is an integer.*

(i) If $\mathcal{K}(S) = w_2 - a - b$, $0 \leq b \leq t-2$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + bq$;

(ii) If $\mathcal{K}(S) = w_2 - a - b$, $b \geq t-1$, then $\widetilde{\mathcal{K}}(\widetilde{S}) \leq t + (t-1)q$.

Proof. (i) Denote by H_i the hyperplanes through S in Σ. Set $\mathcal{K}(H_i) = w_1 - \alpha_i$, $i = 1, \ldots, q$. Note that $\widetilde{\mathcal{K}}(H_i) \equiv n + t - w_1 + \alpha_i \equiv \alpha_i \pmod q$, since $n + t \equiv w_1 \pmod q$. Thus $\widetilde{\mathcal{K}}(H_i) \leq \alpha_i$. Furthermore, we have

$$\begin{aligned} n &= \sum_{i=0}^{q} \mathcal{K}(H_i) - q\mathcal{K}(S) \\ &= (q+1)w_1 - \sum_{i=1}^{q} \alpha_i - aq - q(w_2 - a - b) \\ &= n + t - \sum_{i=1}^{q} \alpha_i + bq. \end{aligned}$$

This implies that $\sum \alpha_i = t + bq$. On the other hand,

$$\begin{aligned} \widetilde{\mathcal{K}}(\widetilde{S}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{H}_i) \\ &\leq \sum_{i=1}^{q} \alpha_i \pmod q \\ &= t + bq. \end{aligned}$$

(ii) This follows by the facts that $\widetilde{\mathcal{K}}(\widetilde{S}) \equiv t \pmod q$, each point is of multiplicity at most t and the line $\widetilde{S}$ is incident with the 0-point $\widetilde{H}_0$. □

Lemma 2. *Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ be as in Lemma 1. Let T be a subspace of codimension 3 in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}(T) = w_3$. Then*

$$\widetilde{\mathcal{K}}(\widetilde{T}) \leq t(q+1) + \varepsilon_1 q.$$

Proof. Denote by S_i, $i = 0, \ldots, q$, the subspaces of codimension 2 through T in a maximal hyperplane H. Set $\mathcal{K}(S_i) = w_2 - \alpha_i$. We have that

$$\begin{aligned} \mathcal{K}(H) = w_1 &= \sum_{i=0}^{q} \mathcal{K}(S_i) - q\mathcal{K}(T) \\ &= (q+1)w_2 - \sum_{i=0}^{q} \alpha_i - qw_3 \\ &= (q+1)(sv_{k-2} - \varepsilon_{k-2}v_{k-3} - \ldots - \varepsilon_3 v_2 - \varepsilon_2 v_1) - \\ &\qquad q(sv_{k-3} - \varepsilon_{k-2}v_{k-4} - \ldots - \varepsilon_3 v_1) - \sum_{i=0}^{q} \alpha_i. \end{aligned}$$

Since $(q+1)v_{j-1} - qv_{j-2} = v_j$, this simplifies to

$$\begin{aligned} w_1 &= sv_{k-1} - \varepsilon_{k-2}v_{k-2} - \ldots - \varepsilon_3 v_3 - (q+1)\varepsilon_2 v_1 - \sum_{i=0}^{q} \alpha_i \\ &= sv_{k-1} - \varepsilon_{k-2}v_{k-2} - \ldots - \varepsilon_3 v_3 - \varepsilon_2 v_2 - \sum_{i=0}^{q} \alpha_i \\ &= w_1 + \varepsilon_1 v_1 - \sum_{i=0}^{q} \alpha_i. \end{aligned}$$

This implies that $\sum_{i=0}^{q} \alpha_i = \varepsilon_1 v_1 = \varepsilon_1 < q$. By Lemma 1, $\widetilde{\mathcal{K}}(\widetilde{S}_i) \le t + \alpha_i q$, whence

$$\begin{aligned} \widetilde{\mathcal{K}}(\widetilde{T}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{S}_i) - q\widetilde{K}(\widetilde{H}) \\ &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(\widetilde{S}_i) \\ &\le \sum_{i=0}^{q} (t + \alpha_i q) \\ &= t(q+1) + q\sum_{i=0}^{q} \alpha_i \\ &\le t(q+1) + \varepsilon_1 q. \end{aligned}$$

