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PARTITIONED GRAPHS AND DOMINATION RELATED PARAMETERS

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Let G be a graph of order $n \geq 2$ and $n_1, n_2, .., n_k$ be integers such that $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ and $n_1 + n_2 + .. + n_k = n$. Let for $i = 1, .., k$: $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$ where $\mathcal{K}_m$ is the set of all pairwise non-isomorphic graphs of order m, $m = 1, 2, \ldots$ In this paper we study when for a domination related parameter μ (such as domination number, independent domination number and acyclic domination number) is fulfilled $\mu(G) = \mu(\cup_{i=1}^{k} < V_i, G >)$ for all vertex partitions $\{V_1, V_2, .., V_k\}$, $k \geq 2$, of a vertex set of G such that $< V_i, G >$ is isomorphic to some a member of $\mathcal{A}_i$, $i = 1, 2, .., k$. In the process several results for acyclic domination vertex critical graphs are presented. Results for independence number of double vertex graphs are obtained.

Keywords: domination number, acyclic domination number, independent domination number, independence number, double vertex graph

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1. NOTATION AND DEFINITIONS

For a graph theory terminology not presented here, we follow Haynes, et al.[8]. All our graphs are finite and undirected with no loops or multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. The subgraph induced by $S \subseteq V(G)$ is denoted by $< S, G >$. We denote by K_n and $\overline{K}_n$ complete graph on n vertices and its complement. If $n \geq 3$ then C_n is a connected 2 - regular graph of order n. P_m is a tree of order m and diameter $m-1$, $m \geq 1$. By $\mathcal{K}_s$ we denote the set of all pairwise non-isomorphic graphs of order s, $s \geq 1$. A subset of vertices A in a graph G is said to be *acyclic* if $< A, G >$

contains no cycles. A subset of vertices I in a graph G is said to be *independent* if $< I, G >$ contains no edges. The *independence number* $\beta_0(G)$ is the maximum cardinality of an independent set in G. A *dominating set* in a graph G is a set of vertices D such that every vertex of G is either in D or is adjacent to an element of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality taken over all dominating sets of G. The *independent domination number* $i(G)$ (*acyclic domination number* $\gamma_a(G)$) of a graph G is the minimum cardinality of an independent dominating (acyclic dominating) set of G.

Throughout this paper, let a property $\mathcal{P}$ of graphs be given and $\mu(G)$ be a numeral invariant of a graph G defined in a such a way that it is the minimum or maximum number of vertices of a set $S \subseteq V(G)$ which has the property $\mathcal{P}$. A set with property $\mathcal{P}$ and with $\mu(G)$ vertices is called a μ *- set* of G. A vertex v of a graph G is μ - critical if $\mu(G - v) \neq \mu(G)$. The graph G is μ - critical if all its vertices are μ - critical. Much has been written about the effects on a parameter(such connectedness, chromatic number, domination number) when a graph is modified by deleting a vertex. μ - critical graphs for $\mu = \gamma, i$ was investigated by Brigham et al.[4] and Ao and MacGillivray (see [9, ch. 16]) respectively. Further properties on these graphs can be found in [6], [7], [8, ch.5], [9, ch. 16], [10].

In this work, by a partition of a graph G into k parts, $k \geq 2$, we mean a family $\boldsymbol{A} = \{G_1, G_2, .., G_k\}$ of pairwise disjoint induced subgraphs of G, with $\cup_{i=1}^{k} V(G_i) = V(G)$ and $1 \leq |V(G_1)| \leq |V(G_2)| \leq .. \leq |V(G_k)|$. We denote by $G[\boldsymbol{A}]$ the graph $\cup_{i=1}^{k} G_i$.

Let G be a graph of order $n \geq 2$ and $n_1, n_2, .., n_k$ be integers such that $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ and $n_1 + n_2 + .. + n_k = n$. Let $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$, $i = 1, .., k$. We say that a partition $\boldsymbol{A} = \{G_1, G_2, .., G_k\}$ of G is of type $[\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k]$ if G_i is isomorphic to some a member of $\mathcal{A}_i$, $i = 1, .., k$. The set of all partitions of a graph G which are of type $[\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k]$ will be denoted by $\mathcal{F}_G\ (\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k)$.

For a graph invariant μ and a family $\{\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k\}$, where $\mathcal{A}_i \subseteq \mathcal{K}_{n_i}$, $i = 1, .., k$ and $1 \leq n_1 \leq n_2 \leq .. \leq n_k$ it is important to characterize/study the graphs G with $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G\ (\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k)$.

