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INSTABILITY OF SOLITARY WAVE SOLUTIONS OF A CLASS OF NONLINEAR DISPERSIVE SYSTEMS[1]

SEVDZHAN HAKKAEV

In this paper the orbital stability and instability properties of solitary wave solutions of a class of nonlinear dispersive systems are studied. By applying the abstract results of Grillakis et al. ([11]), we obtain the stability of the solitary waves.

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1. INTRODUCTION

In the present paper we consider the stability and instability of solitary wave solutions $(\varphi(x-ct), \psi(x-ct))$ for the following system of nonlinear evolution equations:

$$\begin{cases} Mu_t + u_x + (u^p v^{p+1})_x = 0 \\ Mv_t + v_x + (u^{p+1} v^p)_x = 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ are real-valued functions and M is a pseudodifferential operator of order $\mu > 1$ (see (2.1)) and $p > 0$. This system can also be interpreted as a coupled version of the generalized Benjamin-Bona-Mahony (BBM) equation

$$Mu_t + (a(u))_x = 0.$$

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Among the papers devoted to the stability of the BBM equation are [13], [14] and [16]. When $a(u) = u^p$ and $M = 1 - \partial_x^2$, it is obtained in [14] that solitary waves are stable for all p. In [16] this result is extended for a more general class of pseudodifferential operators.

Here, using the same lines of ideas as in [12] and [16], we show that if $p \leq \mu$, then solitary waves are always stable, while if $p > \mu$, there is a critical speed c_0 such that we have instability for $c < c_0$ and stability for $c > c_0$.

System (1.1) has four natural invariants $E(u, v) = -\dfrac{2}{p+1}\displaystyle\int u^{p+1}v^{p+1}dx$, $V(u,v) = \dfrac{1}{2}\displaystyle\int [u^2 + v^2 + uMu + vMv]dx$, $I_1(u,v) = \displaystyle\int udx$, $I_2(u,v) = \displaystyle\int vdx$. Our analysis is based on the invariants E and V, following the proofs of [16], [11] and [9].

This paper is organized as follows. In Section 2 we discuss the existence and the asymptotic behavior of solutions of (1.1). In Section 3 we state our main assumptions and prove the stability and instability results.

Notations:

∘ The norm in $H^s(\mathbb{R})$ will be denoted by $\|\cdot\|_s$, and $\|\cdot\|$ will denote the norm in $L^2(\mathbb{R})$.

∘ We denote $X^s = H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $X = L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $\|\mathbf{f}\|_{X^s} = \|f\|_s^2 + \|g\|_s^2$ for $\mathbf{f} = (f, g)$.

∘ $\widehat{\Lambda^\mu g}(\xi) = |\xi|^\mu \widehat{g}(\xi)$, $L = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$.

2. THE EVOLUTION EQUATION

We begin with a discussion of the existence and uniqueness theory of the initial value problem associated with (1.1). The operator M has the form

$$\widehat{Mu}(\xi) = (1 + |\xi|^\mu)\widehat{u}(\xi). \tag{2.1}$$

We state the basic theorem which guarantees the existence and uniqueness of solutions of (1.1) in $H^{\frac{\mu}{2}}(\mathbb{R})$.

Theorem 2.1. *If* $\mathbf{u}_0 \in X^\nu$, *then there exists a unique global solution* $\mathbf{u}$ *of* (1.1) *in* $C([0,\infty); X^\nu)$ *with* $\mathbf{u}(0) = \mathbf{u}_0$. *Moreover, the functionals* E, V, I_1 *and* I_2 *are constant with respect to* t.

Proof. In order to obtain the existence of weak solutions, we consider the problem

$$\mathbf{u}_t + A\mathbf{u} + G(\mathbf{u}) = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \tag{2.2}$$

where

$$A = \begin{pmatrix} M^{-1}\partial_x & 0 \\ 0 & M^{-1}\partial_x \end{pmatrix} \text{ and } G(\mathbf{u}) = \begin{pmatrix} M^{-1}\partial_x(u^p v^{p+1}) & 0 \\ 0 & M^{-1}\partial_x(u^{p+1}v^p) \end{pmatrix}.$$

