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RELATIVELY INTRINSICALLY ARITHMETICAL SETS

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Sets that have Σ^0_n (Π^0_n, arithmetical) associates in every partial enumeration of given countable abstract structure are considered. Some minimal classes of partial enumerations are obtained such that admissibility in every such class yields the respective definability.

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1. INTRODUCTION

Let $\mathfrak{A} = (B; P_1, \ldots, P_m)$ be a total countable relational structure. *Partial enumeration* of $\mathfrak{A}$ is an ordered pair $(f, \mathfrak{B})$, where f is a partial function from the set of all naturals N onto B, $\mathfrak{B}$ is a total structure over N and the mapping $f \restriction Dom(f)$ is a strong homomorphism from $\mathfrak{B} \restriction Dom(f)$ onto $\mathfrak{A}$. An associate of a set $A \subseteq B$ (in the enumeration $(f, \mathfrak{B})$) is a set $W \subseteq N$ such that $W \cap Dom(f) = f^{-1}(A)$, i.e. the pullback $f^{-1}(A)$ is exactly the set W, restricted to $Dom(f)$. Following [2], say that the set A is *relatively intrinsically* Σ^0_n (Π^0_n, *arithmetical*) if for every partial enumeration $(f, \mathfrak{B})$ it has an associate that is Σ^0_n (Π^0_n, arithmetical) in $\mathfrak{B}$. This is a typical implicit definition of complexity of a set A over an abstract structure and a natural question that arises here is whether the set A could be described also explicitely. And further, if such an explicite characterization does exist, is it necessary to involve the whole class of partial enumerations in order to obtain it? In other words, does there exist some smaller class of partial enumerations such that the fact that A has an appropriate associate in every enumeration in this class yields the respective explicite characterization of A?

Results that answer the last question can be found in [8], where Σ^0_n-admissible sets are considered (although in another context), and in [4], where a minimal class of enumerations for the Σ^0_1-admissible sets is obtained. Here we further extend the investigations from [5], considering also relatively intrinsically Π^0_n and arithmetical sets.

2. PRELIMINARIES

Let us fix a relational abstract structure $\mathfrak{A} = (B; P_1, \ldots, P_m)$, where B is at most denumerable and each $P_i, 1 \leq i \leq m$, is a total predicate of k_i arguments on B. The equality relation is not supposed to be among the initial predicates of $\mathfrak{A}$.

Definition 2.1. *Partial enumeration* (of the structure $\mathfrak{A}$) is an ordered pair $(f, \mathfrak{B})$, where f is a partial function from the set of all natural numbers N onto B and $\mathfrak{B} = (N; Q_1, \ldots, Q_m)$ is a total structure in the signature of $\mathfrak{A}$ such that for every $1 \leq i \leq m$ the equivalence

$$Q_i(x_1, \ldots, x_{k_i}) \iff P_i(f(x_1), \ldots, f(x_{k_i}))$$

holds whenever $x_1, \ldots, x_{k_i}$ are in $Dom(f)$.

The set $Dom(f)$ is the *domain* of the enumeration $(f, \mathfrak{B})$. We shall classify the enumerations of $\mathfrak{A}$ with respect to the complexity of their domains.

Let $D(\mathfrak{B})$ be the atomic diagram of $\mathfrak{B}$, more precisely

$$D(\mathfrak{B}) = \{\langle i, x_1, \ldots, x_{k_i}, \varepsilon\rangle | Q_i(x_1, \ldots, x_{k_i}) = \varepsilon, 1 \leq i \leq m\},$$

where $\langle \ldots \rangle$ is some effective coding of all finite sequences over N, which we shall suppose fixed until the end of this work. (We shall identify the boolean constants `true` and `false` with 0 and 1, respectively).

Definition 2.2. The enumeration $(f, \mathfrak{B})$ is Σ^0_n (Π^0_n) iff the set $Dom(f)$ is Σ^0_n (Π^0_n) in the diagram $D(\mathfrak{B})$ of $\mathfrak{B}$.

Definition 2.3. The enumeration $(f, \mathfrak{B})$ is *arithmetical* iff the set $Dom(f)$ is arithmetical in the diagram $D(\mathfrak{B})$, i.e. $Dom(f)$ is Σ^0_n or Π^0_n in $D(\mathfrak{B})$ for some $n \geq 1$.

Definition 2.4. Let A be a subset of B^k. The set $W \subseteq N^k$ is called an *associate* of A (in the enumeration $(f, \mathfrak{B})$) iff the equivalence

$$(x_1, \ldots, x_k) \in W \iff (f(x_1), \ldots, f(x_k)) \in A$$

holds for all $x_1, \ldots, x_k$ in $Dom(f)$.

Obviously, if f is not total, the set A has many associates.

Definition 2.5. Say that a set $A \subseteq B^k$ is Σ^0_n- (Π^0_n-, *arithmetically*) *admissible in* $(f, \mathfrak{B})$ if A has an associate, which is Σ^0_n (Π^0_n, arithmetical) in $D(\mathfrak{B})$.

Remark. If we stick to the terminology from [1] and [2], kept also in [6] and [7], we should call the above sets *relatively intrinsically* Σ^0_n (Π^0_n, *arithmetical*) in $(f, \mathfrak{B})$. We, however, will use the shorter term "admissible", which come from the LACOMBE'S notion of $\forall$-admissibility [3].

Next we introduce Σ^0_n and Π^0_n, $n \geq 0$, formulas in a recursive fragment of the language $L_{\omega_1,\omega}$ of $\mathfrak{A}$. The definition is by simultaneous induction on n. For that purpose to each formula we assign (at least one) index.

We assume that we have chosen some effective coding κ of all atomic formulas in the first-order language $L_{\mathfrak{A}}$ of $\mathfrak{A}$, extended with the logical constants T and F (denote it by $L^+_{\mathfrak{A}}$). Throughout the paper, we shall suppose also that some effective enumeration $W_0, W_1, \ldots$ of all recursively enumerable (r. e.) subsets of N is fixed.

Definition 2.6. (i) Every atomic formula Φ in $L^+_{\mathfrak{A}}$ is a Σ^0_0 formula with an index $\langle 0, 0, \kappa(\Phi)\rangle$.

Every negated atomic formula $\neg\Phi$ in $L^+_{\mathfrak{A}}$ is a Π^0_0 formula with an index $\langle 1, 0, \kappa(\Phi)\rangle$.

Every finite conjunction $\Phi_1 \& \ldots \& \Phi_l$ of Σ^0_0 or Π^0_0 formulas with indices $v_1, \ldots, v_l$, respectively, is a Δ^0_1 formula with an index $\langle 2, v_1, \ldots, v_l\rangle$.

(ii) If every $v \in W_e$ is an index of a Δ^0_{n+1} formula Φ^v, whose free variables are among $X_1, \ldots, X_k$, then

$$\bigvee_{v \in W_e} \Phi^v$$

is a Σ^0_{n+1} formula with an index $\langle 0, n+1, e\rangle$ (with free variables among $X_1, \ldots, X_k$).

If Φ is $\neg\Psi$, where Ψ is a Σ^0_{n+1} formula with an index $\langle 0, n+1, e\rangle$, then Ψ is a Π^0_{n+1} formula with an index $\langle 1, n+1, e\rangle$.

If Φ is $\Psi_1 \& \ldots \& \Psi_l$ and every Ψ_j is a Σ^0_m or Π^0_m, $0 \leq m \leq n+1$, formula with an index v_j, $1 \leq j \leq l$, then Φ is a Δ^0_{n+2} formula with an index $\langle 2, v_1, \ldots, v_l\rangle$.

Definition 2.7. A set $A \subseteq B^k$ is Σ^0_n (Π^0_n) *definable* on $\mathfrak{A}$ iff there exists some Σ^0_n (Π^0_n) formula Φ with variables among $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ and elements $t_1, \ldots, t_l$ of B such that for every $(s_1, \ldots, s_k) \in B^k$

$$(s_1, \ldots, s_k) \in A \iff \mathfrak{A} \models \Phi(X_1/s_1, \ldots, X_k/s_k, Y_1/t_1, \ldots, Y_l/t_l).$$

Clearly, if a set A is Σ^0_n (Π^0_n) definable on $\mathfrak{A}$, then A is Σ^0_n (Π^0_n) -admissible in every enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$.

In order to save space, from now on we shall consider only subsets of B. All the results can be easily generalized for subsets of B^k for arbitrary $k \geq 1$.

Let $(f, \mathfrak{B})$ be an enumeration of $\mathfrak{A}$. We first introduce a satisfaction relation $(f, \mathfrak{B}) \models_n F_e(x)$. For our purposes, it is suitable to make a slight deviation from the standard satisfaction relation for the Σ_n^0 in $D(\mathfrak{B})$ sets (as it is in [8], for example). Let $U(e, x)$ be an universal function for the class of all unary primitive recursive functions. Using the S_n^m-theorem, we obtain a recursive function h such that for every index e

$$W_{h(e)} = \{U(e, x) | x \in N\}.$$

It is well known that a nonempty set $W \subseteq N$ is r. e. iff $W = W_{h(e)}$ for some index e. We shall suppose that the function h is fixed until the end of this work. It will appear in the definitions of the basic notions of forcing and satisfaction relation.

We begin with the definition of the satisfaction relation $\models_n$, which is by induction on n. As customary, D_v will denote the finite set with canonical index v.

