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REMARK ON THE NON-INTEGRABILITY OF THE PERTURBED MOTION OF THE PARTICLE IN A CENTRAL FIELD IN CONSTANT CURVATURE SPACES

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In the recent article [1] Kozlov and Harin generalize the motion of a particle in a central field to the case of constant curvature spaces. In this remark we show that the problem of the non-integrability of the perturbed motion in a central field on the sphere and on the Lobachevski's space is reduced to the flat case considered by Holmes and Marsden using Melnikov integrals.

Keywords: constant curvature spaces, central field motion, non-integrability, Melnikov integrals

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1. INTRODUCTION

In [1] Kozlov and Harin generalize the motion of a particle in a central field to the case of constant curvature spaces. They study mainly the cases when all orbits are closed. It turns out that these cases are analogous to the gravitational potential and to the potential of an elastic string. Another important result is that the integrability of generalized two-center problem on a constant curvature surface is established and it is shown that the integrability remains even "elastic forces" are added.

It is natural to consider also the non-integrability of perturbed motion of the particle in a central field in constant curvature spaces. More precisely, we consider the interaction potentials which allow separatrices in the dynamics of the unperturbed problem and these separatrices split after small perturbation. As a tool we shall use the so-called Melnikov's integrals [2].

In this note we shall show that the problem of non-integrability of the perturbed motion in a central field on the sphere and on the Lobachevski's spaces can be reduced to the flat case, considered by Holmes and Marsden [3]. We shall first recall briefly their result, which is an example of a more general treatment of Melnikov's theory.

Consider the perturbed Hamiltonian

$$H^{\varepsilon} = H^0(r, p_r, p_\theta) + \varepsilon H^1(r, p_r, \theta, p_\theta)\,, \tag{1}$$

where $m = 1$, (r, θ) are the usual polar coordinates,

$$H^0 = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r)\,. \tag{2}$$

Let $V(r)$ be a potential with a single maximum, so that for suitable values of $p_\theta \neq 0$ the effective potential has a minimum at r_- and a maximum at r_+ $(r_- < r_+)$ and

$$V(r) + p_\theta^2/(2r^2) \to \infty \quad \text{as} \quad r \to 0\,.$$

Thus H^0 has a homoclinic orbit

$$(\overline{r}(t), \overline{p_r}(t),\ \overline{\theta}(t) + \theta_0,\ \overline{p_\theta})\,,$$

where

$$\overline{r}(t) \to r_+, \quad \overline{p_r}(t) \to 0 \quad \text{as} \quad t \to \pm\infty\,,$$

$$\overline{p_r}(0) = 0, \quad \overline{p_\theta} \neq 0, \quad \overline{\theta}(t) = \int_0^t \Omega(t)\,\mathrm{d}t.$$

The derivative $\Omega(t) = \partial H^0/\partial p_\theta$ is evaluated on the homoclinic orbit.

Proposition 1 ([3]). *Let the Melnikov integral*

$$M(\theta_0) = \int_{-\infty}^{\infty} \left\{H^0, \frac{H^1}{\Omega}\right\}(t, \theta_0)\,\mathrm{d}t\,,$$

where $\{\cdot\}$ is the Poisson bracket, have simple zeros as a function of θ_0. Then for a sufficiently small ε, the system (1) has Smale horseshoes on the energy surface $H^{\varepsilon} = h$ and hence it is non-integrable.

The note is organized as follows. In section 2 we consider the motion of a particle in a central field on S^3, following Kozlov and Harin [1]. A simple construction reduces the problem to the flat case, already discussed in the foregoing. In section 3 we consider briefly the situation on Lobachevski's space. We conclude the note with several remarks.

2. MOTION IN A CENTRAL FIELD IN S^3

In this section we consider the analogue of the classical problem in a central field on S^3. We follow closely Kozlov and Harin [1]. Consider the particle P with unit mass moving in a field of force with the potential V, depending only on the distance between the particle and some fixed point M (say the north pole) on the sphere S^3. Let θ be the length of the arc of the great circle connecting the points P and M. It is well-known that the potential of gravitational interaction satisfies the Laplace equation. Then V has to be a function of the angle θ only and the Laplace equation has to be replaced by the Laplace-Beltrami's one

$$\triangle V = \sin^{-2}\theta \frac{\partial}{\partial\theta}\left(\sin^2\theta \frac{\partial V}{\partial\theta}\right) = 0\,.$$

Its solution is

$$V = -\frac{\gamma}{\tan\theta} + \alpha\,,$$

where α, $\gamma > 0$ are constants. It is seen that in addition to the attracting center M, this field has a repulsive center at the antipodal point M'. It is proven also that when V is an arbitrary function of θ, the trajectories of P lie on two-dimensional sphere containing points M and M'.

