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ON A CLASS OF VERTEX FOLKMAN GRAPHS

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Let $a_1, \dots, a_r$ be positive integers and $m = \sum_{i=1}^{r}(a_i - 1) + 1$. For a graph G the symbol $G \to (a_1, \dots, a_r)$ means that in every r-colouring of the vertices of G there exists a monochromatic a_i-clique of colour i, for some i, $1 \le i \le r$. In this paper we consider the graphs $G \to (a_1, \dots, a_r)$ (vertex Folkman graphs) with $\mathrm{cl}(G) < m - 1$.

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1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We say that G is an n-vertex graph, when $|V(G)| = n$. For $v \in V(G)$ we denote by $\mathrm{Ad}(v)$ the set of all vertices adjacent to v. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p such that the graph G contains a p-clique is denoted by $\mathrm{cl}(G)$ (the clique number of G). A set of vertices in a graph G is said to be independent if no two of them are adjacent. The cardinality of any largest independent set of vertices in G is written as $\alpha(G)$ (the independence number of G).

If $W \subseteq V(G)$, then: $G[W]$ is the subgraph induced by W and $G - W$ is the subgraph induced by $V(G) \setminus W$.

In this paper we shall use also the following notations:

$\chi(G)$ — the chromatic number of G;
$\pi(G)$ — the maximum number of independent edges in G (the matching number of G);

$\overline{G}$ — the complement of graph G;
K_n — the complete graph of n vertices;
P_n — the path of n vertices;
C_n — the simple cycle of n vertices.

By $C_n = v_1, \dots, v_n$ we denote that

$$V(C_n) = \{v_1, \dots, v_n\} \quad \text{and} \quad E(C_n) = \{[v_i, v_{i+1}],\ i = 1, \dots, n-1,\ [v_1, v_n]\}.$$

Let G_1 and G_2 be two graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[v_1, v_2],\ v_1 \in V(G_1),\ v_2 \in v(G_2)\}$

2. THE VERTEX FOLKMAN GRAPHS

Definition. Let G be a graph, $a_1, \dots, a_r$, $r \geq 2$, be positive integers and

$$V(G) = V_1 \cup \ldots \cup V_r, \quad V_i \cap V_j = \varnothing,\ i \neq j,$$

be an r-colouring of the vertices of G. This r-coloring is said to be $(a_1, \dots, a_r)$-free if for all $i \in \{1, \dots, r\}$ the set V_i contains no a_i-clique. The symbol $G \to (a_1, \dots, a_r)$ means that every r-coloring of $V(G)$ is not $(a_1, \dots, a_r)$-free.

It is obvious that:

Proposition 1. *If* $m = \sum\limits_{r=1}^{r}(a_i - 1) + 1$, *then* $K_m \to (a_1, \dots, a_r)$.

Proposition 2. *For any* $r \geq 2$

$$G \to (\underbrace{2, \dots, 2}_{r}) \quad \Longleftrightarrow \quad \chi(G) \geq r + 1.$$

Proposition 3. *Let* $G \to (a_1, \dots, a_r)$ *and* $\{b_1, \dots, b_t\} \subseteq \{a_1, \dots, a_r\}$. *Then* $G \to (b_1, \dots, b_t)$.

Proposition 4. *Let* $A \subseteq V(G)$ *be an independent set of* G *and* $G_1 = G - A$. *Let also* $G \to (a_1, \dots, a_r)$, *where* $a_i \geq 2$ *for some* $i \in \{1, \dots, r\}$. *Then* $G_1 \to (a_1, \dots, a_i - 1, \dots a_r)$.

Proof. Assume the opposite and let $V_1 \cup \ldots \cup V_r$ be an $(a_1, \dots, a_i - 1, \dots, a_r)$-free r-colouring of $V(G_1)$. Then $V_1 \cup \ldots \cup (V_i \cup A) \cup \ldots \cup V_r$ is an $(a_1, \dots, a_i, \dots, a_r)$-free r-colouring of $V(G)$, which is a contradiction.

Proposition 5. *For any permutation* φ *of the symmetric group* S_r *we have*

$$G \to (a_1, \dots, a_r) \quad \Longleftrightarrow \quad G \to (a_{\varphi(1)}, \dots, a_{\varphi(r)}).$$

Let $a_1, \dots, a_r$, $r \geq 2$, be positive integers. Then we put

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \quad \text{and} \quad a = \max\{a_1, \dots, a_r\}. \tag{1}$$

We put also

$$H(a_1,\dots,a_r;q)=\{G:G\to(a_1,\dots,a_r) \text{ and } \mathrm{cl}(G)<q\}$$

$$F(a_1,\dots,a_r;q)=\min\{|V(G)|:G\in H(a_1,\dots,a_r;q)\}$$

It is clear that if $\mathrm{cl}(G)<a$, then there exists an $(a_1,\dots,a_r)$-free r-colouring of $V(G)$. Folkman has proved in [3] that if $q\ge a+1$, then $H(a_1,\dots,a_r;q)\neq\varnothing$. The graphs of $H(a_1,\dots,a_r;q)$, $q\ge a+1$, will be called the vertex Folkman graphs. The numbers $F(a_1,\dots,a_r;q)$ are called vertex Folkman numbers.