Definition 3.1. Set

$$\begin{array}{rcl}
(f, \mathfrak{B}) \models u & \iff & \exists i \exists x_1 \ldots \exists x_{k_i} \exists \varepsilon (1 \leq i \leq m \ \& \ u = \langle i, x_1, \ldots, x_{k_i}, \varepsilon \rangle \ \& \\
& & Q_i(x_1, \ldots, x_{k_i}) = \varepsilon), \\
(f, \mathfrak{B}) \models_1 F_e(x) & \iff & \exists v (\langle v, x \rangle \in W_{h(e)} \ \& \ \forall u \in D_v (f, \mathfrak{B}) \models u)), \\
(f, \mathfrak{B}) \models_{n+1} F_e(x) & \iff & \exists v (\langle v, x \rangle \in W_{h(e)} \ \& \ \forall u \in D_v \exists d \exists y (u = \langle d, y, 0 \rangle \& \\
& & (f, \mathfrak{B}) \models_n F_d(y) \ \vee \ u = \langle d, y, 1 \rangle \ \& \ (f, \mathfrak{B}) \not\models_n F_d(y))).
\end{array}$$

Put finally

$$(f, \mathfrak{B}) \models_n \neg F_e(x) \iff (f, \mathfrak{B}) \not\models_n F_e(x).$$

The next fact is a direct consequence of Proposition 3.3 of [8] and our choice of the satisfaction relation $\models_n$.

Proposition 3.1. (i) *If $W \subseteq N$ is Σ_n^0 in $D(\mathfrak{B})$, then there exists an index e such that $W = \{x | (f, \mathfrak{B}) \models_n F_e(x)\}$.*

(ii) *If $W \subseteq N$ is Π_n^0 in $D(\mathfrak{B})$, then $W = \{x | (f, \mathfrak{B}) \models_n \neg F_e(x)\}$ for some index e.*

Definition 3.2. *Finite part* is an $(m+2)$–tuple

$$\tau = (f_\tau, H_\tau, q_1^\tau, \ldots, q_m^\tau),$$

where f_τ is a finite function from N into B, $H_\tau \subseteq N$, $Dom(f_\tau) \cap H_\tau = \emptyset$, $Dom(f_\tau) \cup H_\tau = \{0, \ldots, l-1\}$ for some $l \in N$ and $q_i^\tau, 1 \leq i \leq m$, is a partial predicate of k_i arguments on $\{0, \ldots, l-1\}$ such that for every $x_1, \ldots, x_{k_i}$ in $Dom(f_\tau)$

$$q_i^\tau(x_1, \ldots, x_{k_i}) \iff P_i(f_\tau(x_1), \ldots, f_\tau(x_{k_i})).$$

The set $Dom(f_\tau) \cup H_\tau$, which is in fact the initial segment $[0, l)$ of N, we shall call *domain* of τ ($Dom(\tau)$); l is the *length* of τ (in symbols $|\tau|$). If $l = 0$, τ is the *empty* finite part. We shall use small Greek letters to denote finite parts.

Below we introduce three types of binary relations between finite parts that model in a different way the notion "extention of a finite part".

Definition 3.3. Let $\tau = (f_\tau, H_\tau, q_1^\tau, \ldots, q_m^\tau)$ and $\delta = (f_\delta, H_\delta, q_1^\delta, \ldots, q_m^\delta)$ be finite parts. Set

$$\begin{aligned} \tau \subseteq \delta &\iff f_\tau \subseteq f_\delta \ \&\ H_\tau \subseteq H_\delta \ \&\ q_1^\tau \subseteq q_1^\delta \ \&\ \ldots \ \&\ q_m^\tau \subseteq q_m^\delta, \\ \tau \leqslant \delta &\iff \tau \subseteq \delta \ \&\ f_\tau = f_\delta, \\ \tau \preccurlyeq \delta &\iff \tau \leqslant \delta \ \&\ H_\tau = H_\delta. \end{aligned}$$

Clearly, these three relations are partial orderings. We shall sometimes write $\tau \supseteq \delta, \tau \geqslant \delta$, etc. for $\delta \subseteq \tau, \delta \leqslant \tau$, etc.

Definition 3.4. The enumeration $(f, \mathfrak{B} = (N; Q_1, \ldots, Q_m))$ *extends* τ (in symbols $\tau \subseteq (f, \mathfrak{B})$) iff $f_\tau \subseteq f$, $H_\tau \subseteq N \setminus Dom(f)$ and $q_i^\tau \subseteq Q_i$ for every $i = 1, \ldots, m$.

Next we define the forcing relation $\Vdash_n$ again by induction on n. Notice that in the definition of $\Vdash_n \neg \ldots$ we use the strongest numerical extention $\geqslant$ instead of the usual $\supseteq$. This type of forcing is called a "starred forcing" in [8].

Definition 3.5.

$$\begin{aligned} \tau \Vdash u &\iff \exists i \exists x_1 \ldots \exists x_{k_i} \exists \varepsilon (1 \le i \le m \ \&\ u = \langle i, x_1, \ldots, x_{k_i}, \varepsilon \rangle \ \&\ \\ &\qquad q_i^\tau(x_1, \ldots, x_{k_i}) = \varepsilon), \\ \tau \Vdash_1 F_e(x) &\iff \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v (\tau \Vdash u)), \\ \tau \Vdash_1 \neg F_e(x) &\iff \forall \rho (\rho \geqslant \tau \Rightarrow \rho \nVdash_1 F_e(x)), \\ \tau \Vdash_{n+1} F_e(x) &\iff \exists v (\langle v, x \rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y (u = \langle d, y, 0 \rangle \& \\ &\qquad \tau \Vdash_n F_d(y) \ \vee\ u = \langle d, y, 1 \rangle \ \&\ \tau \Vdash_n \neg F_d(y))), \\ \tau \Vdash_{n+1} \neg F_e(x) &\iff \forall \rho (\rho \geqslant \tau \Rightarrow \rho \nVdash_n F_e(x)). \end{aligned}$$

Lemma 3.1. *Let $n \geq 1$. For every finite part τ :*

(i) $\{(e, x) | \tau \Vdash_n F_e(x)\}$ *is a Σ_n^0 set;*

(ii) $\{(e, x) | \tau \Vdash_n \neg F_e(x)\}$ *is a Π_n^0 set.*

Proof. Straightforward induction on n. The crucial point here is that we consider numerical extentions $\geqslant$ instead of $\supseteq$ in the definition of the forcing relation. □

In what follows, we shall need the following notion of *restriction* of a finite part τ to δ (for $\tau \supseteq \delta$).

Definition 3.6. Let $\tau \supseteq \delta$. Set

$$\tau | \delta = (f_\delta, H_\delta \cup (Dom(f_\tau) \setminus Dom(f_\delta)), q_1^\tau, \ldots, q_m^\tau).$$

It can be easily checked that $\tau|\delta$ is also a finite part and $\tau|\delta \geqslant \delta$.

The next important property of the restrictions will be systematically used in the sequel.

Lemma 3.2. *Let δ be a finite part. For every $n \geq 1$:*

(i) $\forall \tau \supseteq \delta(\tau \Vdash_n F_e(x) \iff \tau|\delta \Vdash_n F_e(x))$;

(ii) $\forall \tau \supseteq \delta(\tau \Vdash_n \neg F_e(x) \iff \tau|\delta \Vdash_n \neg F_e(x))$.

Proof. Induction on n. The validity of (i) for $n = 1$ follows from the obvious equivalence

$$\tau \Vdash u \iff \tau|\delta \Vdash u.$$

Assume that (i) is true for some $n \geq 1$. We shall show first that (ii) is also true for n and then, using this fact and the induction hypothesis, will establish (i) for $n+1$.

Indeed, take some $\tau \supseteq \delta$ such that $\tau \Vdash_n \neg F_e(x)$. We have to see that $\tau|\delta \Vdash_n \neg F_e(x)$. Assuming that this is not true, we will have that for some $\rho \geqslant \tau|\delta$: $\rho \Vdash_n F_e(x)$. We have $\rho \geqslant \tau|\delta \geqslant \delta$, so by induction hypothesis $\rho|\delta \Vdash_n F_e(x)$. Now consider the tuple

$$\rho_1 = (f_\tau, H_\rho \setminus Dom(f_\tau), q_1^\rho, \ldots, q_m^\rho).$$

Let us first check that ρ_1 is a finite part. Obviously, $Dom(f_\tau)$ and $H_\rho \setminus Dom(f_\tau)$ are disjoint. Further, since $f_\delta = f_\rho$, we have that $H_\rho \cap Dom(f_\delta) = \emptyset$ and hence

$$Dom(f_\tau) \cup (H_\rho \setminus Dom(f_\tau)) = (Dom(f_\delta) \cup (Dom(f_\tau) \setminus Dom(f_\delta))) \cup$$

$$(H_\rho \setminus (Dom(f_\tau) \setminus Dom(f_\delta))) = Dom(f_\delta) \cup H_\rho = Dom(f_\rho) \cup H_\rho,$$

which is an initial segment. So,

$$Dom(q_i^{\rho_1}) = Dom(q_i^\rho) \subseteq Dom(f_\rho) \cup H_\rho = Dom(f_{\rho_1}) \cup H_{\rho_1}.$$

Finally, if $x_1, \ldots, x_{k_i}$ are in $Dom(f_{\rho_1}) = Dom(f_\tau)$, then

$$!q_i^\tau(x_1, \ldots, x_{k_i}) \text{ and } q_i^\tau(x_1, \ldots, x_{k_i}) = P_i(f_\tau(x_1), \ldots, f_\tau(x_{k_i})).$$

However, $\rho \geqslant \tau|\delta$, hence $q_i^\rho \supseteq q_i^\tau$, so $q_i^{\rho_1}(x_1, \ldots, x_{k_i}) = q_i^\rho(x_1, \ldots, x_{k_i})$ is defined and is equal to $P_i(f_\tau(x_1), \ldots, f_\tau(x_{k_i}))$, which is actually $P_i(f_{\rho_1}(x_1), \ldots, f_{\rho_1}(x_{k_i}))$, hence ρ_1 is a finite part indeed.

It can be easily checked that $\rho_1 \geqslant \tau$ and $\rho_1|\delta = \rho|\delta$. As we have seen above, $\rho|\delta \Vdash_n F_e(x)$, hence $\rho_1|\delta \Vdash_n F_e(x)$. From here, using again the induction hypothesis and the fact that $\rho_1|\delta \supseteq \delta$, we get $\rho_1 \Vdash_n F_e(x)$, which contradicts the fact that $\tau \Vdash_n \neg F_e(x)$.