Equation (2.2) can be written as an integral equation

$$\mathbf{u} = U(t)\mathbf{u}_0 + \int_0^t U(t-\tau)G(\mathbf{u}(\tau))d\tau,$$

where $U(t)$ is a C_0 group of unitary operators in X^ν generated by a skew adjoint operator A with $D(A) = X^\nu$ and $\mathbf{u}_0 \in D(A)$. We solve the integral equation by the semigroup theory. Since $X^\nu \subset L^\infty \times L^\infty$, it is easy to show that $\mathbf{u} \to G(\mathbf{u})$ carries $Y \to Y$ in locally Lipschitzian manner, where Y denotes a Hilbert product space of $D(A)$ with the graph norm given by $||\mathbf{u}||_Y = ||\mathbf{u}||_{X^\nu} + ||A\mathbf{u}||_{X^\nu}$. By [15], Theorem 6.1.4, for any $\mathbf{u}_0 \in X^\nu$ there is some $T \in (0, \infty)$ so that a unique solution $\mathbf{u}(\cdot, t)$ with initial data $\mathbf{u}_0$ exists for $0 < t \leq T$.

Multiplying (1.1) by (u, v) yields

$$\frac{d}{dt}||\mathbf{u}(t)||_{X^\nu} = 0.$$

This implies that $\mathbf{u}$ is bounded in X^ν and proves the global existence of a weak solution $\mathbf{u}$ for (1.1).

The fact that E and V are constants follows from the local existence. Finally, if $I_1(u_0, v_0)$ and $I_2(u_0, v_0)$ exist, then $I_1(u(t), v(t))$ and $I_2(u(t), v(t))$ do exist and $I_1(u_0, v_0) = I_1(u(t), v(t))$ and $I_2(u_0, v_0) = I_2(u(t), v(t))$. This follows by integrating each equation of (1.1) over (a, b) and letting $a \to -\infty$, $b \to \infty$. This completes the proof of Theorem 2.1. □

Consider the linear initial value problem associated to Eq. (1.1)

$$\begin{cases} Mu_t + u_x = 0 \\ Mv_t + v_x = 0 \\ (u(0), v(0)) = (u_0, v_0) \in X^\nu \end{cases} \tag{2.3}$$

and the related unitary group $V(t)$ which is defined by

$$V(t)f(x) = S_t \star f(x),$$

where S_t is defined by the oscillatory integral

$$S_t(x) = \int_{-\infty}^{\infty} e^{it(\frac{\xi}{1+|\xi|^\mu} - x\xi)} d\xi.$$

Therefore the solution of Eq. (2.3) is given by the unitary group $W(t)$ in X^ν defined for $\mathbf{u}_0 = (u_0, v_0)$ by

$$W(t)\mathbf{u}_0 = (V(t)u_0(x), V(t)v_0(x)).$$

Theorem 2.2. *Let* $\mathbf{u} \in X^\nu \cap (L^1(\mathbb{R}) \times L^1(\mathbb{R}))$ *and let* $\mathbf{u}(x,t)$ *be the solution of* (1.1) *with initial data* $\mathbf{u}_0$. *Then there exists* $0 < \eta < 1$ *such that*

$$\sup_{-\infty < z < \infty} \left| \int_{-\infty}^{z} \mathbf{u}(x,t)dx \right| \leq c(1 + t^\eta), \tag{2.4}$$

where the constant c *depends only on* $\mathbf{u}_0$.

To prove Theorem 2.2, we need the following lemma, which is proved in [16].

Lemma 2.1. *Let* $S(t)$ *be the evolution operator to the linear equation*

$$((1 + \Lambda^\mu)\partial_t + \partial_x)w = 0 \quad (S(t)w(0) = w(t)).$$

Then $S(t) : H^\nu \cap L^1 \to L^\infty$ *for all* $t > 0$. *Moreover, there exist* $\theta \in (0,1)$ *and* $c > 0$ *such that*

$$|S(t)w_0|_\infty \leq ct^{-\theta}(|w_0|_1 + \|w_0\|_\nu), \;\; \theta = \frac{\mu - 1}{2\mu}.$$

From Lemma 2.1 and Young's inequality for convolution we have

$$|W(t)|_{L^\infty \times L^\infty} \leq ct^{-\theta}(|\mathbf{u}_0|_{L^1 \times L^1} + \|\mathbf{u}_0\|_{X^\nu}). \tag{2.5}$$

