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CONNECTION BETWEEN THE LOWER P-FRAME CONDITION AND EXISTENCE OF RECONSTRUCTION FORMULAS IN A BANACH SPACE AND ITS DUAL

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In the present paper it is proved that under an additional assumption (which is automatically satisfied in case $p = 2$) validity of the lower p-frame condition for a sequence $\{g_i\} \subset X^*$ implies that for f in a subset of X there exists a representation $f = \sum g_i(f) f_i$, where $\{f_i\} \subset X$ satisfies the upper q-frame condition, $\frac{1}{q} + \frac{1}{p} = 1$. An example showing that the above representation is not necessarily valid for all f in X (neither reconstruction formula of type $g = \sum g(f_i) g_i$ for all $g \in X^*$) is given. It is shown that when $\mathcal{D}(U)$ is dense in X, $g \in X^*$ can be represented as $g = \sum g(f_i) g_i$ if and only if $\sum g(f_i) g_i$ converges.

Keywords: p-frames, lower bound, reconstructions, Banach spaces, dual spaces

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1. INTRODUCTION

It is well known that if a sequence $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ is a frame for a Hilbert space $\mathcal{H}$, i.e. there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{i=1}^{\infty} | < f, g_i > |^2 \leq B||f||^2, \quad \forall f \in \mathcal{H},$$

then every $f \in \mathcal{H}$ can be represented by a dual frame $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$:

$$f = \sum_{i=1}^{\infty} < f, f_i > g_i = \sum_{i=1}^{\infty} < f, g_i > f_i. \tag{1.1}$$

Sequences, which satisfy the lower frame condition, but may fail the upper one, are used in some applications (for example, in irregular sampling). For this reason the existence of reconstruction formulas like (1.1) when only the lower frame condition is assumed has become a topic of investigation. The first study in this direction may be found in [3]. There an operator is associated to a family $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ and under some assumptions on that operator it is proved that $\{g_i\}_{i=1}^{\infty}$ satisfies the lower frame condition and there exists a Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$ (i.e. sequence satisfying the upper frame condition) such that $f = \sum_{i=1}^{\infty} < f, f_i > g_i$, $\forall f \in \mathcal{H}$. Later, aim of investigation has been to get reconstruction formulas when the lower frame condition is assumed to be valid. In [2] it is proved that if $\{g_i\}_{i=1}^{\infty} \subset \mathcal{H}$ satisfies the lower frame condition, then there exists a Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset \mathcal{H}$ such that

$$f = \sum_{i=1}^{\infty} < f, g_i > f_i, \ \forall f \in \mathcal{D}(U), \tag{1.2}$$

where

$$\mathcal{D}(U) = \{f \in X \mid \sum_{i=1}^{\infty} | < f, g_i > |^2 < \infty\}, \tag{1.3}$$

$$U : \mathcal{D}(U) \subseteq \mathcal{H} \to \ell^2, \ \ Uf := \{< f, g_i >\}_{i=1}^{\infty}. \tag{1.4}$$

Recently, frames in Hilbert spaces have been generalized to p-frames in Banach spaces [1]. A sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$ is called p-frame for X $(1 < p < \infty)$ if there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p} \leq B\|f\|_X, \ \forall f \in X.$$

$\{g_i\}_{i=1}^{\infty}$ is called a p-Bessel sequence for X if it satisfies the upper p-frame inequality for all $f \in X$. In [4], p-frames $\{g_i\}_{i=1}^{\infty} \subset X^*$ in general Banach spaces are considered and necessary and sufficient condition for existence of reconstruction formulas like