Conversely, suppose that $\tau|\delta \Vdash_n \neg F_e(x)$ and towards contradiction assume that $\tau \nVdash_n \neg F_e(x)$. Therefore there exists $\rho \geqslant \tau$ with $\rho \Vdash_n F_e(x)$. We have $\rho \geqslant \tau \supseteq \delta$, so by induction hypothesis $\rho|\delta \Vdash_n F_e(x)$. Further, $\rho|\delta \geqslant \tau|\delta$, which follows immediately from the fact that $\rho \geqslant \tau \supseteq \delta$. However, $\tau|\delta \Vdash_n \neg F_e(x)$ and we could not have $\rho|\delta \Vdash_n F_e(x)$, which is the desired contradiction.

Let us now check the validity of (i) for $n+1$. Indeed, we have that (i) and (ii) are true for n, so

$$\begin{aligned}
\tau \Vdash_{n+1} F_e(x) \iff & \exists v(\langle v, x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d, y, 0\rangle \& \\
& \tau \Vdash_n F_d(y) \ \vee\ u = \langle d, y, 1\rangle \ \&\ \tau \Vdash_n \neg F_d(y))) \\
\iff & \exists v(\langle v, x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d, y, 0\rangle \& \\
& \tau|\delta \Vdash_n F_d(y) \ \vee\ u = \langle d, y, 1\rangle \ \&\ \tau|\delta \Vdash_n \neg F_d(y))) \\
\iff & \tau|\delta \Vdash_{n+1} F_e(x). \square
\end{aligned}$$

Using Lemma 3.2, one can easily get the monotonicity of the forcing relation.

Lemma 3.3. (i) $\delta \Vdash_n F_e(x) \ \& \tau \supseteq \delta \ \Rightarrow \tau \Vdash_n F_e(x)$;

(ii) $\delta \Vdash_n \neg F_e(x) \ \& \tau \supseteq \delta \ \Rightarrow \tau \Vdash_n \neg F_e(x)$.

Proof. Let us first see the validity of (ii). Suppose that $\delta \Vdash_n \neg F_e(x)$, $\tau \supseteq \delta$, but $\tau \nVdash_n \neg F_e(x)$. Then for some $\rho \geqslant \tau, \rho \Vdash_n F_e(x)$. Since $\rho \geqslant \tau \supseteq \delta$, applying Lemma 3.2, we get $\rho|\delta \Vdash_n F_e(x)$. This, together with the fact that $\rho|\delta \geqslant \delta$, contradicts the assumption $\delta \Vdash_n \neg F_e(x)$.

Now (i) is by induction on n. Dropping the obvious case $n = 1$, suppose that (i) is true for some n. We have also that (ii) is true for this n, so

$$\begin{aligned}
\delta \Vdash_{n+1} F_e(x) \iff & \exists v(\langle v, x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d, y, 0\rangle \& \\
& \delta \Vdash_n F_d(y) \vee\ u = \langle d, y, 1\rangle \ \&\ \delta \Vdash_n \neg F_d(y))) \\
\Rightarrow\ & \exists v(\langle v, x\rangle \in W_{h(e)} \ \&\ \forall u \in D_v \exists d \exists y(u = \langle d, y, 0\rangle \& \\
& \tau \Vdash_n F_d(y) \vee\ u = \langle d, y, 1\rangle \ \&\ \tau \Vdash_n \neg F_d(y))) \\
\iff & \tau \Vdash_{n+1} F_e(x). \square
\end{aligned}$$

Let us remind some basic notions from the forcing constructions machinery.

Definition 3.7. (i) Let X be a set of finite parts. The enumeration $(f, \mathfrak{B})$ *meets* X if $\exists \delta(\delta \in X \ \&\ \delta \subseteq (f, \mathfrak{B}))$.

(ii) X is *dense* in $(f, \mathfrak{B})$ if $\forall \delta \subseteq (f, \mathfrak{B}) \exists \tau \supseteq \delta(\tau \in X)$.

(iii) Let $\mathfrak{F}$ be a family of sets of finite parts. The enumeration $(f, \mathfrak{B})$ is $\mathfrak{F}$-*generic* if for every $X \in \mathfrak{F}$ the following condition holds:

if X is dense in $(f, \mathfrak{B})$, then $(f, \mathfrak{B})$ meets X.

Set $X^k_{e,x} = \{\tau | \tau \Vdash_k F_e(x)\}$ and let

$$\mathfrak{F}_n = \bigcup_{e,x \in N, 1 \leq k \leq n} X^k_{e,x}.$$

We have the following Truth Lemma that brings together the forcing and satisfaction relation.

Lemma 3.4. *Let $n \geq 1$. Then*

(i) *If $(f, \mathfrak{B})$ is $\mathcal{F}_{n-1}$-generic enumeration, then*

$$(f, \mathfrak{B}) \models_n F_e(x) \iff \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n F_e(x)).$$

(ii) *If $(f, \mathfrak{B})$ is $\mathcal{F}_{n-1}$-generic enumeration, then*

$$\exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n \neg F_e(x)) \Rightarrow (f, \mathfrak{B}) \models_n \neg F_e(x).$$

(iii) *If $(f, \mathfrak{B})$ is $\mathcal{F}_n$-generic enumeration, then*

$$(f, \mathfrak{B}) \models_n \neg F_e(x) \Rightarrow \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_n \neg F_e(x)).$$

Proof. Induction on n. It is straightforward that for every enumeration $(f, \mathfrak{B})$

$$(f, \mathfrak{B}) \models_1 F_e(x) \iff \exists \tau \subseteq (f, \mathfrak{B})(\tau \Vdash_1 F_e(x)),$$

hence (i) is true for $n = 1$. Now assume that (i) holds for an arbitrary $n \geq 1$. We shall successively check that (ii) and (iii) also hold for this n and after that – that (i) is true for $n + 1$.

Indeed, let $(f, \mathfrak{B})$ be $\mathcal{F}_{n-1}$-generic, $\tau \subseteq (f, \mathfrak{B})$ and $\tau \Vdash_n \neg F_e(x)$. Towards a contradiction, assume that $(f, \mathfrak{B}) \models_n F_e(x)$. By induction hypothesis $\exists \delta \subseteq (f, \mathfrak{B})$: $\delta \Vdash_n F_e(x)$. Now denote by $\tau \cup \delta$ the tuple

$$(f_\tau \cup f_\delta, H_\tau \cup H_\delta, q_1^\tau \cup q_1^\delta, \ldots, q_m^\tau \cup q_m^\delta).$$

Since τ and δ have a common extension — the enumeration $(f, \mathfrak{B})$, it can be easily seen that $\tau \cup \delta$ is a finite part, too. We have $\tau \cup \delta \supseteq \tau$, $\tau \cup \delta \supseteq \delta$, and by Lemma 3.3 $\tau \cup \delta \models_n \neg F_e(x)$ and at the same time $\tau \cup \delta \models_n F_e(x)$, which is impossible.

Now let $(f, \mathfrak{B})$ be $\mathcal{F}_n$-generic and suppose that $(f, \mathfrak{B}) \models_n \neg F_e(x)$. We have to see that there exists $\tau \subseteq (f, \mathfrak{B})$ such that $\tau \Vdash_n \neg F_e(x)$. Indeed, assume that for every finite part $\tau \subseteq (f, \mathfrak{B})$, $\tau \nVdash_n \neg F_e(x)$. This means that

$$\forall \tau \subseteq (f, \mathfrak{B}) \exists \rho \supseteq \tau (\rho \Vdash_n F_e(x)),$$

in other words, $X^n_{e,x}$ is dense in $(f, \mathfrak{B})$. However, $X^n_{e,x}$ is in $\mathcal{F}_n$ and $(f, \mathfrak{B})$ is $\mathcal{F}_n$-generic, hence $(f, \mathfrak{B})$ meets $X^n_{e,x}$, i.e. there exists $\tau \subseteq (f, \mathfrak{B})$ such that $\tau \in X^n_{e,x}$, in other words, $\tau \Vdash_n F_e(x)$, according to our choice of $X^n_{e,x}$. Now applying (i) for n (notice that $(f, \mathfrak{B})$ is $\mathcal{F}_{n-1}$-generic, too), we obtain $(f, \mathfrak{B}) \models_n F_e(x)$ — a contradiction.

It remains to see the validity of (i) for $n + 1$. Again take some $\mathcal{F}_n$-generic enumeration $(f, \mathfrak{B})$ and suppose that $(f, \mathfrak{B}) \models_{n+1} F_e(x)$. Hence there exists $\langle v, x \rangle \in W_{h(e)}$ such that

$$\forall u \in D_v \exists d \exists y (u = \langle d, y, 0 \rangle \,\&\, (f, \mathfrak{B}) \models_n F_d(y) \vee u = \langle d, y, 1 \rangle \,\&\, (f, \mathfrak{B}) \models_n \neg F_d(y))).$$

Since (i) and (iii) are true for n, we have that for every $u = \langle d_u, y_u, \varepsilon_u \rangle$ in D_v there is some $\tau_u \subseteq (f, \mathfrak{B})$ such that $\tau_u \Vdash_n (\neg)^{\varepsilon_u} F_{d_u}(y_u)$. Again $\tau = \cup\{\tau_u | u \in D_v\}$ is a finite part and by the monotonicity of the forcing relation, $\tau \Vdash_n (\neg)^{\varepsilon_u} F_{d_u}(y_u)$ for every $u \in D_v$, hence $\tau \Vdash_{n+1} F_e(x)$.

