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ERROR ESTIMATES OF HIGH-ORDER DIFFERENCE SCHEMES FOR ELLIPTIC EQUATIONS WITH INTERSECTING INTERFACES

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In the work [1] high-order difference schemes (numerical experiments show second and fourth order of convergence) were derived, but with 1-st and 3-d order local truncation error, respectively, compact difference schemes for elliptic equations with intersecting interfaces. Here, for these difference schemes, we provide error estimates in discrete Sobolev norms.

Keywords: High-order finite difference schemes, Compact stencils, Elliptic problems, Intersected interfaces, Error estimates, Discrete Sobolev norms

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1. INTRODUCTION

Many important physical and industrial applications involve material interface problems, described by differential equations with discontinuous coefficients and concentrated sources. A good example of a problem of this type is the two-dimensional elliptic equation with discontinuous coefficients and Dirac-delta right side.

When the interface is smooth, the singularity is not severe , with smooth solutions inside each region. Various methods have been developed for this case and they work well, at least for moderately large contrast [4], [8],[13], [14], [17]. A method that is the simplest conceptually but nontrivial in implementation is aligning the grid with discontinuity. In [4], it is proved that if the boundary is at least C^2 this FEM converges nearly the same optimal way, in both the L^2 and energy norm, as the problems without interfaces.

An efficient method that uses regular grids is the Immersed Interface Method (IIM), [10], [13], [14]. The essence of the IIM includes using uniform or adaptive Cartesian grids and introducing non-zero correction forms in the starting difference approximations near the interfaces. The role of the jump (interface) relations is very important. In fact the idea for using the jump relations was first proposed for an elliptic problem with line interface in [16]. The construction of our difference schemes also relies on this idea.

The standard strategy for generating higher-order difference schemes is to expand the stencil (see for example [15], [16]). It was applied very successfully recently to elliptic interface problems [3]. A such an approach has the obvious disadvantages of creating large matrix bandwidths, complicating the numerical treatment near the boundaries.

Our method of discretization is based on aligning the grid with discontinuities. The construction uses the differential equation and the jump (interface) relations as additional identities which can be differentiated to eliminate lower order local truncation errors.

Our schemes are compact, i.e. they use minimal stencils: 5-points for the second-order scheme and 9-points for the fourth-order one. In order the maximum principle to be satisfied for the 4-th order difference scheme the mesh steps must satisfied very restrictive inequalities. Here we apply the energy method to obtain error estimates for the difference schemes.

The rest of this paper is organized as follows. In Subsection 1.1 the boundary value problem is stated; the notation used is introduced in Subsection 1.2, a second-order difference scheme on five points stencil, while a fourth-order ones derived in [1] are presented. In Section 3 error estimates for the schemes are obtained.

It is noted that there also exist some analytical and numerical studies about problems on intersecting interfaces [5], [7], [11]. Also, almost second order difference scheme for singularly perturbed problem with a line interface parallel to the axis Oy is constructed and studied for uniform convergence in [2]. Convergence of finite difference schemes for elliptic problems with curvilinear interfaces intersected the domain boundary is studied in [6].

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1.1. BOUNDARY VALUE PROBLEM

The more difficult interface problem is with singularities that arise due a non-smooth or an intersecting interface. As a typical example we consider the elliptic equation.

$$Lu := -\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + r(x,y)u = F(x,y), \qquad (1.1)$$
$$(x,y) \in \Omega \equiv (0,X) \times (0,Y),$$

where

$$F(x,y) = f(x,y) - \delta(x-\xi, y)K_x(y) - \delta(x, y-\eta)K_y(x), \quad (\xi,\eta) \in \Omega.$$

We assume that the functions p, q, r, f, K_x, K_y are piecewise continuous (with possible discontinuity on the segments $\Gamma_x \equiv \{(x,y) : x = \xi,\ 0 < y < 1\}$, $\Gamma_y \equiv \{(x,y) : 0 < x < 1,\ y = \eta\}$ for p, q, r, f while in $y = \eta$ for K_x and in $x = \xi$ for K_y) and

$$0 < c_0 \le p(x,y), q(x,y) \le C_0, \quad 0 \le r_0 \le r(x,y) \le C_0 \ \text{ on } \bar{\Omega}. \qquad (1.2)$$