Proof of Theorem 2.2. Let $\mathbf{z}(t) = W(t)\mathbf{u}_0$, that is

$$L\partial_t \mathbf{z} + \partial_x \mathbf{z} = 0, \;\; \mathbf{z}(0) = \mathbf{u}_0.$$

Then

$$\mathbf{u}(t) = \mathbf{z}(t) - \int_0^t W(t-\tau)L^{-1}\partial_x F(\mathbf{u})d\tau$$

$$= \mathbf{z}(t) - \partial_x \int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau,$$

where $F(\mathbf{u}) = (u^p v^{p+1}, u^{p+1}v^p)$.
Let $U(x,t) = \int_{-\infty}^x \mathbf{u}(y,t)dy$ and $Z(x,t) = \int_{-\infty}^x \mathbf{z}(y,t)dy$. Then

$$U(t) = Z(t) - \int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau. \tag{2.6}$$

We estimate the two terms on the right-hand side of (2.6) separately. First, we obtain from the equation for $\mathbf{z}(x,t)$,

$$\mathbf{z}(t) = \mathbf{u}_0 - \partial_x \int_0^t L^{-1}\mathbf{z}(\tau)d\tau,$$

so that

$$Z(T) = U_0 - \int_0^t W(\tau)L^{-1}\mathbf{u}_0 d\tau$$

with $U_0(x) = \int_{-\infty}^{x} \mathbf{u}_0(y)dy$. Using (2.5), we have

$$|Z(x,t)| \leq |\mathbf{u}_0|_{L^1\times L^1} + c\int_0^t (1+\tau)^{-\theta}d\tau(|L^{-1}\mathbf{u}_0|_{L^1\times L^1} + ||L^{-1}\mathbf{u}_0||_{X^\nu})$$

$$\leq c(1+t)^\eta(|L^{-1}\mathbf{u}_0|_{L^1\times L^1} + ||L^{-1}\mathbf{u}_0||_{X^\nu}),$$

where $\eta = 1-\theta$. Noticing that $||L^{-1}\mathbf{u}_0||_{X^\nu} \leq c||\mathbf{u}_0||_{X^\nu}$, then

$$|Z(x,t)| \leq c(1+t)^\eta(|\mathbf{u}_0|_{L^1\times L^1} + ||\mathbf{u}_0||_{X^\nu}).$$

Let $P(x,t)$ denote the second term on the right-hand side of Eq. (2.6). Then by (2.5)

$$|P(x,t)| \leq \left|\int_0^t W(t-\tau)L^{-1}F(\mathbf{u})d\tau\right|$$

$$\leq c\int_0^t (1+t-\tau)^{-\theta}d\tau(|L^{-1}F(\mathbf{u})|_{L^1\times L^1} + ||L^{-1}F(\mathbf{u})||_{X^\nu}).$$

Since $X^\nu \subset L^\infty \times L^\infty \quad (\nu > \frac{1}{2})$, then $|L^{-1}F(\mathbf{u})|_{L^1\times L^1}$ is bounded uniformly in τ by a constant which depends only on $\mathbf{u}_0$. Next observe that $||L^{-1}F(\mathbf{u})||_{X^\nu} \leq (|u|_\infty^p + |v|_\infty^p)||\mathbf{u}||_{X^\nu}$. Thus

$$|P(x,t)| \leq c(1+t)^\eta.$$

This completes the proof of the theorem. □

3. THE SOLITARY WAVE

We consider a smooth solution of (1.1) of the form $(u(x,t),v(x,t)) = (\varphi(x-ct),\psi(x-ct)) = \Phi(x-ct)$ that vanishes at infinity. Substituting Φ in (1.1) and assuming that $\varphi,\psi,\varphi',\psi',\varphi'',\psi'' \to 0$ as $|\zeta| \to \infty$, we obtain

$$\begin{cases} -cM\varphi + \varphi + \varphi^p\psi^{p+1} = 0 \\ -cM\psi + \psi + \varphi^{p+1}\psi^p = 0. \end{cases} \tag{3.1}$$