It is clear that K_{m-1} has an $(a_1,\dots,a_r)$-free r-colouring of $V(K_{m-1})$. It is clear also that from $\chi(G)\le m-1$ it follows that G has an $(a_1,\dots,a_r)$-free vertex r-colouring. Therefore we have the following:

Proposition 6. *If $G\to(a_1,\dots,a_r)$, then $\chi(G)\ge m$.*

Since $K_m\to(a_1,\dots,a_r)$ and $K_{m-1}\not\to(a_1,\dots,a_r)$, if $q\ge m+1$, we have $F(a_1,\dots,a_r;q)=m$.

For the numbers $F(a_1,\dots,a_r;m)$ the following facts are known:

Theorem A ([6]). *Let $a_1,\ a_2,\ \dots,\ a_r,\ r\ge 2$, be positive integers and m and a satisfy* (1). *If $m\ge a+1$, then $F(a_1,\dots,a_r;m)=m+a$.*

Theorem B ([7]). *Let $a_1,\ a_2,\ \dots,\ a_r,\ r\ge 2$, be positive integers and m and a satisfy* (1). *If $m\ge a+1$, $G\in H(a_1,\dots,a_r;m)$ and $|V(G)|=m+a$, then $G=K_{m-a-1}+\overline{C}_{2a+1}$.*

In the present paper we consider the vertex Folkman numbers $F(a_1,\dots,a_r;m-1)$, $m\ge a+2$.

From Proposition 5 follows that $F(a_1,\dots,a_r;q)$ is a symmetric function and thus we may assume that $a_1\le a_2\le\dots\le a_r$. Note that if $a_1=1$, then $F(a_1,\dots,a_r;q)=F(a_2,\dots,a_r;q)$. Hence we may assume also $a_i\ge 2$, $i=1,\dots,r$.

Theorem A yields $F(2,2;3)=5$.

In the special case $a_1=\dots=a_r=2$, $r\ge 3$, we have:

Theorem C. *For any $r\ge 3$ it is true that*

$$F(\underbrace{2,\dots,2}_{r};r)=\begin{cases}11, & r=3 \text{ or } r=4\\ r+5, & r\ge 5.\end{cases}$$

Mycielski, [8], presented an 11-vertex graph $G\in H(2,2,2;3)$, proving that $F(2,2,2;3)\le 11$. Chvatal, [2], show that Mycielski's graph is the unique 11-vertex graph in the class $H(2,2,2;3)$ and hence $F(2,2,2;3)=11$. The inequality $F(2,2,2,2;4)\ge 11$ is proved in [11] and the inequality $F(2,2,2,2,;4)\le 11$ is proved in [10] and [15] (see also [4] and [12]). The equality $F(\underbrace{2,\dots,2}_{r};r)=r+5$, $r\ge 5$, is proved in [10], [15] and later in [7]. If $r\ge 5$, then $K_{r-5}+C_5+C_5$ is the unique $(r+5)$-vertex graph in $H(\underbrace{2,\ldots,2}_{r};r)$, [10]. The class $H(2,2,2,2;4)$ contains 56 11-vertex graphs, [4]. In [4] it is proved also that $F(2,2,2,2;3)=22$. This is the unique known vertex Folkman number $F(a_1,\ldots,a_r;q)$ for which $q\le m-2$.

3. A LOWER BOUND ON THE VERTEX FOLKMAN NUMBERS $F(a_1,\ldots,a_r;m-1)$

Theorem 1. *Let $a_1, \ldots, a_r$ be positive integers. Let a and m satisfy* (1) *and $m\ge a+2$. Then*

$$F(a_1,\ldots,a_r;m-1)\ge m+a+2.$$

Proof. According to Proposition 5 we may assume that $a_1\le a_2\le\cdots\le a_r=a$. Let $G\in H(a_1,\ldots,a_r;m-1)$. Let also A be an independent set of G, $|A|=\alpha(G)$ and $G_1=C-A$. It follows from $m\ge a+2$ that $a_{r-1}\ge 2$. According to Proposition 4, $G_1\in H(a_1,\ldots,a_{r-1}-1,a_r;m-1)$. According to Theorem A, $|V(G_1)|\ge m+a-1$, i.e. $|V(G)|\ge m+a-1+\alpha(G)$. Since $\alpha(G)\ge 2$, it follows that $|V(G)|\ge m+a+1$. We prove that $|V(G)|\ne m+a+1$. Assume the opposite. Then $|V(G_1)|=m+a-1$ and $\alpha(G)=2$. According to Theorem B, $G_1=K_{m-a-2}+\overline{C}_{2a+1}$. Let $A=\{u_1,u_2\}$, $V(K_{m-a-2})=\{z_1,\ldots,z_{m-a-2}\}$ and $C_{2a+1}=v_1,v_2,\ldots,v_{2a+1}$.