We shall solve (1.1) for continuous solution subjected with Dirichlet boundary condition

$$u|_{\partial\Omega} = \varphi(x,y). \qquad (1.3)$$

Then the equation (1.1) is equivalent to the: equation

$$Lu := f(x,y), \quad (x,y) \in \Omega \backslash \Gamma = \bigcup_{s=1}^{4} \Omega_s, \quad \Gamma = \Gamma_x \cup \Gamma_y, \text{ see Fig.1} \qquad (1.4)$$

and the interface relations

$$[u]_{\Gamma_x} \equiv u(\xi+, y) - u(\xi-, y) = 0, \ y \in (0,1), \qquad (1.5)$$
$$[u]_{\Gamma_y} \equiv u(x, \eta+) - u(x, \eta-) = 0, \ x \in (0,1), \qquad (1.6)$$

$$\left[p\frac{\partial u}{\partial x}\right]_{\Gamma_x \backslash \{\eta\}} = K_x(y), \ y \in (0,1)\backslash\{\eta\}, \qquad (1.7)$$
$$\left[q\frac{\partial u}{\partial y}\right]_{\Gamma_y \backslash \{\xi\}} = K_y(x), \ x \in (0,1)\backslash\{\xi\}, \qquad (1.8)$$

$$\left\{\left[p\frac{\partial u}{\partial x}\right]_{\xi}\right\}_{\eta} + \left\{\left[q\frac{\partial u}{\partial y}\right]_{\eta}\right\}_{\xi} = \{K_x\}_{\eta} + \{K_y\}_{\xi}, \qquad (1.9)$$

where, for example, $\{K_x\}_{\eta} = \frac{1}{2}\left(K_x(\eta-) + K_x(\eta+)\right)$.

Similar interface relations can be derived if a finite number of line interfaces $x = \xi_i,\ i = 1, ..., I$ and $y = \eta_j,\ j = 1, ..., J$ are assumed (see the numerical results in Table 6 and Figure 5, 6 for $I = J = 2$ in [1]).

Let m be a nonnegative integer and $\alpha \in (0,1)$. The standard notation $C^m(\bar{\Omega})$ is known for the space of functions where derivatives up to order m are continuous on $\bar{\Omega}$ with maximum norm ([9], [12]). The notation $C^{m+\alpha}(\bar{\Omega})$ is used for the space of

Holder continuous functions with corresponding norm [9],[12]. Finally, by $C^{m+\alpha}(\Omega)$ we denote the space of functions, which belong to $C^{m+\alpha}(\bar{\Omega}')$, where $\Omega' \subset \Omega$.

We also will use the functional space

$$\mathcal{H}_{\Omega}^{s+\alpha} = \bigcap_{k=1}^{4} C^{s+\alpha}(\overline{\Omega}_k),\ s \in N. \tag{1.10}$$

The regularity of the solutions (i.e. the belonging of the solution to appropriate Holder space) depending on the input data smoothness and various compatibility conditions satisfied by the input data. Interface problems with smooth interface curves that do not intersect the domain boundary have been widely investigated in the literature, [4], [12], [14] and the references there. But for problems of type (1.1)-(1.3)(or (1.4)-(1.9)) such results are not known. We will further assume for (1.1)-(1.3) that the solutions have the necessary smoothness (see Propositions 2.1, 3.1).

1.2. GRIDS AND GRID FUNCTIONS

Let introduce the non-uniform mesh $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_k,\quad \omega = \bar{\omega} \cap \Omega,\quad \gamma = \bar{\omega}\backslash\omega$, where

$\bar{\omega}_h = \{x_0 = 0,\ x_i = x_{i-1} + h_i,\quad i = 1, \ldots, N_1 - 1,\ x_{N_1} = x_{N_1 - 1} + h = \xi,\ x_{N_1+1} = \xi + h,\ x_i = x_{i-1} + h_i,\ i = N_1 + 2, \ldots, N,\ x_N = X\}$,

$\bar{\omega}_k = \{y_0 = 0,\ y_j = y_{j-1} + k_j,\ j = 1, \ldots, M_1 - 1,\ y_{M_1} = y_{M_1 - 1} + h = \eta,\ y_{M_1+1} = \eta + h,\ y_j = y_{j-1} + k_j,\ \ j = M_1 + 2, \ldots, M,\ y_M = Y\}$.

A such mesh is designed on Fig.2.