□

Lemma 3. *Let $\mathcal{K}$ be t-quasidivisible Griesmer (n,w)-arc in* PG$(k-1,q)$, $q \ge 3$ *with $d = n - w$ given by (4.1). Let $\widetilde{\mathcal{K}}$ be defined by (3.1). Let further $\varepsilon_0, \varepsilon_1 \le \sqrt{q}$. For every maximal subspace T of codimension 3 in* PG$(k-1,q)$*, i.e. a subspace with $\mathcal{K}(T) = w_3$, there holds*

$$\widetilde{\mathcal{K}}(\widetilde{T}) = t(q+1).$$

Proof. We have that $\widetilde{T}$ is a plane in $\widetilde{\mathrm{PG}}(k-1,q)$. By Lemma 2, $\widetilde{\mathcal{K}}(\widetilde{T}) \le \varepsilon_0(q+1) + \varepsilon_1 q$. Set $\widetilde{\mathcal{K}}(\widetilde{T}) = \varepsilon_0(q+1) + \varepsilon_1' q$, where $0 \le \varepsilon_1' \le \varepsilon_1$.

Assume $\varepsilon_1' > 0$. Set $\widetilde{\mathcal{F}} = \widetilde{\mathcal{K}}|_{\widetilde{T}}$, i.e. $\widetilde{\mathcal{F}}$ is the restriction of $\widetilde{\mathcal{K}}$ to the plane $\widetilde{T}$ in the dual geometry. Define a dual plane arc $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ by

$$\mathcal{F}(\widetilde{L}) = i \quad \text{iff} \quad \widetilde{\mathcal{F}}(L) = t + iq.$$

Denote by (A_i) the spectrum of $\widetilde{\mathcal{F}}$. We have

$$\begin{aligned} \sum A_{t+iq} &= q^2 + q + 1, \\ \sum (t+iq) A_{t+iq} &= (\varepsilon(q+1) + \varepsilon_1' q)(q+1) \end{aligned}$$

for some $\varepsilon_1' \le \varepsilon_1$. This implies $\sum_i iA_{t+iq} = \varepsilon_1'(q+1) + \varepsilon_0$.

Now let us denote by B_i the number of lines L with $\widetilde{\mathcal{F}}(L) = t + iq$ through a fixed point P of multiplicity $c \ge 0$. Then

$$\begin{aligned} \sum B_{t+iq} &= q+1\,, \\ \sum (t+iq) B_{t+iq} &= (q+1)\varepsilon_0 + \varepsilon_1' q + cq\,, \end{aligned}$$

which implies $\sum iB_{t+iq} = \varepsilon_1' + c$. Hence $\mathcal{F}$ is a $(\varepsilon_0'(q+1) + \varepsilon_0, \varepsilon')$-blocking set.

From $\varepsilon_0, \varepsilon_1 < \sqrt{q}$ and $q \geq 3$ we get that $\varepsilon_0 + \varepsilon_1' < \sqrt{\varepsilon_1' q} + 1$ and, consequently, $\varepsilon'(q+1) + \varepsilon_0 < \varepsilon' q + \sqrt{\varepsilon' q} + 1$. By a well-known result by Ball [1] and De Beule-Storme-Metsch [2], $\mathcal{F}$ contains a line. Going back to $\widetilde{\mathcal{F}}$, this implies that all lines L_i in $\widetilde{T}$ through P have multiplicity at least $t + q = \varepsilon_0 + q$. Now we have

$$\begin{aligned} \varepsilon_0(q+1) + \varepsilon_1 q \geq \widetilde{\mathcal{K}}(\widetilde{T}) &= \sum_{i=0}^{q} \widetilde{\mathcal{K}}(L_i) - q\widetilde{\mathcal{K}}(P) \\ &\geq (q+1)(\varepsilon_0 + q) - q\widetilde{\mathcal{K}}(P) \\ &\geq \varepsilon_0(q+1) + q(q+1) - \varepsilon_0 q. \end{aligned}$$

