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ON MUSIELAK–ORLICZ SEQUENCE SPACES WITH AN ASYMPTOTIC ℓ_∞ DUAL [1]

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We investigate Mushielak–Orlicz sequence spaces ℓ_Φ with a dual ℓ_Φ^*, which is stabilized asymptotic ℓ_∞ with respect to the unit vector basis. We give a complete characterization of the bounded relatively weakly compact subsets $K \subset \ell_\Phi$. We prove that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and thus ℓ_Φ fails the fixed point property for closed, bounded convex sets and non–expansive (or contractive) maps on them.

Keywords: Mushielak–Orlicz sequence spaces, asymptotically isometric copy of ℓ_1, asymptotic ℓ_∞ space, fixed point property, weakly compact.

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1. INTRODUCTION

The notion of asymptotic ℓ_p spaces first appeared in [14], where the collection of spaces that are now known as stabilized asymptotic ℓ_p spaces were introduced. Later in [13] more general collection of spaces, known as asymptotic ℓ_p spaces, were introduced. Characterization of the stabilized asymptotic ℓ_∞ MO sequence space was given in [5]. It is found in [17] that if the dual of a MO sequence space ℓ_Φ is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis then ℓ_Φ is saturated with complemented copies of ℓ_1 and has the Schur property.

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A characterization of the relatively weakly compact sets in an Orlicz spaces $L_M[0,1]$, such that the function N complementary to M satisfies $\lim_{t\to\infty} \frac{N(\lambda t)}{N(t)} = \infty$ for some $1 < \lambda < \infty$ is given in [2]. Using the technique of [2] and [17] we generalize this result for MO sequence spaces. More precisely we characterize the relatively weakly compact sets of a MO sequence space ℓ_Φ, and its dual ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis.

In the second part of this note we prove that MO spaces ℓ_Φ with stabilized asymptotic ℓ_∞ dual are saturated with asymptotically isometric copies of ℓ_1. The notion of asymptotically isometric copy of ℓ_1 in a Banach space appeared in [7] and is used to investigate the fpp for non–expansive mappings of the non–reflexive subspaces of $L_1[0,1]$. Using the ideas of [1], [7] and [17] we show that any subspace of ℓ_Φ contains an asymptotically isometric copy of ℓ_1, provided that ℓ_Φ^* is stabilized asymptotic ℓ_∞ space with respect to the unit vector basis and as a consequence of [7] this class of MO sequence spaces fails the fpp for closed, bounded, convex sets in ℓ_Φ and non–expansive maps on them. Let us mention that such a conclusion could have been drawn directly by using the recent characterization of the MO sequence spaces ℓ_Φ having fpp given in [16]: An MO sequence space has fpp for closed bounded convex sets and non–expansive maps on them iff it is reflexive. The examples at the end show that sometimes to check reflexivity is more difficult than to check that ℓ_Φ^* is stabilized asymptotic ℓ_∞ with respect to the unit vector basis, due to the engagement of several constants in the definition of the δ_2–condition for a MO function Φ.

2. PRELIMINARIES

We use the standard Banach space terminology from [11], Let us recall that an Orlicz function M is even, continuous, non-decreasing convex function such that $M(0) = 0$ and $\lim_{t\to\infty} M(t) = \infty$. We say that M is non–degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^\infty$ of Orlicz functions is called a Musielak–Orlicz function or MO function in short.

The MO sequence space ℓ_Φ, generated by a MO function Φ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The Luxemburg's norm in ℓ_Φ is defined by

$$\|x\|_\Phi = \inf\left\{ r > 0 : \sum_{i=1}^\infty \Phi_i(x_i/r) \le 1 \right\}.$$

We denote by h_Φ the closed linear subspace of ℓ_Φ, generated by all $x = \{x_i\}_{i=1}^\infty \in \ell_\Phi$, such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for every $\lambda > 0$.

If the MO function Φ consists of one and the same function M one obtains the Orlicz sequence spaces ℓ_M and h_M.

Let $1 \le p_i$, $i \in \mathbb{N}$ be a sequence of reals. The MO sequence space ℓ_Φ, where $\Phi = \{t^{p_i}\}_{i=1}^\infty$ is called Nakano sequence space and is denoted by $\ell_{\{p_i\}}$. In [4] it was proved that two Nakano sequence spaces $\ell_{\{p_i\}}$, $\ell_{\{q_i\}}$ are isomorphic iff there exists $0 < C < 1$ such that

$$\sum_{i=1}^{\infty} C^{1/|p_i - q_i|} < \infty .$$

An extensive study of Orlicz and MO spaces can be found in [11] and [15].