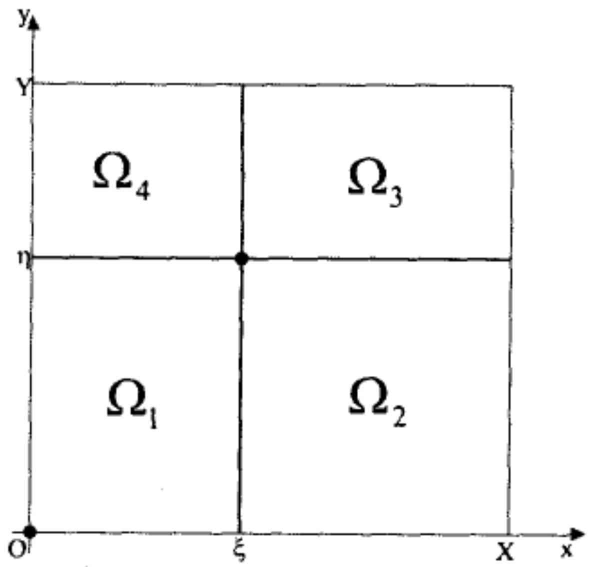

Fig.1. The Domain Ω

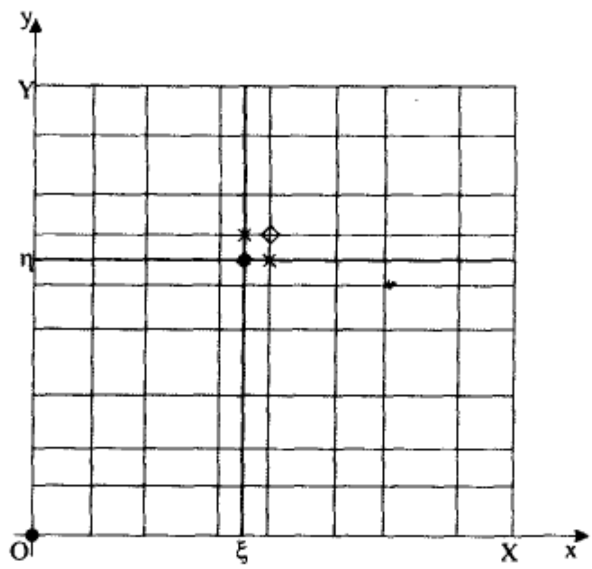

Fig.2. Non-uniform mesh

The finite-difference operators are defined in standard manner by $U(x, y)$:

$$U_{\bar{x}} = U_{\bar{x},i} = (U(x_i, y_j) - U(x_{i-1}, y_j))/h_i,\ U_x = U_{x,i} = U_{\bar{x},i+1},$$
$$U_{\bar{y}} = U_{\bar{y},j} = (U(x_i, y_j) - U(x_i, y_{j-1}))/k_j,\ U_y = U_{y,j} = U_{\bar{y},j+1},$$

$$U_{\hat{x}} = U_{\hat{x},i} = (U(x_{i+1/2}, y_j) - U(x_{i-1/2}, y_j))/\hbar_i,$$

$$\hbar_i = \frac{1}{2}(h_i + h_{i+1}),\ \hbar_0 = h_1/2,\ \hbar_N = h_N/2,$$

$$U_{\hat{y}} = U_{\hat{y},j} = (U(x_i, y_{j+1/2}) - U(x_i, y_{j-1/2}))/\bar{k}_j,$$

$$\bar{k}_j = \frac{1}{2}(k_j + k_{j+1}),\ \bar{k}_0 = k_1/2,\ \bar{k}_M = k_M/2,$$

$$U_{\bar{x}\hat{x}} = U_{\bar{x}\hat{x},i} = \frac{1}{\hbar_i}(U_{x,i} - U_{\bar{x},i}),\ U_{\bar{y}\hat{y}} = U_{\bar{y}\hat{y},j} = \frac{1}{\bar{k}_j}(U_{y,j} - U_{\bar{y},j}).$$

$$U_{\overset{\circ}{x}} = U_{\overset{\circ}{x},i} = \frac{h_{i+1}U_{\bar{x},i} + h_i U_{x,i}}{h_i + h_{i+1}},\ U_{\overset{\circ}{y}} = U_{\overset{\circ}{y},j} = \frac{k_{j+1}U_{\bar{y},j} + k_j U_{y,j}}{k_j + k_{j+1}},$$

$$U_{\bar{x}\hat{x}} = U_{\bar{x}\hat{x},i} = \frac{1}{\hbar_i}(U_{x,i} - U_{\bar{x},i}),\ U_{\bar{y}\hat{y}} = U_{\bar{y}\hat{y},j} = \frac{1}{\bar{k}_j}(U_{y,j} - U_{\bar{y},j}).$$

$$U_{\overset{\circ}{x}\overset{\circ}{y}} = \left(U_{\overset{\circ}{x}}\right)_{\overset{\circ}{y}} = \left(U_{\overset{\circ}{y}}\right)_{\overset{\circ}{x}}.$$