This implies $q + 1 \leq \varepsilon_0 + \varepsilon_1 < 2\sqrt{q}$, i.e. $(\sqrt{q} - 1)^2 < 0$, which is a contradiction. Therefore $\varepsilon_1' = 0$, which proves the lemma. □

Lemma 4. *Let $\mathcal{K}$ be t-quasidivisible Griesmer (n, w)-arc in* $\mathrm{PG}(k-1, q)$*, $q \geq 3$ with $d = n - w$ given by (4.1). Let $\widetilde{\mathcal{K}}$ be defined by (3.1). Let U be a subspace in* $\mathrm{PG}(k-1, q)$ *with* $\operatorname{codim} U = r$*, $1 \leq r \leq k$, which is of maximal multiplicity w_r (if* $\operatorname{codim} U = k$*, $U = \varnothing$). If $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-2} < \sqrt{q}$, then*

$$\widetilde{\mathcal{K}}(\widetilde{U}) = \varepsilon_0 v_{r-1}.$$

Proof. Assume that the result is proved for all subspaces of codimension up to $r - 1$. Note that $\widetilde{U}$ is an $(r-1)$-dimensional subspace of $\widetilde{\mathrm{PG}}(k-1, q)$.

Let $U \subset S$ be maximal subspaces of codimensions r and $r - 2$, respectively. Denote by T_i, $i = 0, \ldots, q$, the subspaces through U of codimension $r - 1$ that are contained in S. Then at most ε_{r-2} of the subspaces T_i are not of maximal multiplicity, i.e. at least $q + 1 - \varepsilon_{r-2}$ of them are of multiplicity w_{r-1}. Indeed, if the number of the maximal subspaces among the T_i's is denoted by γ, then we have $w_{r-2} \leq (q+1)w_{r-1} - qw_r - \gamma$, i.e.

$$\begin{aligned} \gamma &\leq -w_{r-2} + (q+1)w_{r-1} - qw_r \\ &= \varepsilon_{r-1} v_2 + \varepsilon_{r-2} v_1 - (q+1)\varepsilon_{r-1} v_1 \\ &= \varepsilon_{r-2}. \end{aligned}$$

Since U is a subspace of maximal multiplicity, there exists a maximal hyperplane H containing U. Hence $\widetilde{U}$ contains a 0-point with respect to $\widetilde{\mathcal{K}}$, say $\widetilde{P}$. In the case of codimension k, we can take as $\widetilde{P}$ any 0-point in $\widetilde{\mathrm{PG}}(k-1, q)$.

Consider a projection φ from $\widetilde{P}$ onto some hyperplane $\widetilde{V}$ in $\widetilde{U}$ disjoint from $\widetilde{P}$. We have $\widetilde{V} \cong \mathrm{PG}(r-2, q)$. Define a new arc

$$\mathcal{F} = \frac{1}{q}(\widetilde{\mathcal{K}}^{\varphi} - \varepsilon_0).$$

For every point $X \in \widetilde{V}$ we have $0 \leq \mathcal{F}(X) \leq \varepsilon_0 - 1$. $\varphi(\widetilde{U})$ is a subspace of dimension $r - 2$, $\varphi(\widetilde{T}_i)$ are hyperplanes in $\varphi(\widetilde{U})$ (dimension $r - 3$), and $\varphi(\widetilde{S})$ is a subspace of dimension $r - 4$ contained in all $\varphi(\widetilde{T}_i)$. By the induction hypothesis $\mathcal{F}(\varphi(\widetilde{T}_i)) = 0$ for T_i of maximal multiplicity, i.e. $\mathcal{K}(T_i) = w_{r-1}$. Without loss of generality T_i, $i = \varepsilon_{r-2}, \ldots, q$, are maximal. So, the points $X \in \widetilde{V}$ with $\mathcal{F}(X) > 0$ are contained in the subspaces $\varphi(\widetilde{T}_j)$ with $j \in \{0, \ldots, \varepsilon_{r-2} - 1\}$.

