ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ"
ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI"
FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

EQUALITY CASES FOR TWO POLYNOMIAL INEQUALITIES

D. DRYANOV, R. FOURNIER

A complete characterization of the equality cases for two recent polynomial inequalities is given. The proofs are based on simple interpolation and quadrature techniques. We discuss also the meaning and the sharpness of these inequalities.

Keywords: Bernstein and Markov Type Inequalities.
2000 MSC: 41A17.

Let $\mathcal{P}_n$ be the linear space of polynomials $p(z) := \sum_{k=0}^{n} a_k z^k$ of degree at most n with complex coefficients. The following two polynomial inequalities have been a subject of extensive research.

Bernstein polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\|p\|_{\mathbb{D}} := \max_{z\in\partial\mathbb{D}} |p(z)|$ with $\mathbb{D} := \{z : |z| < 1\}$, $p \in \mathcal{P}_n$. Then

$$\| p' \|_{\mathbb{D}} \leq n \| p \|_{\mathbb{D}} \tag{1}$$

with equality only for the monomials $p_n(z) := Kz^n$, where $K \in \mathbb{C}$.

Markov polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\| p \|_{[-1,1]} := \max_{x\in[-1,1]} |p(x)|$, $p \in \mathcal{P}_n$. Then

$$\| p' \|_{[-1,1]} \leq n^2 \| p \|_{[-1,1]} \tag{2}$$

with equality only for multiples of the n^{th} Chebyshev polynomial $T_n \in \mathcal{P}_n$ defined by $T_n(x) := \cos(n \arccos(x)),\ x \in [-1, 1]$.

We refer the reader to the survey paper [1], and to the books [2], [9], [10] for up-to-date references concerning (1) and (2) and their extensions. One of the most striking results [7] on polynomial inequalities is the following discrete improvement of (2).

Duffin and Schaeffer polynomial inequality. Let $\mathcal{P}_n$ be equipped with the norm $\|p\|_{[-1,1]} := \max_{-1\le x\le 1} |p(x)|$, $p \in \mathcal{P}_n$ and let $x_j = \cos(j\pi/n)$, $0 \le j \le n$, be the extremal points of T_n in $[-1,1]$. Then

$$\| p' \|_{[-1,1]} \le n^2 \max_{0\le j\le n} |p(x_j)| \tag{3}$$

with equality only for multiples of the n^{th} Chebyshev polynomial $T_n \in \mathcal{P}_n$.

In this article we give a complete characterization of the equality cases for two polynomial inequalities (see Theorem A and Theorem B), recently published in [5]. The proofs are based on simple interpolation and quadrature techniques. We also discuss the meaning and the sharpness of these inequalities.

We consider the following two inequalities:

Theorem A. *Let $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$. Then*

$$\left| \frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} \right| \le n \max_{0\le j\le n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|, \tag{4}$$

where the inequality is strict for each $\theta \notin \{0,\pi\} \pmod{2\pi}$ and any polynomial $p \not\equiv 0$.

Theorem B. Let $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$. Then

$$\left| p'\left(e^{i\theta}\right) \right| \le n \max_{j\in J_n} \left| \frac{p\left(e^{i(\theta + j\pi/n)}\right) + p\left(e^{i(\theta - j\pi/n)}\right)}{2} \right|, \tag{5}$$

where $J_n := \{0\} \cup \{ j : 1 \le j \le n,\ j \text{ odd} \}$.

Theorem A is a Duffin and Schaeffer type result in the spirit of (3). It gives an upper bound for the uniform norm of the divided difference

$$\frac{p\left(e^{i\theta}\right) - p\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}}$$

of a polynomial $p \in \mathcal{P}_n$.

Remark. Theorem B gives a pointwise estimate for the first derivative of a given polynomial $p \in \mathcal{P}_n$ of degree at most n by using $(n+1)$ functional values of p. Note that $(n+1)$ is the minimal number of functional values for which such an estimate holds. Assume, on the contrary, that for a fixed point $z_0 \in \partial\mathbb{D}$ there exist n distinct complex numbers $z_1, \dots, z_n$ in $\bar{\mathbb{D}} := \{z : |z| \le 1\}$ such that $|p'(z_0)| \le \sum_{k=1}^{n} \beta_k |p(z_k)|$ $(\beta_k \ge 0)$ for any polynomial p of degree at most n. Applying Gauss-Lucas Theorem, the polynomial $p(z) := (z-z_1)(z-z_2)\cdots(z-z_n)$

satisfies $p'(z_0) \neq 0$ and we are led to a contradiction. Furthermore, *Theorem B contains an improvement of Bernstein's inequality (1).* It follows from (5) that

$$|p'(e^{i\theta})| \leq n \max_{j\in J_n} \left| \frac{p\left(e^{i(\theta+j\pi/n)}\right) + p\left(e^{i(\theta-j\pi/n)}\right)}{2} \right| \leq n \, \|p\|_{\mathbb{D}}$$

for any $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$.

