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A GENERALIZATION OF A RESULT OF DIRAC

NEDYALKO NENOV

Let G be a graph, $\chi(G) = r$ and $\mathrm{cl}(G) < r$. Dirac has proved in [2] that for such graph $|V(G)| \geq r+2$ and $|V(G)| = r+2$ only if $G = K_{r-3} + C_5$. The main result in the current article generalizes the proposition mentioned above (Theorem 2.1). As a consequence of Theorem 2.1, some results for Folkman graphs are obtained (Theorems 7.1–7.4, 8.1).

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1. NOTATIONS

We consider only finite, non-oriented graphs, without loops and multiple edges. The vertex set and the edge set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We call a p-clique of G a set of p vertices, each two of which are adjacent. The biggest natural number p, such that the graph G contains a p-clique, is denoted by $\mathrm{cl}(G)$ (the clique number of G).

If $X \subseteq V(G)$, then:

$G[X]$ is the subgraph of G induced by X;
$G - X$ is the subgraph of G induced by $V(G) \setminus X$;
$\Gamma_G(X)$ is the set of vertices in G, adjacent to at least one vertex of X.

In this paper we shall use also the following notations:

$\alpha(G)$ — the independence number of G;
$\chi(G)$ — the chromatic number of G;
$\pi(G)$ — the maximum number of independent edges in G
(the matching number of G);

$\overline{G}$ — the complement of G;
K_n — the complete graph of n vertices;
C_n — the simple cycle of n vertices.

By $G-e$, $e \in E(G)$, we denote the supergraph of G such that $V(G-e) = V(G)$, $E(G-e) = E(G) \setminus \{e\}$ and $G+e$, $e \in E(\overline{G})$, is the supergraph of G for which $V(G+e) = V(G)$, $E(G+e) = E(G) \cup \{e\}$.

Let G_1 and G_2 be graphs without common vertices. We denote by $G_1 + G_2$ the graph G for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[v_1, v_2],\ v_1 \in V(G_1),\ v_2 \in V(G_2)\}$.

2. THE MAIN RESULT

Definition 2.1. The partition $V(G) = V_1 \cup \ldots \cup V_r$ is p-saturated if the union of each p of the sets V_i, $i = 1, \ldots, r$, contains a p-clique of G.

Definition 2.2. The partition $V(G) = V_1 \cup \ldots \cup V_r$ is r-chromatic if the sets V_i, $i = 1, \ldots, r$, are independent.

Definition 2.3. A graph G is p-saturated if each $\chi(G)$-chromatic partition of $V(G)$ is p-saturated.

It is clear that if $\chi(G) \geq 2$, then G is 2-saturated. Dirac has proved in [2] the following proposition:

Let $\chi(G) = r$ and $\mathrm{cl}(G) < r$. Then $|V(G)| \geq r + 2$ and if $|V(G)| = r + 2$, then $G = K_{r-3} + C_5$.

The main result in this paper is the following generalization of the above mentioned proposition:

Theorem 2.1. *Let $\chi(G) = r$, $\mathrm{cl}(G) < r$ and G is p-saturated, but is not $(p+1)$-saturated. Then $|V(G)| \geq r + p$ and $|V(G)| = r + p$ only if $G = K_{r-p-1} + \overline{C}_{2p+1}$.*

We need the next propositions.

Proposition 2.1. *For any graph G*

$$\chi(G) + \pi(\overline{G}) \leq |V(G)|.$$

Proof. Let $|V(G)| = n$, $\pi(\overline{G}) = s$, and $\{x_1, y_1\}$, $\ldots$, $\{x_s, y_s\}$ be a matching of $\overline{G}$. If v_1, $\ldots$, v_{n-2s} are the other vertices of G, then

$$\{x_1, y_1\} \cup \ldots \cup \{x_s, y_s\} \cup \{v_1\} \cup \ldots \cup \{v_{n-2s}\}$$

is an $(n-s)$-chromatic partition of G. Hence, $\chi(G) \leq n - s$. □

Proposition 2.2. *Let $\chi(G) = r$, G be a p-saturated, $2 \le p \le r$, and $V(G) = V_1 \cup \ldots \cup V_r$ be an r-chromatic partition of G. Then for any k, $p \le k \le r$, the graph $G[V_1 \cup \ldots \cup V_k]$ is p-saturated.*

