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# COMPOSITION OF INVERSE PROBLEMS WITH A GIVEN LOGICAL STURCTURE

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The paper presents a method for obtaining problems whose conclusions contain disjunctive propositions. These problems constitute a version of inverse problems with a given logical structure. The logical models in the groups of problems studied have been interpreted comprehensively. Equivalent problems have been given by keeping or not keeping the condition of homogeneity in their conclusion.

**Keywords:** Inverse problems, composition of problems with given logical structures.

**2000 Math. Subject Classification:** Primary 51F20, Secondary 51M15

## 1. INTRODUCTION

In mathematical logic a propositional calculus (also called sentential calculus or sentential logic) is a formal system in which formulas of a formal language may be interpreted to represent propositions. A system of rules and logical statements allows certain formulas to be derived. These derived formulas may be interpreted to be true propositions. Usually in Truth-functional propositional logic, formulas are interpreted as having either a truth value of *true* or a truth value of *false*.

Using the sentential logic in this paper we propose a composing technology of new problems as an interpretation of specific logical models. Our aim is to give suitable logical models for formulation of *equivalent* problems and *generating* problems of a given problem.

## 2. PRELIMINARIES

In logic, a set of symbols is commonly used to express logical representations. Let us recall the basic symbols and logical representations we shall deal with.

Let $p$ and $q$ be two statements.

i) $p \wedge q$ denotes *logical conjunction* (should be read as "*p and q*"). The statement $p \wedge q$ is *true* if $p$ and $q$ are both true; else it is *false*.

ii) $p \vee q$ denotes *logical disjunction* (should be read as "*p or q*"). The statement $p \vee q$ is *true* if $p$ or $q$ (or both) are true; if both are false, the statement is *false*.

iii) $p \veebar q$ denotes *exclusive disjunction* (should be read as "*either p or q*"). The statement $p \veebar q$ is *true* when either $p$ or $q$, but not both, are true.

iv) $\neg p$ denotes *negation* (should be read as "*not p*"). The statement $\neg p$ is *true* if and only if $p$ is false.

v) $p \rightarrow q$ denotes *logical implication* (should be read as "*if p then q*"). The statement $p \rightarrow q$ is *true* just in the case that either $p$ is false or $q$ is true, or both. The statements $p$ and $q$ aren't necessarily related comprehensively to each other.

vi) $p \Rightarrow q$ denotes *material implication* (should be read as "*p implies q*" or "*q follows p*"). The relation $p \Rightarrow q$ means that if $p$ is true then $q$ is also true; if $p$ is false then nothing is said about $q$. The statements $p$ and $q$ are related comprehensively to each other.

vii) $p \leftrightarrow q$ denotes *logical equivalence* (should be read as "*p if and only if q*"). The statement $p \leftrightarrow q$ is *true* just in case either both $p$ and $q$ are false, or both $p$ and $q$ are true. The statements $p$ and $q$ aren't necessarily related comprehensively to each other.

viii) $p \Leftrightarrow q$ denotes *material equivalence* (should be read as "*q is necessary and sufficient for p*"). The relation $p \Leftrightarrow q$ means that $p \Rightarrow q$ and $q \Rightarrow p$. The statements $p$ and $q$ are related comprehensively to each other.

## 3. THEORETICAL BASIS OF THE PROPOSED METHOD FOR GENERATING PROBLEMS

In this section we describe in detail the theoretical basis of the method for generating problems with a given logical structure. In what follows $p_1$, $p_2$; $t$, $p$, $q$, $r$ will stand for statements.

In this paper we deal with a generalization of the formal logical rule [6]

$$(p_1 \to r) \wedge (p_2 \to r) \Leftrightarrow (p_1 \vee p_2 \to r). \tag{$*$}$$

Semantic rules connected with the material implication correspond to the formal derivation rules used in the proofs below. By semantic interpretations the formal derivation rules are called *consequence rules* [1].

This correspondence allows us to formulate and comprehensively use the proposition below.