Remark. The following polynomial inequality

$$\| p' \|_{\mathbb{D}} \leq n \max_{0 \leq j \leq 2n-1} \left| p\left(e^{ij\pi/n}\right) \right| \tag{6}$$

has been published in [8]. The above inequality may be thought as an analogue of (3) on the unit disk $\mathbb{D}$. It is seen from (6) that for $p \in \mathcal{P}_n$ and any $\gamma \in \mathbb{R}$,

$$|z p'(z)| \leq n \max_{0 \leq j \leq 2n-1} \left| p\left(|z| \, e^{i(\gamma + j\pi/n)}\right) \right| \quad (|z| \leq 1)$$

However, for a given $z := re^{i\theta}$ with $|z| \leq 1$, it is not clear at all how to choose $\gamma = \gamma(z)$ in order to minimize the right hand-side in the above inequality. On the other hand, it follows from (5) that for any $r \in (0, 1]$

$$\begin{aligned} |z p'(z)| \quad &\leq \quad n \max_{j \in J_n} \left| \frac{p\left(r\, e^{i(\theta+j\pi/n)}\right) \; + \; p\left(r\, e^{i(\theta-j\pi/n)}\right)}{2} \right| \\ &\leq \quad n \max_{0 \leq j \leq 2n-1} \left| p(r\, e^{i(\theta + j\pi/n)}) \right| \end{aligned}$$

and, because the number of the functional values used in (5) is $(n+1)$, hence smaller than $2n$, the estimate (5) can be considerably better than the estimate (6). We show in this paper that (5) has many extremal polynomials including all extremals of (6). Hence *(5) is more sensitive than (6).* Let us point out that the strength of (6) lies in the fact that it gives an upper-bound for the *uniform* norm $\|p'\|_{\mathbb{D}}$ of a polynomial. However, it is not true that for all $p \in \mathcal{P}_n$

$$\|p'\|_{\mathbb{D}} \leq n \max_{0\leq j\leq n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|.$$

This can be seen by taking the polynomials $p_{n,k}(z) := z^n + iz^k, \;\; 0 < k < n.$ Obviously $\|p'_{n.k}\|_{\mathbb{D}} = n + k$ while

$$n \max_{0\leq j\leq n} \left| \frac{p_{n,k}\left(e^{ij\pi/n}\right) + p_{n,k}\left(e^{-ij\pi/n}\right)}{2} \right| = \sqrt{2}n.$$

The proof of Theorem A is based on the following

Representation Formula 1. Let $\theta \in \mathbb{R}$ be fixed. Then there exist $(n+1)$ numbers $\alpha_0(\theta)$, $\alpha_1(\theta)$, ..., $\alpha_n(\theta)$ such that

$$\frac{p\left(e^{\mathrm{i}\theta}\right) - p\left(e^{-\mathrm{i}\theta}\right)}{e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}} = \sum_{j=0}^{n} (-1)^j \alpha_j(\theta) \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2}$$

holds for all $p \in \mathcal{P}_n$ and $\sum_{j=0}^{n} |\alpha_j(\theta)| \leq n$. More precisely, we have the following explicit expressions for the numbers $\alpha_0(\theta)$, $\alpha_1(\theta)$, ..., $\alpha_n(\theta)$:

$$\alpha_0(\theta) = \frac{1}{2n} \frac{1 - \cos n\theta}{1 - \cos\theta}; \quad \alpha_n(\theta) = \frac{(-1)^{n-1}}{2n} \frac{1 - (-1)^n \cos n\theta}{1 + \cos\theta}$$

and

$$\alpha_j(\theta) = \frac{(-1)^j - \cos n\theta}{n\left(\cos\frac{j\pi}{n} - \cos\theta\right)}, \qquad 1 \leq j \leq n-1.$$

On the other hand, the proof of Theorem B is based on the next representation formula which amounts to the particular case $\theta = 0$ in the representation formula 1.

Representation Formula 2. For all $p \in \mathcal{P}_n$ and $\theta \in \mathbb{R}$,

$$e^{\mathrm{i}\theta} p'(e^{\mathrm{i}\theta}) = \beta_0\, p(e^{\mathrm{i}\theta}) - \sum_{j \in J_n,\, j \geq 1} \beta_j \frac{p(e^{\mathrm{i}(\theta + j\pi/n)}) + p(e^{\mathrm{i}(\theta - j\pi/n)})}{2}$$

where $\beta_0 = n/2$, $\beta_j = (n \sin^2(j\pi/2n))^{-1}$, $j \in J_n$, $1 \leq j < n$, $\beta_n = \frac{1+(-1)^{n-1}}{4n}$, and $\sum_{j \in J_n,\, j \geq 1} \beta_j = n/2$.