Proof. We put $G[V_1 \cup \ldots \cup V_k] = G'$. It is clear that $\chi(G') = k$. Assume the opposite and let $V_1' \cup \ldots \cup V_k'$ be a k-chromatic partition of $V(G')$ which is not p-saturated. Then the r-chromatic partition $V_1' \cup \ldots \cup V_k' \cup V_{k+1} \cup \ldots \cup V_r$ of $V(G)$ is also not p-saturated, which is a contradiction. □

3. EXAMPLES OF p-SATURATED GRAPHS

Lemma 3.1. *Let $V' \subseteq V(\overline{C}_{2p+1})$, $|V'| = m < 2p + 1$ and $G = \overline{C}_{2p+1}[V']$. Then $\mathrm{cl}(G) \ge \left\lceil \frac{m}{2} \right\rceil$.*

Proof. It follows from $m < 2p + 1$ that $\chi(\overline{G}) \le 2$. Let $V(\overline{G}) = V_1 \cup V_2$, where V_1 and V_2 are independent sets of $\overline{G}$. Then $\alpha(\overline{\overline{G}}) \ge \max\{|V_1|, |V_2|\}$. Hence $\alpha(\overline{G}) \ge \left\lceil \frac{m}{2} \right\rceil$, i.e. $\mathrm{cl}(G) \ge \left\lceil \frac{m}{2} \right\rceil$. □

Proposition 3.1. *For any $p \ge 3$ the graph $\overline{C}_{2p+1}$ is p-saturated, but the graph $\overline{C}_{2p+1} - e$ is not p-saturated for any $e \in E(\overline{C}_{2p+1})$.*

Proof. It is clear that $\chi(\overline{C}_{2p+1}) = p + 1$. Let $V_1 \cup \ldots \cup V_{p+1}$ be $(p + 1)$-chromatic partition of $V(C_{2p+1})$ and let $V' = V(G) \setminus V_i$. We put $G' = \overline{C}_{2p+1}[V']$. From $\alpha(\overline{C}_{2p+1}) = 2$ it follows that $2p - 1 \le |V'| \le 2p$. By Lemma 3.1, $\mathrm{cl}(G') \ge p$. Hence $\overline{C}_{2p+1}$ is p-saturated.

Let $e \in E(\overline{C}_{2p+1})$ and $\widetilde{G} = \overline{C}_{2p+1} - e$. Assume that $V(C_{2p+1}) = \{v_1, \ldots, v_{2p+1}\}$ and $E(C_{2p+1}) = \{[v_i, v_{i+1}], i = 1, \ldots, 2p, [v_1, v_{2p+1}]\}$. We may assume that $e = [v_1, v_{2s+1}]$, $1 \le s \le p - 1$.

Case 1. $s = 1$. In this case $\mathrm{cl}(\widetilde{G}) = p - 1$ and hence $\widetilde{G}$ is not p-saturated.

Case 2. $2 \le s \le p - 1$. In this case $\alpha(\widetilde{G}) = 2$. Hence $\chi(\widetilde{G}) = p + 1$. It is clear that

$$\{v_1\} \cup \{v_2, v_3\} \cup \ldots \cup \{v_{2p}, v_{2p+1}\}$$

is a $(p + 1)$-chromatic partition of $V(\widetilde{G})$. Obviously, $\widetilde{G}[v_1, \ldots, v_{2s+1}] = \overline{C}_{2s+1}$. Hence $\{v_1, \ldots, v_{2s+1}\}$ contains no $(s+1)$-clique of $\widetilde{G}$. Thus $\{v_1, \ldots, v_{2p-1}\}$ contains no p-clique and $\widetilde{G}$ is not p-saturated. □

Proposition 3.2. *Let $2 \le p < r$ and $G = K_{r-p-1} + \overline{C}_{2p+1}$. Then the graph G is p-saturated, but for any $e \in E(G)$ the graph $G - e$ is not p-saturated or $\chi(G - e) < r$.*

