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A SIMPLE CHARACTERIZATION OF THE COMPUTABILITY OF REAL FUNCTIONS

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The TTE-approach to computability of real functions uses infinitary names of the argument's and the function's values, computability being defined as the existence of some algorithmic procedure transforming the names of any argument's value into ones of the corresponding value of the function. Two ways to avoid using such names are considered in the present paper. At each of them, the corresponding characterization of computability of real functions is through the existence of an appropriate recursively enumerable set establishing some relation between rational approximations of the argument's value and rational approximations of the corresponding value of the function. The characterizations in question are derived from ones for computability of functions in metric and in topological spaces.

Keywords: Approximation, computable, continuous, enumeration, enumeration operator, real function, real number, recursive operator, recursively enumerable set, TTE

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1. INTRODUCTION

The widely used TTE-approach to computability of real functions (cf. e.g. [6]) uses infinitary names of the argument's and the function's values, and computability is defined as the existence of some algorithmic procedure transforming all such names of any argument's value into ones of the corresponding value of the function. The standard TTE-computability of real functions[1] is a particular instance of

[1] I.e. the (ρ^p, ρ)-computability in the sense of [6] of partial functions from $\mathbb{R}^p$ to $\mathbb{R}$, and, more generally, the (ρ^p, ρ^q)-computability of partial functions from $\mathbb{R}^p$ to $\mathbb{R}^q$.

TTE-computability of functions in metric spaces, which, under some assumptions satisfied in this particular case, was characterized in [4] without using infinitary names. In the case in question, the corresponding characterization is through the existence of an appropriate recursively enumerable set establishing some relation between rational approximations of the argument's value and rational approximations of the corresponding value of the function. In [5, Example 3.10], a simpler similar characterization of the computability of real functions is given, and it is obtained by using the fact that the standard TTE-computability of real functions is a particular instance of TTE-computability of functions in topological spaces.[2] A somewhat more systematic consideration of these two characterizations is done in the present paper by introducing the notions of a metric approximation net and a topological approximation net for a real function. On the whole, the paper follows the slides of the author's talk at the 2013 Spring Scientific Conference of FMI[3], thus some details are omitted.

1.1. TWO CHARACTERIZATIONS OF THE COMPUTABILITY OF A REAL NUMBER

The two above-mentioned characterizations of computability of a real function can be regarded as analogs of the ones for the notion of computable real number which are indicated below.

Theorem 1. *For any real number y, the following three conditions are equivalent:*

A. The number y is computable.

B. A recursively enumerable set E of $\mathbb{Q} \times \mathbb{N}$ exists such that:

1. $\forall (b,n) \in E \left(|b-y| < \dfrac{1}{n+1} \right)$.

2. $\forall n \in \mathbb{N}\ \exists b \left((b,n) \in E \right)$.

C. The set $\left\{ (b,n) \in \mathbb{Q} \times \mathbb{N} \;\middle|\; |b-y| < \dfrac{1}{n+1} \right\}$ *is recursively enumerable.*

The proof of this theorem is straightforward.

Remark 1. *Of course, condition C is equivalent to the existence of a recursively enumerable subset E of $\mathbb{Q} \times \mathbb{N}$ such that*

$$|b-y| < \frac{1}{n+1} \Leftrightarrow (b,n) \in E$$

for any $b \in \mathbb{Q}$ and any $n \in \mathbb{N}$.

[2]TTE-computability in the topological case is considered, for instance, in [6, Section 3.2] and in [1,2,3,7].

[3]Held in Sofia on March 16, 2013.

Remark 2. *It can be non-constructively proved that Theorem 1 holds also with "recursive" instead of "recursively enumerable".*

1.2. SOME NOTATIONS, ASSUMPTIONS AND DEFINITIONS

For $z = (z_1, \ldots, z_N) \in \mathbb{R}^N$, where $N \in \mathbb{N}^+$, we set

$$\|z\| = \max(|z_1|, \ldots, |z_N|)$$

For $t \in \mathbb{N}$, we set $r_t = \dfrac{1}{t+1}$.

Throughout the paper, it will be supposed that

$$p, q \in \mathbb{N}^+, \ M \subseteq \mathbb{R}^p, \ f : M \to \mathbb{R}^q.$$

Two definitions follow. The notion introduced in the first one is a particular instance of a notion introduced in [4]. The second definition introduces a similar, but simpler notion. Some similarity can be observed between the conditions of these definitions and the conditions B and C in Theorem 1.