We can repeat the argument from the last two paragraphs to another subspace S' of codimension $r - 2$ containing U. We get that the points $X \in \widetilde{V}$ are contained in another ε_{r-2} subspaces of $\varphi(\widetilde{U})$, say $\varphi(\widetilde{T}_j')$ with $j \in \{0, \ldots, \varepsilon_{r-2} - 1\}$. So the non-zero points of $\mathcal{F}$ are contained $\varphi(\widetilde{T}_i) \cap \varphi(\widetilde{T}_j')$, where $i, j \in \{0, \ldots, \varepsilon_{r-2} - 1\}$. Hence the number of points X with $\mathcal{F}(X) > 0$ does not exceed

$$\varepsilon_{r-2}^2 v_{r-3} \leq q v_{r-3} = v_{r-2} - 1.$$

Let $X \in \widetilde{V}$ with $\mathcal{F} = c \geq 0$. Every point in $\varphi(\widetilde{U})$ is incident with v_{r-2} lines. Thus, there is a line $L \in \widetilde{V}$ through X which contains apart from X just 0-points. This line is the image of a plane π which has q-lines of multiplicity ε_0 and one line of multiplicity $\varepsilon_0 + cq$, where $c \leq \varepsilon_0 - 1$ (with respect to $\widetilde{\mathcal{K}}$). Thus, $\widetilde{\mathcal{K}}(\pi) = \varepsilon_0(q+1) + cq$ and, by Lemma 3, we should have $c = 0$.

Thus, $\mathcal{F}(X) = 0$ for all $X \in \widetilde{V}$ and all lines through P in $\widetilde{U}$ are t-lines. This proves the lemma. □

Now we can prove our main theorem.

Theorem 4. *Let $\mathcal{K}$ be a Griesmer $(n, n-d)$-arc which is t-quasidivisible modulo q with d given by (4.1). Let $t = \varepsilon_0, \dots \varepsilon_{k-2} < \sqrt{q}$. Then $\mathcal{K}$ is t-extendable.*

Proof. By Lemma 4, $\widetilde{\mathcal{K}}$ is a (tv_{k-1}, tv_{k-2})-minihyper. By Corollary 3.5 from [8], every (xv_{k-1}, xv_{k-2}) minihyper in $\mathrm{PG}(k-1, q)$ with $x \leq q - \frac{q}{p}$ is the sum of hyperplanes. Since $t < \sqrt{q}$, the result follows. □

We conclude with an example illustrating our approach to the extendability of incomplete caps. Let $\mathcal{K}$ be a (q^2+1-t)-cap in $\mathrm{PG}(3, q)$ with $t < \sqrt{q}$. Assume the largest hyperplane (plane) has multiplicity $q+1$. This is obviously always the case for odd q. The code $C_{\mathcal{K}}$ associated with $\mathcal{K}$ has parameters $[q^2+1-t, 4, q^2-q-t]_q$ and $d = q^2 - q - t = q^3 - (q-1)q^2 - q - t$, i.e. $s = 1$, $\varepsilon_2 = q - 1$, $\varepsilon_1 = 1$, $\varepsilon_0 = t < \sqrt{q}$. The admissible multiplicities of planes are $q+1, \dots, q+1-t, 1$ and 0. Since $\varepsilon_2 \geq \sqrt{q}$, we cannot apply Theorem 4 directly. We can state only that if L is a 2-line, then $\widetilde{\mathcal{K}}(\widetilde{L}) = t$. Nevertheless, we can prove the t-extendability of $\mathcal{K}$.

At first, we prove that every point of $\mathcal{K}$ is incident with an 1-plane. Consider a projection from such 1-point P onto a plane π not incident with P. The induced arc $\mathcal{K}^{\varphi}$ is a (q^2-t, q)-arc and its complement is a $(q+1-t, 1)$-blocking set. Since $t < \sqrt{q}$, it contains a line L and the plane $\langle L, P \rangle$ is an 1-plane in $\mathrm{PG}(3, q)$. Now, by Lemma 1, $\widetilde{\mathcal{K}}(\widetilde{L}) = t$.