Proof. If $r = p+1$, Proposition 3.2 follows from Proposition 3.1. Let $r \ge p+2$. Obviously, $\chi(G) = r$. We put $V(K_{r-p-1}) = \{z_1, \ldots, z_{r-p-1}\}$. Let $V_1 \cup \ldots \cup V_{p+1}$ be a $(p+1)$-chromatic partition of $V(\overline{C}_{2p+1})$. Then $\{z_1\}\cup\ldots\cup\{z_{r-p-1}\}\cup V_1\cup\ldots\cup V_{p+1}$ is an r-chromatic partition of $V(G)$. It is clear that each r-chromatic partition of $V(G)$ has this form. Let V be the union of p subsets of this r-chromatic partition, $V' = V(K_{r-p-1}) \cap V$, $V'' = V(\overline{C}_{2p+1}) \cap V$ and $|V'| = q$. Then V' is a q-clique. Since $\overline{C}_{2p+1}$ is p-saturated (Proposition 3.1), V'' contains a $(p-q)$-clique. Hence V contains a p-clique. This proves that G is p-saturated.

Consider the graph $\widetilde{G} = G - e$, $e \in E(G)$.

Case 1. $e \notin E(\overline{C}_{2p+1})$. In this case obviously $\chi(\widetilde{G}) < r$.

Case 2. $e \in E(\overline{C}_{2p+1})$. By Proposition 3.1, the graph $\overline{C}_{2p+1} - e$ is not p-saturated. Hence $\widetilde{G} = K_{p-r-1} + (\overline{C}_{2p+1} - e)$ is also not p-saturated. □

4. α-CRITICAL GRAPHS

Definition 4.1. A graph G is said to be α-critical if $\alpha(G-e) > \alpha(G)$ for all $e \in E(G)$.

For the α-critical graphs the following facts are known:

Theorem A ([4], see also [1, Th. 8, p. 290]). *In an α-critical graph G without isolated vertices, each independent set A satisfies $|\Gamma_G(A)| \geq |A|$.*

Theorem B ([5, p. 58, exercise 25]). *Let G be a connected α-critical graph with $|V(G)| = 2\alpha(G)+1$. Then G is the simple cycle with $2\alpha(G)+1$ vertices.*

5. THE LEMMAS

Lemma 5.1. *Let G be a graph and $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$. If the graph H is such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(G)$ and $E(H) \supseteq E(G)$, then $\mathrm{cl}(H-v) = \mathrm{cl}(H)$ for all $v \in V(H)$.*

Proof. We have

$$\mathrm{cl}(H-v) \leq \mathrm{cl}(H) = \mathrm{cl}(G) = \mathrm{cl}(G-v) \leq \mathrm{cl}(H-v).$$

Hence $\mathrm{cl}(H) = \mathrm{cl}(H-v)$ for all $v \in V(H)$. □

Lemma 5.2. *Let G be a graph such that $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$. Then:*

(a) $|\Gamma_{\overline{G}}(Q)| \geq |Q|$ *for each clique Q of G;*

(b) $\pi(\overline{G}) \geq \mathrm{cl}(G)$;

(c) $|V(G)| \geq \chi(G) + \mathrm{cl}(G)$.

Proof. Let the graph H be such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(G)$, $E(H) \supseteq E(G)$ and $\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$. From Lemma 5.1, $\mathrm{cl}(H) = \mathrm{cl}(H-v)$ for all $v \in V(G)$. Hence $\overline{H}$ is a graph without isolated vertices. It follows from

$\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$ that $\alpha(\overline{H}-e) > \alpha(\overline{H})$ for all $e \in E(\overline{H})$. So, $\overline{H}$ is an α-critical graph without isolated vertices. By Theorem A, $|\Gamma_{\overline{H}}(Q)| \geq |Q|$ for each independent set Q of $\overline{H}$, i.e. for each clique Q of H. Since $\Gamma_{\overline{H}}(Q) \subseteq \Gamma_{\overline{G}}(Q)$, $|\Gamma_{\overline{G}}(Q)| \geq |Q|$.

Let Q be a clique of G such that $|Q| = \mathrm{cl}(G)$. From (a) and Hall's theorem it follows that $\pi(\overline{G}) \geq \mathrm{cl}(G)$. This inequality together with Proposition 2.1 imply (c). □

Remark. The proposition (a) of Lemma 5.2 is essentially the same as exercise 8, p. 302 in [1]. Another proof of (b) is obtained in [17].

Lemma 5.3. *Let G be a graph such that $\chi(G) = p+1$, $\mathrm{cl}(G) = p$ and let G be p-saturated. Then*:

(a) $\mathrm{cl}(G-v) = \mathrm{cl}(G)$, $\forall v \in V(G)$;

(b) $\pi(\overline{G}) \geq p$.