**Proposition 3.1.** *The following equivalence is true:*

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow t \wedge (p \vee q) \to r. \tag{1}$$

*Proof.* Let statement $p_1$ in (*) have structure $t \wedge p$ and statement $p_2$ in (*) have structure $t \wedge q$. Then

$$(t \wedge p \to r) \wedge (t \wedge q \to r) \Leftrightarrow (t \wedge p) \vee (t \wedge q) \to r \Leftrightarrow t \wedge (p \vee q) \to r,$$

i. e. the conjunction of the problems

$$t \wedge p \to r \tag{2}$$

and

$$t \wedge q \to r \tag{3}$$

is equivalent to the problem

$$t \wedge (p \vee q) \to r. \tag{4}$$

□

Any true proposition could have more than one inverse proposition. However, not every inverse proposition is a true statement. The truth value of an inverse proposition of a given true proposition depends essentially on its composition principle.

According to [6], if a given proposition has the logical structure $p_1 \wedge p_2 \to r$, then each one of the following propositions could be considered to be its inverse: $r \to p_1 \wedge p_2$, $p_1 \wedge r \to p_2$ and $r \wedge p_2 \to p_1$.

The most interesting and important inverse propositions are those that are true as well as independent from the other possible inverse propositions, i. e. the *strongest* inverse propositions.

Equivalence (1) formally describes a method for composing new problems with a given logical structure and for formulating their inverse problems.

According to Proposition 3.1 problems with logical structures (2) and (3) generate a problem with a logical structure (4).

In this paper we consider only problems inverse to problems of type (4) with structure

$$t \wedge r \;\rightarrow\; p \vee q. \tag{5}$$

Problems with logical structures (2) and (3) are said to be *generating* problems with structure (4) and their inverse problems with structure (5).

To change the logical structure in the conclusion of the inverse problem from *logical disjunction* to *exclusive disjunction* we need a dichotomic decomposition of the considered set of geometric objects with respect to any remarkable property and its negation. Such a decomposition guarantees the *homogeneity* [4] of the statements (based on one and the same equivalence relation) in the conclusion of the problem.

**Proposition 3.2.** *The following equivalence is true:*

$$t \wedge r \;\rightarrow\; p \vee q \;\Leftrightarrow\; t \wedge r \;\rightarrow\; p \veebar (\neg p \wedge q). \tag{6}$$

Proposition 3.2 gives the equivalence between problems with a logical structure (5) and problems with a logical structure

$$t \wedge r \;\rightarrow\; p \veebar (\neg p \wedge q). \tag{7}$$

Any problem with a logical structure (7) satisfies the condition of *homogeneity in the conclusion.*

## 4. APPLICATION OF THE METHOD TO SPECIFIC GROUPS OF PROBLEMS

We discuss four groups of problems to illustrate the described generating method. In each of the groups we formulate suitable *generating* problems for the corresponding equivalent and inverse problems.

The problems in each of the proposed groups are comprehensively related to each other.

### 4.1. PROBLEMS OF GROUP I

The statements used for the formulation of the problems in this group are

$t$:=\{*The straight line* $AD$, $D \in BC$, *is a median in* $\triangle ABC$.\}

$p$:= $\{AC = AB\}$

$q$:=$\{\angle BAC = 90^0\}$

$r$:=$\{\angle DAC + \angle ABC = 90^0\}$

First we formulate and solve the *generating* problems.

**Problem 4.1.** *Let the straight line $AD$, $D \in BC$, be a median in $\triangle ABC$. Prove that if $AC = AB$, then $\angle DAC + \angle ABC = 90^0$.*

This problem has a logical structure $t \wedge p \ \rightarrow \ r$. Its proof follows immediately from Fig. 1.

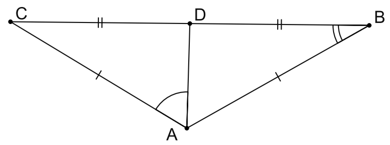


Fig. 1.

**Problem 4.2.** *Let the straight line $AD$, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle BAC = 90^0$, then $\angle DAC + \angle ABC = 90^0$.*

Problem 4.2 has a logical structure $t \wedge q \ \rightarrow \ r$. The proof follows easily from Fig. 2.