Definition 2.1. *We say that the MO function Φ satisfies the δ_2 condition at zero if there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$ such that for every $n \in \mathbb{N}$*

$$\Phi_n(2t) \leq K\Phi_n(t) + c_n$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide iff Φ satisfies the δ_2 condition at zero.

Recall that given MO functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms iff Φ is equivalent to Ψ, i.e. there exist constants $K, \beta > 0$ and a non–negative sequence $\{c_n\}_{n=1}^{\infty} \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$, [9] and [12].

Throughout this paper M will always denote Orlicz function while Φ - an MO function. As the properties we are dealing with are preserved by isomorphisms without loss of generality we may assume that Φ consists entirely of non–degenerate Orlicz functions, such that for every $i \in \mathbb{N}$ the Orlicz function Φ_i is differentiable, $\Phi_i'(0) = 0$ and $\Phi_i(1) = 1$ [17]

Definition 2.2. *For an Orlicz function M, such that $\lim_{t \to 0} M(t)/t = 0$ the function*

$$N(x) = \sup\{t|x| - M(t) : t \geq 0\},$$

is called function complementary to M.

Definition 2.3. *The MO function $\Psi = \{\Psi_j\}_{j=1}^{\infty}$, defined by*

$$\Psi_j(x) = \sup\{t|x| - \Phi_j(t) : t \geq 0\}, j = 1, 2, ..., n, ...$$

is called complementary to Φ.

Let us note that the condition $\lim_{t \to 0} M(t)/t = 0$ secures that the complementary function N is always non-degenerate. Observe that if N is function complementary to M, then M is complementary to N and if the MO function Ψ is complementary to the MO function Φ, then Φ is function complementary to Ψ. Throughout this paper the function complementary to the MO function Φ is denoted by Ψ.

It is well known that $h_M^* \cong \ell_N$ and $h_\Phi^* \cong \ell_\Psi$. Well known equivalent norm in ℓ_Φ is the Orlicz norm $\|x\|_\Phi^O = \sup\left\{\sum_{j=1}^\infty x_j y_j : \sum_{j=1}^\infty \Psi_j(y_j) \le 1\right\}$, which satisfies the inequalities (see e.g.[10])

$$\|\cdot\|_\Phi \le \|\cdot\|_\Phi^O \le 2\|\cdot\|_\Phi .$$

We will use the Hölder's inequality: $\sum_{j=1}^\infty |x_j y_j| \le \|x\|_\Phi^O \|y\|_\Psi$, which holds for every $x = \{x_j\}_{j=1}^\infty \in \ell_\Phi$ and $y = \{y_j\}_{j=1}^\infty \in \ell_\Psi$, where Φ and Ψ are complementary MO functions.

By $\{e_j\}_{j=1}^\infty$ and $\{e_j^*\}_{j=1}^\infty$ we denote the unit vector basis in h_Φ and h_Ψ respectively. For a Banach space X with a basis $\{v_i\}_{i=1}^\infty$ and element $x \in X$, $x = \sum_{i=1}^\infty x_i v_i$ we define $\mathrm{supp} x = \{i \in \mathbb{N} : x_i \ne 0\}$. We write $n \le x$ if $n \le \min\{\mathrm{supp} x\}$ and $x < y$ if $\max\{\mathrm{supp} x\} < \min\{\mathrm{supp} y\}$. We say that x is a block vector with respect to the basis $\{v_i\}_{i=1}^\infty$ if $x = \sum_{i=p}^q x_i v_i$ for some finite p and q and we say that x is a normalized block vector if it is a block vector and $\|x\| = 1$.