The verification of the opposite direction of (i) is very similar – this time use the validity of (i) and (ii) for our n and the monotonicity of the satisfaction relation. Notice that in this direction of (i) it is sufficient to have $\mathfrak{F}_{n-1}$-genericity of the enumeration $(f, \mathfrak{B})$ (as it is in the case of the relation $\tau \Vdash_n \neg F_e(x)$, point (ii)). We, however, will not need this refinement for the positive case of the forcing relation. □

4. NORMAL FORMS

Suppose that $\tau = (f_\tau, H_\tau, q_1^\tau, \ldots, q_m^\tau)$ is a finite part, $x \in N$ is the first not in $Dom(\tau)$ (i.e. $x = |\tau|$) and $s \in B$. Then by $\tau * s$ we shall denote the tuple $(g, H_\tau, r_1, \ldots, r_m)$, where g is the function with a graph $G_{f_\tau} \cup \{(x, s)\}$ and for each $1 \leq i \leq m$, r_i is the predicate with a graph

$$G_{q_i} \cup \{(x_1, \ldots, x_{k_i}, \varepsilon) | (x_1, \ldots, x_{k_i}) \in Dom(g) \ \& \ P_i(g(x_1), \ldots, g(x_{k_i})) = \varepsilon\}.$$

Clearly, $\tau * s$ is a finite part, too.

Definition 4.1. (i) A set $A \subseteq B$ has a Σ_n^0 *normal form* if there exist a finite part δ and a natural number e such that for $x = |\delta|$ the equivalence

$$s \in A \iff \exists \rho (\rho \geqslant \delta * s \ \& \ \rho \Vdash_n F_e(x)) \tag{4.1}$$

holds for every $s \in B$.

(ii) A set $A \subseteq B$ has a Π_n^0 *normal form* if there exist a finite part δ and a natural number e such that for $x = |\delta|$ the equivalence

$$s \in A \iff \delta * s \Vdash_n \neg F_e(x))$$

holds for every $s \in B$.

Clearly, if the set A has a Σ_n^0 normal form, then $B \setminus A$ has a Π_n^0 normal form and vice versa.

Now we are in a position to prove a series of auxiliary propositions that make a connection between the implicit notion of admissibility and the explicit notion of normal form. Their proofs make use of generic enumerations and in essence follow the general scheme used in such type of constructions (in particular, the proof of Proposition 4.1 can be found in [8]). We formulate and prove them here not for the results themselves but rather for the precise constructions of the generic enumerations in their proofs. In the next section we shall explain how to refine these constructions, in order to obtain the main results in this work.

Proposition 4.1. *Let $n \geq 1$. If $A \subseteq B$ is Σ_n^0-admissible in every enumeration, then A has a Σ_n^0 normal form.*

Proof. Assume that A does not have a Σ_n^0 normal form. We shall construct an enumeration $(f, \mathfrak{B})$ (a refuting enumeration) such that A does not have a Σ_n^0 associate in it. The construction of $(f, \mathfrak{B})$ will be carried out in steps. Using induction on a, we shall define a sequence

$$\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_a \subseteq \ldots$$

of finite parts such that the set A is not admissible in any enumeration $(f, \mathfrak{B})$ that extends τ_a for every a. We shall make three types of steps. The first type (when $a \equiv 0(mod3)$) will ensure that s is onto B, the second type is for $\mathcal{F}_{n-1}$-genericity and the third type of steps will guarantee that A is not admissible in $(f, \mathfrak{B})$.

Let us fix an enumeration $s_0, s_1, \ldots$ of the elements of the basic set B. Set τ_0 to be the empty finite part and suppose that we have built τ_{3a} for some $a \geq 0$. We are going to explain how to define τ_{3a+1}. Let $a = \langle e, x, j \rangle$ and put $k = \min(j+1, n-1)$ (so we always have $1 \leq k < n$ for $n > 1$ and $k = 0$ if $n = 1$). If $k = 0$, set $\tau_{3a+1} = \tau_{3a}$ (since in this case $n = 1$ and no genericity is needed), otherwise ask the question "$\exists\rho(\rho \geqslant \tau_{3a} : \rho \Vdash_k F_e(x)$?". If yes, set $\tau_{3a+1} = \rho$ (take an arbitrary $\rho \geqslant \tau_{3a}$ such that $\rho \Vdash_k F_e(x)$), otherwise set $\tau_{3a+1} = \tau_{3a}$.

In order to define τ_{3a+2}, we will use the fact that the set A does not have a Σ_n^0 normal form. Hence the equivalence (4.1) is not true for $\delta = \tau_{3a+1}$ and $e = a$. This means that for $x = |\tau_{3a+1}|$ there exists $s \in B$ such that one of the following two conditions is true:

i) $s \in A$, but for every $\rho \geqslant \tau_{3a+1} * s$ we have $\rho \nVdash_n F_a(x)$;

ii) $s \notin A$, but there exists $\rho \geqslant \tau_{3a+1} * s$ such that $\rho \Vdash_n F_a(x)$.

In the first case put $\tau_{3a+2} = \tau_{3a+1} * s$, in the second case take an arbitrary $\rho : \rho \geqslant \tau_{3a+1} * s \ \& \rho \Vdash_n F_a(x)$ and set $\tau_{3a+2} = \rho$. Finally, set $\tau_{3a+3} = \tau_{3a+2} * s_a$.

Now define the tuple $(f, \mathfrak{B} = (N; Q_1, \ldots, Q_m))$ as follows:

$$f = \bigcup_a f_{\tau_a},$$

and for every $1 \leq i \leq m$ and $(x_1, \ldots, x_{k_i}) \in N^{k_i}$:

$$Q_i(x_1, \ldots, x_{k_i}) = \begin{cases} q_i^{\tau_a}(x_1, \ldots, x_{k_i}), & \text{if } \exists a \ !q_i^{\tau_a}(x_1, \ldots, x_{k_i}), \\ \text{arbitrary,} & \text{otherwise.} \end{cases}$$

Obviously, (f,$\mathfrak{B}$)$\supseteq \tau_a$ for every a. Since for every $s \in B$ there exists a such that $s \in Range(\tau_a)$, we have that $Range(f) = B$. The definition of the notion of finite part guarantees that f is a strong homomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$, i. e. $(f, \mathfrak{B} = (N; Q_1, \ldots, Q_m))$ is an enumeration of $\mathfrak{A}$. Let us see now that $(f, \mathfrak{B})$ is $\mathcal{F}_{n-1}$-generic if $n > 1$. Indeed, take some $X_{e,x}^k$, $1 \leq k < n$, and suppose that $X_{e,x}^k$ is dense in $(f, \mathfrak{B})$, i. e. $\forall \tau \subseteq$(f,$\mathfrak{B}$)$\exists \rho \supseteq \tau : \rho \Vdash_k F_e(x)$. Take $a = \langle e, x, k-1 \rangle$

and consider the step $3a+1$. From the density of $X^k_{e,x}$ it follows that for $\tau = \tau_{3a}$ there exists $\rho \supseteq \tau_{3a}$ such that $\rho \Vdash_k F_e(x)$. Hence, putting $\rho* = \rho|\tau_{3a}$, we will have, using Lemma 3.2, that $\rho* \geqslant \tau_{3a}$ and $\rho* \Vdash_k F_e(x)$. This means, according to our construction of $\{\tau_a\}_a$, that $\tau_{3a+1} \Vdash_k F_e(x)$ and $\tau_{3a+1} \subseteq (f, \mathfrak{B})$, i. e. $(f, \mathfrak{B})$ meets $X^k_{e,x}$, hence $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic.

Towards a contradiction, assume that A is Σ^0_n-admissible in this $(f, \mathfrak{B})$. Hence A has an associate W, which is Σ^0_n in $\mathfrak{B}$. Therefore, according to Proposition 3.1, $W = \{x | (f, \mathfrak{B}) \models_n F_a(x)\}$ for some a. We have that for every $x \in Dom(f)$

$$(f, \mathfrak{B}) \models_n F_a(x) \iff f(x) \in A. \tag{4.2}$$

Now have a look at step $3a+2$. If the case i) holds at this step, then for some $s \in A$ we will have that $\tau_{3a+2} = \tau_{3a+1} * s$ and $\tau_{3a+2} \Vdash_n \neg F_a(x)$. By definition $\tau_{3a+2}(x) = s$ for $x = |\tau_{3a+1}|$, hence $x \in Dom(f)$ and $f(x) = s \in A$. So according to (4.2), $(f, \mathfrak{B}) \models_n F_a(x)$. On the other hand, $\tau_{3a+2} \Vdash_n \neg F_a(x)$, which, combined with the $\mathfrak{F}_{n-1}$-genericity of $(f, \mathfrak{B})$ and Lemma 3.4, gives us $(f, \mathfrak{B}) \models_n \neg F_a(x)$ — a contradiction.

If it is the case ii) at the step $3a+2$, then we will have $\tau_{3a+2} \Vdash_n F_a(x)$, $\tau_{3a+2}(x) = s$ and $f(x) = s \notin A$. On the other hand, since $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic, according to Lemma 3.4, $(f, \mathfrak{B}) \models_n F_a(x)$, hence by (4.2), $f(x) = s \in A$ — again a contradiction. □

As a consequence we obtain the following proposition.

Proposition 4.2. *Let $n \geq 1$. If $A \subseteq B$ is Π^0_n-admissible in every enumeration, then A has a Π^0_n normal form.*

Proof. Take an arbitrary enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$. If W is an associate of A in $(f, \mathfrak{B})$, then, clearly, $N \setminus W$ is an associate of $B \setminus A$ in $(f, \mathfrak{B})$. Hence $B \setminus A$ is Σ^0_n-admissible in $(f, \mathfrak{B})$ and, according to the previous proposition, $B \setminus A$ has a Σ^0_n normal form, therefore A has a Π^0_n normal form. □

Proposition 4.3. *Let $n \geq 1, A \subseteq B$ and for every enumeration $(f, \mathfrak{B})$ of $\mathfrak{A}$ the set A is Σ^0_n- or Π^0_n-admissible in $(f, \mathfrak{B})$. Then A has Σ^0_n normal form or Π^0_n normal form (and hence the set A is Σ^0_n-admissible in every $(f, \mathfrak{B})$ or is Π^0_n-admissible in every $(f, \mathfrak{B})$).*

Proof. We shall follow the proof of Proposition 4.1. Assuming that A does not have neither Σ^0_n nor Π^0_n normal form, we will construct an enumeration $(f, \mathfrak{B})$, in which A does not have an appropriate (Σ^0_n or Π^0_n) associate. We shall use four types of steps here in order to define the sequence $\{\tau_a\}_a$. The first three ones will be just as in the proof of Proposition 4.1. At the steps $4a+4$ we do the following. According to our assumption that A does not have a Π^0_n normal form, we have that there exists $s \in B$ such that, putting $x = |\tau_{4a+3}|$, one of the following two cases hold:

i) $s \in A$, but $\tau_{4a+3} * s \nVdash_n \neg F_a(x)$;

ii) $s \notin A$, but $\tau_{4a+3} * s \Vdash_n \neg F_a(x)$.