Case 1. $\mathrm{Ad}(u_i)\not\supseteq V(K_{m-a-2})$, $i=1,2$. In this case $\chi(G)=\chi(G_1)=m-1$, which contradicts Proposition 6.

Case 2. $\mathrm{Ad}(u_1)\not\supseteq V(K_{m-a-2})$ and $\mathrm{Ad}(u_2)\supseteq V(K_{m-a-2})$. Let u_1 and z_1 be not adjacent. It follows from $\mathrm{cl}(G)\le m-2$ that $\mathrm{Ad}(u_2)\not\supseteq V(\overline{C}_{2a+1})$. Hence we may assume that u_2 and v_1 are not adjacent. The equality

$$V(G)=\{z_1,u_1\}\cup\{z_2\}\cup\ldots\cup\{z_{m-a-2}\}\cup\{u_2,v_1\}\cup\{v_2,v_3\}\cup\ldots\cup\{v_{2n},v_{2n+1}\}$$

implies $\chi(G)\le m-1$, which contradicts Proposition 6.

Case 3. $\mathrm{Ad}(u_i)\supseteq V(K_{m-a-2})$, $i=1,2$. We put

$$M=\{v_{2i-1}:i=1,\ldots,p-1\}\subseteq V(\overline{C}_{2a+1}).$$

It is clear that M is an $(a-1)$-clique. We prove that

$$M\not\subseteq \mathrm{Ad}(u_i),\quad i=1,2. \tag{2}$$

Assume the opposite and let $M\subseteq \mathrm{Ad}(u_1)$. From $\mathrm{cl}(G)\le m-2$, $\{u_1,v_{2a-1},v_{2a}\}$ is an independent set, which contradicts $\alpha(G)=2$.

We put

$$V'=V(K_{m-a-2})\cup\{v_{2a+1},v_{2a},v_{2a-1},v_{2a-2}\},\quad G'=G[V'],$$
$$V_r'=V(\overline{C}_{2a+1})-\{v_{2a+1},v_{2a},v_{2a-1},v_{2a-2}\},\quad V_r=V_r'\cup\{u_1,u_2\}.$$

Obviously, $\chi(G')=m-a=\sum\limits_{i=1}^{r-1}(a_i-1)$. This equality implies that there exists an $(a_1,\ldots,a_{r-1})$-free $(r-1)$-colouring $V_1\cup\ldots\cup V_{r-1}$ of $V(G')$. Since M is the unique $(a-1)$-clique in V_r', from (2) follows that V_r contains no a-cliques. Hence $V_1 \cup \ldots \cup V_r$ is an $(a_1, \ldots, a_r)$-free r-colouring of $V(G)$, which is a contradiction.

Corollary. $F(4,4;6) \geq 13$.

In [13] it is proved that $F(4,4;6) \leq 14$, but the exact value of $F(4,4;6)$ is unknown.

4. ON THE NUMBERS $F(3,p;p+1)$ AND $F(2,2,p;p+1)$

Lemma 1. *Let* $V' \subseteq V(\overline{C}_{2p+1})$, $|V'| = m$ *and* $G = \overline{C}_{2p+1}[V']$. *If* $m < 2p+1$, *then* $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$.

Proof. It follows from $m < 2p+1$ that $\overline{G}$ is a subgraph of the graph P_{2p} (the path of $2p$ vertices). Hence $\chi(\overline{G}) \leq 2$. Let $V(\overline{G}) = V_1 \cup V_2$, where V_1 and V_2 are independent sets of $\overline{G}$. Then $\alpha(\overline{G}) \geq \max\{|V_1|, |V_2|\}$. Hence $\alpha(\overline{G}) \geq \left\lceil \frac{m}{2} \right\rceil$, i.e. $\mathrm{cl}(G) \geq \left\lceil \frac{m}{2} \right\rceil$.