We proceed as follows. In Section 2, we deals with critical vertices in a graph with respect to the acyclic domination number and give a necessary and sufficient condition for a graph to be γ_a- critical. In Section 3 we study when $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G\ (\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k)$ for some families $\{\mathcal{A}_1, \mathcal{A}_2, , ., \mathcal{A}_k\}$.

2. ACYCLIC DOMINATION NUMBER

The concept of acyclic domination was introduced by Hedetniemi et al.[11]. In this section some properties of critical vertices with respect to γ_a will be given.

Theorem 2.1. *Let G be a graph of order $n \geq 2$ and $u, v \in V(G)$.*

(i) *Let $\gamma_a(G - v) < \gamma_a(G)$.*

(i.1) [15] *If* $uv \in E(G)$ *then* u *belongs to no* γ_a *- set of* $G-v$;

(i.2) *If* M *is a* γ_a *- set of* $G-v$ *then* $M \cup \{v\}$ *is a* γ_a *- set of* G;

(i.3) [15] $\gamma_a(G-v) = \gamma_a(G) - 1$;

(ii) *Let* $\gamma_a(G-v) > \gamma_a(G)$. *Then* v *belongs to every* γ_a *- set of* G;

(iii) *If* $\gamma_a(G-v) < \gamma_a(G) < \gamma_a(G-u)$ *then* $uv \notin E(G)$;

(iv) *If* v *belongs to no* γ_a *- set then* $\gamma_a(G-v) = \gamma_a(G)$.

Proof. (i) For reason of completeness, we shall give here the proofs of (i.1) and (i.3).

(i.1): Let $uv \in E(G)$ and M be a γ_a - set of $G-v$. If $u \in M$ then M will be an acyclic dominating set of G with $|M| < \gamma_a(G)$ - a contradiction.

(i.2)and (i.3): If M is a γ_a - set of $G-v$ then (i.1) implies that $M_1 = M \cup \{v\}$ is an acyclic dominating set of G with $|M_1| = \gamma_a(G-v) + 1 \leq \gamma_a(G)$. Hence M_1 is a γ_a - set of G and $\gamma_a(G-v) = \gamma_a(G) - 1$.

(ii) If $M$is a γ_a - set of G and $v \notin M$ then M is an acyclic dominating set of $G-v$. But then $\gamma_a(G) = |M| \geq \gamma_a(G-v) > \gamma_a(G)$ and the result follows.

(iii) Let $\gamma_a(G-v) < \gamma_a(G)$ and M be a γ_a - set of $G-v$. Then by (i.2), $M \cup \{v\}$ is a γ_a-set of G. Let $\gamma_a(G-u) > \gamma_a(G)$. Now (ii) implies that $u \in M$ and by (i.1) - $uv \notin E(G)$.

(iv) By (ii), $\gamma_a(G-v) \leq \gamma_a(G)$. Assume $\gamma_a(G-v) < \gamma_a(G)$. It follows from (i.2) that $M \cup \{v\}$ is a γ_a – set of G, where M is a γ_a – set of $G-v$ - a contradiction. □

Theorem 2.2. Let G be a graph of order at least two. Then

(i) [3, 10] G *is* γ *- critical if and only if* $\gamma(G-v) = \gamma(G) - 1$ *for all* $v \in V(G)$;

(ii) (Ao and MacGillivray (see the bibliography in [9, ch.16])) G *is* i *- critical if and only if* $i(G-v) = i(G) - 1$ *for all* $v \in V(G)$.

Analogously result is valid and for γ_a - critical graphs.

Theorem 2.3. *Let* G *be a graph of order* $n \geq 2$. *Then* G *is a* γ_a *- critical graph if and only if* $\gamma_a(G-v) = \gamma_a(G) - 1$ *for all* $v \in V(G)$.

Proof. Necessity is obvious.