Proof. Let $V_1 \cup \ldots \cup V_{p+1}$ be a $(p+1)$-chromatic partition of $V(G)$. Since this partition is p-saturated, $\mathrm{cl}(G - V_i) = p$, $i = 1, \ldots, p+1$. From these equalities it follows that $\mathrm{cl}(G-v) = \mathrm{cl}(G) = p$ for all $v \in V(G)$. Lemma 5.2(b) implies the inequality $\pi(\overline{G}) \geq p$. □

Lemma 5.4. *Let G be a graph such that $|V(G)| = 2p+1$, $\chi(G) = p+1$, $\mathrm{cl}(G) = p$, and let G be p-saturated. Then the graph $\overline{G}$ is connected.*

Proof. According to Lemma 5.3(b), $\pi(\overline{G}) \geq p$. Let $V(G) = \{v_1, \ldots, v_{2p+1}\}$ and let $\{v_1, v_2\}, \ldots, \{v_{2p-1}, v_{2p}\}$ be a matching of $\overline{G}$. Then

$$\{v_1, v_2\} \cup \ldots \cup \{v_{2p-1}, v_{2p}\} \cup \{v_{2p+1}\}$$

is a $(p+1)$-chromatic partition of G. The connected component of $\overline{G}$, which contains v_{2p+1}, will be denoted by M. By Lemma 5.3(a), $\overline{G}$ has no isolated vertices. Hence $|M| \geq 2$. Obviously, if one of the vertices v_{2k-1}, v_{2k} belongs to M, then $\{v_{2k-1}, v_{2k}\} \subseteq M$. Hence we may assume that

$$M = \{v_1, v_2, \ldots, v_{2s-1}, v_{2s}, v_{2p+1}\}, \quad 1 \leq s \leq p.$$

Suppose that $\overline{G}$ is not connected. Then $s < p$. Since G is p-saturated, M contains an $(s+1)$-clique Q of G. It is clear that $\Gamma_{\overline{G}}(Q) \subseteq M$. Thus, $|\Gamma_{\overline{G}}(Q)| \leq s$. Since $\mathrm{cl}(G-v) = \mathrm{cl}(G)$ for all $v \in V(G)$ (see Lemma 5.3(a)), this contradicts Lemma 5.2(a) and proves Lemma 5.4. □

Lemma 5.5. *Let G be a graph such that $\chi(G) = p+1$, $\mathrm{cl}(G) = p$, and let G be also p-saturated. Then $|V(G)| \geq 2p+1$ and $|V(G)| = 2p+1$ only if $G = \overline{C}_{2p+1}$.*

Proof. It follows from Lemma 5.3(a) that

$$\mathrm{cl}(G-v) = \mathrm{cl}(G), \quad \forall v \in V(G). \tag{5.1}$$

By Lemma 5.2(c), $|V(G)| \geq 2p+1$. Let $|V(G)| = 2p+1$. Consider the graph H such that $V(H) = V(G)$, $\mathrm{cl}(H) = \mathrm{cl}(H)$, $E(H) \supseteq E(G)$ and $\mathrm{cl}(H+e) > \mathrm{cl}(H)$ for all $e \in E(\overline{H})$. According to (5.1), Lemma 5.1 and Lemma 5.2(c), $\chi(H) \leq p+1$. Since $\chi(H) \geq \chi(G) = p+1$, we have $\chi(H) = p+1$. Obviously, each $(p+1)$-chromatic partition of H is also a $(p+1)$-chromatic partition of G. Hence H is also p-saturated. By Lemma 5.4, $\overline{H}$ is connected. It follows from $\mathrm{cl}(H+e) > \mathrm{cl}(H)$, $\forall e \in E(\overline{H})$, that $\alpha(\overline{H}-e) > \alpha(\overline{H})$, $\forall e \in E(\overline{H})$. So, $\overline{H}$ is an α-critical and connected graph. According to Theorem B, $\overline{H} = C_{2p+1}$. Thus G is a subgraph of $\overline{C}_{2p+1}$. By Proposition 3.1, $G = \overline{C}_{2p+1}$. □