Let $C_{2p+1} = v_1, v_2, \ldots, v_{2p+1}$, $p \geq 3$, and $M_1 = V(C_{2p+1}) - \{v_1, v_{2p-1}, v_{2p-2}\}$. The map σ defined by $\sigma(v_i) = v_{i+1}$, $i = 1, \ldots, 2p$, and $\sigma(v_{2p+1}) = v_1$ is obviously an automorphism of $\overline{C}_{2p+1}$. We put $M_i = \sigma^{i-1}(M_1)$, $i = 1, \ldots, 2p+1$. We denote by Γ_p the extension of $\overline{C}_{2p+1}$ by adding the new vertices $u_1, \ldots, u_{2p+1}$, each two of which are not adjacent and such that $\mathrm{Ad}(u_i) = M_i$, $i = 1, \ldots, 2p+1$. The graph Γ_3 is given on Fig. 1. This graph is published in [9].

Theorem 2. *For any* $p \geq 3$ *we have* $\Gamma_p \in H(3,p;p+1)$.

Proof. Since $\overline{C}_{2p+1}[M_i] = \overline{K}_2 + \overline{P}_{2p-4}$, $\mathrm{cl}(\overline{C}_{2p+1}[M_i]) = p-1$. Hence $\mathrm{cl}(\Gamma_p) = p$.

Let $V_1 \cup V_2$ be the 2-colouring of $V(\Gamma_p)$. We put $V_i' = V(\overline{C}_{2p+1}) \cap V_i$, $G_i = \overline{C}_{2p+1}[V_i']$, $i = 1, 2$. Assume that $\mathrm{cl}(G_1) < 3$ and $\mathrm{cl}(G_2) < p$. Lemma 1 and $\mathrm{cl}(G_1) < 3$ imply $|V_1'| \leq 4$. Lemma 1 and $\mathrm{cl}(G_2) < p$ yield $|V_2'| \leq 2p-2$, i.e. $|V_1'| \geq 3$. So, $|V_1'| = 3$ or $|V_1'| = 4$.

Case 1. $|V_1'| = 3$. Since $\mathrm{cl}(G_1) < 3$, V_1' contains two non adjacent vertices. Hence we may assume that $v_1, v_2 \in V_1'$. We put $w = V_1' - \{v_1, v_2\}$ and $Q = \{v_{2k+1} : k = 1, \ldots, p\}$. Since Q is a p-clique and $\mathrm{cl}(G_2) < p$, we have $w \in Q$.

Subcase 1a. $w \in Q - \{v_{2p-1}, v_{2p+1}\}$. If $u_2 \in V_1$, then $\{u_2, v_1, w\}$ is a 3-clique in V_1. Let $u_2 \in V_2$. We put $Q' = \{v_{2k} : k = 2, \ldots, p-1\}$. It is clear that Q' is a $(p-2)$-clique. Since $Q' \subseteq \mathrm{Ad}(u_2)$ and $v_{2p+1} \in \mathrm{Ad}(u_2)$, $Q' \cup \{v_{2p+1}, u_2\}$ is a p-clique in V_2.

Subcase 1b. $w = v_{2p-1}$. If $u_{2p} \in V_1$, then $\{v_1, u_{2p}, v_{2p-1}\}$ is a 3-clique in V_1. Let $u_{2p} \in V_2$. We put $Q'' = Q - \{v_{2p-1}, v_{2p-3}\}$. Since $Q'' \cup \{v_{2p-2}\} \subseteq \mathrm{Ad}(u_{2p})$, $Q'' \cup \{v_{2p-2}, u_{2p}\}$ is a p-clique in V_2.