Here U_{ij} is any discrete function. Note that when it is clear that $u(x, y)$ is a continuous function, we shall sometimes use the notation $u_{ij} := u(x_i, y_j)$, while when it is clear that U_{ij} is a discrete function, we shall sometimes use the notation $U(x_i, y_j) := U_{ij}$.

Let $g(x, y)$ be a piecewise continuous function define in Ω.

$$g_{\breve{x}} = g_{\breve{x}_i}(y) = \frac{h_i g(x_i-, y) + h_{i+1} g(x_i+, y)}{h_i + h_{i+1}},$$

$$g_{\breve{y}} = g_{\breve{y}_j}(x) = \frac{k_j g(x, y_j-) + k_{j+1} g(x, y_j+)}{k_j + k_{j+1}},$$

$$g_{\breve{x}\breve{y}} = g_{\breve{x}_i\breve{y}_j} = (g_{\breve{x}_i})_{\breve{y}_j} = (g_{\breve{y}_j})_{\breve{x}_i}.$$

If $h_i = h_{i+1}$, then $g_{\breve{x}} = \{g\}_{x_i}$. If $k_j = k_{j+1}$, then $g_{\breve{y}} = \{g\}_{y_j}$.

Throughout the paper, C sometimes subscribed, denotes a generic positive constant that is independent of any mesh used.

2. THE DIFFERENCE SCHEMES

In this section we present the finite difference schemes for the problem (1.1)-(1.9) derived in [1].

The following difference scheme is derived in [1]:

$$\begin{aligned}\Lambda U_{ij} &= -(aU_{\bar{x}})_{\hat{x}_i\breve{y}_j} - (bU_{\bar{y}})_{\hat{y}_j\breve{x}_i} + c_{\breve{x}_i\breve{y}_j}U_{ij} \\ &= \varphi_{\breve{x}_i\breve{y}_j} - \frac{1}{\hbar_i}\left(\left[p\frac{\partial u}{\partial x}\right]_{x_i}\right)_{\breve{y}_j} - \frac{1}{\bar{k}_j}\left(\left[q\frac{\partial u}{\partial y}\right]_{y_j}\right)_{\breve{x}_i} \\ &= \begin{cases} \varphi, & x_i \neq \xi,\ y_j \neq \eta; \\ \varphi_{\breve{\xi}} - \frac{1}{h}K_x(y), & x_i = \xi,\ y_j \neq \eta; \\ \varphi_{\breve{\eta}} - \frac{1}{h}K_y(x), & x_i \neq \xi,\ y_j = \eta; \\ \varphi_{\breve{\xi}\breve{\eta}} - \frac{1}{h}\{K_x(y)\}_\eta - \frac{1}{h}\{K_y(x)\}_\xi, & x_i = \xi,\ y_j = \eta. \end{cases}\end{aligned} \tag{2.1}$$

The following assertion was proved in [1].

Proposition 2.1. *Suppose that* $p,\ q \in \mathcal{H}_\Omega^{3+\alpha}$, $r,\ f \in \mathcal{H}_\Omega^{2+\alpha}$, $K_x \in C^\alpha[0,\eta]\bigcup[\eta,1]$, $K_y \in C^\alpha[0,\xi]\bigcup[\xi,1]$, $\alpha \in (0,1)$, $u \in C(\overline{\Omega})\bigcap\mathcal{H}_\Omega^{4+\alpha}$. *Then the truncation error of the scheme (2.1) is of order one.*

2.2. FOURTH ORDER DIFFERENCE SCHEME

In this subsection we consider the problem (1.4)-(1.9) in the case of piecewise constant coefficients

$$p(x,y) = p_s,\ q(x,y) = q_s,\ r(x,y) = r_s,\ (x,y) \in \Omega_s;\ s = 1,2,3,4.$$