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \ \forall f \in X, \tag{1.5}$$

$$g = \sum_{i=1}^{\infty} g(f_i) g_i, \ \forall g \in X^* \tag{1.6}$$

via a dual q-frame is found, namely the condition "the range of the operator U is complemented in ℓ^p", where $U : X \to \ell^p$, $Uf = \{g_i(f)\}_{i=1}^{\infty}$. In the present paper we are interested in reconstruction formulas when only the lower p-frame condition is assumed. In Section 3, generalization of (1.2) to the case when a family $\{g_i\}_{i=1}^{\infty} \subset X^*$ is assumed to satisfy the lower p-frame condition is investigated; for this case a necessary and sufficient condition for validity of formula like (1.2) via a q-Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is given. Section 4 concerns the question whether the lower p-frame condition implies existence of reconstruction formulas not only in $\mathcal{D}(U)$, but in the whole spaces X and X^* (like (1.5) and (1.6)). In Section 5 we investigate the lower p-frame inequality in Banach spaces in case the corresponding operator U is assumed to be densely defined. This is motivated by the work by Christensen and Li, who investigated the lower frame inequality in Hilbert spaces in case the operator U, given by (1.3) and (1.4), is densely defined, with the aim of obtaining reconstruction formulas in the weak sense. Our aim in this section is to obtain representation in X^* with convergence in norm-sense. In Proposition 5.1 a necessary and sufficient condition for an element $g \in X^*$ to be represented via a formula like (1.6) is given.

2. NOTATIONS AND BASIC FACTS

Throughout the paper $(\mathcal{H}, <\cdot,\cdot>)$ denotes a separable Hilbert space; X denotes a separable Banach space and X^* denotes its dual space; p and q are assumed to satisfy $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$; the canonical basis of ℓ^p $(1 < p < \infty)$ is the basis consisting of the elements (1,0,0,0,...), (0,1,0,0,...), (0,0,1,0,...),...

A sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition when there exists a constant $A > 0$ such that

$$A\|f\|_X \leq \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p}, \ \forall f \in X. \tag{2.1}$$

To such a sequence the (possibly unbounded) linear operator

$$U : \mathcal{D}(U) \subseteq X \to \ell^p, \ \ Uf := \{g_i(f)\}_{i=1}^{\infty}, \tag{2.2}$$

where $\mathcal{D}(U) = \{f \in X \mid \sum_{i=1}^{\infty} |g_i(f)|^p < \infty\}$, is associated. $\mathcal{R}(U)$ denotes the range of U.

Recall that a linear operator $U : \mathcal{D}(U) \subseteq X \to Y$, whose domain is a linear subset of a Banach space X and whose range lies in a Banach space Y, is closed if the conditions $\{x_j\} \subset \mathcal{D}(U), x_j \to x$ in X and $Ux_j \to y$ in Y when $j \to \infty$ imply $x \in \mathcal{D}(U)$ and $Ux = y$ or, equivalently, if the graph of U is closed in the product space $X \times Y$ [5, p. 57].

The following known results are needed:

Lemma 2.1 [7, p.156]. *Let E, F be linear normed spaces and $U : E \to F$ be a linear operator. Then, for $A > 0$, the inequality $\|Uf\|_F \geq A\|f\|_E$ holds for all $f \in \mathcal{D}(U)$ if and only if U has a bounded inverse $U^{-1} : \mathcal{R}(U) \to E$ for which $\|U^{-1}\| \leq \frac{1}{A}$.*

Lemma 2.2 [4]. *$\{g_i\}_{i=1}^{\infty} \subset X^*$ is a p-Bessel sequence for X with bound B if and only if*

$$T : \{d_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} d_i g_i$$

is a well defined (hence bounded) operator from ℓ^q into X^* and $\|T\| \leq B$.