In the first case we have that for some $\rho \geqslant \tau_{4a+3} * s : \rho \Vdash_n F_a(x)$. Put in this case $\tau_{4a+4} = \rho$. In the second case put $\tau_{4a+4} = \tau_{4a+3} * s$.

Now let $(f, \mathfrak{B})$ be an enumeration that extends τ_a for every $a \in N$. As we have established in the proof of Proposition 4.1, $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic and A is not Σ_n^0-admissible in it. Let us see that A is not Π_n^0-admissible in $(f, \mathfrak{B})$ as well. Assume the contrary and take a Π_n^0 set W that is an associate of A. According to Proposition 3.1, $W = \{x | (f, \mathfrak{B}) \models_n \neg F_a(x)\}$ for some a. Consider the step $4a + 4$. If the case i) holds at this step, then $\tau_{4a+4} \Vdash_n F_a(x)$ for $x = |\tau_{4a+3}|, \tau_{4a+4}(x) = s$ and $s \in A$. Since $(f, \mathfrak{B})$ is $\mathfrak{F}_{n-1}$-generic, using Lemma 3.4 we get $(f, \mathfrak{B}) \models_n F_a(x)$, hence $x \notin W$, whereas $f(x) \in A$ — a contradiction. In the case ii) we put $\tau_{4a+4} = \tau_{4a+3} * s$ with $\tau_{4a+4} \Vdash_n \neg F_a(x)$ and $f(x) = s \notin A$. Now again by Lemma 3.4 we get $(f, \mathfrak{B}) \models_n \neg F_a(x)$, hence $x \in W$, whereas $f(x) = s \notin A$. □

Proposition 4.4. *Let the set $A \subseteq B$ be arithmetically admissible in every enumeration $(f, \mathfrak{B})$. Then there exists $n \geq 1$ such that A has Σ_n^0 or Π_n^0 normal form.*

Proof. Assume the contrary. We generalize the idea used in the proof of Proposition 4.3 in such a way that n is now a parameter of the construction. Again we will make four types of steps. With the first type of steps (of the form $4a + 1$) we shall ensure $\mathfrak{F}_n$-genericity of $(f, \mathfrak{B})$ for every $n \geq 1$; with the second and the third types — that A does not have neither Σ_n^0, nor Π_n^0 associate in $(f, \mathfrak{B})$ for every $n \geq 1$. The fourth type of steps will guarantee that the mapping $f = \bigcup_a f_{\tau_a}$ is onto B.

Let τ_0 be the empty finite part and suppose that we have constructed τ_{4a} for some a. Let $a = \langle e, x, n \rangle$. If there exists $\rho \geqslant \tau_{4a} : \rho \Vdash_{n+1} F_e(x)$, put $\tau_{4a+1} = \rho$, otherwise put $\tau_{4a+1} = \tau_{4a}$. In order to determine τ_{4a+2}, we represent a as $\langle e, n-1 \rangle$ for some e and $n \geq 1$ and use the fact that A does not have a Σ_n^0 normal form. So putting $x = |\tau_{4a+1}|$, we will have that there exists some $s \in B$ such that one of the next two possibilities holds:

i) $s \in A$, but $\forall \rho \geqslant \tau_{4a+1} * s, \rho \nVdash_n F_e(x)$;

ii) $s \notin A$, but $\exists \rho \geqslant \tau_{4a+1} * s, \rho \Vdash_n F_e(x)$.

Set $\tau_{4a+2} = \tau_{4a+1} * s$ or $\tau_{4a+2} = \rho$ if it is the case i) or ii), respectively.

At the step $4a + 3$ with $a = \langle e, n-1 \rangle$ for some e and $n \geq 1$ we proceed in a similar way, taking into account this time the fact that A does not have a Π_n^0 normal form and hence for $x = |\tau_{4a+2}|$ there is an $s \in B$ such that:

i) $s \in A$, but $\tau_{4a+2} * s \nVdash_n \neg F_e(x)$;

ii) $s \notin A$, but $\tau_{4a+2} * s \Vdash_n \neg F_e(x)$.

If it is the case i), then for some $\rho \geqslant \tau_{4a+2} * s$ we will have $\rho \Vdash_n F_e(x)$, so put $\tau_{4a+3} = \rho$ in this case. In the second case put $\tau_{4a+3} = \tau_{4a+2} * s$.

At the step $4a + 4$ we put $\tau_{4a+4} = \tau_{4a+3} * s_a$. To complete the proof, proceed just as in the proof of Proposition 4.3. □

5. THE MAIN RESULT

In this section we will introduce a step-wise refinement $\overset{t}{\Vdash}_n$ of the forcing relation $\Vdash_n$ that will allow us to define more precise construction of the generic enumerations $(f, \mathfrak{B})$, built in the proofs of the propositions in the previous section. As a result, we will obtain a refined versions of these propositions that will bring us to our final results.

The definition of $\overset{t}{\Vdash}_n$ will follow the step-wise enumeration of the sets $W_{h(e)}$ by the function $\lambda t.U(e,t)$.

Set for brevity

$$\tau \Vdash_0 D_v \iff \tau \Vdash u \text{ for every } u \in D_v$$

and for $n \geq 1$

$$\begin{aligned}\tau \Vdash_n D_v \iff & \forall u \in D_v \exists d \exists y((u = \langle d, y, 0\rangle \& \tau \Vdash_n F_d(y)) \vee \\ & (u = \langle d, y, 1\rangle \ \& \ \tau \Vdash_n \neg F_d(y)))).\end{aligned}$$

Let $\lambda x, i.(x)_i$ be a recursive function that returns the i-th component of the sequence with a code x (if it exists). So we have for $n \geq 1$:

$$\tau \Vdash_n F_e(x) \iff \exists v(\langle v, x\rangle \in W_{h(e)} \ \& \ \tau \Vdash_{n-1} D_v) \iff$$

$$\exists t \exists v(U(e,t) = \langle v, x\rangle \ \& \ \tau \Vdash_{n-1} D_v)) \iff \exists t((U(e,t))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t))_0}).$$

Definition 5.1. Put

$$\tau \overset{t}{\Vdash}_n F_e(x) \iff \exists t_0(t_0 \leq t \ \& \ (U(e,t_0))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t_0))_0}).$$

The first t with $\tau \overset{t}{\Vdash}_n F_e(x)$ may be thought of as the first step at which the validity of $\tau \Vdash_n F_e(x)$ is established.

Here are the main properties of the relation $\overset{t}{\Vdash}_n$ that we will need.

Lemma 5.1. (i) $\tau \Vdash_n F_e(x) \iff \exists t(\tau \overset{t}{\Vdash}_n F_e(x))$;

(ii) $\tau \overset{t}{\Vdash}_n F_e(x) \ \& \ t' > t \Rightarrow \tau \overset{t'}{\Vdash}_n F_e(x)$;

(iii) $\tau \overset{t}{\Vdash}_n F_e(x) \ \& \delta \supseteq \tau \Rightarrow \delta \overset{t}{\Vdash}_n F_e(x)$;

(iv) *The set* $\{(e,x,t) | \tau \overset{t}{\Vdash}_n F_e(x)\}$ *is recursive in* $\emptyset^{(n-1)}$.

Proof. (i) and (ii) are straightforward; the proof of (iii) is by a routine induction on n. In order to establish (iv), notice that according to Lemma 3.1 and the Post's theorem the set $M = \{(e,x,t) | (U(e,t))_1 = x \ \& \ \tau \Vdash_{n-1} D_{(U(e,t))_0}\}$ is recursive in $\emptyset^{(n-1)}$, hence the set $\{(e,x,t) | \exists t_0(t_0 \leq t \ \& \ (e,x,t_0) \in M\}$ is recursive in $\emptyset^{(n-1)}$ as well. □

Let $D(\tau)$ – the *diagram* of τ – be the set $\{\langle i, x_1, \ldots, x_{k_i}, \varepsilon\rangle | q_i^\tau(x_1, \ldots, x_{k_i}) = \varepsilon, 1 \leq i \leq m\}$. Clearly $D(\tau)$ is a finite subset of N.

Definition 5.2. The *code* of τ (in symbols, $\|\tau\|$) is the canonical index of the diagram $D(\tau)$ of τ.

In fact, the code $\|\tau\|$ of τ does not code τ completely since it preserves no information about $Dom(f_\tau)$ and H_τ. We, however, will consider codes $\|\tau\|$ only for finite parts τ such that $\tau \succcurlyeq \tau_0$ for some fixed τ_0. In such case, clearly, $\|\tau\|$ identifies completely τ. If $\|\tau\| \leq \|\delta\|$, we shall say that τ is *less* than δ.

Let $t \in N$. Denote by τ^{+t} the finite part

$$\tau = (f_\tau, H_\tau \cup \{|\tau|, \ldots, |\tau| + t - 1\}, q_1^\tau, \ldots, q_m^\tau)$$

($\tau^{+0} \stackrel{df}{=} \tau$). Clearly, $\tau^{+t} \geqslant \tau$, $|\tau^{+t}| = |\tau| + t$ and $\|\tau^{+t}\| = \|\tau\|$. The next simple observation will be of use when constructing special generic enumerations.