Definition 2.4. *A Banach space X is said to be stabilized asymptotic ℓ_∞ with respect to a basis $\{v_i\}_{i=1}^\infty$. if there exists a constant $C \ge 1$, such that for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$, so that whenever $N \le x_1 < \ldots < x_n$ are successive normalized block vectors. then $\{x_i\}_{i=1}^n$ are C-equivalent to the unit vector basis of ℓ_∞^n, i.e.*

$$\frac{1}{C} \max_{1\le i\le n} |a_i| \le \left\| \sum_{i=1}^n a_i x_i \right\| \le C \max_{1\le i\le n} |a_i|.$$

The following characterization of the stabilized asymptotic ℓ_∞ MO sequence spaces is due to Dew:

Proposition 2.1. *(Proposition 4.5.1 [5]) Let $\Phi = \{\Phi_j\}_{j=1}^\infty$ be a MO function. Then the following are equivalent:*
(i) h_Φ is stabilized asymptotic ℓ_∞ (with respect to its natural basis $\{e_j\}_{j=1}^\infty$);
(ii) there exists $\lambda > 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \le p \le q$ and $\sum_{j=p}^q \Phi_j(a_j) \le 1$. then

$$\sum_{j=p}^q \Phi_j(a_j/\lambda) \le \frac{1}{n}.$$

An easy sufficient condition for h_Φ to be stabilized asymptotic ℓ_∞ with respect to the unit vector basis is the following

Proposition 2.2. *(Proposition 4.5.3 [5]) Let $\varphi_\lambda(j) = \inf\{\Phi_j(\lambda t)/\Phi_j(t) : t > 0\}$. If $\lim_{j\to\infty} \varphi_\lambda(j) = \infty$ for some $\lambda > 1$ then h_Φ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis.*

Let X be a Banach space. By $Y \hookrightarrow X$ we denote that Y is isomorphic to a subspace of X.

Definition 2.5. *We say that a collection $K \subset h_\Phi$ has equi–absolutely continuous norms if*
for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\sup\{\| \sum_{k=n}^{\infty} x_k e_k\| : x = \{x_k\}_{k=1}^{\infty} \in K\} < \varepsilon$ *for every $n \geq N$.*

Definition 2.6. *We say that a Banach space $(X, \|\cdot\|)$ is asymptotically isometric to ℓ_1 if it has a normalized basis $\{v_n\}_{n=1}^{\infty}$ such that for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ increasing to 1 we have*

$$\sum_{n=1}^{\infty} \lambda_n |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n v_n \right\| \tag{1}$$

for all $x = \sum_{n=1}^{\infty} t_n v_n \in X$.

Whenever $(X, \|\cdot\|)$ contains a normalized sequence $\{x^{(n)}\}_{n=1}^{\infty}$ satisfying (1) then the closed linear span of $\{x^{(n)}\}_{n=1}^{\infty}$ is asymptotically isometric to ℓ_1

We say that X is saturated with subspaces with the property (*) if in every infinite dimensional subspace Z of X there is an infinite dimensional subspace Y of Z isomorphic to a space with the property (*).

3. WEAKLY COMPACT SETS OF MO SEQUENCE SPACES

Lemma 3.1. *Let Φ be a MO function. which has δ_2 condition at zero and $K \subset h_\Phi$. Suppose that K fails to have equi–absolutely continuous norms. Then there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^{\infty} \subset K$. $\{p_n, q_n\}_{n=1}^{\infty}$. $p_n, q_n \in \mathbb{N}$, $p_n \leq q_n < p_{n+1}$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ such that*

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0 \tag{1}$$

for every $n \in \mathbb{N}$.

Proof. Since K does not have equi–absolutely continuous norms there are $\varepsilon > 0$, $\{\alpha_n\}_{n\in\mathbb{N}}$, $\alpha_n \in \mathbb{N}$ and $\{z^{(n)}\} \subset K$ such that

$$\left\| \sum_{i=\alpha_n}^{\infty} z_i^{(n)} e_i \right\| > \varepsilon.$$

Let $n_1 = 1$. We choose $n_2 > n_1$ such that

$$\left\| \sum_{i=\alpha_{n_1}}^{\alpha_{n_2}-1} z_i^{(n_1)} e_i \right\| > \varepsilon/2.$$

Put $p_1 = \alpha_{n_1}$, $q_1 = \alpha_{n_2} - 1$, $x^{(1)} = z^{(n_1)}$. We choose $n_3 > n_2$ such that

$$\left\| \sum_{i=\alpha_{n_2}}^{\alpha_{n_3}-1} z_i^{(n_2)} e_i \right\| > \varepsilon/2.$$

Put $p_2 = \alpha_{n_2}$, $q_2 = \alpha_{n_3} - 1$, $x^{(2)} = z^{(n_2)}$.