Although the representation formula 2 follows easily from the representation formula 1, it is an interesting result by its own. It implies for example that

$$\left|e^{\mathrm{i}\theta} p'(e^{\mathrm{i}\theta}) - \frac{n}{2} p(e^{\mathrm{i}\theta})\right| \leq \frac{n}{2} \max_{j \in J_n,\, j \geq 1} \left|\frac{p(e^{\mathrm{i}(\theta + j\pi/n)}) + p(e^{\mathrm{i}(\theta - j\pi/n)})}{2}\right|,$$

($p \in \mathcal{P}_n$, $\theta \in \mathbb{R}$). This is clearly a Duffin-Schaeffer type extension of the following classical result:

$$\left|zp'(z) - \frac{n}{2} p(z)\right| \leq \frac{n}{2} \|p\|_{\mathbb{D}} \quad (p \in \mathcal{P}_n,\ z \in \mathbb{D}).$$

The representation formula 2 can be used also to obtain *a refinement of Bernstein trigonometric inequality* in the form

$$|t'(\theta)| \leq n \max_{1 \leq k \leq n} \left|\frac{t(\theta + (2k-1)\pi/(2n)) - t(\theta - (2k-1)\pi/(2n))}{2}\right|$$

for $\theta \in \mathbb{R}$ and any trigonometric polynomial t of degree $\leq n$ with complex coefficients. It is easily seen that $2n$ is the minimal number of functional values for which such pointwise estimate for the first derivative of a trigonometric polynomial t of degree $\le n$ is possible (see [6, Theorem 4.1]). For any such t, we define an algebraic polynomial $p_t \in \mathcal{P}_{2n}$ by $p_t(e^{i\theta}) := e^{in\theta}t(\theta)$, $\theta \in \mathbb{R}$. Then simple calculations show that $t'(\theta) = ie^{-in\theta}[e^{i\theta}p_t'(e^{i\theta}) - np_t(e^{i\theta})]$. Applying representation formula 2 to $p_t \in \mathcal{P}_{2n}$ yields

$$t'(\theta) = \sum_{k=1}^{n} \lambda_{n,k} \frac{t(\theta + (2k-1)\pi/(2n)) - t(\theta - (2k-1)\pi/(2n))}{2}, \quad \theta \in \mathbb{R},$$

where $\lambda_{n,k} = (-1)^{k-1}[2n \sin^2((2k-1)\pi/(4n))]^{-1}$, $1 \le k \le n$ with $\sum_{k=1}^{n} |\lambda_{n,k}| = n$. This is *a variant of M. Riesz interpolation formula* that implies the above *refinement of Bernstein trigonometric inequality.*

We present a complete characterization of the equality cases in (4) and (5). The following quadrature formula is useful in studying polynomial inequalities: Let $\mathcal{T}_n$ denote the linear space of all complex trigonometric polynomials of degree at most n, $n \in \mathbb{N}$. The quadrature formula (we mention [6, Theorem 2.1] as a ready reference)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} t(\theta)\, d\theta = \frac{1}{m} \sum_{j=0}^{m-1} t\left(\frac{2j\pi}{m} + \gamma\right) \quad (\gamma \in \mathbb{R}) \tag{7}$$

holds for all $t \in \mathcal{T}_{m-1}$. The quadrature (7) is the unique, up to a real translation γ of the nodes, quadrature formula based on m nodes which is exact in $\mathcal{T}_{m-1}$, i.e., a quadrature formula with trigonometric degree of precision $m-1$. There is no quadrature formula with m nodes and having a trigonometric degree of precision greater than $m-1$.

The equality cases in Theorem B. Let a polynomial $p_{\theta_0} \in \mathcal{P}_n$ be extremal for (5) at a fixed number $\theta = \theta_0 \in \mathbb{R}$. Then, for an arbitrarily chosen $\theta_1 \in \mathbb{R}$, the polynomial $p_{\theta_1}(z) := p_{\theta_0}\left(e^{i(\theta_0 - \theta_1)}z\right)$ $(z = e^{i\theta})$ is extremal for (5) at $\theta = \theta_1$.

Let $E_{\theta_0,n}$ denote the class of all polynomials from $\mathcal{P}_n$ extremal for (5) at θ_0. Then

$$E_{\theta_0,n} := \left\{ p\left(e^{-i\theta_0} z\right) \, : \, p \in E_{0,n} \right\}.$$

Hence, in order to determine all extremal polynomials in Theorem B, it is sufficient to describe the class $E_{0,n}$ of all polynomials that are extremal for (5) at $\theta = 0$.

Now, suppose that $p \in \mathcal{P}_n$, $p(z) := \sum_{k=0}^{n} a_k z^k$ is extremal for (5) at $\theta = 0$, i.e., $p \in E_{0,n}$. Let $h_n(z)$ be the Lagrange interpolating polynomial of degree at most n which is uniquely determined by the interpolation conditions

$$h_n(1) = 2n\, a_0,\ h_n(\cos(l\pi/n)) = n\, a_l,\ (1 \le l \le n-1),\ h_n(-1) = 2n\, a_n.$$

Then, $r_n(\theta) := h_n(\cos\theta)$ is the unique *even* trigonometric polynomial of degree at most n which satisfies the interpolation conditions

$$r_n(0) = 2n\, a_0,\ r_n(l\pi/n) = n\, a_l\ (1 \le l \le n-1),\ r_n(\pi) = 2n\, a_n\,.$$

Let

$$M := \max_{j \in J_n} \left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right|.$$

The representation formula 2 implies that equality in (5) holds for $\theta = 0$ and the polynomial $p \in \mathcal{P}_n$ if and only if for some $\gamma \in \mathbb{R}$

$$(-1)^j \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} = Me^{\mathrm{i}\gamma}, \qquad j \in J_n. \tag{8}$$

The linear system (8) whose unknowns are the coefficients of the extremal polynomial p is in general greatly undetermined because of the small cardinality of J_n.