Lemma 5.2. *Suppose that $\exists \delta \geqslant \tau(\delta \Vdash_n F_e(x))$. Then there exist $t \in N$ and ρ such that $\rho \succcurlyeq \tau^{+t}$ and $\rho \overset{t}{\Vdash}_n F_e(x)$.*

Proof. Let $\delta \Vdash_n F_e(x)$ and $\delta \geqslant \tau$. Then there exists $t_0 : \delta \overset{t_0}{\Vdash}_n F_e(x)$. Put $t = \max(t_0, k)$, where $k = |\delta| - |\tau|$, and consider the finite part $\rho = \delta^{+(t-k)}$. Clearly, $\rho \geqslant \delta \geqslant \tau$ and $|\rho| = |\delta| + t - k = |\tau| + t = |\tau^{+t}|$, hence $\rho \succcurlyeq \tau^{+t}$. We have $\delta \overset{t_0}{\Vdash}_n F_e(x)$, hence $\delta \overset{t}{\Vdash}_n F_e(x)$ and, by monotonicity, $\rho \overset{t}{\Vdash}_n F_e(x)$. □

Now put

$$\mu_0(\tau, n, e, x) \simeq \begin{cases} \min\{t | t > 0 \ \& \ \exists \delta \succcurlyeq \tau^{+t}(\delta \overset{t}{\Vdash}_n F_e(x))\}, & \text{if } \exists \delta \geqslant \tau(\delta \Vdash_n F_e(x)), \\ \neg !, & \text{otherwise,} \end{cases}$$

$$\mu(\tau, n, e, x) \simeq \begin{cases} \min\{\rho | \exists t (t \simeq \mu_0(\tau, n, e, x) \& \rho \succcurlyeq \tau^{+t} \& \rho \overset{t}{\Vdash}_n F_e(x))\}, & \text{if } !\mu_0(\tau, n, e, x), \\ \neg !, & \text{otherwise.} \end{cases}$$

Here by $\min\{\rho | \ldots\}$ we mean the least finite part ρ with the respective property.

Using Lemma 5.2, we easily get

$$\exists \delta \geqslant \tau(\delta \Vdash_n F_e(x)) \implies !\mu_0(\tau, n, e, x) \ \& \ !\mu(\tau, n, e, x).$$

Let us notice also that, according to Lemma 5.1 and the fact that there exist finitely many δ: $\delta \succcurlyeq \tau$, both functions μ_0 and μ are computable in $\emptyset^{(n-1)}$.

Proposition 5.1. *Let $n \geq 1$. If $A \subseteq B$ is Σ_n^0-admissible in every Π_n^0 enumeration, then A has a Σ_n^0 normal form.*

Proof. Assume that A does not have a Σ_n^0 normal form. We have to construct a Π_n^0 enumeration $(f, \mathfrak{B})$ in which A is not Σ_n^0-admissible. The construction of the enumeration $(f, \mathfrak{B})$ will follow the scheme, described in the proof of Proposition 4.1. We will, however, be more careful at the positive cases of the steps $k = 3a + 1$ and $k = 3a + 2$, i.e. when we put τ_k to be an arbitrary $\rho \geqslant \tau_{k-1}$ with the respective property. Now we will choose this ρ more precisely. In addition, we will ensure that every τ_k is *total*, i.e. every $q_i^{\tau_k}, 1 \le i \le m$, is a total predicate over $Dom(\tau_k)$. (In fact, the last requirement is not essential for our construction. We will support it only to facilitate the proof that $(f, \mathfrak{B})$ is a Π_n^0 enumeration.)

If $\delta = (f_\delta, H_\delta, q_1^\delta, \ldots, q_m^\delta)$ is a finite part, let

$$\delta^+ = (f_\delta, H_\delta, q_1, \ldots, q_m),$$

where $q_i \supseteq q_i^\delta$ and $q_i(x_1, \ldots, x_{k_i}) = 0$ whenever $(x_1, \ldots, x_{k_i}) \in Dom(\delta)^{k_i}$ and $\neg ! q_i^\delta(x_1, \ldots, x_{k_i}), 1 \le i \le m$. Clearly $\delta^+ \succcurlyeq \delta$ and δ is total.

Now we are ready to explain how to define τ_k for each step k. Indeed, assume that τ_{3a} is defined. Following the construction, described in the proof of Proposition 4.1, we present a as $a = \langle e, x, j \rangle$. So putting $k(a) = \min(j+1, n-1) = \min((a)_2 + 1, n-1)$, we set

$$\tau_{3a+1} = \begin{cases} \mu(\tau_{3a}, k(a), (a)_0, (a)_1)^+, & \text{if } \exists \rho \succcurlyeq \tau_{3a}(\rho \Vdash_{k(a)} F_{(a)_0}((a)_1)), \\ \tau_{3a}, & \text{otherwise.} \end{cases}$$

In order to explain how to proceed at step $3a+2$, look again at the construction in the proof of Proposition 4.1. At this step we have that at least one $s \in B$ with some special property exists. (Take an arbitrary s with this property, for example take the first one in the enumeration $s_0, s_1, \ldots$ of B). Since we will need to cite this s in the future, let us denote it by r_a.

$$\tau_{3a+2} = \begin{cases} \mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1)^+, & \text{if } \exists \rho \succcurlyeq (\tau_{3a+1} * r_a)^+, \\ & \rho \Vdash_n F_a(|\tau_{3a+1}| + 1) \\ (\tau_{3a+1} * r_a)^+, & \text{otherwise.} \end{cases}$$

Finally, put $\tau_{3a+3} = (\tau_{3a+2} * s_a)^+$.

Now set

$$f = \bigcup_k f_{\tau_k}, \; Q_i = \bigcup_k = q_i^{\tau_k}, \; 1 \le i \le m.$$

The fact that A is not Σ_n^0-admissible in $(f, \mathfrak{B})$ follows immediately from the proof of Proposition 4.1. What we claim here is that $(f, \mathfrak{B})$ is a Π_n^0 enumeration, i.e. that $Dom(f)$ is a Π_n^0 in $D(\mathfrak{B})$, or equivalently, that $N \setminus Dom(f)$ is r. e. in $D(\mathfrak{B})^{(n-1)}$. Below we will see that $N \setminus Dom(f)$ is in fact r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, hence $(f, \mathfrak{B})$ is a Π_n^0 enumeration indeed.

Remark. Let us notice that we do not achieve more than we claim for the complexity of $Dom(f)$, since as is well known, for the $\mathfrak{F}_n$-generic enumerations $(f, \mathfrak{B})$, $D(\mathfrak{B})^{(n-1)}$ is Turing equivalent to $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$.

Let $l \in N$. Denote by α_l the finite part $(f|[0,l), (N \setminus Dom(f)) \cap [0,l), q_1, \ldots, q_m)$, where each q_i is the predicate Q_i of $\mathfrak{B}$, restricted to the interval $[0, l)$. Clearly, $\alpha_l \subseteq (f, \mathfrak{B})$ and α_l is total. The problem here is that we have at our disposal only the structure $\mathfrak{B}$, not the whole enumeration $(f, \mathfrak{B})$, so we cannot construct α_l. We, however, can determine the finite part

$$\beta_l = (\emptyset; [0, l)); q_1, \dots, q_m).$$

Clearly, β_l is $\alpha_l|\emptyset$, hence, using Lemma 3.2, we get

$$\beta_l \Vdash_n F_e(x) \iff \alpha_l \Vdash_n F_e(x). \tag{5.1}$$

From here for every p, e, x we get

$$\mu_0(\alpha_l, p, e, x) \simeq \mu_0(\beta_l, p, e, x) \ \& \ \mu(\alpha_l, p, e, x) \simeq \mu(\beta_l, p, e, x).$$

Set

$$H_0 = \emptyset, \ H_k = H_{\tau_k} \setminus H_{\tau_{k-1}} \text{ for } k > 0.$$

Clearly, H_k are disjoint and $N \setminus Dom(f) = \bigcup_k H_k$. Hence

$$x \in N \setminus Dom(f) \iff \exists k(x \in H_k).$$

Let us look closely how the sets H_k are constructed. We have $H_{3a} = \emptyset$,

$$H_{3a+1} = \begin{cases} \{|\tau_{3a}|, \dots, |\rho| - 1\}, & \text{if } \rho \simeq \mu(\tau_{3a}, k(a), (a)_0, (a)_1)^+, \\ \emptyset, & \text{if } \neg!\mu(\tau_{3a}, k(a), (a)_0, (a)_1), \end{cases}$$

$$H_{3a+2} = \begin{cases} \{|\tau_{3a+1}| + 1, \dots, |\rho| - 1\}, & \text{if } \rho \simeq \ \mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1)^+, \\ \emptyset, & \text{if } \neg!\mu((\tau_{3a+1} * r_a)^+, n, a, |\tau_{3a+1}| + 1). \end{cases}$$

Clearly, $x \in N \setminus Dom(f)$ iff $x \in H_{3a+1}$ or $x \in H_{3a+2}$. Consider, for example, the set H_{3a+1} (the case with H_{3a+2} is similar). Suppose also that H_{3a+1} is not empty. We have $\tau_{3a} \subseteq (f, \mathfrak{B})$, $\tau_{3a} = \alpha_l$ for $l = |\tau_{3a}|$, so if we knew the length of τ_{3a}, we could compute $t = \mu_0(\alpha_l, k(a), (a)_0, (a)_1) = \mu_0(\beta_l, k(a), (a)_0, (a)_1)$, using the oracles $\emptyset^{(n-1)}$ and $D(\mathfrak{B})$. Hence (the canonical index of) $H_{3a+1} = \{|\tau_{3a}|, \dots, |\tau_{3a}| + t - 1\}$ would be computable in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$.