Definition 1. A metric approximation net *(abbr.* m.a.n.*) for the function f is a subset S of $\mathbb{Q}^p \times \mathbb{N} \times \mathbb{Q}^q \times \mathbb{N}$ such that the following conditions are satisfied for any $x \in M$:*

1. $\forall (a, m, b, n) \in S \big(\|a - x\| < r_m \Rightarrow \|b - f(x)\| < r_n\big)$.
2. $\forall n \in \mathbb{N}\, \exists m \in \mathbb{N}\, \forall a \in \mathbb{Q}^p \big(\|a - x\| < r_m \Rightarrow \exists b \big((a, m, b, n) \in S\big)\big)$.

Definition 2. A topological approximation net *(abbr.* t.a.n.*) for the function f is a subset S of $\mathbb{Q}^p \times \mathbb{N} \times \mathbb{Q}^q \times \mathbb{N}$ such that*

$$\|b - f(x)\| < r_n \Leftrightarrow \exists a\, \exists m \big((a, m, b, n) \in S \,\&\, \|a - x\| < r_m\big) \tag{1.1}$$

for all $x \in M$, $b \in \mathbb{Q}^p$, $n \in \mathbb{N}$.

The two notions are different. The function f can be chosen so that a m.a.n. for f exists which is not a t.a.n. for it, and a t.a.n. for f exists which is not a m.a.n. for it.

Example 1. *Let $p = q = 1$, $M = \{0\}$, $f(0) = 0$, and let us set*

$$S_1 = \{(a, m, 0, n) \mid a \in \mathbb{Q},\ m, n \in \mathbb{N}\},$$
$$S_2 = \{(0, m, b, n) \mid b \in \mathbb{Q},\ m, n \in \mathbb{N},\ |b| < r_n\}.$$

Then S_1 is a m.a.n. for the function f without being a t.a.n. for it, and S_2 is a t.a.n. for the function f without being a m.a.n. for it.

Remark 3. *Definitions 1, 2 imply immediately that, whenever S is a t.a.n. for the function f, and some subset of S is a m.a.n. for it, the set S is also a m.a.n. for f.*

Despite the difference between the notions of m.a.n. and t.a.n., some essential properties of them are similar. The next theorem is a particular instance of a result from [4].

Theorem 2. *A m.a.n. for the function f exists if and only if f is continuous. Then the following set is a m.a.n. for f containing as subsets all such ones:*

$$\big\{(a,m,b,n)\in\mathbb{Q}^p\times\mathbb{N}\times\mathbb{Q}^q\times\mathbb{N}\,\big|\,\forall x\in M\big(\|a-x\|<r_m\Rightarrow\|b-f(x)\|<r_n\big)\big\}.$$

It is easily seen that Theorem 2 remains true after replacing m.a.n. with t.a.n. in its statement.

2. M.A.N., T.A.N. AND STANDARD TTE-COMPUTABILITY OF REAL FUNCTIONS

From now on, let $\alpha:\mathbb{N}\to\mathbb{Q}^p$ be a computable enumeration of $\mathbb{Q}^p$, and $\beta:\mathbb{N}\to\mathbb{Q}^q$ be a computable enumeration of $\mathbb{Q}^q$. In the terminology of [5], *an α-name* of an element x of $\mathbb{R}^p$ is any function $u:\mathbb{N}\to\mathbb{N}$ such that $\|\alpha(u(m))-x\|<r_m$ for all $m\in\mathbb{N}$, and similarly is defined what is a β-name of an element of $\mathbb{R}^q$. The function f is called (α,β)*-computable* if a recursive operator exists which transforms all α-names of any $x\in M$ into β-names of $f(x)$.

Clearly, the (α,β)-computability of f does not depend of the choice of the computable enumerations α and β, and it is equivalent to the (ρ^p,ρ^q)-computability of f.

The next theorem follows immediately from the main theorem in [4].

Theorem 3. *The function f is (ρ^p,ρ^q)-computable if and only if a recursively enumerable m.a.n. for f exists.*

In [5], another computability notion was considered besides (α,β)-computability. In the case considered here, it looks as follows. Suppose a computable bijective mapping of $\mathbb{N}^2$ of $\mathbb{N}$ is chosen, and let $\langle s,t\rangle$ denote the image of the pair (s,t) under this mapping. We consider the indexed base $\mathcal{U}=\{U_i\}_{i\in\mathbb{N}}$ of the space $\mathbb{R}^p$ and the indexed base $\mathcal{V}=\{V_j\}_{j\in\mathbb{N}}$ of the space $\mathbb{R}^q$, which are defined by means of the equalities

$$U_{\langle k,m\rangle}=\{x\in\mathbb{R}^p\,|\,\|\alpha(k)-x\|<r_m\},$$
$$V_{\langle l,n\rangle}=\{y\in\mathbb{R}^q\,|\,\|\beta(l)-y\|<r_n\}.$$

The function f is called $(\mathcal{U},\mathcal{V})$*-computable* if an enumeration operator exists which, for any $x\in M$, transforms the set $\{i\in\mathbb{N}\,|\,x\in U_i\}$ into the set $\{j\in\mathbb{N}\,|\,f(x)\in V_j\}$.