Lemma 2.3 [6, 8]. *For every $1 \leq r < p < \infty$, ℓ^r is a linear subset of ℓ^p and $\|\{c_i\}_{i=1}^{\infty}\|_{\ell^p} \leq \|\{c_i\}_{i=1}^{\infty}\|_{\ell^r}$ for all $\{c_i\}_{i=1}^{\infty} \in \ell^r$. Furthermore, no space of the family ℓ^p, $1 \leq p < \infty$, is isomorphic to a subspace of another member of this family.*

Corollary 2.1. *For every $1 \leq r < p < \infty$, the space ℓ^r, considered as a subset of ℓ^p, is not closed in ℓ^p.*

3. CONSEQUENCES OF THE LOWER P-FRAME CONDITION IN THE GENERAL CASE

We begin with a consequence of the lower p-frame condition concerning the associated operator U, which is a generalization of a result concerning the lower frame condition in Hilbert spaces [2]:

Lemma 3.1. *Suppose that $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition* (2.1). *Then the operator U given by* (2.2) *is an injective closed operator with closed range. Furthermore, the inverse $U^{-1} : \mathcal{R}(U) \to \mathcal{D}(U)$ is bounded and $\|U^{-1}\| \leq \dfrac{1}{A}$.*

Proof. To prove that U is closed, consider a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathcal{D}(U)$ for which

$$x_j \to x \text{ in } X \text{ and } Ux_j \to \{c_i\}_{i=1}^{\infty} \text{ in } \ell^p \text{ when } j \to \infty.$$

Since all g_i are continuous functionals and since convergence in ℓ^p implies convergence by coordinates, the assumptions imply that for all i,

$$g_i(x_j) \to g_i(x) \text{ as } j \to \infty$$

and

$$g_i(x_j) \to c_i \text{ as } j \to \infty.$$

Thus $\{g_i(x)\}_{i=1}^{\infty} = \{c_i\}_{i=1}^{\infty}$, i.e. $x \in \mathcal{D}(U)$ and $Ux = \{c_i\}_{i=1}^{\infty}$, and hence U is closed.

To prove that U has closed range, consider again a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathcal{D}(U)$, and assume that $Ux_j \to y$ as $j \to \infty$. Thus $\{Ux_j\}_{j=1}^{\infty}$ is a Cauchy sequence, which implies by (2.1) that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Thus $x_j \to x$ for a certain element x of the Banach space X. Since U is closed, one can now conclude that $x \in \mathcal{D}(U)$ and $y = Ux$, i.e. y belongs to the range of U.

The rest follows by Lemma 2.1 $\square$.

Note that when $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\mathcal{D}(U) = X$, then $\{g_i\}_{i=1}^{\infty}$ is a p-frame for X. Indeed, in this case Lemma 3.1 implies the existence of a bounded inverse U^{-1} from the closed subspace $\mathcal{R}(U)$ of ℓ^p onto X, which by the Inverse Mapping Theorem implies boundedness of U, i.e. validity of the upper p-frame condition. Similarly, if $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\mathcal{D}(U)$ is a closed subspace of X, then $\{g_i|_{\mathcal{D}(U)}\}_{i=1}^{\infty}$ is a p-frame for the Banach space $\mathcal{D}(U)$. Reconstruction formulas when both the lower and the upper p-frame conditions are satisfied have been studied in [4]. In this paper we are mostly interested in cases when $\{g_i\}_{i=1}^{\infty} \subset X$ satisfies the lower p-frame condition and $\mathcal{D}(U) \subsetneq X$ is not closed in X (i.e. $\{g_i\}_{i=1}^{\infty} \subset X^*$ fails to be a p-Bessel sequence for X or for $\mathcal{D}(U)$). For examples of this kind see 4.1, 5.1 and 3.1.

When (2.1) is satisfied, the above lemma assures that the operator U given by (2.2) has a bounded inverse $U^{-1} : \mathcal{R}(U) \to \mathcal{D}(U)$. The next theorem shows that the existence of a bounded extension of U^{-1} on ℓ^p is a necessary and sufficient condition for existence of representations of the elements in $\mathcal{D}(U)$ via a q-Bessel sequence:

Theorem 3.1. *Suppose that $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition* (2.1). *Then the following are equivalent:*

(i) *there exists a q-Bessel sequence $\{f_i\}_{i=1}^{\infty} \subset X (\subseteq X^{**})$ for X^* such that*