The problems here are two. The first one is that we cannot decide recursively in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$ whether $H_{3a+1} \neq \emptyset$. The second problem is that we cannot compute the length $|\tau_a|$, using the oracles $\emptyset^{(n-1)}$ and $D(\mathfrak{B})$. So our idea is to start a recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$ procedure that for every a computes consecutive approximations $l_a^0, l_a^1, \dots$, leading to the "real" length $|\tau_a|$. Using the approximate lengths l_a^s, we will define finite sets H_a^s that will be already recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$. Not all of the sets H_a^s are approximations of our sets H_a, but as we will see, their union $\cup_{a,s} H_a^s$ coincides with $\cup_a H_a$, i. e. with $N \setminus Dom(f)$. The rest of the proof of the proposition consists in precise definitions of these approximations and their properties, and is gathered in the next four lemmas.

We define by simultaneous recursion the functions l_k^s and t_s^k as follows:

$$l_{-1}^s = 0,\ l_k^0 = k \text{ for every } k \in N, s \in N,$$

$$t_k^0 = 1 \text{ for every } k \in N,$$

and for every k, s in N

$$l_k^{s+1} = \begin{cases} l_{k-1}^{s+1} + 1, & \text{if } l_{k-1}^s \neq l_{k-1}^{s+1}, \\ l_k^s + t_k^s, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_{k(a)}^{t_k^s} F_{(a)_0}((a)_1), \\ l_k^s + t_k^s, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a+1 \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_n^{t_k^s} F_a(l_k^s), \\ l_k^s, & \text{in the remained cases,} \end{cases}$$

$$t_k^{s+1} = \begin{cases} 1, & \text{if } l_{k-1}^s \neq l_{k-1}^{s+1}, \\ 0, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_{k(a)}^{t_k^s} F_{(a)_0}((a)_1), \\ 0, & \text{if } l_{k-1}^s = l_{k-1}^{s+1} \ \&\ t_k^s > 0 \ \&\ k = 2a+1 \ \& \\ & \exists \rho \succcurlyeq \beta_{l_k^s}^{+t_k^s} : \rho \Vdash_n^{t_k^s} F_a(l_k^s), \\ t_k^s + 1, & \text{in the remained cases.} \end{cases}$$

Our first lemma establishes some basic properties of l_k^s and t_k^s.

Lemma 5.3. *For every $k \in N$ and $s \in N$*

(i) $l_k^s \leq l_k^{s+1}$;

(ii) $l_k^s \leq k + s$;

(iii) $l_k^s < l_k^{s+1} \Rightarrow l_k^{s+1} = k + s + 1$;

(iv) $l_k^s + t_k^s \leq k + s + 1$;

(v) $t_k^s > 0 \Rightarrow l_k^s + t_k^s = k + s + 1$.

Proof. Induction on k. The case $k = 0$ is by a straightforward induction on s. Assume now that for some $k > 0$

$$\forall s (l_{k-1}^s \leq l_{k-1}^{s+1} \ \&\ l_{k-1}^s \leq k - 1 + s \ \&\ (l_{k-1}^s < l_{k-1}^{s+1} \Rightarrow l_{k-1}^{s+1} = k + s), \tag{5.2}$$
$$l_{k-1}^s + t_{k-1}^s \leq k + s \ \&\ (t_{k-1}^s > 0 \Rightarrow l_{k-1}^s + t_{k-1}^s = k + s)).$$

In order to establish (i) – (v) for k, we shall proceed by induction on s. For $s = 0$ the only points that are not obvious are (i) and (iii). We consider the cases in the definition of l_k^1. If $l_{k-1}^0 \neq l_{k-1}^1$, then $l_k^1 = l_{k-1}^1 + 1 = (k-1)+1+1 = k+1$, according to (5.2). Hence $l_k^1 > l_k^0 = k$, i. e. (i) and (iii) hold. If $l_{k-1}^0 = l_{k-1}^1$, but $l_k^1 \neq l_k^0$ (i. e. it is the second or the third case of the definition of l_k^1), then $l_k^1 = l_k^0 + t_k^0 = l_k^0 + 1 = k+1$, hence again (i) and (iii) are true. They are evidently true if $l_k^1 = l_k^0$.

Now suppose that the conditions (i) — (v) hold for some s. We are going to check them for $s+1$. If $l_k^s < l_k^{s+1}$, then by induction hypothesis for s, $l_k^{s+1} = k+s+1$, i. e. (ii) is true for $s+1$. We check similarly conditions (iv) and (v), using the fact that (iv) and (v) hold for s. To see that (i) and (iii) also hold for $s+1$, we consider separately the cases in the definition of l_k^{s+2}. If $l_{k-1}^{s+1} \neq l_{k-1}^{s+2}$, then $l_k^{s+2} = l_{k-1}^{s+2} + 1$, which is by (5.2) exactly $(k-1)+(s+2)+1 = k+s+2$, so we checked (i) and (iii) for $s+1$. The next two cases of the definition of l_k^{s+2} are treated similarly to the case $s=0$. The last case of the definition is again obvious. □

Clearly, the functions $\lambda s.l_k^s$ have finitely many different values (since they depend on the switches in the values of $\lambda s.l_m^s$ for $m < k$). This fact, combined with the previous lemma, means that for every k there exists least S_k such that

$$l_k^0 \leq l_k^1 \leq \cdots \leq l_k^{S_k} = l_k^{S_k+1} \ldots \tag{5.3}$$

The following additional properties follow directly or by an easy induction from the definition of l_k^s, Lemma 5.3 and the choice of S_k.

Lemma 5.4. (i) $0 \leq S_0 \leq S_1 \leq \ldots$;

(ii) $S_k = 0 \Rightarrow L_k = k$;

(iii) $S_k > 0 \Rightarrow l_{k+1}^{S_k} = l_k^{S_k} + 1$;

(iv) $S_k > 0 \Rightarrow l_{k+1}^{S_k} = l_{k+1}^{S_{k+1}-1}$;

(v) $l_k^{s-1} < l_k^s \;\&\; s < S_k \Rightarrow \exists m < k \exists s' (l_m^{s'-1} < l_m^{s'})$;

(vi) $l_k^{s-1} < l_k^s \Rightarrow \forall m > k (l_m^{s-1} + 1 = l_m^s)$;

(vii) $k \leq m \Rightarrow l_k^s \leq l_m^s$.

Put $L_k = l_k^{S_k}$.

Our next lemma makes connection between (the lengths of) the finite parts τ_k and the function l_k^s (in fact, it clarifies the definition of this function).

Lemma 5.5. *For every $a \in N$ we have that $|\tau_{3a+1}| = L_{2a}$, $|\tau_{3a+2}| = L_{2a+1}$ and $|\tau_{3a+3}| = L_{2a+1} + 1$.*

Proof. We have by definition that $|\tau_{3a+3}| = |\tau_{3a+2}| + 1$, hence we have to check the first two equalities. We shall proceed by induction on a. Let $a = 0$. We shall see in turn that $|\tau_1| = L_0$ and $|\tau_2| = L_1$ ($|\tau_0| = 0$ by definition).

Case 1. $\exists \rho \geqslant \tau_0 (\rho \Vdash_{k(0)} F_{(0)_0}((0)_1))$. In this case $\tau_1 = \mu(\tau_0, k(0), (0)_0, (0)_1)^+$ and the length $|\tau_1|$ of τ_1 is $T = \mu_0(\tau_0, k(0), (0)_0, (0)_1)$. Then clearly, $l_0^s = 0, t_0^s = s+1$

for $s < T$, and $l_0^T = l_0^{T-1} + t_0^{T-1} = T$, $t_0^T = 0$. Therefore $l_0^s = T, t_0^s = 0$ for $s > T$, so $L_0 = T$, i. e. $|\tau_1| = L_0$.

Case 2. $\neg\exists\rho \geqslant \tau_0(\rho \Vdash_{k(0)} F_{(0)_0}((0)_1))$. Then by definition $\tau_1 = \tau_0$, the length $|\tau_1|$ is 0, and in this case $L_0 = 0$, too.

Let us now see that $|\tau_2| = L_1$. If $|\tau_1| = 0$, then clearly $|(\tau_1 * r_0)^+| = 1$, hence $(\tau_1 * r_0)^+ = \alpha_1$ and τ_2 is an appropriate extention of α_1.

Case 1. $\exists\rho \geqslant (\tau_1 * r_0)^+(\rho \Vdash_n F_0(|\tau_1| + 1))$. Taking into account (5.1) and the fact that $|\tau_1| = 0$, we can rewrite equivalently this condition as $\exists\rho \geqslant \beta_1(\rho \Vdash_n F_0(1))$, which is closer to the definition of l_1^s and t_1^s. Clearly, $|\tau_2| = T + 1$, where $T = \mu_0((\tau_1 * r_0)^+, n, 0, 1)$, and since in this case $l_0^s = 0$ for every s, using the appropriate definitions, we notice that $l_1^s = 1, t_1^s = s + 1$ for every $s < T$ and $l_1^T = l_1^{T-1} + t_1^{T-1} = T + 1$, $t_1^T = 0$. Hence $l_1^s = T + 1, t_1^s = 0$ for $s > T$, so $L_1 = T + 1$, which means that $|\tau_2| = L_1$.

Case 2. $\neg\exists\rho \geqslant (\tau_1 * r_0)^+(\rho \Vdash_n F_0(|\tau_1| + 1))$, or equivalently $\neg\exists\rho \geqslant \beta_1(\rho \Vdash_n F_0(1))$. It can be easily checked that in this case $l_1^s = 1$ for every s, hence $L_1 = 1$, which is exactly the length of τ_2.

Suppose now that $|\tau_1| = L_0 > 0$. We have $L_0 = l_0^{S_0}$, hence $l_0^{S_0-1} < l_0^{S_0}$ and $l_1^{S_0} = l_0^{S_0} + 1 = |\tau_1| + 1$. So $|(\tau_1 * r_0)^+| = l_1^{S_0}$ and $(\tau_1 * r_0)^+$ is in fact $\beta_{l_1^{S_0}}$. Now using this fact and having in mind the respective definitions, proceeding as in the case $|\tau_1| = 0$, we see that $|\tau_2| = L_1$.