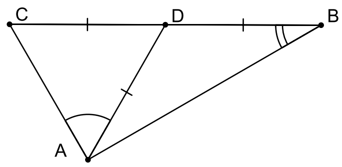


Fig. 2.

According to the logical structures of Problems 4.1 and 4.2 and in view of Proposition 3.1, we construct the following inverse problem with logical structure $t \wedge r \ \rightarrow \ p \vee q$.

**Problem 4.3.** ([3, Problem 3]) *Let the straight line $AD$, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$, then $AC = AB$ or $\angle BAC = 90^0$.*

The next two problems are equivalent to Problem 4.3.

**Problem 4.4.** ([3, Problem 2]) *Let the straight line $AD$, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$ and $\angle BAC \neq 90^0$, then $AC = AB$.*

**Problem 4.5.** ([3, Problem 1]) *Let the straight line $AD$, $D \in BC$, be a* *median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$ and $AC \neq AB$, then $\angle BAC = 90^0$.*

Another version of Problem 4.5 is Problem 246, p. 211 in [10].

In view of Proposition 3.2, Problem 4.3 can be reformulated as follows by keeping the condition of homogeneity in its conclusion (compare also with [3, Problem 4]; [9, p. 24, Problem 6]; [8, p. 22, Problem 1]; [11, p. 265, Problem 312]).

**Problem 4.6.** *Let the straight line $AD$, $D \in BC$, be a median in $\triangle ABC$. Prove that if $\angle DAC + \angle ABC = 90^0$, then $\triangle ABC$ is either isosceles ($AC = AB$), or not isosceles but right-angled ($\angle BAC = 90^0$).*

### 4.2. PROBLEMS OF GROUP II

The statements used for the formulation of the problems in this group are

$t$:={*In $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, is the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, is the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$.*}

$p := \{AC = BC\}$

$q := \{\angle ACB = 60^0\}$

$r := \{JA_1 = JB_1\}$

First we formulate and solve the *generating* problems.

**Problem 4.7.** *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $AC = BC$, then $JA_1 = JB_1$.*

Problem 4.7 has a logical structure $t \wedge p \rightarrow r$.

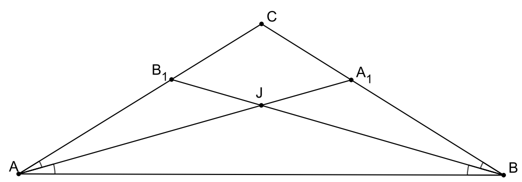


Fig. 3.

*Proof.* Since $AC = BC$, then $\angle CAB = \angle CBA$ and hence $\angle A_1AB = \angle B_1BA$ (Fig. 3). From the Criteria for congruence of triangles we have $\triangle A_1AB \cong \triangle B_1BA$.

As a consequence it follows that $AA_1 = BB_1$, $\triangle AJB$ is isosceles, $AJ = BJ$ and $JA_1 = JB_1$. ☐

**Problem 4.8.** *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $\angle ACB = 60^0$, then $JA_1 = JB_1$.*

This problem has a logical structure $t \wedge q \rightarrow r$.

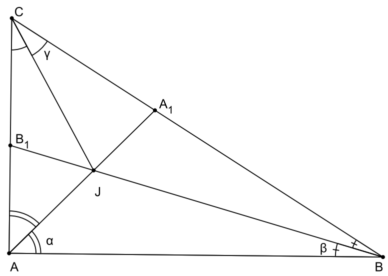


Fig. 4.

*Proof.* Let us denote $\angle BAA_1 = \angle CAA_1 = \alpha$, $\angle ABB_1 = \angle CBB_1 = \beta$ (Fig. 4). Since $J$ is the intersection point of the bisectors $AA_1$ and $BB_1$ of $\triangle ABC$, then $CJ$ is the bisector of $\angle ACB$ and $\angle JCA = \angle JCB = \gamma = 30^0$. Since $\alpha+\beta+\gamma = 90^0$, then $\alpha + \beta = 60^0$, $\angle AJB = 120^0$ and the quadrilateral $CA_1JB_1$ can be inscribed in a circle. Hence, $JA_1 = JB_1$ as chords corresponding to equal angles (arcs) in a circle. ☐

According to the logical structures of Problems 4.7 and 4.8 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \rightarrow p \vee q$.