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \ \forall f \in \mathcal{D}(U); \tag{3.1}$$

(ii) *the operator $U^{-1} : \mathcal{R}(U) \to X$ can be extended to a linear bounded operator on ℓ^p.*

Proof. Assume (i). By Lemma 2.2, the operator $V : \{c_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} c_i f_i$ is a well defined linear bounded operator from ℓ^p into X. For every $f \in \mathcal{D}(U)$ we have

$$V(Uf) = \sum_{i=1}^{\infty} g_i(f) f_i = f = U^{-1}Uf$$

and hence V is an extension of U^{-1}.

Assume now (ii). Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis for ℓ^p and let $f_i := Ve_i$ for all i. Then, by construction, for all $f \in \mathcal{D}(U)$ we have

$$f = VUf = \sum_{i=1}^{\infty} g_i(f) f_i.$$

Now let $g \in X^*$. Considering the functional $gV \in (\ell^p)^*$, the natural isometrical isomorphism between $(\ell^p)^*$ and ℓ^q implies that the sequence $\{g(f_i)\}_{i=1}^{\infty} = \{gV(e_i)\}_{i=1}^{\infty}$ belongs to ℓ^q and

$$\left(\sum_{i=1}^{\infty}|g(f_i)|^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{\infty}|gV(e_i)|^q\right)^{\frac{1}{q}} = \|gV\|_{(\ell^p)^*} \leq \|V\| \cdot \|g\|_{X^*}, \ \forall g \in X^*.$$

Hence $\{f_i\}_{i=1}^{\infty}$, considered as a family in X^{**}, is a q-Bessel sequence for X^* $\square$.

Note that when $\{g_i\}_{i=1}^{\infty} \subset X^*$ is a p-frame for X, then the above conditions (i) and (ii) are equivalent to the condition

(iii) *$\mathcal{R}(U)$ is complemented in ℓ^p*

(see [4]). When only the lower p-frame condition is assumed, (iii) implies (ii). Indeed, if P is a bounded projection from ℓ^p onto $\mathcal{R}(U)$, then, clearly, $U^{-1}P$ is a linear bounded extension of U^{-1} on ℓ^p. In special cases the inverse implication is also true. For example, if $p = 2$ and $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower 2-frame condition, then $\mathcal{R}(U)$ is closed in the Hilbert space ℓ^2 and hence (iii) and (ii) are satisfied; thus (i) is always valid in this case. Example 3.1 and Example 5.1 are examples of cases, when $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition for X, $\mathcal{D}(U) \subsetneqq X$ is not closed in X and (i), (ii) and (iii) are satisfied. It is still an open question whether there exists an example of a family, which satisfies the lower p-frame condition, $\mathcal{D}(U) \subsetneqq X$ is not closed in X, (i) and (ii) are satisfied, but (iii) fails.

Example 3.1. Let $1 < p < s < \infty$. Consider the Banach space $X = \ell^s$. Let $\{e_i\}_{i=1}^{\infty}$ be the canonical basis for ℓ^s and let $\{E_i\}_{i=1}^{\infty} \subset (\ell^s)^*$ be the associated coefficient functionals. By Lemma 2.3, the set $\mathcal{D}(U) = \{\{d_i\}_{i=1}^{\infty} \in \ell^s : \{E_j(\{d_i\}_{i=1}^{\infty})\}_{j=1}^{\infty} \in \ell^p\}$ is actually $\ell^p \subsetneqq X$ and for all $\{d_i\}_{i=1}^{\infty} \in \mathcal{D}(U)$ we have $\|U(\{d_i\}_{i=1}^{\infty})\|_{\ell^p} = \|\{d_i\}_{i=1}^{\infty}\|_{\ell^p} \geq \|\{d_i\}_{i=1}^{\infty}\|_{\ell^s}$; for the elements $\{d_i\}_{i=1}^{\infty} \in \ell^s \setminus \ell^p$ the lower p-frame inequality is clearly satisfied. By Corollary 2.1, $\mathcal{D}(U)$ is not closed in X. The range of the operator U is $\mathcal{R}(U) = \ell^p$ and thus (iii), and hence (ii) and (i) are valid.