If we have selected $x^{(1)}, x^{(2)}, \dots, x^{(k)}$ by $x^{(s)} = z^{(n_s)}$, $p_s = \alpha_{n_s}$, $q_s = \alpha_{n_{s+1}} - 1$ for $1 \le s \le k$, then we choose $n_{k+1} > n_k$ such that

$$\left\| \sum_{i=\alpha_{n_{k+1}}}^{\alpha_{n_{k+2}}-1} z_i^{(n_{k+1})} e_i \right\| > \varepsilon/2.$$

Now we put $p_{k+1} = \alpha_{n_{k+1}}$, $q_{k+1} = \alpha_{n_{k+2}} - 1$ $x^{(k+1)} = z^{(n_{k+1})}$.

Obviously the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ verifies (1) with $\varepsilon_0 = \varepsilon/2$. □

Lemma 3.2.([2]) *Let X be a Banach space. Suppose that $\{x_n\} \subset X$ is weakly null and $\{x_n^*\} \subset X^*$ is weakly* null. Then for each $\varepsilon > 0$ there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of positive integers so that for each $k \in \mathbb{N}$ holds:*

$$\sum_{j \neq k} \left| x_{n_j}^*(x_{n_k}) \right| < \varepsilon.$$

Theorem 1. *Let Φ be a MO function, which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then any weakly null sequence in ℓ_Φ has equi-absolutely continuous norms.*

Proof. Suppose the contrary. There is a weakly null sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset \ell_\Phi$ that fails to have equi–absolutely continuous norms. By Lemma 3.1 there exist $\varepsilon_0 > 0$ and strongly increasing sequences $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$, $p_n, q_n \in \mathbb{N}$, $p_n \le q_n < p_{n+1}$ such that

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0.$$

Choose $y^{(n)} \in h_\Psi$ such that supp $y^{(n)} = \{i\}_{i=p_n}^{q_n}$, $\sum_{k=p_n}^{q_n} \Psi_k(y_k^{(n)}) \le 1$ and $\left| \sum_{k=p_n}^{q_n} y_k^{(n)} x_k^{(n)} \right| > \frac{3}{4}\varepsilon_0$. For a fixed $x \in \ell_\Phi$ by Holder's Inequality:

$$\left| \sum_{k=1}^{\infty} x_k y_k^{(n)} \right| = \left| \sum_{k=p_n}^{q_n} x_k y_k^{(n)} \right| \le \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi \left\| y^{(n)} \right\|_\Psi^O.$$

As x is fixed and $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$ it follows that

$$\lim_{n\to\infty} \left\| \sum_{k=p_n}^{q_n} x_k e_k \right\|_\Phi = 0.$$

Thus $\{y^{(n)}\}_{n=1}^{\infty}$ is weak* null sequence. By Lemma 3.2 there is a subsequence of naturals $\{n_k\}_{k=1}^{\infty}$ so that

$$\sum_{j\neq k}\left|\sum_{i=p_{n_j}}^{q_{n_j}} y_i^{(n_j)}x_i^{(n_k)}\right| < \varepsilon_0/2.$$

We claim that

$$\lim_{k\to\infty}\sum_{j=1}^{\infty}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = \lim_{k\to\infty}\sum_{j=p_{n_k}}^{q_{n_k}}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = 0, \tag{2}$$

where $\lambda > 1$ is the constant from Proposition 2.1. Indeed, by assumption h_Ψ is a stabilized asymptotic ℓ_∞ space and there exists $\lambda > 1$ such that for every $m \in \mathbb{N}$ there is $N \in \mathbb{N}$ so that whenever $\sum_{j=p_{n_k}}^{q_{n_k}} \Psi_j(y_j^{(n_k)}) \leq 1$ then the inequality $\sum_{j=p_{n_k}}^{q_{n_k}}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \leq 1/m$ holds for every $q_{n_k} \geq p_{n_k} \geq N$.

Thus $\lim_{n_k\to\infty}\sum_{j=p_{n_k}}^{q_{n_k}}\Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) = 0$.