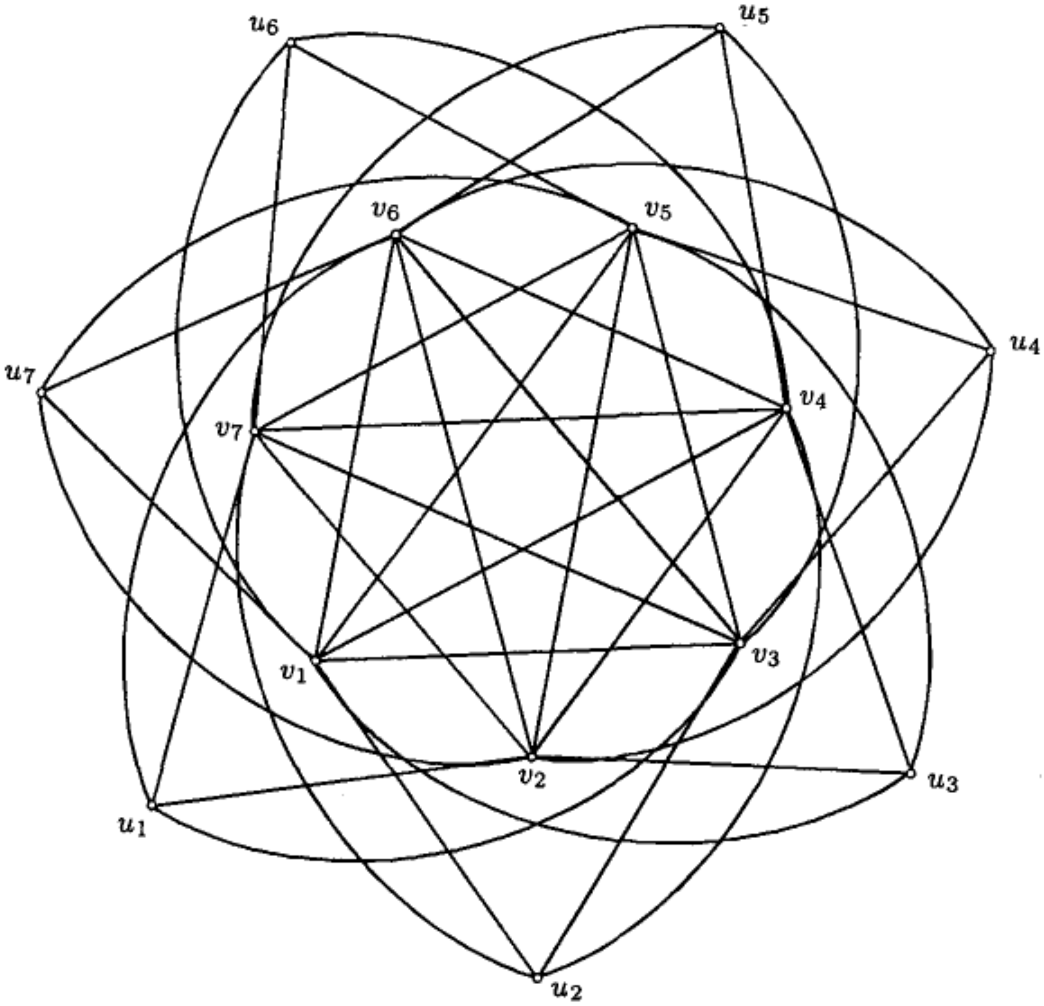

Fig. 1. Graph Γ_3

Subcase 1c. $w = v_{2p+1}$. This subcase is equivalent to $w = v_3$ in subcase 1a.

Case 2. $|V_1'| = 4$. From $\mathrm{cl}(G_1) < 3$ and $\alpha(G_1) = 2$ it follows that $E(\overline{G}_1)$ contains two edges e_1, e_2 without common vertex. Hence, we may assume that $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_i, v_{i+1}\}$ for some $i \in \{3, \ldots, 2p\}$.

Subcase 2a. $i = 2k$, $2 \leq k \leq p$. Let $u_4 \in V_1$. If $k = 2$, then $\{u_4, v_2, v_5\}$ is a 3-clique in V_1. If $3 \leq k \leq p$, then $\{u_4, v_{2k}, v_2\}$ is a 3-clique in V_1. Let $u_4 \in V_2$. We put $Q_1 = \{v_{2l+1} : l = 1, \ldots, k-1\}$ and $Q_2 = \{v_{2l} : l = k+1, \ldots, p\}$, $k < p$. If $k = p$, then $Q_2 = \varnothing$. It is clear that $\widetilde{Q} = Q_1 \cup Q_2$ is a $(p-1)$-clique. Since $\widetilde{Q} \subseteq \mathrm{Ad}(u_4)$, $\widetilde{Q} \cup \{u_4\}$ is a p-clique in V_2.

Subcase 2b. $i = 2k-1$, $2 \leq k \leq p$. Let $m = 2p - 2k + 3$. Then $\sigma^m(v_{2k-1}) = v_1$, $\sigma^m(v_{2k}) = v_2$, $\sigma^m(v_1) = v_{m+1}$, $\sigma^m(v_2) = v_{m+2}$ (the map σ is defined above). Since m is an odd number, the subcase 2b is equivalent to the subcase 2a.

Theorem 3. *If $p \geq 3$, then*

$$2p + 4 \leq F(3, p; p+1) \leq 4p + 2. \tag{3}$$

Proof. From Theorem 2, $F(3, p; p+1) \leq |V(\Gamma_p)| = 4p + 2$. The lower bound in (3) has been proved in Theorem 1.

The inequality $F(3, 3; 4) \leq 14$ is proved in [9]. The work [16] provides a computer proof of the inequality $F(3, 3; 4) \geq 14$ and thus $F(3, 3; 4) = 14$. In [13] it is proved that $F(3,4;5)=13$. The exact value of $F(3,p;p+1)$, $p\geq 5$, is unknown.

Theorem 4. *If $p\geq 3$, then*

$$2p+4\leq F(2,2,p;p+1)\leq 4p+2. \tag{4}$$

Proof. The lower bound in (4) has been proved in Theorem 1. Since $\Gamma_p\to(3,p)$ implies $\Gamma_p\to(2,2,p)$, we have $F(2,2,p;p+1)\leq 4p+2$.