4. A COUNTEREXAMPLE

As it was shown in the previous section, under the additional assumption on complementability of $\mathcal{R}(U)$ in ℓ^p or the existence of a bounded extension of U^{-1} on ℓ^p, the lower p-frame condition implies the existence of reconstruction formulas in $\mathcal{D}(U)$. This section concerns the question whether the same assumptions imply existence of reconstruction formulas in the whole space X or in the whole X^*. Example 4.1 below answers negative; it shows a case when there are no reconstruction formulas neither in the whole space X^* via the sequence $\{g_i\}_{i=1}^{\infty} \subset X^*$, satisfying the lower p-frame condition, nor in the whole space X via a dual family $\{f_i\}_{i=1}^{\infty} \subset X$

(a q-Bessel sequence satisfying (3.1)). The example concerns a case when X is a Hilbert space and $p = 2$ (in this case the assumption "$\mathcal{R}(U)$-complemented in ℓ^2" is automatically satisfied) and thus it shows that the answer is negative even for this special most considered case. Note that in a recent paper [2], concerning the lower frame condition in Hilbert spaces, it has been shown that the representation in (1.2) is not necessarily valid for all $f \in \mathcal{H}$; the counterexample given in [2] and the one given in the present paper are obtained independently; the counterexample given in [2] is more complicated than the one below.

Example 4.1. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $(\mathcal{H}, <\cdot,\cdot>)$ and consider the family $\{g_i\}_{i=2}^{\infty} := \{i(e_1 + e_i)\}_{i\geq 2} \subset \mathcal{H}$. The family $\{g_i\}_{i=2}^{\infty}$ has the following properties:

(i) $\{g_i\}_{i=2}^{\infty}$ satisfies the lower frame inequality, but it is not a frame for $\mathcal{H}$;

(ii) e_1 can not be written as $\sum_{i=2}^{\infty} c_i g_i$ for any numbers $\{c_i\}_{i=2}^{\infty}$;

(iii) if $\{f_i\}_{i=2}^{\infty}$ is a Bessel sequence, satisfying (1.2), e_1 can not be written as $\sum_{i=2}^{\infty} c_i f_i$ for any numbers $\{c_i\}_{i=2}^{\infty}$.

Proof. (i) Let $x \in H$ be arbitrary fixed. If $< x, e_1 >= 0$, then

$$\begin{aligned}\sum_{i=2}^{\infty} |< x, g_i >|^2 &= \sum_{i=2}^{\infty} i^2 |< x, e_i >|^2 \geq \sum_{i=2}^{\infty} |< x, e_i >|^2 \\ &= \sum_{i=1}^{\infty} |< x, e_i >|^2 = \|x\|^2.\end{aligned}$$

Let now $< x, e_1 > \neq 0$. Since $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$,

$$\sum_{i=2}^{\infty} |< x, e_i >|^2 < \infty$$

and hence $< x, e_i > \to 0$ when $i \to \infty$. Therefore

$$\sum_{i=2}^{\infty} |< x, e_i > + < x, e_1 >|^2 = \infty, \tag{4.1}$$

because otherwise $< x, e_i >$ would converge to $(- < x, e_1 >) \neq 0$, which is a contradiction. Now (4.1) implies that

$$\sum_{i=2}^{\infty} |< x, g_i >|^2 = \sum_{i=2}^{\infty} i^2 |< x, e_i > + < x, e_1 >|^2 = \infty$$

and hence the inequality

$$\sum_{i=2}^{\infty} |< x, g_i >|^2 \geq \|x\|^2$$

is satisfied.