From (3.1) we see that if E' and V' represent the Frechet derivatives of E, V, then

$$E'(\varphi_c,\psi_c) + cV'(\varphi_c,\psi_c) = 0. \tag{3.2}$$

Moreover, if H_c is the linearized operator of $E' + cV'$ around Φ_c, namely

$$H_c = E''(\Phi_c) + cV''(\Phi_c) \tag{3.3}$$

$$= \begin{pmatrix} c\Lambda^\mu + (c-1) - p\varphi^{p-1}\psi^{p+1} & -(p+1)\varphi^p\psi^p \\ -(p+1)\varphi^p\psi^p & c\Lambda^\mu + (c-1) - p\varphi^{p+1}\psi^{p-1} \end{pmatrix},$$

then $H_c(\partial_x\varphi_c, \partial_x\psi_c) = 0$.
We now establish our main assumptions on Φ_c and H_c under which we solve the problem of stability and instability. They are as follows.

Assumption 1. There is an interval $(c_1, c_2) \subset \mathbb{R}$ such that for every $c \in (c_1, c_2)$ there exists a solution $\Phi_c = (\varphi_c, \psi_c)$, $\varphi > 0$, $\psi > 0$ of (3.2) in $X^{\nu+3}$. The curve $c \to \Phi_c$ is C^1 with values in $X^{\nu+1}$. Moreover, $(1+|\xi|)^{\frac{1}{2}}\dfrac{d\Phi_c}{dc} \in L^1 \times L^1$.

Assumption 2. The zero eigenvalue of the operator H_c is simple. H_c has a unique negative simple eigenvalue with an eigenfunction χ_c. Besides the negative and the zero eigenvalues, the rest of the spectrum of H_c is positive and bounded away from zero. Moreover, the mapping $c \to \chi_c$ is continuous with values in $X^{\nu+1}$ and $(1+|\xi|)^{\frac{1}{2}}\chi_c \in L^1 \times L^1$, $\chi_1 > 0$, $\chi_2 > 0$.

Denote
$$d(c) = E(\Phi_c) + cV(\Phi_c).$$
After a differentiation with respect to c, we have
$$d'(c) = \langle E'(\Phi_c) + cV'(\Phi_c), \frac{d\Phi_c}{dc}\rangle + V(\Phi_c) = V(\Phi_c), \tag{3.4}$$
$$d''(c) = \langle V'(\Phi_c), \frac{d\Phi_c}{dc}\rangle. \tag{3.5}$$
Next we examine the relation between the convexity properties of the function $d(c)$ and the properties of the functional E near the critical point Φ_c under the constraint $V = \text{const}$.

Theorem 3.3. *Let $c > 0$ be fixed. If $d''(c) < 0$, then there is a curve $w \to \Psi_w$ which satisfies $V(\Phi_c) = V(\Psi_w)$. $\Phi_c = \Psi_c$, and on which $E(\mathbf{u})$ has a strict local maximum at $\mathbf{u} = \Phi_c$.*

Proof. Following the ideas of Souganidis and Strauss [16], we note that for $G(w,s) = V(\Phi_w + s\chi_c)$, $G(c,0) = V(\Phi_c)$ and $\dfrac{\partial}{\partial s}V(\Phi_w + s\chi_c)(c,0) = \langle V'(\Phi_c), \chi_c\rangle = (L\Phi_c, \chi_c) \neq 0$. Therefore, it follows from the implicit function theorem that there is a C^1 function $s(w)$ for w near c such that $s(c) = 0$ and $G(w, s(w)) = V(\Phi_c)$. Next we define $\Psi_w = \Phi_c + s(w)\chi_x$. It is easy to be seen that $\dfrac{d}{dw}E(\Psi_w)_{|w=c} = 0$ and
$$\frac{d^2}{dw^2}E(\Psi_w)_{|w=c} = \langle H_c\mathbf{y}, \mathbf{y}\rangle,$$
where $\mathbf{y} = \dfrac{d\Psi_w}{dw}_{|w=c} = \dfrac{d}{dc}\Phi_c + s'(c)\chi_c$. So it suffices to show that $\langle H_c\mathbf{y}, \mathbf{y}\rangle < 0$. We have
$$0 = \frac{d}{dw}V(\Psi_w)_{|w=c} = \langle V'(\Phi_c), \frac{d}{dw}_{|w=c}\rangle$$

$$= (L\Phi_c, \mathbf{y}) = (L\Phi_c, \frac{d}{dc}\Phi_c) + s'(c)(L\Phi_c, \chi_c).$$