Sufficiency: Let G be a γ_a - critical graph. Clearly for every isolated vertex $v \in V(G)$, $\gamma_a(G-v) = \gamma_a(G) - 1$. Hence if G is isomorphic to $\overline{K}_n$ then $\gamma_a(G-v) = \gamma_a(G) - 1$ for all $v \in V(G)$. So, let G have a component of order at least two, say Q. Because of Theorem 2.1 (iii), either for all $v \in V(Q)$, $\gamma_a(Q-v) > \gamma_a(Q)$ or for all $v \in V(Q)$, $\gamma_a(Q-v) < \gamma_a(Q)$. Suppose, for all $v \in V(Q)$, $\gamma_a(Q-v) > \gamma_a(Q)$. It follows by Theorem 2.1 (ii) that $V(Q)$ is the unique acyclic dominating set of Q. Since $V(Q)$ is an acyclic set then Q is a tree which implies $\gamma_a(Q) = \gamma(Q) = |V(Q)|$

- a contradiction with the well known Ore's theorem [12] that for every connected graph H of order at least two, $\gamma(H) \le |V(H)|/2$. □

Theorem 2.4. *Let G_1 and G_2 be two connected graphs both of order at least two with $V(G_1) \cap V(G_2) = \{x\}$. If $\gamma_a(G_1 - x) < \gamma_a(G_1)$ and $\gamma_a(G_2 - x) < \gamma_a(G_2)$ then $\gamma_a(G) = \gamma_a(G_1) + \gamma_a(G_2) - 1$ and $\gamma_a(G - x) = \gamma_a(G) - 1$.*

Proof. It follows from Theorem 2.1 (i.2) that there exist a γ_a - set U_1 of G_1 and a γ_a - set U_2 of G_2 such that $x \in U_1 \cap U_2$. Hence $U_1 \cup U_2$ is an acyclic dominating set of G of cardinality $\gamma_a(G_1) + \gamma_a(G_2) - 1$. So we prove $\gamma_a(G) \le \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Let M be a γ_a - set of G and $M_i = M \cap V(G_i)$, $i = 1, 2$. There exist three possibilities:

(*) $x \notin M$ and M_i is an acyclic dominating set of G_i, $i = 1, 2$;

(**) $x \notin M$ and there are i, j such that $\{i, j\} = \{1, 2\}$, M_i is an acyclic dominating set of G_i and M_j is an acyclic dominating set of $G_j - x$;

(***) $x \in M$.

If (*) holds, then $\gamma_a(G) = |M| = |M_1| + |M_2| \ge \gamma_a(G_1) + \gamma_a(G_2)$ - a contradiction. If (**) holds, then $\gamma_a(G) = |M| = |M_1| + |M_2| \ge \gamma_a(G_i) + \gamma_a(G_j - x) = \gamma_a(G_1) + \gamma_a(G_2) - 1$. If (***) holds then $\gamma_a(G) = |M| = |M_1| + |M_2| - 1 \ge \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Thus we have $\gamma_a(G) = \gamma_a(G_1) + \gamma_a(G_2) - 1$.

Clearly $\gamma_a(G - x) = \gamma_a(G_1 - x) + \gamma_a(G_2 - x)$ and by Theorem 2.1 (i.3) it follows $\gamma_a(G - x) = \gamma_a(G_1) + \gamma_a(G_2) - 2$. Hence $\gamma_a(G - x) = \gamma_a(G) - 1$.□

Corollary 2.5. *Let G be a connected graph with blocks $G_1, G_2, .., G_n$. If the all $G_1, G_2, .., G_n$ are γ_a - critical then $\gamma_a(G) = \Sigma_{i=1}^{n} \gamma_a(G_i) - n + 1$.*

Proof. We proceed by induction on the number of blocks n. The statement is immediate if $n = 1$. Let the blocks of G be $G_1, G_2, .., G_n, G_{n+1}$ and without loss of generality let G_{n+1} contain only one cut-vertex of G. Hence Theorem 2.4 implies that $\gamma_a(G) = \gamma_a(G_{n+1}) + \gamma_a(Q) - 1$ where $Q = < \cup_{i=1}^{n} V(G_i), G >$. The result now follows from the inductive hypothesis. □

It is not possible to characterize γ - critical graphs in terms of forbidden graphs as it is shown in [3]. We shall prove a similar result for γ_a - critical graphs. We need the following example which is analogous to the one used in the proof of Theorem 6 in [3].

Example 2.6. Let G be a graph. If $\gamma_a(G) \ge 3$ then let $T = G$, otherwise $T = G \cup K_1 \cup K_1$. Let $V(T) = \{v_1, v_2, .., v_n\}$. Define the graph H as follows: $V(H) = \cup_{i=1}^{n} \{v_i, u_i, w_i\}$ and $E(H) = E(G) \cup \{v_i u_j, u_i w_j, w_i v_j \mid 1 \le i, j \le n, j \ne i\}$. It is straightforward to verify that no two vertices dominate H. Hence $\gamma_a(H) \ge 3$.