By using the interpolating trigonometric polynomial r_n, the linear system (8) can be represented in the following equivalent form

$$\begin{cases} \frac{1}{2n} r_n(0) + \frac{1}{n}\sum_{l=1}^{n-1} r_n(l\pi/n)\cos(jl\pi/n) + \frac{1}{2n} r_n(\pi)\cos(jn\pi/n) \\ = -Me^{\mathrm{i}\gamma} \quad (j \text{ odd}, \ j \le n) \\ \frac{1}{2n} r_n(0) + \frac{1}{n}\sum_{l=1}^{n-1} r_n(l\pi/n) + \frac{1}{2n} r_n(\pi) = Me^{\mathrm{i}\gamma} \\ M > 0 \text{ and } \gamma \in \mathbb{R}. \end{cases} \tag{9}$$

Let n be even. Define $E^e_{0,n} := E_{0,n}$. Then $J_n = \{0\} \cup \{j = 1, 3, \ldots, n-1\}$. By using the quadrature (7) with $m = 2n$, the system (9) is equivalent to the following integral system:

$$\begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\cos j\theta \, d\theta = -Me^{\mathrm{i}\gamma}, \qquad j = 1, 3, \ldots, n-1 \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\, d\theta = Me^{\mathrm{i}\gamma}. \end{cases}$$

Hence, the interpolating trigonometric polynomial r_n must have the form

$$r_n(\theta) = Me^{\mathrm{i}\gamma}\left(1 - 2\sum_{l=0}^{(n-2)/2} \cos((2l+1)\theta)\right) + \sum_{k=1}^{n/2} b_{2k}\cos(2k\theta) \tag{10}$$

where $b_{2k} \in \mathbb{C}$.

Let us denote by Ω^e_n the class of all *even* trigonometric polynomials $r_n(\theta)$ of the form (10), where the parameters $M \ge 0$, γ real, b_{2k} complex, $k = 0, 1, \ldots, n/2$, are arbitrary. We describe the class $E^e_{0,n}$ of all polynomials, extremal for (5) at $\theta = 0$, n even, through Ω^e_n. The following holds:

$$E^e_{0,n} \equiv \left\{ p \in \mathcal{P}_n : p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n}\sum_{l=1}^{n-1} r_n(l\pi/n)\, z^l + \frac{1}{2n} r_n(\pi) z^n, \ r_n \in \Omega^e_n, \ n \text{ even} \right\}.$$

In view of this, for n even, the class $E^e_{0,n}$ *of all polynomials that are extremal for (5) at* $\theta = 0$ *is completely determined by the trigonometric polynomial class* Ω^e_n *via a simple interpolation procedure:*

Let $a_0, \ldots, a_n$ *be the coefficients of an extremal polynomial* $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *from* $E^e_{0,n}$. *Then, the* $(n+1)$ *numbers in the second row of the table*

0	π/n	$2\pi/n$	$\ldots$	$(n-2)\pi/n$	$n-1)\pi/n$	$n\pi/n$
$(2n)\, a_0$	$n\, a_1$	$n\, a_2$	$\ldots$	$n\, a_{n-2}$	$n\, a_{n-1}$	$(2n)\, a_n$

are interpolation functional values at the interpolation nodes given in the first row for the even trigonometric polynomial $r_n \in \Omega^e_n$. *The trigonometric polynomial* $r_n \in \Omega^e_n$ *is uniquely determined by the above* $(n+1)$ *interpolation conditions.*

Conversely, let $r_n \in \Omega^e_n$ *with arbitrary* $M \geq 0$, γ *real, and complex* b_{2k}, $k = 1, \ldots, n/2$. *Then*

$$a_0 = \frac{r_n(0)}{2n},\ a_1 = \frac{r_n(\pi/n)}{n},\ \ldots,\ a_{n-1} = \frac{r_n((n-1)\pi/n)}{n},\ a_n = \frac{r_n(\pi)}{2n}$$

are the coefficients of an extremal polynomial $p(z) := a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ *from the class* $E^e_{0,n}$. *In other words,* $E_{0,n}$ *is in one-to-one correspondence to* Ω^e_n *through* $(n+1)$ *interpolation conditions at the equally spaced points* $k\pi/n$, $k = 0, 1, \ldots, n$.

Example 1. Let $n = 2$. Then, the class Ω^e_2 consists of only one even trigonometric polynomial $r_2(\theta) = Me^{i\gamma}(1 - 2\cos\theta) + b_2\cos(2\theta)$ and $r_2(0) = b_2 - Me^{i\gamma}$, $r_2(\pi/2) = Me^{i\gamma} - b_2$, $r_2(\pi) = 3Me^{i\gamma} + b_2$. Following our description of the extremal polynomial set $E^e_{0,n}$ we conclude that the class $E^e_{0,2}$ consists of the three-parametric (M, b_2, γ) set of polynomials

$$p_{M,b_2,\gamma}(z) := \left(3Me^{i\gamma} + b_2\right)\frac{z^2}{4} + \left(Me^{i\gamma} - b_2\right)\frac{z}{2} + \left(-Me^{i\gamma}/4 + b_2/4\right),$$

where $M \geq 0$, γ real, b_2 complex, are arbitrary.