6. A PROOF OF THEOREM 2.1

Let $V_1 \cup \ldots \cup V_r$ be an r-chromatic partition of G such that $V' = V_1 \cup \ldots \cup V_{p+1}$ contains no $(p+1)$-clique of G. Let $G' = G[V']$ and $V'' = V(G) \backslash V'$. By Proposition 2.2, the graph G' is p-saturated. Hence $\mathrm{cl}(G') = p$. Obviously, $\chi(G') = p+1$. According to Lemma 5.5, $|V'| \geq 2p+1$. Since $|V''| \geq r-p-1$, we have $|V(G)| \geq r+p$. Let $|V(G)| = r+p$. Then $|V(G')| = 2p+1$ and $|V''| = r-p-1$. By Lemma 5.5, $G' = \overline{C}_{2p+1}$. Thus G is a subgraph of $K_{r-p-1} + \overline{C}_{2p+1}$. It follows from Proposition 3.2 that $G = K_{r-p-1} + \overline{C}_{2p+1}$.

7. ON THE VERTEX FOLKMAN GRAPHS

Definition 7.1. Let G be a graph and let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers. The r-partition $V_1 \cup \ldots \cup V_r$ of $V(G)$ is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the set V_i contains no a_i-clique of G. The symbol $G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$ means that every r-partition of $V(G)$ is not $(a_1, \ldots, a_r)$-free.

Let $m = \sum_{i=1}^{r}(a_i - 1) + 1$. Consider an r-partition $V(K_{m-1}) = V_1 \cup \ldots \cup V_r$, where $|V_i| = a_i - 1$. Obviously, this r-partition is $(a_1, \ldots, a_r)$-free. Hence $K_{m-1} \stackrel{v}{\not\longrightarrow} (a_1, \ldots, a_r)$. It is clear that $K_m \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$. Thus, from $\mathrm{cl}(G) \geq m$ it follows that $G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$. Clearly, $G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. Folkman proves in [3] that for every $a_1, \ldots, a_r$ there exists a graph $G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$ with $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$. The graph G, such that $G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r)$, is called a vertex $(a_1, \ldots, a_r)$-Folkman graph.

It is clear that

Proposition 7.1. *For any permutation φ of the symmetric group S_r we have*

$$G \stackrel{v}{\longrightarrow} (a_1, \ldots, a_r) \iff G \stackrel{v}{\longrightarrow} (a_{\varphi(1)}, \ldots, a_{\varphi(r)}).$$

For the positive integers $a_1, \ldots, a_r$, $r \geq 2$, we put

$$m = \sum_{i=1}^{r}(a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \ldots, a_r\}. \tag{7.1}$$

Theorem 7.1. *Let positive integers $a_1, \ldots, a_r$, $r \geq 2$, m and p satisfy* (7.1) *and $G \xrightarrow{v} (a_1, \ldots, a_r)$. Then $\chi(G) \geq m$ and if $\chi(G) = m$, the graph G is p-saturated.*

Proof. Suppose $\chi(G) \leq m-1$ and $V(G) = V_1 \cup \ldots \cup V_{m-1}$ is an $(m-1)$-chromatic partition of G. Let $V(K_{m-1}) = \{z_1, \ldots, z_{m-1}\}$ and let $W_1 \cup \ldots \cup W_r$ be an r-partition of $V(K_{m-1})$ such that $|W_i| = a_i - 1$. Consider the map $V(G) \xrightarrow{\varphi} V(K_{m-1})$, where $v \xrightarrow{\varphi} z_i$ for all $v \in V_i$. We put $V'_k = \varphi^{-1}(W_k)$, $k = 1, \ldots, r$. Since V'_k is an union of $a_k - 1$ independent sets of G, V'_k contains no a_k-clique, $k = 1, \ldots, r$. So, $V'_1 \cup \ldots \cup V'_r$ is an $(a_1, \ldots, a_r)$-free partition of G, which is a contradiction.