**Problem 4.9.** ([3, Problem 7]) *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$, then $AC = BC$ or $\angle ACB = 60^0$.*

The next two problems are equivalent to Problem 4.9.

**Problem 4.10.** ([3, Problem 5]) *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$ and $AC \neq BC$, then $\angle ACB = 60^0$.*

**Problem 4.11.** ([3, Problem 6]) *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of*

*$\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$ and $\angle ACB \neq 60^0$, then $AC = BC$.*

In view of Proposition 3.2, Problem 4.9 can be reformulated by keeping the condition of homogeneity in its conclusion.

**Problem 4.12.** ([3, Problem 8]) *Let in $\triangle ABC$ the straight line $AA_1$, $A_1 \in BC$, be the bisector of $\angle CAB$, the straight line $BB_1$, $B_1 \in AC$, be the bisector of $\angle CBA$ and $AA_1 \cap BB_1 = J$. Prove that if $JA_1 = JB_1$, then $\triangle ABC$ is either isosceles $(CA = CB)$ or not isosceles but $\angle ACB = 60^0$.*

### 4.3. PROBLEMS OF GROUP III

The statements used for the formulation of the problems in this group are

$t :=$ {*The straight line $CH$, $H \in AB$, is the altitude and the straight line $CM$, $M \in AB$, is the median of $\triangle ABC$.*}

$p := \{AC = BC\}$

$q := \{\angle ACB = 90^0\}$

$r := \{\angle ACM = \angle BCH\}$

First we formulate and solve the *generating* problems.

**Problem 4.13.** *Let the straight line $CH$, $H \in AB$, be the altitude and the straight line $CM$, $M \in AB$, be the median of $\triangle ABC$. Prove that if $AC = BC$, then $\angle ACM = \angle BCH$.*

Problem 4.13 has a logical structure $t \wedge p \to r$.

*Proof.* In any isosceles triangle the altitude and the median to its base are congruent. Hence, $M \equiv H$ and $\angle ACM = \angle BCH$. □

**Problem 4.14.** *Let the straight line $CH$, $H \in AB$, be the altitude and the straight line $CM$, $M \in AB$, be the median of $\triangle ABC$. Prove that if $\angle ACB = 90^0$, then $\angle ACM = \angle BCH$ (and also $\angle ACH = \angle BCM$).*

This problem has a logical structure $t \wedge q \to r$.

*Proof.* In the right-angled not isosceles $\triangle ABC$ the location of the collinear points $B$, $H$ and $M$ is either $H/BM$ or $M/BH$. Let, for instance, $H/BM$ (Fig. 5). Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. Then $\alpha + \beta = 90^0$. Since $AM = MC$ $(= MB)$, then $\triangle AMC$ is isosceles and $\angle ACM = \alpha$. In the right-angled $\triangle BHC$ we have $\angle BCH = 90^0 - \beta = \alpha$. Hence, $\angle ACM = \angle BCH$ (and also $\angle ACH = \angle BCM$).

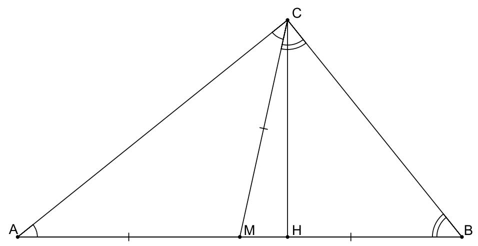


Fig. 5.

For a right-angled isosceles triangle see Problem 4.13. □

According to the logical structures of problems 4.13 and 4.14 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \rightarrow p \vee q$.