In [14] it is proved that $F(2,2,4;5)=13$. From Theorem 3, $10\leq F(2,2,3;4)\leq 14$. The exact value of $F(2,2,3;4)$ is unknown.

5. ON THE NUMBERS $F(\underbrace{2,\ldots,2}_{r},p;p+r-1)$

We put

$$F(\underbrace{2,\ldots,2}_{r},p;p+r-1)=F_r(2,p),$$

$$H(\underbrace{2,\ldots,2}_{r},p;p+r-1)=H_r(2,p).$$

Theorem 5. *Let $G\in H(2,2,p;p+1)=H_2(2,p)$. Then for any $r\geq 2$, $K_{r-2}+G\in H_r(2,p)$.*

Proof. It follows from $\mathrm{cl}(G)<p+1$ that $\mathrm{cl}(K_{r-2}+G)<p+r-1$. We prove that

$$K_{r-2}+G\to(\underbrace{2,\ldots,2}_{r},p) \tag{5}$$

by induction on r. The base $r=2$ is clear, since $G\in H_2(2,p)$. Assume that $r\geq 3$ and

$$K_{r-3}+G\to(\underbrace{2,\ldots,2}_{r-1},p). \tag{6}$$

Let $V_1\cup\ldots\cup V_{r+1}$ be an $(r+1)$-colouring of $V(K_{r-2}+G)$. Let $w\in V(K_{r-2})$ and $K_{r-2}+G=\{w\}+(K_{r-3}+G)$. If $V_i\cap V(K_{r-3}+G)=\varnothing$ for some i, then from (6) it follows that $V_1\cup\ldots\cup V_{r+1}$ is not $(2,\ldots,2,p)$-free. Let

$$V_i\cap V(K_{r-3}+G)\neq\varnothing,\quad i=1,\ldots,r+1. \tag{7}$$

Assume that V_i, $i=1,\ldots,r$, are independent sets. From (7), $w\notin V_i$, $i=1,\ldots,r$. Hence $w\in V_{r+1}$. Let $V'_{r+1}=V_{r+1}\setminus\{w\}$. Then $V_1\cup\ldots\cup V_{r-1}\cup(V_r\cup V'_{r+1})$ is an r-colouring of $V(K_{r-3}+G)$. From (6), $V_r\cup V'_{r+1}$ contains a p-clique. Since V_r is an independent set, V'_{r+1} contains a $(p-1)$-clique. Hence V_{r+1} contains a p-clique. Thus, (5) holds.

Theorem 6. *For any $p\geq 3$ and $r\geq 2$, one has*

$$2p+r+2\leq F_r(2,p)\leq 4p+r. \tag{8}$$

Proof. Theorem 2 yields $\Gamma_p \in H(2,2,p;p+1)$. From Theorem 5 it follows that $K_{r-2}+\Gamma_p \in H_r(2,p)$. Hence $F_r(2,p) \leq 4p+r$. The lower bound in (8) follows from Theorem 1.

Theorem 7. *For any $r \geq 2$, one has*

$$r+10 \leq F_r(2,4) \leq r+11. \tag{9}$$

Proof. Consider the 13-vertex graph Q (the complementary graph $\overline{Q}$ is given on Fig. 2). It is proved in [14] that $Q \in H(2,2,4;5)$. From Theorem 5, $K_{r-2}+Q \in H_r(2,4)$. Hence $F_r(2,4) \leq r+11$. The lower bound in (9) follows from Theorem 1.

6. ON THE NUMBERS $F(\underbrace{3,\ldots,3}_{r},p;2r+p-1)$

We put

$$F(\underbrace{3,\ldots,3}_{r},p;2r+p-1) = F_r(3,p),$$

$$H(\underbrace{3,\ldots,3}_{r},p;2r+p-1) = H_r(3,p).$$

Theorem 8. *Let $G \in H(3,p;p+1) = H_1(3,p)$. Then for any $r \geq 1$, $K_{2r-2}+G \in H_r(3,p)$.*

Proof. From $\mathrm{cl}(G) < p+1$ we have $\mathrm{cl}(K_{2r-2}+G) < 2r+p-1$. We prove

$$K_{2r-2}+G \to (\underbrace{3,\ldots,3}_{r},p) \tag{10}$$

by induction on r. The base $r=1$ is clear, since $G \in H_1(3,p)$. Assume that $r \geq 2$ and

$$K_{2r-4}+G \to (\underbrace{3,\ldots,3}_{r-1},p). \tag{11}$$

Let $V_1 \cup \ldots \cup V_{r+1}$ be an $(r+1)$-colouring of $V(K_{2r-2}+G)$ and suppose that

$$\text{each } V_i,\ i=1,\ldots,r, \text{ contains no 3-cliques.} \tag{12}$$

Let $K_{2r-2}+G = K_2+(K_{2r-4}+G)$, where $V(K_2)=\{a,b\}$. If $V_i \cap V(K_{2r-4}+G) = \varnothing$ for some i, then from (11) it follows that $V_1 \cup \ldots \cup V_{r+1}$ is not $(3,\ldots,3,p)$-free. Suppose that

$$V_i \cap V(K_{2r-4}+G) \neq \varnothing, \quad i=1,\ldots,r+1. \tag{13}$$

Case 1. $a,b \in V_i$ for some $i \in \{1,\ldots,r\}$. It follows from (13) that V_i contains a 3-clique, which contradicts (12).