Let $\chi(G) = m$. Suppose that G is not p-saturated and let $V_1 \cup \ldots \cup V_m$ be an m-chromatic partition of G such that $V' = V_1 \cup \ldots \cup V_p$ contains no p-clique of G. By Proposition 7.1, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_r = p$. We put $G' = G - V'$. Obviously, $\chi(G') = m - p = m - a_r = \sum_{i=1}^{r-1}(a_i - 1)$. From these equalities it follows that G' has an $(a_1, \ldots, a_{r-1})$-free $(r-1)$-partition $W_1 \cup \ldots \cup W_{r-1}$. But then $W_1 \cup \ldots \cup W_{r-1} \cup V'$ is an $(a_1, \ldots, a_r)$-free r-partition of G, which is a contradiction. This ends the proof of Theorem 7.1. □

Theorem 7.2. *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers and let m and p satisfy* (7.1). *Let the graph G be such that $G \xrightarrow{v} (a_1, \ldots, a_r)$ and* $\mathrm{cl}(G) < m$. *Then $\pi(\overline{G}) \geq p$.*

Proof. We prove the inequality $\pi(\overline{G}) \geq p$ by induction on m. It follows from $G \xrightarrow{v} (a_1, \ldots, a_r)$ that $\mathrm{cl}(G) \geq p$. Since $\mathrm{cl}(G) < m$, $m \geq p+1$. By this inequality, the minimal admissible value of m is $p+1$.

1. Let $m = p+1$. According to Proposition 7.1, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_r = p$. From $m = p+1$ it follows that $a_1 = \cdots = a_{r-2} = 1$, $a_{r-1} = 2$ and $\mathrm{cl}(G) = p$. Hence $G \xrightarrow{v} (a_1, \ldots, a_r)$ implies $G \xrightarrow{v} (2, p)$. From $G \xrightarrow{v} (2, p)$ it follows $\mathrm{cl}(G - v) \geq p$ for all $v \in V(G)$. So, $\mathrm{cl}(G - v) = \mathrm{cl}(G) = p$ for all $v \in V(G)$. According to Lemma 5.2(b), $\pi(\overline{G}) \geq p$.

2. Let $m \geq p+2$. If $\mathrm{cl}(G - v) = \mathrm{cl}(G)$, $\forall v \in V(G)$, from Lemma 5.2(b) it follows that $\pi(\overline{G}) \geq \mathrm{cl}(G)$. Hence $\pi(\overline{G}) \geq p$. Suppose $\mathrm{cl}(G - v_0) < \mathrm{cl}(G)$ for some $v_0 \in V(G)$. Since $\mathrm{cl}(G) < m$, $\mathrm{cl}(G - v_0) < m - 1$. We may assume that $a_1 \leq \cdots \leq a_r = p$. It follows from $m \geq p+2$ that $a_{r-1} \geq 2$. Obviously, $G \xrightarrow{v} (a_1, \ldots, a_r)$ implies $G - v_0 \xrightarrow{v} (a_1, \ldots, a_{r-2}, a_{r-1} - 1, a_r)$. Applying the inductive hypothesis for $G - v_0$, we conclude that $\pi(\overline{G} - v_0) \geq p$.

Hence, $\pi(\overline{G}) \geq p$. □

Theorem 7.3 ([7]). *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers and m and p satisfy* (7.1). *If $G \xrightarrow{v} (a_1, \ldots, a_r)$ and* $\mathrm{cl}(G) < m$, *then $|V(G)| \geq m + p$.*

Another proof of Theorem 7.3. According to Theorem 7.1, $\chi(G) \geq m$, and accordingly to Theorem 7.2, $\pi(\overline{G}) \geq p$. It follows from Proposition 2.1 that $|V(G)| \geq m + p$. □

Theorem 7.4 ([8]). *Let $a_1, \ldots, a_r$, $r \geq 2$, be positive integers, and let m and p satisfy* (7.1). *If $G \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$, $\mathrm{cl}(G) < m$ and $|V(G)| = m + p$, then $G = K_{m-p-1} + \overline{C}_{2p+1}$.*

Another proof of Theorem 7.4. It follows from Proposition 2.1 and Theorem 7.2 that $|V(G)| \geq \chi(G) + p$. Since $|V(G)| = m + p$, we conclude that $\chi(G) \leq m$. By Theorem 7.1, $\chi(G) = m$ and G is p-saturated. It follows from Theorem 2.1 and $|V(G)| = m + p$ that G is not $(p + 1)$-saturated and $G = K_{m-p-1} + \overline{C}_{2p+1}$.

It is proved in [6] that $K_{m-p-1} + \overline{C}_{2p+1} \overset{v}{\longrightarrow} (a_1, \ldots, a_r)$.