**Problem 4.15.** *Let the straight line $CH$, $H \in AB$, be the altitude and the straight line $CM$, $M \in AB$, be the median of $\triangle ABC$. Prove that if $\angle ACM = \angle BCH$, then $AC = BC$ (i. e. $\triangle ABC$ is isosceles) or $\angle ACB = 90^0$ (i. e. $\triangle ABC$ is right-angled).*

*Proof.* Let $\angle CAB = \alpha$ and $\angle CBA = \beta$. In any triangle at least two of the angles must be acute angles. Hence, in $\triangle ABC$ at least one of the angles $\alpha$ and $\beta$ is acute. Let, for instance, $\beta < 90^0$. If we assume that $\alpha \geq 90^0$ then the location of the collinear points $A$, $H$ and $M$ is either $A/HM$, or $A \equiv H$ (Fig. 6). Then for

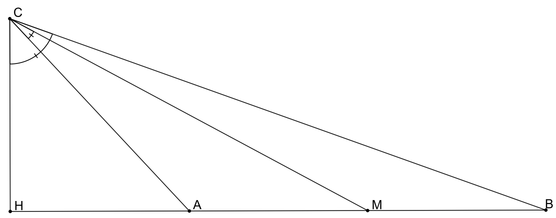


Fig. 6.

the right-angled $\triangle BCH$ is valid $\angle ACM < \angle BCH$, which contradicts the given condition $\angle ACM = \angle BCH$. Hence, $\alpha < 90^0$ and the points $H$ and $M$ lie between the points $A$ and $B$.

There are two possibilities for the points $H$ and $M$ - they either coincide or not.

($i$) Let $H \equiv M$. In this case the median $CM$ in $\triangle\, ABC$ coincides with the altitude $CH$, i. e. $\triangle\, ABC$ is isosceles. If in addition $\angle ACB = 90^0$, then $\triangle\, ABC$ is isosceles right-angled.

($ii$) Let $H \neq M$ and $H/BM$ (the considerations in the case $M/BH$ are analogous). In the considered case $\alpha < \beta$ (Fig. 7).

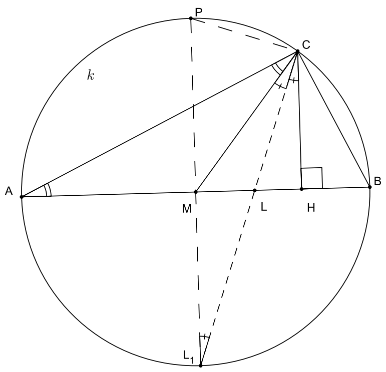


Fig. 7.

Let $CL$, $L \in AB$, be the bisector of $\angle ACB$. It follows that $CL$ is also the bisector of $\angle MCH$ (see also [2, p. 184, problem 29]; [5, p. 41, problem 2.32]).

Let $k$ be the circumscribing circle of $\triangle\, ABC$ and $CL \cap k = L_1$. The point $L_1$ is the middle point of the arc $\widehat{AL_1B}$. The points $C$ and $L_1$ lie on alternate sides of $AB$. The perpendicular projection of $L_1$ onto the chord $AB$ is the middle point $M$. Then the straight line $L_1M$ is the perpendicular bisector of $AB$.

The straight line $CL_1$ cuts the parallel lines $CH$ ($CH \perp AB$) and $L_1M$ ($L_1M \perp AB$) and hence the alternate angles $\angle HCL$ and $\angle ML_1L$ are equal, i. e. $\triangle CML_1$ is isosceles. Thus the point $M$ also lies on the the perpendicular bisector of the chord $CL_1$.

Since the perpendicular bisectors of any two non parallel chords of a circle cut at its center, the point $M$ is the center of $k$, the chord $AB$ is a diameter of $k$ and $\angle ACB = 90^0$.

**Remark 4.16.** Let $P = ML_1 \cap k$. Then $PL_1$ is a diameter of $k$ and $\angle PCL_1 = 90^0$. It is easily seen that $\triangle MPC$ is isosceles and the point $M$ is the center of $k$. $\square$

We reformulate Problem 4.15 by keeping the condition of homogeneity of the conclusion.

**Problem 4.17.** *Let the straight line $CH$, $H \in AB$, be the altitude and the straight line $CM$, $M \in AB$, be the median of $\triangle\, ABC$. Prove that if $\angle ACM = \angle BCH$, then $\triangle ABC$ is either isosceles ($AC = BC$), or not isosceles but right-angled ($\angle ACB = 90^0$).*

The statements used for the formulation of the problems in this group are

$t :=$ {*The middle points of the sides* $BC$, $CA$ *and* $AB$ *of* $\triangle ABC$ *are* $F$, $D$, *and* $E$ *respectively.*}

$p := \{AC = BC\}$

$q := \{\angle ACB = 60^0\}$

$r :=$ { *The center* $G$ *of the circumscribing circle* $k$ *of* $\triangle FDE$ *lies on the bisector of* $\angle ACB$}.