Let $n = 4$. Then, the class Ω^e_4 consists of the even trigonometric polynomials $r_4(\theta) = Me^{i\gamma}(1 - 2\cos\theta - 2\cos(3\theta)) + b_2\cos(2\theta) + b_4\cos(4\theta)$ and $r_4(0) = b_2 + b_4 - 3Me^{i\gamma}$, $r_4(\pi/4) = Me^{i\gamma} - b_4$, $r_2(\pi/2) = Me^{i\gamma} - b_2 + b_4$, $r_4(3\pi/4) = Me^{i\gamma} - b_4$, $r_4(\pi) = 5Me^{i\gamma} + b_2 + b_4$. Following our description of the extremal polynomial set $E^e_{0,n}$ we conclude that the class $E^e_{0,4}$ consists of the four-parametric (M, b_2, b_4, γ) set of polynomials

$$\begin{aligned} p_{M,b_2,b_4,\gamma}(z) \;:=\; & \left(5Me^{i\gamma} + b_2 + b_4\right)\frac{z^4}{8} + \left(Me^{i\gamma} - b_4\right)\frac{z^3}{4} \\ & + \left(Me^{i\gamma} - b_2 + b_4\right)\frac{z^2}{4} + \left(Me^{i\gamma} - b_4\right)\frac{z}{4} + \frac{b_2 + b_4 - 3Me^{i\gamma}}{8} \end{aligned}$$

where $M \geq 0$, γ real, b_2, b_4 complex, are arbitrary.

Example 2. By Theorem B

$$\left|p'\left(e^{\mathrm{i}k\pi/n}\right)\right| \le n \max_{j\in J_n}\left|p\left(e^{\mathrm{i}((k\pm j)\pi/n)}\right)\right| \quad (k\in\mathbb{Z}).$$

We take for simplicity $k=0$. Let p^* be extremal for the above inequality when $k=0$. By the representation formula 2, the polynomial $p^*\in\mathcal{P}_n$ must satisfy $p^*(e^{\mathrm{i}j\pi/n}) = p^*(e^{-\mathrm{i}j\pi/n}) = -Me^{\mathrm{i}\gamma}$ $(j\in J_n,\ j\ge 1)$, $p^*(1)=Me^{\mathrm{i}\gamma}$ for some $M\ge 0$ and $\gamma\in\mathbb{R}$. Taking into account that the cardinality of $\{e^{\pm \mathrm{i}j\pi/n},\ j\in J_n\}$ is $(n+1)$, we conclude that the unique polynomial of degree at most n which satisfies the above $(n+1)$ interpolation conditions is $p^*(z)=Me^{\mathrm{i}\gamma}z^n$, i.e., the only equality cases in the above inequality are constant multiples of z^n. It is easily seen that the same holds for arbitrary $k\in\mathbb{Z}$. From here, the only extremals of the inequality

$$\left|p'\left(e^{\mathrm{i}k\pi/n}\right)\right| \le n \max_{0\le j\le 2n-1}\left|p\left(e^{\mathrm{i}j\pi/n}\right)\right| \quad (k\in\mathbb{Z})$$

are constant multiples of z^n. *Now, taking into account that (5) has many extremal polynomials including the constant multiples of z^n which are the only extremals (see [5] for details) of (6), we conclude that (5) is a much more sensitive estimate than (6).* Following our description for the extremal polynomials in (5), the polynomial $p^*\in E^e_{0,n}$ corresponds to the *even* trigonometric polynomial

$$r_n^*(\theta) := Me^{\mathrm{i}\gamma}(-1)^{n-1}\frac{\sin n\theta}{\cos\theta/2}\sin\theta/2 \in \Omega_n^e$$

which satisfies the interpolation conditions $r_n^*(l\pi/n):=0$, $l=0,\dots,n-1$, $r_n^*(\pi):=(2n)Me^{\mathrm{i}\gamma}$ and this agrees with our description of $E^e_{0,n}$.

Remark. From the fact that the monomials z^k, $0\le k\le n-1$ are evidently not extremal for (5), in other words they do not belong to $E^e_{0,n}$, one may conclude that for fixed k, $0\le k\le n-1$, there is no trigonometric polynomial $r_n\in\Omega_n^e$ which satisfies the following interpolation conditions: $r_n(l\pi/n)=\delta_{k,l}$, $0\le l\le n$.