As seen from [5], standard TTE-computability and $(\mathcal{U}, \mathcal{V})$-computability of f are equivalent.

In the general case studied in [5], some topological spaces $\mathbf{X}$ and $\mathbf{Y}$ with countable bases are considered instead of $\mathbb{R}^p$ and $\mathbb{R}^q$, and $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$, $\mathcal{V} = \{V_j\}_{j\in\mathbb{N}}$ can be any indexed countable bases of these spaces. Under some assumptions, it is shown that the $(\mathcal{U}, \mathcal{V})$-computability of f is equivalent to the existence of a recursively enumerable subset R of $\mathbb{N}^2$ with the following property:

$$\forall x \in M\ \forall j \in \mathbb{N}\,\big(\,f(x) \in V_j \Leftrightarrow \exists i\,\big(\,(i,j) \in R\ \&\ x \in U_i\,\big)\,\big) \tag{2.1}$$

(this is an improvement of a result from [3]).

The above-mentioned assumptions are satisfied in the case considered here thanks to the recursive enumerability of the sets

$$\{(a_1, a_2, r) \in \mathbb{Q}^p \times \mathbb{Q}^p \times \mathbb{Q} \mid \|a_1 - a_2\| < r\},$$
$$\{(b_1, b_2, r) \in \mathbb{Q}^q \times \mathbb{Q}^q \times \mathbb{Q} \mid \|b_1 - b_2\| < r\}$$

(these sets are even recursive). In this case, the property (2.1) is obviously equivalent to the following one:

$$\|\beta(l) - f(x)\| < r_n \Leftrightarrow \exists k, m \in \mathbb{N}\big((\langle k, m\rangle, \langle l, n\rangle) \in R\ \&\ \|\alpha(k) - x\| < r_m\big) \tag{2.2}$$

for any $x \in M$ and all $l, n \in \mathbb{N}$. Making use of (2.2), one easily gets the following result.

Theorem 4. *The function f is (ρ^p, ρ^q)-computable if and only if a recursively enumerable t.a.n. for f exists.*

Proof. Cf. Example 3.10 in [5]. □

2.1. SOME EXAMPLES OF RECURSIVELY ENUMERABLE T.A.N.'S

Example 2. *Let $p = q = 1$, $M = \mathbb{R} \setminus \{0\}$, $f(x) = \dfrac{1}{x}$ for any $x \in M$, and let*

$$S = \left\{ (a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N} \;\middle|\; r_m < |a|,\quad \left|b - \frac{1}{a}\right| + \frac{r_m}{|a|(|a| - r_m)} \le r_n \right\}.$$

We will show that S is a recursively enumerable t.a.n. for f. The recursive enumerability of this set is clear (it is even recursive). To prove that S is a t.a.n. for f, we have to show that, whenever $x \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{Q}$ and $n \in \mathbb{N}$, the inequality $\left|b - \dfrac{1}{x}\right| < r_n$ holds if and only if

$$r_m < |a|, \quad \left|b - \frac{1}{a}\right| + \frac{r_m}{|a|(|a| - r_m)} \le r_n, \quad |a - x| < r_m \tag{2.3}$$

for some $a \in \mathbb{Q}$ *and some* $m \in \mathbb{N}$*. Let* $x \in \mathbb{R} \setminus \{0\}$*,* $b \in \mathbb{Q}$*,* $n \in \mathbb{N}$*. If* $a \in \mathbb{Q}$*,* $m \in \mathbb{N}$ *and the inequalities* (2.3) *hold, then* $|x| > |a| - r_m$ *and therefore*

$$\left|b - \frac{1}{x}\right| \le \left|b - \frac{1}{a}\right| + \frac{|x-a|}{|a|\,|x|} < \left|b - \frac{1}{a}\right| + \frac{r_m}{|a|(|a| - r_m)} \le r_n.$$

Suppose now that $\left|b - \frac{1}{x}\right| < r_n$*. Then*

$$r_m < |x|, \quad \left|b - \frac{1}{x}\right| + \frac{r_m}{|x|(|x| - r_m)} < r_n$$

for some $m \in \mathbb{N}$*. At such a choice of* m*, the inequalities* (2.3) *will be satisfied by any rational number* a*, which is sufficiently close to* x*.*

Remark 4. *It can be shown that* $\{(a, m, b, n) \in S \mid ab = 1\}$ *is a m.a.n. for* f*. Making use of Remark 3, we conclude that* S *is also a m.a.n. for* f*.*