The fact, that $\{g_i\}_{i=2}^{\infty}$ does not satisfy the upper frame inequality follows from the equalities

$$\sum_{i=2}^{\infty} | < e_k, g_i > |^2 = k^2 = k^2 \|e_k\|, \quad \forall k \geq 2.$$

(ii) If there exist constants $c_2, c_3, c_4, ...$ such that $e_1 = \sum_{i=2}^{\infty} c_i i(e_1 + e_i)$, then the orthogonality $< e_k, e_1 >= 0$, $\forall k \geq 2$, implies that all c_i are zero, which is a contradiction.

(iii) Let now $\{f_i\}_{i=2}^{\infty}$ be a Bessel sequence, satisfying (1.2). For every $k \geq 2$, e_k belongs to $\mathcal{D}(U)$ and thus, by (1.2),

$$e_k = \sum_{i=2}^{\infty} < e_k, i(e_1 + e_i) > f_i = k f_k.$$

If we assume that $e_1 = \sum_{i=2}^{\infty} c_i f_i$ for some numbers $\{c_i\}_{i=2}^{\infty}$, this would imply that $e_1 = \sum_{i=2}^{\infty} \frac{c_i}{i}\, e_i$, which is a contradiction. □

5. THE LOWER P-FRAME CONDITION IN A SPECIAL CASE

Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition. In this section we are interested in representation of elements in the dual space X^*. In the previous section we have seen an example of a case when $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfies the lower p-frame condition and $\{f_i\}_{i=1}^{\infty} \subset X$ is a q-Bessel sequence satisfying (3.1), but not all g in X^* can be represented as $g = \sum_{i=1}^{\infty} g(f_i) g_i$. Here the elements $g \in X^*$ which allow such representations are investigated. We consider the special case when the given sequence $\{g_i\}_{i=1}^{\infty}$ satisfies one more assumption, namely that the domain of the associated operator U, defined by (2.2), is a dense subset of X. The following result holds true:

Theorem 5.1. *Let $\{g_i\}_{i=1}^{\infty} \subset X^*$ satisfy the lower p-frame condition, $\mathcal{D}(U)$ be dense in X and $\{f_i\}_{i=1}^{\infty} \subset X$ be a q-Bessel sequence satisfying* (3.1). *Then an element $g \in X^*$ can be represented as*

$$g = \sum_{i=1}^{\infty} g(f_i) g_i$$

if and only if

$$\text{the sequence } \left\{ \sum_{i=1}^{n} g(f_i) g_i \right\}_{n=1}^{\infty} \text{ is convergent.}$$

Proof. Fix an arbitrary $g \in X^*$.

It is only needed to prove that if $\{\sum_{i=1}^n g(f_i)g_i\}_{n=1}^\infty$ is convergent, then it converges to g. Suppose that $\sum_{i=1}^\infty g(f_i)g_i = \lim_{n\to\infty} \sum_{i=1}^n g(f_i)g_i$ exists. Denote the canonical basis of ℓ^p by $\{e_i\}_{i=1}^\infty$ and the canonical basis of ℓ^q by $\{z_i\}_{i=1}^\infty$ $(\frac{1}{p}+\frac{1}{q}=1)$. Let $V : \ell^p \to X$ be the linear bounded extension of U^{-1} defined in the proof of Theorem 3.1; then $f_i = V(e_i)$, $\forall i$. By the isometrical isomorphism of $(\ell^p)^*$ and ℓ^q, $\{gV(e_i)\}_{i=1}^\infty = \{g(f_i)\}_{i=1}^\infty \in \ell^q$ can be identified with $V^*(g) = gV \in (\ell^p)^*$ and thus

$$\sum_{i=1}^n g(f_i)z_i \underset{n\to\infty}{\longrightarrow} \sum_{i=1}^\infty g(f_i)z_i = V^*g. \tag{5.1}$$