Remark. It deserves to be mentioned that *there are (many for $n\ge 4$) extremal polynomials for (5) of degree strictly less than n.* It is easily seen that $p\in E^e_{0,n}\cap\mathcal{P}_{n-1}$ if and only if the trigonometric polynomial $r_n\in\Omega_n^e$ corresponding to p satisfies $\sum_{k=1}^{n/2} b_{2k} = -Me^{\mathrm{i}\gamma}(n+1)$ $\quad$ (n even). In Example 1 for $n=2$, the above equality is $b_2=-3Me^{\mathrm{i}\gamma}$ and an extremal polynomial in $E^e_{0,2}$ of degree less than 2 is $p(z)=2Me^{\mathrm{i}\gamma}z - Me^{\mathrm{i}\gamma}$. Analogously, for $n=4$ we have $b_2+b_4=-5Me^{\mathrm{i}\gamma}$ and the extremal polynomials in $E^e_{0,4}$ of degree less than 4 are given by the three-parametric (M,b_2,γ) set

$$p(z)=\left(\frac{3Me^{\mathrm{i}\gamma}}{2}+\frac{b_2}{4}\right)z^3-\left(Me^{\mathrm{i}\gamma}+\frac{b_2}{2}\right)z^2+\left(\frac{3Me^{\mathrm{i}\gamma}}{2}+\frac{b_2}{4}\right)z-Me^{\mathrm{i}\gamma}$$

where $M\ge 0$, γ real, and b_2 complex, are arbitrary.

Let n be odd. Define $E_{0,n} := E^o_{0,n}$ and let $r_n(\theta) := t_{n-1}(\theta) + A\cos n\theta \in \mathcal{T}_{n,e}$, where $t_{n-1} \in \mathcal{T}_{n-1,e}$ is of degree at most $n-1$. Then, applying the quadrature (7), we see that the system (9) is equivalent to the following system:

$$\begin{cases} \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\cos k\theta\, d\theta = -M e^{i\gamma} \qquad (k = 1, 3, \dots, n-2) \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} r_n(\theta)\, d\theta = M e^{i\gamma} \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} t_{n-1}(\theta)\cos n\theta\, d\theta + A = -M e^{i\gamma} \Rightarrow A = -M e^{i\gamma} \end{cases}$$

and therefore

$$r_n(\theta) = M e^{i\gamma}\left(1 - 2\sum_{l=0}^{(n-3)/2}\cos\left((2l+1)\theta\right) - \cos n\theta\right) + \sum_{j=1}^{(n-1)/2} b_{2j}\cos(2j\,\theta) \tag{11}$$

with $b_{2j} \in \mathbb{C}$, $M > 0$, and $\gamma \in \mathbb{R}$. Let us denote by Ω^o_n the class of all *even* trigonometric polynomials of the form (11). Then

$$E^o_{0,n} \equiv \left\{ p \in \mathcal{P}_n \;:\; p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n}\sum_{l=1}^{n-1} r_n(l\pi/n)\, z^l + \frac{1}{2n} r_n(\pi)\, z^n,\ r_n \in \Omega^o_n,\ n \text{ odd} \right\}.$$

In view of this and as in the case n even, the extremal set $E^o_{0,n}$ is in one-to-one *interpolation* correspondence with the class of trigonometric polynomials Ω^o_n.

Example 3. Let $n = 3$. Then $E^o_{0,3}$ is the three-parametric (M, b_2, γ) set of polynomials

$$p_{M,b_2,\gamma}(z) := \frac{2Me^{i\gamma} + b_2/2}{3} z^3 + \frac{Me^{i\gamma} - b_2/2}{3} z^2 + \frac{Me^{i\gamma} - b_2/2}{3} z - \frac{Me^{i\gamma} - b_2/2}{3},$$

for arbitrary $M \geq 0$, $\gamma \in \mathbb{R}$ and a complex number b_2.

Remark. It is easily seen that $p \in E^o_{0,n} \cap \mathcal{P}_{n-1}$ if and only if the unique $r_n \in \Omega^o_n$, which corresponds to p, satisfies $\sum_{j=1}^{(n-1)/2} b_{2j} = -Me^{i\gamma}(n+1)$. In the particular case of Example 3 we have $b_2 = -4Me^{i\gamma}$. Hence, an extremal polynomial in $E_{0,3}$ of degree less than 3 is $p(z) = Me^{i\gamma}\left(z^2 + z - 1\right)$. In the case n odd, the *even* trigonometric polynomial r^*_n from Example 2 belongs to Ω^o_n and this agrees with the fact that $Me^{i\gamma}z^n \in E^o_{0,n}$.

The equality case in Theorem A. First of all, let us mention that we have extremal polynomials in (4) only for $\theta \in \{0, \pi\}$ (mod 2π). In view of the explicit form of (5) and (4), the class of all extremal polynomials in (4) is a sub-set of the class of all extremal polynomials in (5) for $\theta = 0$ (mod 2π) and $\theta = \pi$ (mod 2π). Let $\tilde{E}_{0,n}$ denote the class of all polynomials extremal for (4) in the case $\theta = 0$. Then, $\tilde{E}_{\pi,n} = \left\{ p(-z) : p \in \tilde{E}_{0,n} \right\}$, and all extremal polynomials in Theorem A are given by $\tilde{E}_{0,n} \cup \tilde{E}_{\pi,n}$. Hence, in order to determine the class of all extremal polynomials in Theorem A, it is enough to describe the subclass $\tilde{E}_{0,n}$ of all extremal polynomials in (5) for $\theta = 0$ satisfying the following additional inequalities on the set $\{0, 1, \ldots, n\} \setminus J_n$:

$$\left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right| \leq M \quad (j \notin J_n, 1 \leq j \leq n). \tag{12}$$

Surprisingly, there are also many extremal polynomials for the inequality (4) which amounts to

$$|p'(1)| \leq n \max_{0 \leq j \leq n} \left| \frac{p\left(e^{\mathrm{i}j\pi/n}\right) + p\left(e^{-\mathrm{i}j\pi/n}\right)}{2} \right|,$$

in spite of the fact that these extremals must satisfy not only (8) but also (12).