Case 2. $a \in V_i$, $b \in V_j$, $i \neq j$, $i,j \in \{1,\ldots,r\}$. We may assume that $a \in V_1$, $b \in V_2$. We put $V_1' = V_1 - \{a\}$, $V_2' = V_2 - \{b\}$. From (12), V_1' and V_2' are independent sets. Hence $V_1' \cup V_2'$ contains no 3-cliques. Consider an r-colouring

$(V_1' \cup V_2') \cup V_3 \cup \ldots \cup V_{r+1}$ of $V(K_{2r-4} + G)$. It follows from (11) and (12) that V_{r+1} contains a p-clique.

Case 3. $a \in V_i$, $i \neq r+1$ and $b \in V_{r+1}$. We may assume that $a \in V_r$. We put $V_r' = V_r - a$, $V_{r+1}' = V_{r+1} - \{b\}$. From (12), V_r' is an independent set. Consider an r-colouring $V_1 \cup \ldots \cup V_{r-1} \cup (V_r' \cup V_{r+1}')$ of $V(K_{2r-4}+G)$. By (11) and (12), $V_r' \cup V_{r+1}'$ contains a p-clique. Since V_r' is independent, V_{r+1}' contains a $(p-1)$-clique. Hence V_{r+1} contains a p-clique.

Case 4. $a, b \in V_{r+1}$. We put $V_{r+1}' = V_{r+1} - \{a, b\}$. Consider an r-colouring $V_1 \cup \ldots \cup (V_r \cup V_{r+1}')$. From (11) and (12), $V_r \cup V_{r+1}'$ contains a p-clique. By (12), V_{r+1}' contains a $(p-2)$-clique. Hence V_{r+1} contains a p-clique. Thus, (10) holds.

Theorem 9. *Let $p \geq 3$ and $r \geq 1$. Then*

$$2p + 2r + 2 \leq F_r(3,p) \leq 4p + 2r. \tag{14}$$

Proof. By Theorem 2 and Theorem 8, $K_{2r-2} + \Gamma_p \in H_r(3,p)$. Hence $F_r(3,p) \leq 4p + 2r$. The lower bound in (14) follows from Theorem 1.

Theorem 10. *There holds*

$$2r + 10 \leq F_r(3,4) \leq 2r + 11, \quad r \geq 1. \tag{15}$$

Proof. The lower bound in (15) follows from Theorem 1. Consider the 13-vertex graph Q (see Fig. 2). It is proved in [13] that $Q \in H_1(3,4)$. According to Theorem 8, $K_{2r-2} + Q \in H_r(3,4)$. Hence $F_r(3,4) \leq 2r + 11$.

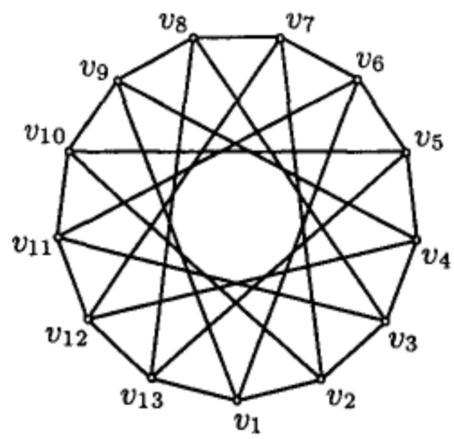

Fig. 2. Graph $\overline{Q}$

Theorem 11. *Let $r \geq 2$. Then*

$$F_r(3,3) = F(\underbrace{3,\ldots,3}_{r+1}; 2r+2) \leq 2r + 10.$$

Proof. We consider the graph Q, which complementary graph $\overline{Q}$ is given on Fig. 2. Obviously, $\alpha(Q) = 2$ and it is true that $\mathrm{cl}(Q) = 4$, [18]. We prove $K_1 + Q \in H(3,3,3;6) = H_2(3,3)$. From $\mathrm{cl}(Q) = 4$ it follows that $\mathrm{cl}(K_1 + Q) = 5$. Let $V_1 \cup V_2 \cup V_3$ be a 3-colouring of $V(K_1 + Q)$ and $V(K_1) = \{w\}$. We may assume $w \in V_1$. Assume also that V_1 contains no 3-cliques. Then $V_1' = V_1 - \{w\}$ is an independent set of Q. From $\alpha(Q) = 2$ it follows $|V_1'| \leq 2$. Hence either $|V_2| \geq 6$ or $|V_3| \geq 6$. Let $|V_2| \geq 6$ and $G = Q[V_2]$. It is clear that $\alpha(G) = 2$. From $\alpha(G) = 2$ and $|V_2| \geq 6$ it follows $\mathrm{cl}(G) \geq 3$, [18], i.e. V_2 contains a 3-clique.