8. EDGE FOLKMAN GRAPHS

Definition 8.1. Let $a_1, \ldots, a_r$, $a_i \geq 2$, $r \geq 2$, be integers. Let G be a graph and let

$$E(G) = E_1 \cup \ldots \cup E_r$$

be an r-colouring of $E(G)$. This r-colouring is said to be $(a_1, \ldots, a_r)$-free if for all $i \in \{1, \ldots, r\}$ the graph G contains no monochromatic a_i-clique of colour i. The symbol $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$ means that every r-colouring of $E(G)$ is not $(a_1, \ldots, a_r)$-free.

Obviously, if $\mathrm{cl}(G) \geq R(a_1, \ldots, a_r)$, where $R(a_1, \ldots, a_r)$ is the Ramsey number, then $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$. It is clear that $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$ implies $\mathrm{cl}(G) \geq \max\{a_1, \ldots, a_r\}$. The existence of a graph $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$ with $\mathrm{cl}(G) = \max\{a_1, \ldots, a_r\}$ was proved in the case $r = 2$ by Folkman in [3] and for arbitrary r by Nesetril and Rodl in [16].

Theorem 8.1. *Let $a_1, \ldots, a_r$, $a_i \geq 2$, $r \geq 2$, be integers and let $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$. Then*

(a) *$\chi(G) \geq R$, where $R = R(a_1, \ldots, a_r)$;*

(b) *suppose that $\chi(G) = R$, $\mathrm{cl}(G) < R$ and there exists an r-colouring*

$$E(K_R) = E_1 \cup \ldots \cup E_r \tag{8.1}$$

with the unique monochromatic a_i-clique P of colour i and without monochromatic a_j-clique of colour j, $j \neq i$. Then G is a_i-saturated and if $K_{R-a_i-1} + \overline{C}_{2a_i+1} \overset{e}{\not\longrightarrow} (a_1, \ldots, a_r)$, then $|V(G)| > R + a_i$.

Proof. The proof of the inequality (a) is due to Lin in [5]. To prove the proposition (b) of Theorem 8.1, suppose to the contrary that $V_1 \cup \ldots \cup V_R$ is an R-chromatic partition of $V(G)$ such that $V_1 \cup \ldots \cup V_{a_i}$ contains no a_i-clique. Let $V(K_R) = \{z_1, \ldots, z_R\}$ and $P = \{z_1, \ldots, z_{a_i}\}$. Consider the map $V(G) \overset{\varphi}{\longrightarrow} V(K_R)$, where $v \overset{\varphi}{\longrightarrow} z_i$, $\forall v \in V_i$. Let $E'_1 \cup \ldots \cup E'_r$ be the r-colouring of $E(G)$, where $[u, v] \in E'_i \iff [\varphi(u), \varphi(v)] \in E_i$ of (8.1). From $G \overset{e}{\longrightarrow} (a_1, \ldots, a_r)$ it follows that in this r-colouring there exists a monochromatic a_k-clique Q of colour k. Obviously, $\varphi(Q)$ is a monochromatic a_k-clique of colour k in (8.1). By the properties of the

r-colouring (8.1) it follows that $i = k$ and $\varphi(Q) = P = \{z_1, \ldots, z_{a_i}\}$. Hence $Q \subseteq V_1 \cup \ldots \cup V_{a_i}$. This contradicts the assumption that $V_1 \cup \ldots \cup V_{a_i}$ contains no a_i-clique and proves that G is a_i-saturated.

According to Theorem 2.1, $|V(G)| \geq \chi(G) + a_i = R + a_i$. Since $K_{R-a_i-1} + \overline{C}_{2a_i+1} \overset{e}{\not\longrightarrow} (a_1, \ldots, a_r)$, $G \neq K_{R-a_i-1} + \overline{C}_{2a_i+1}$. From Theorem 2.1, $|V(G)| > R + a_i$. The proof of Theorem 8.1 is completed. □

Theorem 8.1 generalizes the results from [12].

Consider the graphs G such that $G \overset{e}{\longrightarrow} (3,4)$ and $\mathrm{cl}(G) < 9$. We put $N(3,4;9) = \min\{|V(G)| : G \overset{e}{\longrightarrow} (3,4) \text{ and } \mathrm{cl}(G) < 9\}$.