Let n be even. Let $\tilde{E}^e_{0,n} := \tilde{E}_{0,n}$ and let $r_n \in \Omega^e_n$. Then, applying the quadrature (7) we see that (12) is equivalent to

$$|b_{2k}| \leq 2M, \ k = 1, \ldots, (n-2)/2 \ (n > 2) \quad \text{and} \quad |b_n| \leq M \ (n \geq 2).$$

Let $\tilde{\Omega}^e_n := \{r_n \in \Omega^e_n, \ |b_{2k}| \leq 2M, \ 1 \leq k \leq (n-2)/2, \ |b_n| \leq M\}$. Then we have

$$\tilde{E}^e_{0,n} \equiv \left\{ p \in \mathcal{P}_n \ : \ p(z) = \frac{1}{2n} r_n(0) + \frac{1}{n} \sum_{l=1}^{n-1} r_n(l\pi/n) z^l \right.$$
$$\left. + \frac{1}{2n} r_n(\pi) z^n, \ r_n \in \tilde{\Omega}^e_n, \ n \text{ even} \right\}.$$

Let n be odd. Let $\tilde{E}^o_{0,n} := \tilde{E}_{0,n}$ and let $r_n \in \Omega^o_n$. Then (13) is equivalent to $|b_{2j}| \leq 2M, \ j = 1, \ldots, (n-1)/2$. In view of this we define

$$\tilde{\Omega}^o_n := \{r_n \in \Omega^o_n, \ |b_{2k}| \leq 2M, \ 1 \leq k \leq (n-1)/2\}$$

to conclude that

$$\tilde{E}^o_{0,n} \equiv \left\{ p \in \mathcal{P}_n \ : \ p(z) = \frac{r_n(0)}{2n} + \sum_{l=1}^{n-1} \frac{r_n(l\pi/n)}{n} z^l \right.$$
$$\left. + \frac{r_n(\pi)}{2n} z^n, \ r_n \in \tilde{\Omega}^o_n, \ n \text{ odd} \right\}.$$

Remark. We point out that *all extremal polynomials $p \not\equiv 0$ in Theorem A are of exact degree n.* Let n be even and let us assume to the contrary. If $p \in \mathcal{P}_{n-1} \cap \tilde{E}_{0,n}$ is extremal, then the corresponding $r_n \in \tilde{\Omega}_n^e$ must satisfy $r_n(\pi) = 0$ which amounts to $\sum_{k=1}^{n/2} b_{2k} = -Me^{i\gamma}(n+1)$ together with $|b_{2k}| \le 2M$, $k = 1, \ldots, (n-2)/2$, $|b_n| \le M$. Obviously, this is impossible for $M > 0$ and $n \ge 2$. Analogously, the same conclusion holds for $n \ge 3$ odd (the case $n = 1$ is a trivial one).

An extension of Theorem A. Let $p \in \mathcal{P}_n$ and define a sequence of $\{p_k\} \in \mathcal{P}_n$ by $p_0 := p$ and $p_{k+1}(z) = zp_k'(z)$, $k \ge 0$. The following generalization of Theorem A was obtained in [3]:

Theorem C. *Let $p \in \mathcal{P}_n$, $k \ge 0$, and $\theta \in \mathbf{R}$. Then*

$$\left| \frac{p_k\left(e^{i\theta}\right) - p_k\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} \right| \le n^{1+k} \max_{0 \le j \le n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right| \tag{13}$$

where the inequality is strict for each $\theta \notin \{0, \pi\}$ (mod 2π) and for any polynomial $p \not\equiv 0$.

Clearly, Theorem C amounts to Theorem A for $k = 0$. We now discuss cases of equality in (13) for $k \ge 1$. It is readily seen that for $k = 1$, (13) is equivalent to the Duffin and Schaeffer result and in particular [7] equality holds in (13) for $k = 1$ if and only if $\theta = 0, \pi$ (mod 2π) and $p(z) = Kz^n$, $K \in \mathbf{C}$.