So, $K_1 + Q \to (3,3,3)$ and $\mathrm{cl}(K_1 + Q) = 5$. Hence $K_1 + Q \in H_2(3,3)$. By induction on r it follows $K_{2r-4} + (K_1 + Q) \in H_r(3,3)$ (see the proof of Theorem 8). Hence $F_r(3,3) \leq 2r + 10$.

B. Toft has conjectured that if G is a $(2p+1)$-vertex graph, $\alpha(G) = p$ and $\alpha(G - \{u, v\}) = \alpha(G)$ for all $u, v \in V(G)$, then $G = C_{2p+1}$. This conjecture is verified in [17] and [5] (see problem 8.26, p. 58). The proof in [5], actually, establishes the following stronger statement:

Theorem D. *Let G be a $(2p+1)$-vertex graph, $\alpha(G) = p$, $\alpha(G - v) = \alpha(G)$ for all $v \in V(G)$, and $\alpha(G - \{u, v\}) = \alpha(G)$ for any pair u, v adjacent vertices. Then $G = C_{2p+1}$.*

Theorem D is proved also in [7]. In the proof of Theorem B we shall use the following:

Lemma 2. *Let the graph G be such that $\mathrm{cl}(G - v) = \mathrm{cl}(G)$ for all $v \in V(G)$. Then $\pi(\overline{G}) \geq \mathrm{cl}(G)$.*

This lemma is proved in [17] (see also problem 8, p. 302, in [1]).

The proof of Theorem B. According to Proposition 5 we may assume that $a_1 \leq \cdots \leq a_r = a$. We prove Theorem B by induction on m. By the inequality $m \geq a+1$, the minimal admissible value of m is $a+1$. The base of the induction is then $m = a+1$. From $m = a+1$ it follows that $a_1 = \cdots = a_{r-2} = 1$, $a_{r-1} = 2$, and $\mathrm{cl}(G) = a$. According to Proposition 3, $G \to (2, a)$. By $G \to (2, a)$, $\mathrm{cl}(G) = \mathrm{cl}(G - v)\ \forall v \in V(G)$ and $\mathrm{cl}(G - \{u, v\}) = \mathrm{cl}(G)$ for each pair u, v non adjacent vertices, i.e. the graph $\overline{G}$ satisfies the conditions of Theorem D. Hence $\overline{G} = C_{2a+1}$, i.e. $G = \overline{C}_{2a+1}$.

Let $m \geq a+2$. Let L be a graph such that $V(L) = V(G)$, $E(L) \supseteq E(G)$ and $\mathrm{cl}(L) = m-1$. It is clear that $L \to (a_1, \ldots, a_r)$. We prove that $\mathrm{cl}(L - v_0) < \mathrm{cl}(L)$ for some $v_0 \in V(L)$. Assume the opposite. According to Lemma 2, we have $\pi(\overline{L}) \geq m-1$. Hence

$$\chi(L) \leq m - 1 + \big(|V(L)| - 2(m-1)\big) = a + 1.$$

From $m \geq a+2$ it follows $\chi(L) \leq m-1$. This contradicts Proposition 6.

So, $\exists v_0 \in V(L)$ such that $\mathrm{cl}(L - v_0) < \mathrm{cl}(L) = m-1$. By $m \geq a+2$, $a_{r-1} \geq 2$. According to Proposition 4, $L - v_0 \to (a_1 \ldots, a_{r-1} - 1, a_r)$. Hence $L - v_0 \in H(a_1, \ldots, a_{r-1} - 1, a_r; m-1)$. By the inductive hypothesis, $L - v_0 = K_{m-a-2} + \overline{C}_{2a+1}$. The vertex v_0 is adjacent to each vertex of $V(K_{m-a-2} + \overline{C}_{2a+1})$ (otherwise, $\chi(L) < m$, which contradicts Proposition 6). Therefore, $L = K_{m-a-1} + \overline{C}_{2a+1}$. Since each proper subgraph of $K_{m-a-1} + \overline{C}_{2a+1}$ has an $(a_1, \ldots, a_r)$-free r-colouring of the vertices (see [7], Proposition 3), we have $G = K_{m-a-1} + \overline{C}_{2a+1}$.

The proof of Theorem B is complete.

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