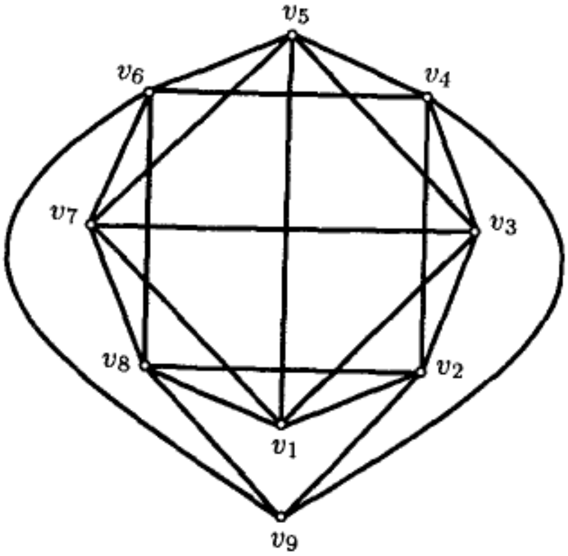

Fig. 1. The graph F

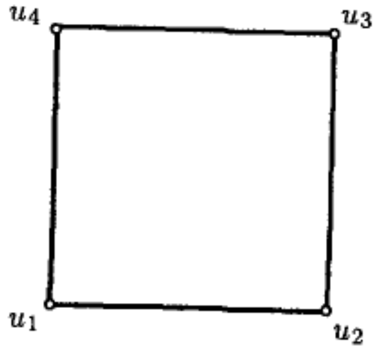

Fig. 2. The graph F_1

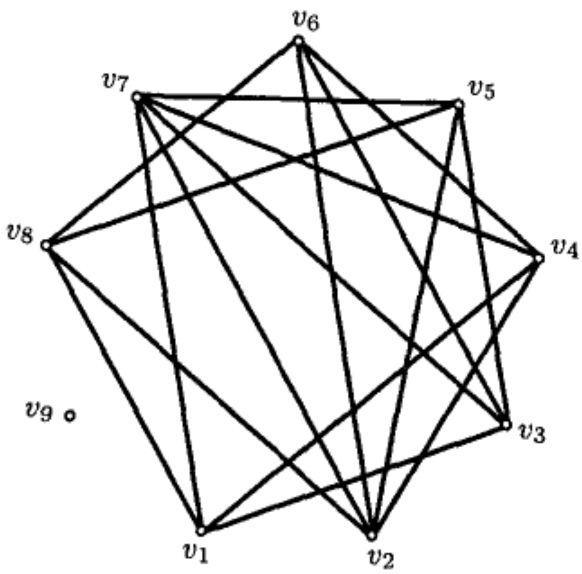

Fig. 3. The graph F_2

Corollary 8.1 ([10]). $N(3,4;9) = 14$.

Proof. It is proved in [11] and [15] that $K_4 + C_5 + C_5 \overset{e}{\longrightarrow} (3,4)$. Hence $N(3,4;9) \leq 14$. We prove the inequality $N(3,4;9) \geq 14$. Since $R(3,4) = 9$, from Theorem 8.1 follows $\chi(G) \geq 9$.

Case 1. $\chi(G) \geq 10$. Since $\mathrm{cl}(G) \leq 8$, Theorem 1 in [13] implies $|V(G)| \geq 14$.

Case 2. $\chi(G) = 9$. By F, F_1 and F_2 we denote the graphs which are given in Fig. 1, Fig. 2 and Fig. 3, respectively. In Fig. 1 is given the unique 9-vertex graph F with $\alpha(F) = 2$ and containing an unique 4-clique ($[v_1, v_3, v_5, v_7]$), [14]. Hence the 2-colouring $E(K_9) = E_1 \cup E_2$, where $E_2 = E(F)$, contains an unique 4-clique of 2nd colour and contains no 3-cliques of 1st colour. Let

$$A = (a_{ij}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}.$$

Consider the 2-colouring $E(K_4 + \overline{C}_9) = E_1 \cup E_2$, where $E(K_4) \cap E_2 = E(F_1)$, $E(\overline{C}_9) \cap E_2 = E(F_2)$ and $[u_i, v_j] \in E_2 \iff a_{ij} = 2$. This 2-colouring is $(3,4)$-free, [10]. By Theorem 8.1, $|V(G)| \geq 14$.

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Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier blvd., 1164 Sofia
BULGARIA
E-mail: nenov@fmi.uni-sofia.bg