First we formulate and solve the *generating* problems.

**Problem 4.18.** *Let the middle points of the sides* $BC$, $CA$ *and* $AB$ *of* $\triangle ABC$ *be* $F$, $D$, *and* $E$ *respectively. Prove that if* $AC = BC$, *then the center* $G$ *of the circumscribing circle* $k$ *of* $\triangle FDE$ *lies on the bisector of* $\angle ACB$.

This problem has a logical structure $t \wedge p \to r$.

*Proof.* The median $CE$ of the isosceles $\triangle ABC$ is the perpendicular bisector of $AB$ and $DF$ and the bisector of $\angle ACB$. Hence, the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$. □

**Problem 4.19.** *Let the middle points of the sides* $BC$, $CA$ *and* $AB$ *of* $\triangle ABC$ *be* $F$, $D$, *and* $E$ *respectively. Prove that if* $\angle ACB = 60^0$, *then the center* $G$ *of the circumscribing circle* $k$ *of* $\triangle FDE$ *lies on the bisector of* $\angle ACB$.

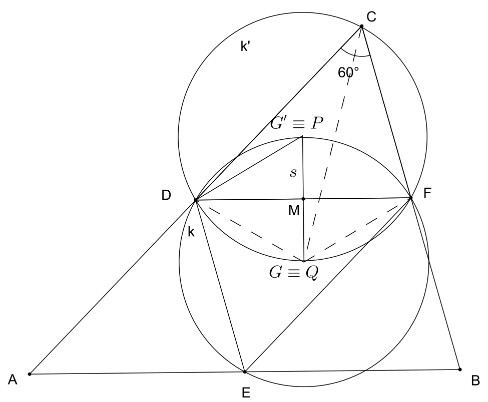


Fig. 8.

This problem has a logical structure $t \wedge q \;\to\; r$.

*Proof.* The quadrilateral $EFCD$ (Fig. 8) is a parallelogram with $\angle DCF = 60^0$. Hence, $\triangle EFD \cong \triangle CDF$ and the circumscribing circles $k$ and $k'$ of $\triangle EFD$ and $\triangle CDF$ respectively have equal radii. The centers $G$ and $G'$ of these circles lie on the perpendicular bisector $s$ of $DF$.

Let $P = s \cap k,\ Q = s \cap k'$. It is easy to be seen that the quadrilateral $FPDQ$ is a rhombus with $\angle PDQ = 60^0$ and $QD = QP = QF$, i. e. the point $Q$ coincides with the center $G$ of $k$. Consequently, the point $P$ coincides with the center $G'$ of $k'$.

The point $Q$ is also the middle point of the arc $\widehat{DQF}$ of $k'$ and then lies on the bisector of $\angle DCF \equiv \angle ACB$. □

According to the logical structures of problems 4.18 and 4.19 and in view of Proposition 3.1 we construct the following inverse problem with logical structure $t \wedge r \;\to\; p \vee q$ (a formulation with a different logical structure is given in [7, Problem 12]):

**Problem 4.20.** *Let the middle points of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$ be $F$, $D$, and $E$ respectively. Prove that if the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$, then $AC = BC$ or $\angle ACB = 60^0$.*

The next two problems are equivalent to Problem 4.20.

**Problem 4.21.** *Let the middle points of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$ be $F$, $D$, and $E$ respectively. Prove that if the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$ and $BC \neq AC$, then $\angle ACB = 60^0$.*

**Problem 4.22.** *Let the middle points of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$ be $F$, $D$, and $E$ respectively. Prove that if the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$ and $\angle ACB \neq 60^0$, then $BC = AC$.*

We reformulate Problem 4.20 by keeping the condition of homogeneity of the conclusion.