But by the definition of H, for each $i = 1, 2, .., n$, $\{u_i, v_i, w_i\}$ is a dominating and independent set (hence and an acyclic set) of H. So, $\gamma_a(H) \leq 3$. Thus $\gamma_a(H) = 3$. Clearly $\{u_i, v_i\}$ is a γ_a - set of $H - w_i$, $\{u_i, w_i\}$ is a γ_a - set of $H - v_i$ and $\{w_i, v_i\}$ is a γ_a - set of $H - u_i$. Therefore H is a γ_a - critical graph and G is its own induced subgraph.

From the above example we immediately have:

Theorem 2.7. *There does not exist a forbidden subgraph characterization of the class of γ_a - critical graphs.*

3. PARTITIONED GRAPHS

We begin with the family $\{\mathcal{A}_1 = \mathcal{K}_1, \mathcal{A}_2 = \mathcal{K}_{n-1}\}$ and $\mu \in \{\gamma, \gamma_a, i\}$.
From Theorem 2.2 and Theorem 2.3 we immediately have:

Theorem 3.1. *Let G be a graph of order $n \geq 2$ and $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_{n-1})$ if and only if G is a μ - critical graph.*

Now, let us consider the family $\{\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2}\}$, $n \geq 3$ and $\mu \in \{\gamma, \gamma_a, i\}$.

Theorem 3.2. *Let G be a graph of order $n \geq 3$ and $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$ if and only if $G = \overline{K}_n$.*

Proof. Clearly if $G = \overline{K}_n$ then $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$. So, let we have $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\mathcal{K}_1, \mathcal{K}_1, \mathcal{K}_{n-2})$ and suppose $G \neq \overline{K}_n$. Note that if H is a graph of order at least two and $u \in V(H)$ then $\mu(H - u) \geq \mu(H) - 1$, which follows from [3, 5], [9, ch.16] and Theorem 2.1.(i) for $\mu = \gamma$, $\mu = i$ and $\mu = \gamma_a$ respectively. Choose $x, y \in V(G)$ to be adjacent and let $\boldsymbol{A} = \{\{x\}, \{y\}, V(G) - \{x, y\}\}$. If $\mu(G - x) \geq \mu(G)$ then $\mu(G - \{x, y\}) \geq \mu(G - x) - 1 \geq \mu(G) - 1$ which implies $\mu(G[\boldsymbol{A}]) \geq 1 + 1 + \mu(G) - 1 > \mu(G)$. Hence $\mu(G - x) = \mu(G) - 1$ and therefore if M is a μ - set of $G - x$ then M does not dominate x in G. Hence y belongs to no μ - set of $G - x$. But if a vertex u of a graph H belongs to no μ - set of H then $\mu(H) = \mu(H - u)$, which follows from [5, 13], [14] and Theorem 2.1 (iv) for $\mu = \gamma$, $\mu = i$ and $\mu = \gamma_a$ respectively. Therefore $\mu(G[\boldsymbol{A}]) = 1 + 1 + \mu(G - \{x, y\}) = 2 + \mu(G - x) = 1 + \mu(G)$, which is a contradiction. □

The next family is $\{\{P_2\}, \mathcal{K}_{n-2}\}$, $n \geq 4$ and again $\mu \in \{\gamma, \gamma_a, i\}$.

Theorem 3.3. *Let G be a μ - critical graph of order $n \geq 4$ and size at least 1, where $\mu \in \{\gamma, \gamma_a, i\}$. Then $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\{P_2\}, \mathcal{K}_{n-2})$.*

Proof. As we have seen, $\mu(G - x) = \mu(G) - 1$ for all $x \in V(G)$. By the proof of Theorem 3.2, if $yx \in E(G)$ then y belongs to no μ - set of $G - x$ which implies $\mu(G - \{x, y\}) = \mu(G - x)$. Hence if $xy \in E(G)$ and $\boldsymbol{A} = \{\{x, y\}, V(G - \{x, y\})\}$ then $\mu(G[\boldsymbol{A}]) = 1 + \mu(G - \{x, y\}) = 1 + \mu(G - x) = \mu(G)$. □

Let G be a graph of order $n \geq 2$. The *double vertex graph* $U_2(G)$ of G is the graph whose vertex set consists of all 2-subsets of $V(G)$ such that two distinct vertex $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $|\{x, y\} \cap \{u, v\}| = 1$ and if $x = u$, they y and v are adjacent in G. The concept of double vertex graphs was introduced by Alavi et al. [1]. For this class of graphs, there are many results about regularity, eulerian, hamiltonian, and bipartite properties of these graphs. For a survey of double vertex graphs see [2]. Here we deal with the independence number of double vertex graphs.