From (3.5), $d''(c) = -s'(c)(L\Phi_c, \chi_c)$, therefore

$$(H_c\mathbf{y}, \mathbf{y}) = s'(c)(H_c\chi_c, \mathbf{y}) - (L\Phi_c, \mathbf{y}) = d''(c) + s'^2(c)(H_c\chi_c, \chi_c) < 0.$$

This proves the theorem. □

We continue our study by specifying the precise form in which stability and instability are to be interpreted. Denoting by τ_s, $s \in \mathbb{R}$, the translation operator $\tau_s f(x) = f(x+s)$ for all $x \in \mathbb{R}$, we define $T(s)\mathbf{f} = (\tau_s f, \tau_s g)$ for $\mathbf{f} = (f, g)$. For $\varepsilon > 0$ consider the tubular neighborhood

$$U_\varepsilon = \{\mathbf{g} \in X^\nu \mid \inf_{s\in\mathbb{R}} ||\mathbf{g} - T(s)\Phi_c||_{X^\nu} < \varepsilon\}.$$

Definition 3.1. The solitary wave Φ_c is X^ν stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\mathbf{u}_0 \in U_\delta$, then (1.1) has a unique solution $\mathbf{u}(t) \in C([0,\infty); X^\nu)$ with $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(t) \in U_\varepsilon$ for all $t \in \mathbb{R}$. Otherwise, Φ_c is called unstable.

The stability assertion (when $d''(c) > 0$) is a special case of [11], so that we omit the proof. For the instability, we need a series of preliminary results which can be proved as in the analogous cases of [9]. For this reason we only state them without proof.

Lemma 3.2. *There are an $\varepsilon > 0$ and a unique C^1 map $\alpha : U_\varepsilon \to \mathbb{R}$ such that for $\mathbf{u} \in U_\varepsilon$ and $r \in \mathbb{R}$:*

(i) $\langle \mathbf{u}(\cdot + \alpha(\mathbf{u})), \partial_x\Phi_c\rangle = 0, \qquad \alpha(\Phi_c) = 0;$

(ii) $\alpha(\mathbf{u}(\cdot + r)) = \alpha(\mathbf{u}) - r;$

(iii) $\alpha'(\mathbf{u}) = \dfrac{\partial_x\Phi_c(\cdot - \alpha(\mathbf{u}))}{\langle \mathbf{u}, \partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))\rangle}.$

Next we define an auxiliary operator B which will play a crucial role in the proof of instability. If $\mathbf{y}$ is as in Theorem 3.3, then $(H_c\mathbf{y}, \mathbf{y}) < 0$ and $(\mathbf{y}, L\Phi_c) = 0$.

Definition 3.2. For $\mathbf{u} \in U_\varepsilon$, define $B(\mathbf{u})$ by the formula

$$B(\mathbf{u}) = \mathbf{y}(\cdot - \alpha(\mathbf{u})) - \frac{(L\mathbf{u}, \mathbf{y}(\cdot - \alpha(\mathbf{u})))}{\langle \mathbf{u}, \partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u}))\rangle} L^{-1}\partial_x^2\Phi_c(\cdot - \alpha(\mathbf{u})).$$

Lemma 3.3. *B is a C^1 function from U_ε into X^ν. Moreover, B commutes with translations, $B(\Phi_c) = \mathbf{y}$ and $\langle B(\mathbf{u}), L\mathbf{u}\rangle = 0$ for every $\mathbf{u} \in U_\varepsilon$.*

Lemma 3.4. *There is a C^1 functional $\Upsilon : D_\varepsilon \to \mathbb{R}$, where $D_\varepsilon = \{\mathbf{v} \in U_\varepsilon : V(\mathbf{v}) = V(\Phi_c)\}$, such that if $\mathbf{v} \in D_\varepsilon$ and $\mathbf{v}$ is not a translate of Φ_c, then*

$$E(\Phi_c) < E(\mathbf{v}) + \Upsilon(\mathbf{v})\langle E'(\mathbf{v}), B(\mathbf{u})\rangle.$$