Let (θ, φ) be the spherical coordinates on the above mentioned two-dimensional sphere. Then the Lagrangian is

$$L = \frac{1}{2}\left(\dot\theta^2 + \sin^2\theta\dot\varphi^2\right) - V(\theta)\,, \quad V(\theta) = U(\tan\theta)\,.$$

Introduce the polar coordinates by

$$\varphi = \varphi, \quad r = \operatorname{cotan}\frac{\theta}{2}\,,$$

see Fig. 1. (A slightly different construction was used in Dubrovin *et al.* [4] or Kozlov *et al.* [1].) In these variables the Lagrangian becomes

$$L = \frac{1}{2}\left(\frac{4\dot r^2}{(1+r^2)^2} + \frac{4r^2\dot\varphi^2}{(1+r^2)^2}\right) - \tilde U(r)\,.$$

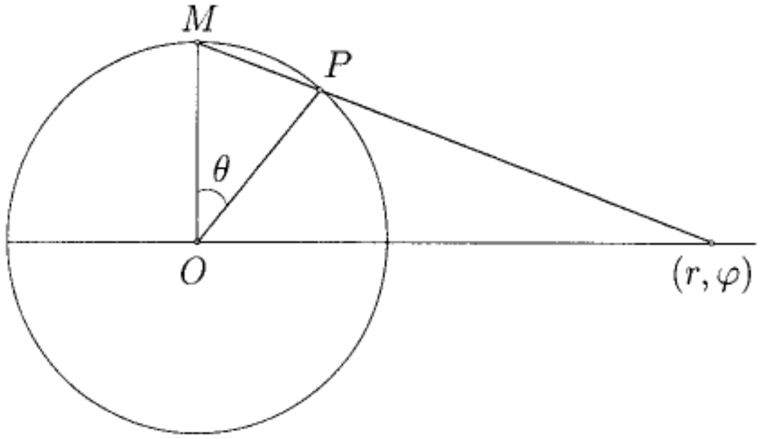

Fig. 1

Next, introduce the "new" time by

$$ds = (1 + r^2)dt/2\,.$$

The 'prime' denotes differentiation with respect to this new time variable, $' = d/ds$. Then

$$L = \frac{1}{2}\left(r'^2 + r^2(\varphi')^2\right) - \tilde{U}(r)\,.$$

Passing to the Hamiltonian, we get

$$H^0 = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \tilde{U}(r)\,, \tag{3}$$

which has *exactly* the form (2) and hence it is integrable.

Consider a small perturbation of the Hamiltonian (4), namely

$$H^\varepsilon = H^0 + \varepsilon H^1(r, p_r, \varphi, p_\varphi). \tag{4}$$

The following proposition is true:

Proposition 2. *Given a potential $V(\theta)$ such that $\tilde{U}(r)$ has a single maximum and for suitable values of $p_\varphi \neq 0$, the effective potential has minimum at r_- and maximum at r_+ $(r_- < r_+)$ and $\tilde{U}(r) + p_\varphi^2/(2r^2) \to \infty$ as $r \to 0$. Thus, $H^0 = h$ has a homoclinic orbit. Then, if*

$$M(\varphi_0) = \int_{-\infty}^{\infty} \left\{H^0, \frac{H^1}{\Omega}\right\}(t, \varphi_0)\, dt\,,$$

evaluated on the homoclinic orbit, has simple zeros, the system (4) *is non-integrable.*

Remark 1. For instance, the class of the potentials of the form

$$\tilde{U}(r) = ar^2 - br^4\,,$$

with positive constants a, b, satisfies the requirements we need.

3. MOTION IN A CENTRAL FIELD IN LOBACHEVSKI'S SPACE

This case is similar to that of the previous section and therefore it will be briefly discussed. Let (χ, φ) be the polar geodesic coordinates. Then the potential $V(\chi)$, analogous to the gravitational potential, has to satisfy the Laplace-Beltrami equation

$$\Delta V = \sinh^{-2}\chi \frac{\partial}{\partial \chi}\left(\sinh^2\chi \frac{\partial V}{\partial \chi}\right) = 0\,,$$

see [5]. Its obvious solution is

$$V = -\frac{\gamma}{\tanh \chi} + \alpha\,,$$

where α, $\gamma > 0$ are constants.

Consider now the Lagrangian of the point with unit mass with more general potential $V(\chi) = U(\tanh \chi)$:

$$L = \frac{1}{2}\left(\dot{\chi}^2 + \sinh^2\chi\dot{\varphi}^2\right) - U(\tanh\chi)\,.$$

Introduce the polar coordinates

$$\varphi = \varphi, \quad r = \tanh\frac{\chi}{2} \quad (r < 1)\,.$$

Then L reads

$$L = \frac{1}{2}\left(\frac{4\dot{r}^2}{(1-r^2)^2} + \frac{4r^2\dot{\varphi}^2}{(1-r^2)^2}\right) - \tilde{U}(r)\,.$$

Let us introduce the "new" time, whose definition in the case under study is

$$ds = (1-r^2)dt/2\,.$$

Passing to the Hamiltonian, we get now

$$H^0 = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + \tilde{U}(r)\,, \tag{5}$$

which has *exactly* the form (2).

Consider, once again, the perturbed system with small Hamiltonian perturbation

$$H^\varepsilon = H^0 + \varepsilon H^1\left(r, p_r, \varphi, p_\varphi\right)\,. \tag{6}$$

The following proposition is true.

Proposition 3. *In the conditions of the Proposition* 2 (*note that here* $r < 1$), *if the Melnikov function* $M(\varphi_0)$ *has simple zeros, the system* (6) *is non-integrable.*

We shall conclude the note with several remarks.

Remark 2. Similarly to Holmes and Marsden [3], it is to be noted that for almost all choices of $\tilde{U}(r)$, the function $M(\varphi_0)$ has simple zeros.

Remark 3. The foregoing problem can be considered as well in higher dimensions. Then the Melnikov's vector can be used (see Wiggins [6]), but certain KAM conditions are needed.

Remark 4. It is seen that the analogue of the classical Kepler problem does not fall in our cases, since it does not possess a homoclinic orbit. We believe that the methods, used by Yoshida [7], can be applied to it for certain classes of perturbations, see also [8].

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