It has been proved in [3] that for $k \ge 0$, there exist real numbers $\beta_{l,k}(\theta)$, $0 \le l \le n$, such that for all $p \in \mathcal{P}_n$

$$\frac{p_k\left(e^{i\theta}\right) - p_k\left(e^{-i\theta}\right)}{e^{i\theta} - e^{-i\theta}} = \sum_{l=0}^{n} \beta_{l,k}(\theta) \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2}$$

with $\sum_{l=0}^{n} |\beta_{l,k}(\theta)| < n^{1+k}$ for $\theta \notin \{0, \pi\}$ (mod 2π). Moreover, the following representation formula (see [3] for details) holds:

$$\begin{aligned} p_{k+1}'(1) &= \sum_{j=0}^{n} (-1)^j \alpha_j(0) \left\{ \sum_{l=0}^{n} \frac{\beta_{l,k}(0)}{2} \left[\frac{p\left(e^{i(j+l)\pi/n}\right) + p\left(e^{-i(j+l)\pi/n}\right)}{2} \right.\right. \\ &\qquad \left.\left. + \frac{p\left(e^{i(j-l)\pi/n}\right) + p\left(e^{-i(j-l)\pi/n}\right)}{2} \right] \right\}. \end{aligned} \tag{14}$$

Let us assume that for some $p \in \mathcal{P}_n$, $p_{k+1}{}'(1) = n^{2+k}M$, where

$M := \max_{0 \le j \le n} \left| \frac{p\left(e^{ij\pi/n}\right) + p\left(e^{-ij\pi/n}\right)}{2} \right|$.

Then, by (14)

$$n^{2+k}M = \left| \sum_{j=0}^{n} (-1)^j \alpha_j(0) \left\{ \sum_{l=0}^{n} \frac{\beta_{l,k}(0)}{2} \left[\frac{p\left(e^{i(j+l)\pi/n}\right) + p\left(e^{-i(j+l)\pi/n}\right)}{2} \right.\right.\right.$$

$$+\frac{p\left(e^{\mathrm{i}(j-l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j-l)\pi/n}\right)}{2}\Bigg]\Bigg\}\Bigg|$$

$$\le \sum_{j=0}^{n}|\alpha_j(0)|\left\{\frac{1}{2}\sum_{l=0}^{n}|\beta_{l,k}(0)|\left|\frac{p\left(e^{\mathrm{i}(j+l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j+l)\pi/n}\right)}{2}\right|\right.$$

$$\left.+\frac{1}{2}\sum_{l=0}^{n}|\beta_{l,k}(0)|\left|\frac{p\left(e^{\mathrm{i}(j-l)\pi/n}\right)+p\left(e^{-\mathrm{i}(j-l)\pi/n}\right)}{2}\right|\right\}$$

$$\le Mn^{2+k}$$

and equality must hold everywhere above. In particular, the modulus of

$$\sum_{l=0}^{n}\beta_{l,k}(0)\frac{p\left(e^{\mathrm{i}l\pi/n}\right)+p\left(e^{-\mathrm{i}l\pi/n}\right)}{2}$$

must be equal to $M\,n^{1+k}$, i.e., $|p_k'(1)| = M\,n^{1+k}$. This shows by induction on $k \ge 1$ that equality can hold in (13) for $\theta = 0$ only when $p(z) = Kz^n$ with $K \in \mathbf{C}$. The case $\theta = \pi$ can be treated in a similar way. *Hence, for $k = 0$, the inequality (13), being equivalent to (4), has many extremal polynomials. However, for $k \ge 1$, the only extremal polynomials in (13) are constant multiples of z^n.*

REFERENCES

1. Bojanov, B. D. Markov-type inequalities for polynomials and splines. In: *Approximation theory X, Innov. Appl. Math*, eds. Ch. Chui, L. Schumaker and J. Stöckler, Vanderbilt Univ. Press, 2002, 31–90.
2. Borwein, P., T. Erdelyi. Polynomials and polynomial inequalities. Springer-Verlag, New York, 1995.
3. Dryanov, D., R. Fournier, St. Ruscheweyh. Some Extensions of the Markov Inequality for Polynomials. *Rocky Mount. J. Math.*, accepted for publication.
4. Dryanov, D., R. Fournier. A Note on Bernstein and Markov Type Inequalities. *Analysis*, **25**, 2005, 73–77.
5. Dryanov, D., R. Fournier. On a discrete variant of Bernstein's polynomial inequality. *J. Approx. Theory*, **136**, 2005, 84–90.
6. Dryanov, D. Quadrature formulae with free nodes for periodic functions. *Numer. Math.*, **67**, 1994, 441–464.
7. Duffin, R. J., A. C. Schaeffer. A refinement of an inequality of the brothers Markoff. *Trans. Amer. Math. Soc.*, **50**, 1941, 517-528.
8. Frappier, C., Q.I. Rahman, St. Ruscheweyh. New inequalities for polynomials. *Trans. Amer. Math. Soc.*, **288**, 1985, 69-99.
9. Milovanovic, G. V.; D. S. Mitrinovic, Th. M. Rassias. Topics in Polynomials: Extremal Problems, Inequalities, Zeros. World Sci. Publ., Singapore, 1994.

10. Rahman, Q. I., G. Schmeisser. Analytic Theory of Polynomials. Oxford University Press, Oxford, 2002.

Received on September 30, 2006

D. Dryanov
Department of Mathematics and Statistics
Concordia University
Montreal, H3G 1M8
E-mail: ddryanov@alcor.concordia.ca

R. Fournier
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, H3C 3J7
E-mail: fournier@dms.umontreal.ca