Therefore there is subsequence $\{n_{k_m}\}_{m=1}^{\infty}$ such that

$$\sum_{m=1}^{\infty}\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}}\Psi_i\left(\frac{y_i^{(n_{k_m})}}{\lambda}\right) \leq 1.$$

Let $y = \sum_{m=1}^{\infty} y^{(n_{k_m})}$. Obviously $y \in h_\Psi$ and since $\{x^{(n)}\}_{n=1}^{\infty}$ is weakly null we must have

$$\lim_{m\to\infty} y(x^{(n_{k_m})}) = \lim_{m\to\infty}\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})}x_i^{(n_{k_m})} = 0.$$

But

$$\left|\sum_{j=1}^{\infty}\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})}x_i^{(n_{k_m})}\right| \geq \left|\sum_{i=p_{n_{k_m}}}^{q_{n_{k_m}}} y_i^{(n_{k_m})}x_i^{(n_{k_m})}\right| - \sum_{j\neq m}\left|\sum_{i=p_{n_{k_j}}}^{q_{n_{k_j}}} y_i^{(n_{k_j})}x_i^{(n_{k_m})}\right|$$

$$\geq \frac{3}{4}\varepsilon_0 - \frac{1}{2}\varepsilon_0 = \frac{1}{4}\varepsilon_0,$$

a contradiction. □

Let us recall that C is weakly sequentially compact if every sequence of points in C has a subsequence weakly convergent to a point of C.

For the proof of the next result we need:

Theorem 2. *(Eberlein–Smulian, see e.g. [8]) Let X be a separable Banach space and C be a weakly closed subset of X. Then C is weakly compact if and only if C is weakly sequentially compact.*

By Theorem 1 it follows immediately:

Corollary 3.1. *Let Φ be a MO function which has δ_2 condition at zero and with a complementary function Ψ such that h_Ψ is stabilized asymptotic ℓ_∞ with respect to the basis $\{e_j^*\}_{j=1}^\infty$. Then a bounded set $K \subset \ell_\Phi$ is relatively weakly compact iff K has equi–absolutely continuous norm.*

Proof. Necessity. Suppose that $K \subset h_\Phi$ is relatively weakly compact. If K fails to have equi–absolutely continuous norms then by Lemma 3.1 there are $\varepsilon_0 > 0$ and sequences $\{x^{(n)}\}_{n=1}^\infty \subset K$, $\{p_n, q_n\}_{n=1}^\infty$, $p_n, q_n \in \mathbb{N}$, $p_n \le q_n < p_{n+1}$ such that

$$\left\| \sum_{i=p_n}^{q_n} x_i^{(n)} e_i \right\| > \varepsilon_0$$

for every $n \in \mathbb{N}$.

By Eberlein–Smulian theorem there are $x \in \ell_\Phi$ and a subsequence $\{x^{(n_k)}\}_{n=1}^\infty$ such that $x^{(n_k)} \longrightarrow x$ weakly in ℓ_Φ. Thus by Theorem 1 $\{x^{(n_k)} - x\}_{k=1}^\infty$ has equi–absolutely continuous norms. Hence $\lim_{k\to\infty} \left\| \sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i) e_i \right\| = 0$ and obviously $\lim_{k\to\infty} \left\| \sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i \right\| = 0$. But

$$\varepsilon_0 < \left\| \sum_{i=p_{n_k}}^{q_{n_k}} x_i^{(n_k)} e_i \right\| \le \left\| \sum_{i=p_{n_k}}^{q_{n_k}} x_i e_i \right\| + \left\| \sum_{i=p_{n_k}}^{q_{n_k}} (x_i^{(n_k)} - x_i) e_i \right\| \underset{k\to\infty}{\longrightarrow} 0,$$

which is a contradiction.

Sufficiency. Let K be a bounded set with equi–absolutely continuous norms. Let $\{x^{(n)}\}_{n=1}^\infty$ be an arbitrary sequence of elements in K. Obviously there exists L such that $|x_k^{(n)}| \le L$ for every $n, k \in \mathbb{N}$. Thus there exists a subsequence $\{x^{(n_i)}\}_{i=1}^\infty$ such that $\lim_{i\to\infty} x_k^{(n_i)} = x_k$ for every $k \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for every $s \ge N$ and every $i \in \mathbb{N}$ the inequality holds $\left\| \sum_{k=s}^\infty x_k^{(n_i)} e_k \right\| < \varepsilon/3$. Fix $s \ge N$. There is $M \in \mathbb{N}$ such that for every $n_i, n_j \ge M$ and every $k = 1, 2, \ldots, s$ the inequality $|x_k^{(n_i)} - x_k^{(n_j)}| \le \frac{\varepsilon}{3s}$ holds. Thus we can write the inequalities:

$$\|x^{(n_i)} - x^{(n_j)}\| = \left\| \sum_{k=1}^\infty x_k^{(n_i)} e_k - \sum_{k=1}^\infty x_k^{(n_j)} e_k \right\|$$

$$\le \left\| \sum_{k=1}^s x_k^{(n_i)} e_k - \sum_{k=1}^s x_k^{(n_j)} e_k \right\| + \left\| \sum_{k=s+1}^\infty x_k^{(n_i)} e_k - \sum_{k=s+1}^\infty x_k^{(n_j)} e_k \right\|$$

$$\le \left\| \sum_{k=1}^s |x_k^{(n_i)} - x_k^{(n_j)}| e_k \right\| + \left\| \sum_{k=s+1}^\infty x_k^{(n_i)} e_k \right\| + \left\| \sum_{k=s+1}^\infty x_k^{(n_j)} e_k \right\|$$

$$< \frac{\varepsilon}{3s} s + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Consequently $\{x^{(n_i)}\}_{n=1}^{\infty}$ is a Cauchy sequence and thus it is norm convergent to $x \in \ell_\Phi$ and thus it is weakly convergent. □

Remark. Let us mention that for the proof of the sufficiency in Corollary 3.1 we do not need that ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$.

4. FIXED POINT PROPERTY FOR MO SEQUENCE SPACES

The next Lemma is similar to that in [17], where it is shown that for every normalized block basis $\{x^{(n)}\}_{n=1}^{\infty}$ of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ contains a subsequence such that $[x^{(n_i)}]_{i=1}^{\infty}$ is isomorphic to ℓ_1.

Lemma 4.1. *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then every normalized block basis $\{x^{(n)}\}_{n=1}^{\infty}$ of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ contains a subsequence $\{x^{(n_i)}\}_{i=1}^{\infty}$ such that $[x^{(n_i)}]_{i=1}^{\infty}$ is asymptotically isometric to ℓ_1.*

Proof. Let $\{x^{(n)}\}_{n=1}^{\infty}$ be a normalized block basis of the unit vector basis $\{e_j\}_{j=1}^{\infty}$ in ℓ_Φ, where $x^{(n)} = \sum_{j=m_n+1}^{m_{n+1}} x_j^{(n)} e_j$, $\{m_n\}_{n=1}^{\infty}$ strictly increasing sequence of naturals. Let $\{\lambda_n\}_{n=1}^{\infty}$ be an increasing sequence, such that $\lim_{n\to\infty} \lambda_n = 1$. For every $n \in \mathbb{N}$ there exists $y^{(n)} = \sum_{j=1}^{\infty} y_j^{(n)} e_j^* \in h_\Psi$ such that

$$\sum_{j=1}^{\infty} \Psi_j(y_j^{(n)}) \le 1 \quad \sum_{j=1}^{\infty} y_j^{(n)} x_j^{(n)} \ge \lambda_n.$$

WLOG we may assume that supp $y^{(n)} \equiv$ supp $x^{(n)}$.

For the sequence $\{y^{(n)}\}_{n=1}^{\infty}$ and the constant $\lambda > 1$ from Proposition 2.1 holds:

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = \lim_{n\to\infty} \sum_{j=m_n+1}^{m_{n+1}} \Psi_j\left(\frac{y_j^{(n)}}{\lambda}\right) = 0.$$

The proof is essentially the same as for (2).

Now passing to a subsequence we get a sequence

$$\{y^{(n_k)}\}_{k\in\mathbb{N}}, \, y^{(n_k)} = \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^*$$

such that

$$\sum_{k=1}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} \Psi_j\left(\frac{y_j^{(n_k)}}{\lambda}\right) \le 1.$$

Denote $y = \sum_{k=1}^{\infty} y^{(n_k)} = \sum_{k=1}^{\infty} \left(\sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right)$. Obviously $y \in \ell_\Psi$ and $\|y\|_\Psi \le \lambda$. As

$$\lim_{s\to\infty} \left\| \sum_{k=s}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi = 0$$

there exists $s_0 \in \mathbb{N}$ such that

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \le \frac{1}{2}.$$

Consequently

$$\left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi^O \le 2 \left\| \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} e_j^* \right\|_\Psi \le 1.$$