Under the assumptions of the theorem we can consider the adjoint operator

$$U^* : \mathcal{D}(U^*) \to X^*,$$

where

$$\mathcal{D}(U^*) = \{G \in (\ell^p)^* |\text{ the functional } G \circ U \text{ is continuous on } \mathcal{D}(U)\}.$$

By definition, U^*G is the unique extension of GU to a continuous functional on X (the continuous extension is unique, because $\mathcal{D}(U)$ is assumed to be dense in X). It is not difficult to see that U^* is a densely defined closed operator. Every z_i belongs to $\mathcal{D}(U^*)$ (considered as a subset of ℓ^q) and $U^*z_i = g_i$, because $(g_i - U^*z_i)(f) = g_i(f) - g_i(f) = 0$ for all f in $\mathcal{D}(U)$, which is dense in X. Then for every $n \in \mathbb{N}$, the finite sum $\sum_{i=1}^n g(f_i)z_i$ belongs to $\mathcal{D}(U^*)$ and

$$U^*\left(\sum_{i=1}^n g(f_i)z_i\right) = \sum_{i=1}^n g(f_i)U^*z_i = \sum_{i=1}^n g(f_i)g_i \;\to\; \sum_{i=1}^\infty g(f_i)g_i. \tag{5.2}$$

Now (5.1), (5.2) and the closeness of U^* imply that V^*g belongs to $\mathcal{D}(U^*)$ and

$$U^*V^*g = \sum_{i=1}^\infty g(f_i)g_i.$$

Since $U^*V^*(g)(f) - g(f) = gVU(f) - g(f) = 0$ for all f in $\mathcal{D}(U)$, which is dense in X, one can conclude that $U^*V^*(g) = g$. Therefore $g = \sum_{i=1}^\infty g(f_i)g_i$. □

As a consequence of Theorem 5.1, for the Hilbert frame case we get:

Corollary 5.1. *Let $\mathcal{H}$ be a Hilbert space and assume that $\{g_i\}_{i=1}^\infty \subset \mathcal{H}$ satisfies the lower frame condition with $\mathcal{D}(U)$ dense in $\mathcal{H}$. Let $h \in \mathcal{H}$ and $\{f_i\}_{i=1}^\infty \subset \mathcal{H}$ be a Bessel sequence satisfying* (1.2). *Then*

$$h = \sum_{i=1}^\infty < h, f_i > g_i$$

if and only if

$$\text{the sequence } \left\{ \sum_{i=1}^{n} < h, f_i > g_i \right\}_{n=1}^{\infty} \text{ is convergent.}$$

Below an example of a sequence satisfying the assumptions of Theorem 5.1 is given.

Example 5.1. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$ and let $\{g_i\}_{i=1}^{\infty} := \{ie_i\}_{i=1}^{\infty}$. Since

$$\sum_{i=1}^{\infty} | < h, g_i > |^2 = \sum_{i=1}^{\infty} i^2 | < h, g_i > |^2 \geq \sum_{i=1}^{\infty} | < h, e_i > |^2 = \|h\|_{\mathcal{H}}^2, \ \forall h \in \mathcal{H},$$

$\{g_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. Clearly,

$$\mathcal{D}(U) = \{c = \sum_{i=1}^{\infty} c_i e_i \in H \ : \ \sum_{i=1}^{\infty} |ic_i|^2 < \infty\}.$$

Since $span\{e_i\} \subseteq \mathcal{D}(U)$, but $\overline{span}\{e_i\} = \mathcal{H} \nsubseteq \mathcal{D}(U)$ (for example $\sum_{i=1}^{\infty} \frac{1}{i} e_i \in \mathcal{H} \setminus \mathcal{D}(U)$), $\mathcal{D}(U)$ is dense, but not closed in $\mathcal{H}$. For every $g \in H^* = H$, the sequence $\left\{ \sum_{i=1}^{n} < g, \frac{1}{i} e_i > g_i \right\}_{n=1}^{\infty}$ converges to g and $\{\frac{1}{i} e_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$.

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