Theorem 3.4. *Let G be a graph and $V(G) = \{v_1, v_2, .., v_n\}$, $n \geq 3$. Then $\beta_0(U_2(G)) \leq \Sigma_{k=1}^{n-1} \beta_0(< \{v_{k+1}, v_{k+2}, .., v_n\}, G >)$.*

Proof. Let for each $k \in \{1, 2, .., n-1\}$, $V_k = \{v_{k+1}, v_{k+2}, .., v_n\}$, $W_k = \{\{v_k, v_j\} | k < j \leq n\}$, $H_k =< V_k, G >$ and $Q_k =< W_k, U_2(G) >$. Certainly $\{Q_{n-1}, Q_{n-2}, .., Q_1\}$ is a partition of $U_2(G)$. For all $k \in \{1, 2, .., n-1\}$ define the map $\pi_k : W_k \to V_k$ by $\pi_k(\{v_k, v_j\}) = v_j$, where $j = k + 1, .., n$. Clearly π_k is a bijection and if $k < j \leq n$, $k < s \leq n$, $j \neq s$ then $\{v_k, v_j\}\{v_k, v_s\} \in E(Q_k)$ if and only if $\pi_k(\{v_k, v_j\})\pi_k(\{v_k, v_s\}) = v_j v_s \in E(H_k)$ which follows by the definition of the double vertex graph. Then the graphs Q_k and H_k are isomorphic, $k = 1, 2, .., n-1$. Combining this with the well known fact that if T is a graph and $e \in E(T)$ then $\beta_0(T - e) \geq \beta_0(T)$ [8], we obtain $\beta_0(U_2(G)) \leq \beta_0(\cup_{k=1}^{n-1} Q_k) = \Sigma_{k=1}^{n-1} \beta_0(Q_k) = \Sigma_{k=1}^{n-1} \beta_0(H_k)$. □

Corollary 3.5 *If G is hamiltonian graph of order n then $\beta_0(U_2(G)) \leq \lfloor n^2/4 \rfloor$.*

Proof. Let $v_1, v_2, .., v_n, v_1$ be a hamiltonian cylle in G. Since $H_k =< \{v_{k+1}, v_{k+2}, .., v_n\}, G >$ has a spanning subgraph isomorphic to P_{n-k} then Theorem 3.4 implies $\beta(U_2(G)) \leq \Sigma_{k=1}^{n-1} \beta_0(H_k) \leq \Sigma_{k=1}^{n-1} \beta_0(P_{n-k})$. Clearly $\beta_0(P_s) = \lceil s/2 \rceil$ for all positive integers s. Hence $\beta_0(U_2(G)) \leq \Sigma_{k=1}^{n-1} \lceil (n-k)/2 \rceil$. It is easy to see that $\Sigma_{k=1}^{n-1} \lceil (n-k)/2 \rceil = \lfloor n^2/4 \rfloor$. □

In the next theorem we will find $\beta_0(U_2(C_n))$.

Theorem 3.6. $\beta_0(U_2(C_n)) = \lfloor n^2/4 \rfloor$.

Proof. By the definition of double vertex graph it immediately follows that the set $M = \{\{v_i, v_{i+1+2r}\} \in V(U_2(C_n)) \mid 1 \leq i \leq n-1, 0 \leq r \leq (n-i-1)/2\}$ (r is an integer) is independent. Hence $\beta_0(U_2(C_n)) \geq |M| = \Sigma_{i=1}^{n-1}\lceil (n-i)/2 \rceil = \lfloor n^2/4 \rfloor$. The result now follows because of Corollary 3.5. □

Theorem 3.7. $\beta_0(U_2(C_n)[\boldsymbol{A}]) = \beta_0(U_2(C_n))$ *for all* $\boldsymbol{A} \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, ..., \{P_{n-1}\})$.