Denote $\overline{y} = \sum_{k=s_0}^{\infty} \sum_{j=p_{n_k}}^{q_{n_k}} y_j^{(n_k)} e_j^*$. Then $\|\overline{y}\|_\Psi^O \le 1$. Now using Hölder's inequality for any sequence $\{t_n\}_{n=1}^{\infty}$, such that $\sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \in \ell_\Phi$ we get

$$\begin{aligned} \left\| \sum_{k=s_0}^{\infty} t_{k-s_0+1} x^{(n_k)} \right\|_\Phi &\ge \frac{1}{\|\overline{y}\|_\Psi^O} \sum_{k=s_0}^{\infty} \sum_{j=m_{n_k}+1}^{m_{n_k+1}} |t_{k-s_0+1} y_j^{(n_k)} x_j^{(n_k)}| \\ &\ge \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \sum_{j=m_{n_k}+1}^{m_{n_k+1}} y_j^{(n_k)} x_j^{(n_k)} \ge \sum_{k=s_0}^{\infty} |t_{k-s_0+1}| \lambda_k. \end{aligned}$$

□

Theorem 3. *Let Φ be a MO function, which satisfies the δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^{\infty}$. Then ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1.*

Proof. According to a well known result of Bessaga and Pelczinski [3] every infinite dimensional closed subspace Y of ℓ_Φ has a subspace Z isomorphic to a subspace of ℓ_Φ, generated by a normalized block basis of the unit vector basis of ℓ_Φ. Now to finish the proof it is enough to observe that by Lemma 4.1 the space Z contains an asymptotically isometric copy of ℓ_1. □

By using a result from [7] that states that a Banach spaces containing an asymptotically isometric copy of ℓ_1 fail the fixed point property for closed, bounded, convex sets and non–expansive (contractive) maps on them, we easily get

Corollary 4.1. *Let Φ be a MO function, which has δ_2 condition at zero and h_Ψ, generated by the MO function Ψ, complementary to Φ, is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$. Then ℓ_Φ fails the fixed point property (fpp) for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.*

We give at the end some examples of MO sequence space, saturated with asymptotically isometric copies of ℓ_1.

Example 1.([17]) Sometimes we know only the complementary function Ψ. For example let the MO function $\Psi = \{\Psi_j\}_{j=1}^\infty$ be defined by $\Psi_j = e^{\alpha_j} e^{-\frac{\alpha_j}{|x|^{c_j}}}$, where $\lim_{j\to\infty} \alpha_j = \infty$ and $0 < c_j$. Then ℓ_Ψ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ because

$$\lim_{j\to\infty} \inf \left\{ \frac{\Psi_j(2x)}{\Psi_j(x)} : 0 \le x \le 1 \right\}$$

$$= \lim_{j\to\infty} \inf \left\{ e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}|x|^{c_j}}} : 0 \le x \le 1 \right\} = \lim_{j\to\infty} e^{\alpha_j \frac{2^{c_j}-1}{2^{c_j}}} = \infty.$$

Thus we conclude that ℓ_Φ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

Example 2.([5]) Consider the Nakano sequence space $\ell_{\{p_n\}}$, where $p_n = \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)}$. It is well known that $\ell_{\{p_n\}}^* \cong \ell_{\{q_n\}}$, where $1/p_n + 1/q_n = 1$, i.e. $q_n = \log_2(n+1)$. It is easy to see that $\lim_{n\to\infty} p_n = \lim_{n\to\infty} \dfrac{\log_2(n+1)}{\log_2\left(\dfrac{n+1}{2}\right)} = 1$ and thus according to [4] and [12] $\ell_{\{p_n\}}$ is saturated with spaces isomorphic to ℓ_1. Moreover according to [5] $\ell_{\{q_n\}}$ is stabilized asymptotic ℓ_∞ with respect to the unit vector basis $\{e_j^*\}_{j=1}^\infty$ and thus $\ell_{\{p_n\}}$ is saturated with asymptotically isometric copies of ℓ_1 and fails fpp for closed, bounded, convex sets in ℓ_Φ and non-expansive (or contractive) maps on them.

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