Lemma 3.5. *The curve $w \to \Psi_w$, constructed in Theorem 3.3, satisfies $E(\Psi_w) < E(\Phi_c)$ for $w \neq c$, $V(\Psi_w) = V(\Phi_c)$ and $\langle E'(\Psi_w), B(\Psi_w)\rangle$ changes its sign as w passes through c.*

Theorem 3.4. *Assume that Assumptions 1 and 2 hold and $d''(c) < 0$. Then the solitary wave Φ_c is unstable.*

Proof. Let $\varepsilon > 0$ be small enough such that Lemma 3.2 and its consequences apply with U_ε. To prove instability of Φ_c, it suffices to show that there are some elements $\mathbf{u}_0 \in X^\nu$ which are arbitrary close to Φ_c, but for which the solution $\mathbf{u}$ of Eq. (1.1) with initial data $\mathbf{u}_0$ leaves U_ε in finite time.

By Lemma 3.5, we can find $\mathbf{u}_0 \in X^\nu$ which is close to Φ_c and satisfies $V(\mathbf{u}_0) = V(\Phi_c)$, $E(\mathbf{u}_0 < E(\Phi_c)$ and $\langle E'(\mathbf{u}_0), B(\mathbf{u}_0)\rangle > 0$. For a fixed $\mathbf{u}_0$, let $[0, t_1)$ denote the maximal interval for which $\mathbf{u}(\cdot, t)$ lies continuously in U_ε. It suffices to show that $t_1 < \infty$.

In view of Theorems 2.1 and 2.2 $\mathbf{u}$ has the following properties:

$$\mathbf{u} \in C([0, t_1); X^\nu), \quad \mathbf{u}(x, 0) = \mathbf{u}_0,$$

$$\sup_{x\in\mathbf{R}} \left| \int_{-\infty}^{x} \mathbf{u}(z, t)dz \right| \leq c_0(1 + t^\eta), \quad t \in [0, t_1),$$

$$\sup_{t\in[0,t_1)} \|\mathbf{u}(t)\|_{X^\nu} \leq c_1.$$

Let us take $\beta(t) = \alpha(\mathbf{u}(t))$, $\mathbf{Y}(x) = \int_{-\infty}^{x} L\mathbf{y}(\rho)d\rho = \int_{-\infty}^{x} \mathbf{y}(\rho)d\rho + N\mathbf{y}(x)$, where $N = \begin{pmatrix} \frac{|\xi|^\mu}{i\xi} & 0 \\ 0 & \frac{|\xi|^\mu}{i\xi} \end{pmatrix}$, and define

$$A(t) = \int_{-\infty}^{\infty} \mathbf{Y}(x - \beta(t))\mathbf{u}(x, t)dx. \tag{3.6}$$

Let H be the Heaviside function and $\gamma = \int_{-\infty}^{\infty} \mathbf{u}_0(x)dx$. We note that by Assumptions 1 and 2, $\int_{-\infty}^{\infty} (1 + |x|)^{\frac{1}{2}} |\mathbf{y}(x)|dx < \infty$ and the function $R(x) = \int_{-\infty}^{\infty} \mathbf{y}(\rho)d\rho - \gamma H(x)$ belongs to $L^2 \times L^2$. Therefore we obtain from Eq. (3.6) that

$$A(t) = \int_{-\infty}^{\infty} R(x - \beta(t))\mathbf{u}(x, t)dx + \gamma \int_{\beta(t)}^{\infty} \mathbf{u}(x, t)dx + \int_{-\infty}^{\infty} N\mathbf{y}(x - \beta(t))\mathbf{u}(x, t)dx.$$

Hence,

$$|A(t)| \leq |R|_2 \|\mathbf{u}\|_{X^\nu} + (c_0(1 + t^\eta) + \|N\mathbf{u}\|_{X^\nu} \| \|\mathbf{u}\|_{X^\nu}. \tag{3.7}$$