Proof. Let $V(C_n) = \{v_1, v_2, .., v_n\}$, $E(C_n) = \{v_1v_2, v_2v_3, .., v_{n-1}v_n, v_nv_1\}$ and for $k = 1, 2, .., n-1$: $Q_k =< \{\{v_k, v_j\} | k < j \leq n\}, U_2(C_n) >$. By the proof of Theorem 3.4 we have that $\boldsymbol{A} = \{Q_{n-1}, Q_{n-2}, .., Q_1\}$ is a partition of $U_2(C_n)$ and for $k = 1, 2, .., n-1$, the graph Q_k is isomorphic to $H_k =< \{v_{k+1}, v_{k+2}, .., v_n\}, C_n >$. But obviously H_k is isomorphic to P_{n-k}. Thus we obtain $\boldsymbol{A} \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, .., \{P_{n-1}\})$. Now, choose an arbitrary $\boldsymbol{B} \in \mathcal{F}_{U_2(C_n)}(\{P_1\}, \{P_2\}, .., \{P_{n-1}\})$. Hence $\beta_0(U_2(C_n)[\boldsymbol{B}]) = \Sigma_{m=1}^{n-1}\beta_0(P_m) = \Sigma_{k=1}^{n-1}\beta_0(P_{n-k}) = \Sigma_{k=1}^{n-1}\lceil (n-k)/2 \rceil = \lfloor n^2/4 \rfloor = \beta_0(U_2(C_n))$. □

4. OPEN QUESTIONS

We close with a list of open problems and questions.

1. Which graphs are γ - critical and γ_a - critical (or one but not the other).
2. Characterize/study those graphs achieving equality in Theorem 3.4.
3. Characterize/study the all graphs G with $\mu(G) = \mu(G[\boldsymbol{A}])$ for all $\boldsymbol{A} \in \mathcal{F}_G(\{P_s\}, \mathcal{K}_{n-s})$, $s \geq 2$ where $\mu \in \{\gamma, \gamma_a, i, ..\}$.

5. REFERENCES

1. Alavi, Y., Behzad, M., Simpson, J. E. Planarity of double vertex graphs. In: *Graph Theory, Combinatorics, Algorithms, and Applications*, eds. Y. Alavi et al., SIAM, Philadelphia, 1991, 472-485.
2. Alavi, Y., Lick, D. R., Liu, J. Survey of double vertex graphs. *Graphs and Combina torics*, **18**, 2002, 709-715.
3. Bauer, D., Harary, F., Nieminen, J. and Suffel, C. I. Domination alteration sets in graphs. *Discrete Mathematics*, **47**, 1983, 153-161.
4. Brigham, R. C., Chinn, P. Z. and Dutton, R. D. Vertex domination-critical graphs. *Networks*, **18**, 1988, 173-179.
5. Carrington, J. R., Harary, F. and Haynes, T. W. Changing and unchanging the domination number of a graph. *J. Combin. Math. Combin. Comput.*, **9**, 1991, 57-63.
6. Fulman, J., Hanson, D., and MacGillivray,G. Vertex domination-critical graphs. *Networks*, **25**, 1995, 41-43.

7. Grobler,P. J. P., Mynhardt, C. M. Vertex criticality for upper domination and irredundance. *Graph theory,* **37** , 2001, 205-212
8. Haynes, T. W., Hedetniemi, S. T. and Slater, P. J. Fundamentals of Domination in Graphs. Marsel Dekker, New York, 1998.
9. Haynes, T. W., Hedetniemi, S. T. and Slater, P. J. Domination in Graphs (Advanced topics). Marsel Dekker, New York, 1998.
10. Haynes, T. W. and Henning, M. A. Changing and unchanging domination: a classifi cation. *Discrete mathematics*, **272**, 2003, 65-79.
11. Hedetniemi, S. M., Hedetniemi, S. T. and Rall, D.F. Acyclic domination. *Discrete Math.*, **222**, 2000, 151-165.
12. Ore, O., Theory of Graphs. Amer. Math. Soc. Colloq. Publ., 1960
13. Sampathkumar, E., Neerlagi, P. S. Domination and neighborhood critical, fixed, free and totally free points. *Sankhya*, **54**, 1992, 403-407.
14. Samodivkin, V. D., I - fixed not i - critical vertices. In: *Proceedings of Thirty First Spring Conference of the Union of Bulgarian Mathematicians*, Borovets, April 3-6, 2002, 177-180.
15. Samodivkin, V. D., Minimal acyclic dominating sets and cut-vertices. *Mathematica Bohemica*, Accepted for publication May 2004. (Ref. No. MB 15/04).

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