**Problem 4.23.** *Let the middle points of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$ be $F$, $D$, and $E$ respectively. Prove that if the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$, then the $\triangle ABC$ is either isosceles ($AC = BC$), or not isosceles but $\angle ACB = 60^0$.*

*Proof.* Let $G'$ be the center of the circumscribing circle $k'$ of $\triangle FDC$ (Fig. 9). In view of the Criteria for congruence of triangles we get that $\triangle FDE \cong \triangle DFC$. It follows that the circumscribing circles $k$ and $k'$ of $\triangle FDE$ and $\triangle DFC$ respectively have equal radii.

Let $M$ be the middle point of $DF$ and $L = GM \cap k'$. The point $G'$ lies on the perpendicular bisector $GM$ of $DF$. Hence, the point $L$ is the middle point of the arc $\widehat{DLF}$ of $k'$ and $CL$ is the bisector of $\angle DCF \equiv \angle ACB$.

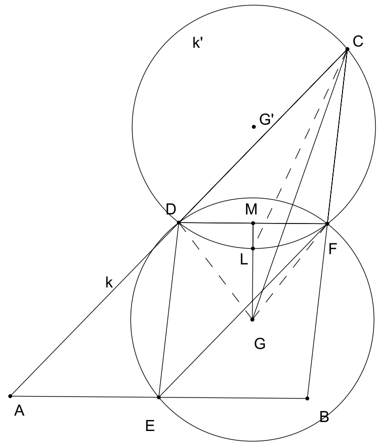


Fig. 9.

Since the center $G$ of $k$ lies on the bisector $CL$ (according to the condition of the Problem), then the straight lines $CL$ and $GM$ either cut at $G$ (have no other common points), or coincide (all of their points are common).

$(i)$ Let $CL \cap GM = L \equiv G$.

In this case $G' \in k$ (Fig. 8) and $\triangle G'DG$ is equilateral, the central $\angle DG'F$ of $k'$ has a measure $120^0$ and hence $\angle ACB = 60^0$.

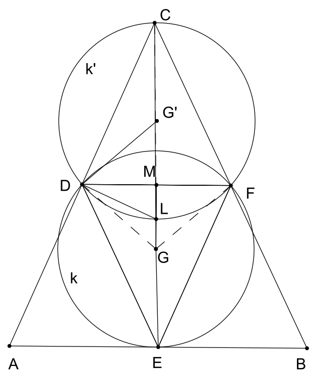


Fig. 10.

$(ii)$ Let $CL \equiv GM$ (Fig. 10).

In this case the bisector $CL$ of $\angle DCF$ coincides with the perpendicular bisector of $DF$. Then $\triangle DCF$ and also $\triangle ABC$ are isosceles, i. e. $AC = BC$. $\square$

## 5. SUMMARY

In this section we formulate a new problem whose proof emphasizes the importance and significance of the described method for generating problems.

The similar conclusions of Problems 4.9 and 4.20 lead to

**Problem 5.1.** *Let the middle points of the sides $BC$, $CA$ and $AB$ of $\triangle ABC$ be $F$, $D$ and $E$ respectively. Let further the straight lines $AA_1$, $A_1 \in BC$, and $BB_1$, $B_1 \in AC$, be the bisectors of $\angle CAB$ and $\angle CBA$, respectively, and let $AA_1 \cap BB_1 = J$.*

*Prove that the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$ if and only if $JA_1 = JB_1$.*

*Proof.* (i) Let the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lie on the bisector of $\angle ACB$.

From Problem 4.20 it follows that $AC = BC$ or $\angle ACB = 60^0$.

- If $AC = BC$ then from Problem 4.7 it follows that $JA_1 = JB_1$.
- If $\angle ACB = 60^0$ then from Problem 4.8 it follows that $JA_1 = JB_1$.

(ii) Let $JA_1 = JB_1$. From Problem 4.9 it follows that either $AC = BC$ or $\angle ACB = 60^0$.

- If $AC = BC$ then from the generating Problem 4.18 it follows that the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$.
- If $\angle ACB = 60^0$ then from the generating Problem 4.19 it follows that the center $G$ of the circumscribing circle $k$ of $\triangle FDE$ lies on the bisector of $\angle ACB$.

□

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