Now

$$\frac{dA}{dt} = -\langle \alpha'(\mathbf{u}), \frac{d\mathbf{u}}{dt} \langle L\mathbf{y}, \mathbf{u}\rangle + \langle \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle$$

$$= \langle -\langle \mathbf{y}(\cdot - \beta), L\mathbf{u}\rangle \alpha'(\mathbf{u}) + \mathbf{Y}(\cdot - \beta), \frac{d\mathbf{u}}{dt}\rangle$$

$$= -\langle B(\mathbf{u}), E'(\mathbf{u})\rangle.$$

As $0 < E(\Phi_c) - E(\mathbf{u}_0) = E(\Phi_c) - E(\mathbf{u})$, Lemma 3.3 implies that

$$0 < \Upsilon(\mathbf{u})\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle.$$

Moreover, since $\mathbf{u}(t) \in U_\varepsilon$ and $\Upsilon(\Phi_c) = 0$, we may assume that $\Upsilon(\mathbf{u}(t)) < 1$ by choosing ε even smaller if necessary.

Therefore for all $t \in [0, t_1)$, $\langle E'(\mathbf{u}(t)), B(\mathbf{u}(t))\rangle > E(\Phi_c) - E(\mathbf{u}_0) > 0$. Hence for $0 < t < t_1$

$$-\frac{dA}{dt} \geq E(\Phi_c) - E(\mathbf{u}_0) > 0.$$

Comparing this with (3.7), we conclude that $t_1 < \infty$. □

Lemma 3.6. *One has* $d(c) = \frac{\mu c}{2}[\langle \Lambda^\mu \varphi_c, \varphi_c\rangle + \langle \Lambda^\mu \psi_c, \psi_c\rangle]$.

Proof. For $\lambda > 0$, let $\Phi^\lambda(x) = \Phi(\frac{x}{\lambda})$. Then

$$E(\Phi^\lambda) + cV(\Phi^\lambda) = \int [-F(\Phi^\lambda) + \frac{c}{2}\Phi^\lambda L\Phi^\lambda]dx$$

$$= \int [-F(\Phi^\lambda) + \frac{c}{2}(\Phi^\lambda)^2 + \frac{c}{2}\Phi^\lambda \Lambda^\mu \Phi^\lambda]dx$$

$$= \int \lambda[-F(\Phi) + \frac{c}{2}\Phi^2]dx + \lambda^{1-\mu}\frac{c}{2}\int \Phi\Lambda^\mu\Phi dx.$$

Next we differentiate this expression with respect to λ and evaluate it at $\lambda = 1$, observing that the left-hand side becomes zero, because $E'(\Phi^\lambda) + cV'(\Phi^\lambda) = 0$. Thus

$$0 = \int [-F(\Phi) + \frac{c}{2}\Phi^2 + (1-\mu)\frac{c}{2}\Phi\Lambda^\mu\Phi]dx,$$

so that

$$d(c) = \frac{\mu c}{2}\int \Phi\Lambda^\mu\Phi dx.$$

Theorem 3.5. *Let Assumptions* 1 *and* 2 *hold:*

a) *if* $p \leq \mu$, *then* Φ_c *is stable for all* $c > 1$;

b) *if* $p > \mu$, *there is a* $c_0 > 1$ *such that* Φ_c *is stable for* $c > c_0$ *and unstable for* $1 < c < c_0$.

Proof. Using the homogeneity of M, we can write the solution Φ_c as

$$\varphi(x) = (c-1)^{\frac{1}{2p}}\varphi_1\left(\left(\frac{c-1}{c}\right)^{\frac{1}{\mu}} x\right),$$

$$\psi(x) = (c-1)^{\frac{1}{2p}}\psi_1\left(\left(\frac{c-1}{c}\right)^{\frac{1}{\mu}} x\right),$$

where (φ_1, ψ_1) is a solution of the system

$$\Lambda^\mu \varphi_1 + \varphi_1 - \varphi_1^p \psi_1^{p+1} = 0$$

$$\Lambda^\mu \psi_1 + \psi_1 - \varphi_1^{p+1} \psi_1^p = 0,$$

which is independent on c. Then

$$d(c) = \frac{\mu c}{2} \left[\int \varphi \Lambda^\mu \varphi + \int \psi \Lambda^\mu \psi \right]$$

$$= \frac{\mu b}{2} (c-1)^{\frac{1}{p}+1-\frac{1}{\mu}} c^{\frac{1}{\mu}},$$

where $b = \int \varphi_1 \Lambda^\mu \varphi_1 + \int \psi_1 \Lambda^\mu \psi_1$. Differentiating twice with respect to c yields