Now suppose that for some a the lemma is true. We have to check that $|\tau_{3a+4}| = L_{2a+2}$ and $|\tau_{3a+5}| = L_{2a+3}$. Indeed, by induction hypothesis, $|\tau_{3a+3}| = L_{2a+1} + 1 = l_{2a+1}^{S_{2a+1}} + 1$. We consider separately the cases $S_{2a+1} = 0$ (hence $L_{2a+1} = 2a + 1$) and $S_{2a+1} > 0$, and obtain that $l_{2a+2}^{S_{2a+1}} = l_{2a+1}^{S_{2a+1}} + 1 = L_{2a+1} + 1$, which is exactly $|\tau_{3a+3}|$. Hence $\alpha_{l_{2a+2}^{S_{2a+1}}}$ is in fact τ_{3a+3}. Now the condition $\exists\rho \geqslant \tau_{3a+3}(\rho \Vdash_{k(a+1)} F_{(a+1)_0}((a+1)_1))$, which is used in the definition of τ_{3a+4}, is equivalent to $\exists\rho \geqslant \beta_{l_{2a+2}^{S_{2a+1}}}(\rho \Vdash_{k(a+1)} F_{(a+1)_0}((a+1)_1))$. If $S_{2a+2} = S_{2a+1}$, then such ρ does not exist, hence $\tau_{3a+4} = \tau_{3a+3}$, $L_{2a+2} = L_{2a+1} + 1$, therefore $|\tau_{3a+4}| = L_{2a+2}$. If $S_{2a+2} > S_{2a+1}$, then $L_{2a+2} = l_{2a+2}^{S_{2a+2}} = l_{2a+2}^{S_{2a+1}} + t_{2a+2}^{S_{2a+1}}$. However $t_{2a+2}^{S_{2a+1}}$ is in fact $\mu_0(\tau_{3a+3}, k(a+1), (a+1)_0, (a+1)_1)$ (the verification is as in the case $a = 0$). So $L_{2a+2} = |\tau_{3a+3}| + \mu_0(\tau_{3a+3}, k(a+1), (a+1)_0, (a+1)_1)$, which is, according to our construction, the length of τ_{3a+4}. In order to prove the equality $L_{2a+3} = |\tau_{3a+5}|$ we proceed in a similar way. □

Set

$$H_k^s = \begin{cases} \{l_k^{s-1}, \dots, l_k^s - 1\}, & \text{if } l_k^{s-1} < l_k^s \ \&\ \forall m < k(l_m^{s-1} = \dots = l_m^{s-1+k-m}), \\ \emptyset, & \text{else.} \end{cases}$$

Let us notice that (for $k > 0$) if $H_k^s \neq \emptyset$, then $l_{k-1}^{s-1} = l_{k-1}^s$, hence the change of the value of l_k^{s-1} is due to the existence of an appropriate $\rho \geqslant \beta_{l_k^{s-1}}$.

Since the function $\lambda k, s.l_k^s$ is recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, the function $H(k, s) =$ the canonical index of H_k^s is recursive in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, too. Hence the set $H =$

$\cup_{k,s} H_k^s$ is r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$. We shall see below that $N \setminus Dom(f)$ coincides with H, hence $N \setminus Dom(f)$ is r. e. in $\emptyset^{(n-1)} \oplus D(\mathfrak{B})$, which will bring us to the end of the proof of the proposition.

Lemma 5.6. $N \setminus Dom(f) = \cup_{k,s} H_k^s$.

Proof. For the first inclusion, let us see that $H_{3a+1} = H_{2a}^{S_{2a}}$ and $H_{3a+2} = H_{2a+1}^{S_{2a+1}}$. We may suppose that H_{3a+1} and H_{3a+2} are nonempty. We shall consider separately the cases $a = 0$ and $a > 0$. If $a = 0$, then by Lemma 5.5, $|\tau_1| = L_0 = l_0^{S_0}$ and since $H_1 \neq \emptyset$, $l_0^{S_0} > 0$. Therefore $l_0^{S_0-1} = \cdots = l_0^0 = 0$, hence

$$H_0^{S_0} = \{l_0^{S_0-1}, \ldots, l_0^{S_0} - 1\} = \{0, \ldots |\tau_1| - 1\} = H_1.$$

If $a > 0$, again by Lemma 5.5, $|\tau_{3a}| = L_{2a-1} + 1 = l_{2a-1}^{S_{2a-1}} + 1$. Now using Lemma 5.4 we get $l_{2a}^{S_{2a-1}} = l_{2a-1}^{S_{2a-1}} + 1 = |\tau_{3a}|$ and $l_{2a}^{S_{2a-1}} = l_{2a}^{S_{2a}-1}$. Hence $l_{2a}^{S_{2a}-1} = |\tau_{3a}|$, while $l_{2a}^{S_{2a}} = |\tau_{3a+1}|$. So $H_{3a+1} = \{l_{2a}^{S_{2a}-1}, \ldots, l_{2a}^{S_{2a}} - 1\}$. If $m < 2a$, then $l_m^s = l_m^{S_m}$ for every $s \geq S_m$, in particular, for every $s \geq S_{2a}$, hence $H_{2a}^{S_{2a}} = \{l_{2a}^{S_{2a}-1}, \ldots, l_{2a}^{S_{2a}} - 1\}$. The verification of the equality $H_{3a+2} = H_{2a+1}^{S_{2a+1}}$ is similar.

Conversely, take some $H_k^s \neq \emptyset$. Clearly $s \leq S_k$ (otherwise $l_k^{s-1} = l_k^s$ and $H_k^s = \emptyset$). If $s = S_k$, as we saw above, $H_k^s = H_{3a+1}$ or H_{3a+2}, depending on whether $k = 2a$ or $k = 2a+1$. Suppose now that $s < S_k$. We cannot claim anymore that H_k^s coincides with some H_a. We shall see, however, that $H_k^s \subseteq H_m^{S_m}$ for some $m < k$, hence if $x \in H_k^s$, then $x \in N \setminus Dom(f)$ again. Indeed, since $H_k^s \neq \emptyset$, we have $l_k^{s-1} < l_k^s$ and $l_k^s < l_k^{S_k}$. Hence, by Lemma 5.4, there exist $m < k$ and s' such that $l_m^{s'-1} = l_m^{s'}$. We may suppose that m is the minimal one with this property. Using the definition of H_k^s, we get that $s' \leq s - 1$ and $s' - 1 \geq s - 1 + k - m$. From the last equality, $s' + m \geq s + k$ hence $l_m^{s'} = s' + m \geq s + k = l_k^s$. On the other hand, $s' \leq s - 1$, i. e. $s' - 1 < s - 1$, so by Lemma 5.4, $l_m^{s'-1} \leq l_k^{s-1}$. From here, $\{l_m^{s'-1}, \ldots, l_m^{s'} - 1\} \leq_k^{s-1}, \ldots, l_k^s - 1\}$. Using the minimality of m, one can easily get that s' is in fact S_m, hence $\{l_m^{s'-1}, \ldots, l_m^{s'} - 1\} = H_m^{S_m}$. □

We can apply this idea of refined refuting enumerations to the constructions used in the proofs of Proposition 4.2, Proposition 4.3 and Proposition 4.4. Thus we obtain that the following is true:

Proposition 5.2. (i) *Let $n \geq 1$. If $A \subseteq B$ is Π_n^0-admissible in every Π_n^0 enumeration of $\mathfrak{A}$, then A has a Π_n^0 normal form.*

(ii) *Let $n \geq 1$. If the set $A \subseteq B$ is Σ_n^0- or Π_n^0-admissible in every Π_n^0 enumeration of $\mathfrak{A}$, then A has Σ_n^0 normal form or Π_n^0 normal form.*

(iii) *Let the set $A \subseteq B$ be arithmetically admissible in every arithmetical enumeration of $\mathfrak{A}$. Then there exists $n \geq 1$ such that A has Σ_n^0 or Π_n^0 normal form.*

In order to formulate our final results, we use a syntactical characterizations of the sets that have Σ_n^0 (Π_n^0) normal form, obtained in [8], Lemma 6.3, namely, that a set has a Σ_n^0 (Π_n^0) normal form iff it is Σ_n^0 (Π_n^0) definable on $\mathfrak{A}$. This statement, combined with the above two propositions, brings us to our final result.

Theorem 5.1. *Let $n \geq 1$. Then the following is true:*

(i) *If $A \subseteq B$ is Σ_n^0-admissible in every Π_n^0 enumeration, then A is Σ_n^0 definable on $\mathfrak{A}$.*

(ii) *If $A \subseteq B$ is Π_n^0-admissible in every Π_n^0 enumeration, then A is Π_n^0 definable on $\mathfrak{A}$.*

(iii) *If $A \subseteq B$ is Σ_n^0-admissible or Π_n^0-admissible in every Π_n^0 enumeration, then A is Σ_n^0 definable or Π_n^0 definable on $\mathfrak{A}$.*

(iv) *If $A \subseteq B$ is arithmetically admissible in every arithmetical enumeration, then there exists $n \geq 1$ such that A is Σ_n^0 or Π_n^0 definable on $\mathfrak{A}$.*

Let us notice that the class of all Π_n^0 enumerations in points (i) – (iii) of the above theorem cannot be reduced anymore. Indeed, let us take a set A, which is definable by means of existential Σ_n^0 formula (cf. [8)], but is not Σ_n^0 definable on $\mathfrak{A}$ (it can be easily seen that such A does exist, if the structure $\mathfrak{A}$ is interesting enough). Clearly, A has a Σ_n^0 associate in every Σ_n^0 enumeration. Hence A is Σ_n^0 admissible in every class of enumerations, that is included in the class of all Σ_n^0 enumerations, and at the same time A is not Σ_n^0 definable on $\mathfrak{A}$.

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