Example 3. *Let* $p = q = 1$*,* $M = \mathbb{R}$*,* $f(x) = \cos x$ *for all* $x \in M$*. For any* $k \in \mathbb{N}$*, let* S_k *be the set of all* $(a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N}$ *satisfying the inequalities*

$$a^2 \le (2k+1)(2k+2), \quad |b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \le r_n, \tag{2.4}$$

where

$$\sigma_k(a) = (-1)^k \frac{a^{2k}}{2(2k)!} + \sum_{i<k} (-1)^i \frac{a^{2i}}{(2i)!}.$$

Let $S = \bigcup_{k=0}^{\infty} S_k$*. The set* S *is recursively enumerable. We will show that it is a t.a.n. for the function* f*. Indeed, let* $x \in M$*,* $b \in \mathbb{Q}$*,* $n \in \mathbb{N}$*. We will prove that the equivalence* (1.1) *holds. Suppose first that* $(a, m, b, n) \in S$ *for some* a *and* m *such that* $|a - x| < r_m$*. Then there exists some* $k \in \mathbb{N}$ *which satisfies the inequalities* (2.4)*, and, using it, we get*

$$\begin{aligned} |b - f(x)| &\le |b - \sigma_k(a)| + |\sigma_k(a) - \cos a| + |\cos a - \cos x| \\ &< |b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \le r_n\,. \end{aligned}$$

Conversely, let $|b - f(x)| < r_n$*. Natural numbers* k *and* m *can be chosen which satisfy the inequalities*

$$x^2 < (2k+1)(2k+2), \quad |b - f(x)| + \frac{x^{2k}}{(2k)!} + r_m < r_n,$$

and then

$$a^2 < (2k+1)(2k+2), \quad |b - f(a)| + \frac{a^{2k}}{(2k)!} + r_m < r_n, \quad |a - x| < r_m$$

for any rational number a sufficiently close to x. At such a choice of k, m and a, the quadruple (a, m, b, n) will belong to S_k, and therefore also to S, because then

$$|b - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m \leq |b - f(a)| + |\cos a - \sigma_k(a)| + \frac{a^{2k}}{2(2k)!} + r_m$$
$$\leq |b - f(a)| + \frac{a^{2k}}{(2k)!} + r_m < r_n.$$

Remark 5. *The same set S is shown in [4] to be a m.a.n. for f.*

Example 4. *Let $p = q = 1$, $M = \mathbb{R} \setminus \mathbb{Z}$, $f(x) = \lfloor x \rfloor$ for any $x \in M$. Then the recursive set*

$$S = \left\{ \left(k + \frac{1}{2}, 1, b, n\right) \middle| \, k \in \mathbb{Z} \;\&\; b \in \mathbb{Q} \;\&\; n \in \mathbb{N} \;\&\; |b - k| < r_n \right\}$$

is a t.a.s. for f. Indeed, let $x \in M$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. If $(a, m, b, n) \in S$ and $|a - x| < r_m$, then $a = k + \frac{1}{2}$, $|b - k| < r_n$ for some integer k, and $r_m = \frac{1}{2}$, thus $\left|k + \frac{1}{2} - x\right| < \frac{1}{2}$, hence $f(x) = k$ and therefore $|b - f(x)| < r_n$. Conversely, if $|b - f(x)| < r_n$ then $(a, m, b, n) \in S$ and $|a - x| < r_m$ for $a = f(x) + \frac{1}{2}$ and $m = 1$.

Remark 6. *The set S from Example 4 is not a m.a.n. for f, since condition 2 of Definition 1 is violated.*

Example 5. *Let p, q, M, f be the same as in Example 4, but S be the set of all $(a, m, b, n) \in \mathbb{Q} \times \mathbb{N} \times \mathbb{Q} \times \mathbb{N}$ which satisfy the inequalities*

$$a + r_m \leq \lfloor a - r_m \rfloor + 1, \quad |b - \lfloor a - r_m \rfloor| < r_n. \tag{2.5}$$

This set is recursive too. We will show that it is also a t.a.n. for f. Let $x \in M$, $b \in \mathbb{Q}$, $n \in \mathbb{N}$. If some $a \in \mathbb{Q}$ and $m \in \mathbb{N}$ satisfy the inequalities (2.5) *and the inequality $|a - x| < r_m$, then $f(x) = \lfloor a - r_m \rfloor$ and therefore $|b - f(x)| < r_n$. Conversely, if $|b - f(x)| < r_n$, then the inequalities* (2.5) *and the inequality $|a - x| < r_m$ can be satisfied by choosing some $m \in \mathbb{N}$ with $x - r_m > \lfloor x \rfloor$ and $x + r_m < \lfloor x \rfloor + 1$, and then choosing the rational number a sufficiently close to x.*

Remark 7. *The set S from Example 5 is a m.a.n. for f, since so is the set $\{(a, m, b, n) \in S \,|\, b = \lfloor a - r_m \rfloor\}$.*

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