$$d''(c) = \frac{\mu b}{2} (c-1)^{\frac{1}{p}-\frac{1}{\mu}-1} c^{\frac{1}{\mu}-2} q(c),$$

where $q(c) = (r+s+1)(r+s+2)c^2 - 2(r+1)(r+s+1)c + r(r+1)$, $r = \dfrac{1}{\mu} - 1$, $s = \dfrac{1}{p} - \dfrac{1}{\mu}$. Whether $d''(c)$ is positive or negative depends on the sign of $q(c)$. This is a quadratic function of c with one negative and one positive root, since $r(r+1) < 0$ and $r + s + 1 > 0$. We call the positive root c_0. Since $q(1) = (\dfrac{1}{p} - \dfrac{1}{\mu})(\dfrac{1}{p} - \dfrac{1}{\mu} + 1)$, then if $p \le \mu$, $d''(c) > 0$ for $c > 1$, and if $p > \mu$, $d''(c) < 0$ for $1 < c < c_0$ and $d''(c) > 0$ for $c > c_0$. Theorem 3.3 is proved. □

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REFERENCES

1. Albert, J. Positivity property and stability of solitary-waves solutions of model equations for long waves. *Comm. PDE*, **17**, 1992, 1-22.
2. Albert, J., J. Bona. Total positivity and the stability of internal waves in stratified fluids in finite depth. *IMA J. of Applied Mathematics*, **46**, 1991, 1-19.
3. Albert, J., J. Bona, D. Henry. Sufficient conditions for stability of solitary wave solutions of model equations for long waves. *Physica*, **24D**, 1987, 343-366.
4. Alarcon, E., J. Angulo, J. Montenegro. Stability and instability of solitary waves for a nonlinear dispersive system. *Nonlinear Analysis*, **36**, 1999, 1015-1035.
5. Benjamin, T. The stability of solitary waves. *Proc. Royal Soc. London, Ser. A*, **328**, 1972, 153-183.
6. Benjamin, T., J. Bona, D. Bose. Solitary-wave solutions of nonlinear problems. *Phil. Trans. Roy. Soc. London, Ser. A* , 1990.

7. Benjamin, T., J. Bona, J. Mahony. Model equations for long waves in nonlinear dispersive system. *Phil. Trans. Roy. Soc. London, Ser. A*, **272**, 1972, 447-478.

8. Bona, J. On the stability theory of solitary waves. *Proc. Roy. Soc. London, Ser. A*, **347**, 1975, 377-430.

9. Bona, J., P. Souganidis, W. Strauss. Stability and instability of solitary waves of Korteweg-de Vries type. *Proc. Roy. Soc. London, Ser. A*, **411**, 1987, 395-412.

10. Hakkaev, S. Stability and instability of solitary wave solutions of nonlinear dispersive system of BBM type (to appear).

11. Grillakis, M., J. Shatah, W. Strauss. Stability theory of solitary waves in the presence of symmetry I. *J. of Functional Analysis*, **94**, 1990, 308-348.

12. Iliev, I., E. Khristov, K. Kirchev. Spectral Methods in Soliton Equations. *Pitman Monographs and Surveys in Pure and Applied Mathematics*, **73**, 1994.

13. Iliev, I., K. Kirchev. Stability of traveling waves for nonlinear equations of Benjamin-Bona-Mahony equation type. *Commun. J. Inst. Nuclear Research, Dubna*, **P5-87-508** , 1987 (in Russian).

14. Iliev, I., K. Kirchev. The stability of traveling waves for nonlinear equations of Benjamin-Bona-Mahony type. *Math. Nachr.*, **141**, 1989, 313–324 (in Russian).

15. Pazy, A. Semigroup of linear operators and applications to partial differential equations. Springer, New York, 1983.

16. Souganidis, P., W. Strauss. Instability of a class of dispersive solitary waves. *Proc. Roy. Soc. of Edinburgh*, **114A**, 1990, 195-212.

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Faculty of Mathematics and Informatics
Shumen University
9712 Shumen
BULGARIA
E-mail: shakkaev@fmi.shu-bg.net