Now consider an 1-line L_0 and assume it is incident only with planes of multiplicity at least $q+1-t$. Consider one such plane π with $\mathcal{K}(\pi) = q - b$, $b \leq t-1$. Let P be the 1-point on L_0 and denote the other 1-lines in π by $L_1, \dots, L_b$. One of them is on the 1-plane through P. Consider the plane $\widetilde{P}$ in the dual geometry. Now $\widetilde{\pi}$ is a 0-point and the $q+1-b$ of the lines through it are t-lines, while the remaining b lines are t or $(t+q)$-lines. This implies that $\widetilde{\mathcal{K}}(\widetilde{P}) \leq t(q+1) + bq$ and, by Lemma 3, we have $b = 0$. So, we have proved that for every 1-line L, $\widetilde{\mathcal{K}}(\widetilde{L}) = t$.

Now consider a 1-plane π. Let L be a 0-line in π which is contained in another 1-plane (different from π). Counting the multiplicities of the planes through L, we get $\widetilde{\mathcal{K}}(\widetilde{L}) = t$. There are $q^2 - t$ such lines. Hence $q^2 + q + 1 - t$ of the lines through $\widetilde{\pi}$ are t-lines, and the remaining lines have multiplicity t or $t + q$. Now $|\widetilde{\mathcal{K}}| = t(q+1) + tq$ and, again by Lemma 3, $|\widetilde{\mathcal{K}}| = t(q+1)$. This implies that $\widetilde{\mathcal{K}}$ is a sum of planes and $\mathcal{K}$ is t-extendable.

ACKNOWLEDGEMENTS. This research has been supported by the Science Research Fund of Sofia University under Contract No. 37/28.04.2014.

5. REFERENCES

1. Ball, S.: Multiple blocking sets and arcs in finite planes. *J. London Math. Soc.*, **54**, no. 2, 1996, 581–593.
2. De Beule, J., K. Metsch, L. Storme: Characterization results on weighted minihypers and on linear codes meeting the Griesmer bound. *Adv. in Math. of Comm.*, **2**, no. 3, 2008, 261–272.

3. Dodunekov, S., J. Simonis: Codes and projective multisets. *Electronic Journal of Combinatorics*, **5**, 1998, R37.
4. Hill, R.: An extension theorem for linear codes. *Des. Codes and Crypt.*, **17**, 1999, 151–157.
5. Hill, R., P. Lizak: Extensions of linear codes. In: Proc. Int. Symp. on Inf. Theory, Whistler, BC, Canada 1995.
6. Landjev, I.: The geometric approach to linear codes. In: Finite Geometries, (eds. A. Blokhuis et al.), Kluwer Acad. Publ., 2001, 247–256.
7. Landjev, I., A. Rousseva: An extension theorem for arcs and linear codes, *Probl. Inform. Transmission*, **42**, 2006, 65–76.
8. Landjev, I., P. Vandendriessche: A study of (xv_t, xv_{t-1})-minihypers in $\mathrm{PG}(t,q)$. *J. Comb. Theory* Ser. A, **119**, 2012, 1123-1131.
9. Maruta, T.: On the extendability of linear codes. *Finite Fields and Appl.*, **7**, 2001, 350–354.
10. Maruta, T.: Extendability of linear codes over $\mathrm{GF}(q)$ with minimum distance, d, $\gcd(d,q) = 1$. *Discrete Math.*, **266**, 2003, 377–385.
11. Maruta, T.: A new extension theorem for linear codes. *Finite Fields and Appl.*, **10**, 2004, 674–685.
12. Maruta, T.: Extension theorems for linear codes over finite fields. *J. of Geometry*, **101**, 2011, 173–183.
13. Yoshida, Y., T. Maruta: An extension theorem for $[n,k,d]_q$-codes with $\gcd(d,q) = 2$. *Australas. J. Combin..*, **48**, 2010, 117–131.

Received on 22 March, 2014

Ivan N. Landjev

New Bulgarian University
21 Montevideo Str.
BG-1618 Sofia
BULGARIA
e-mail: i.landjev@nbu.bg

Institute of Mathematics and Informatics
Bulgarian Academy of Science
bl. 8 Acad. G. Bonchev Str.
BG-1164 Sofia
BULGARIA
e-mail: ivan@math.bas.bg

Assia P. Rousseva
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5 blvd. J. Bourchier, BG-1164 Sofia
BULGARIA
e-mail: assia@fmi.uni-sofia.bg