Then (1.4) reads as follows:

$$-p_s\frac{\partial^2 u}{\partial x^2} - q_s\frac{\partial^2 u}{\partial y^2} + r_s u = f_s(x,y),\ \ (x,y) \in \Omega_s,\ s = 1,2,3,4, \tag{2.2}$$

Further, for simplicity we shall describe our construction in the case

$$\left[\frac{r}{p}\right]_{\Gamma_x} = \left[\frac{r}{q}\right]_{\Gamma_y} = \left[\frac{q}{p}\right]_{\Gamma_x} = \left[\frac{p}{q}\right]_{\Gamma_y} = 0. \tag{2.3}$$

In this case the following difference scheme is derived in [1].

Case 2.1. *Point of type* $\diamond$

$$\begin{aligned}\Lambda' U &= -(pU_{\bar{x}})_{\hat{x}} - (qU_{\bar{y}})_{\hat{y}} + rU - \frac{1}{12}\left(\left(k^2(pU_{\bar{x}})_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2(qU_{\bar{y}})_{\hat{y}\bar{x}}\right)_{\hat{x}}\right) \\ &- \frac{1}{6}\left([k]_{y_j}(pU_{\bar{x}})_{\hat{x}\overset{\circ}{y}} + [h]_{x_i}(qU_{\bar{y}})_{\hat{y}\overset{\circ}{x}}\right) + \frac{1}{12}\left((h^2rU_{\bar{x}})_{\hat{x}} + (k^2rU_{\bar{y}})_{\hat{y}}\right) \\ &+ \frac{1}{6}r\left([h]_{x_i}U_{\overset{\circ}{x}} + [k]_{y_j}U_{\overset{\circ}{y}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}U_{\overset{\circ}{x}\overset{\circ}{y}}\right) \\ = f &+ \frac{1}{12}\left((h^2f_{\bar{x}})_{\hat{x}\bar{y}} + (k^2f_{\bar{y}})_{\hat{y}\check{x}}\right) + \frac{1}{6}\left([h]_{x_i}f_{\overset{\circ}{x}} + [k]_{y_j}f_{\overset{\circ}{y}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}f_{\overset{\circ}{x}\overset{\circ}{y}}\right).\end{aligned} \tag{2.4}$$

If $[h]_{x_i} = [k]_{y_j} = 0$ (2.4) reduces to the Samarskii's famous scheme [15]:

$$\begin{aligned}\Lambda' U &= -(pU_{\bar x})_x - (qU_{\bar y})_y + rU - \frac{k^2}{12}(pU_{\bar x})_{x\bar y y} - \frac{h^2}{12}(qU_{\bar y})_{y\bar x x} \\ &+ \frac{h^2}{12}(rU_{\bar x})_x + \frac{k^2}{12}(rU_{\bar y})_y = f + \frac{h^2}{12} f_{\bar x x} + \frac{k^2}{12} f_{\bar y y}.\end{aligned}$$

Case 2.2. *Points $x_i = \xi$, $y_j \neq \eta$ of type $*$.*

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar x})_x - (qU_{\bar y})_{\hat y \breve x} + r_{\breve x} U - \frac{1}{12}\left(k^2 (pU_{\bar x})_{x\bar y}\right)_{\hat y} - \frac{h^2}{12}(qU_{\bar y})_{\hat y \bar x x} \\ &+ \frac{h^2}{12}(rU_{\bar x})_x + \frac{1}{12}(k^2 rU_{\bar y})_{\hat y \breve x} + \frac{[k]_{y_j}}{6}\left(\{r\}_{x_i} U_{\mathring y} - (pU_{\bar x})_{x\mathring y}\right) \qquad (2.5) \\ &= f_{\breve x} + \frac{h^2}{12} f_{\bar x x} + \frac{1}{12}(k^2 f_{\bar y})_{\hat y \breve x} + \frac{[k]_{y_j}}{6} f_{\mathring y \breve x} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i} \\ &- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right) K_x(y_j) - \frac{1}{12h}\left(k^2 K_x(y)\right)_{\bar y \hat y} + \frac{h}{12}\left\{\frac{q}{p}\right\}_{x_i} (K_x(y))_{\bar y \hat y} \\ &- \frac{[k]_{y_j}}{18}\left(\frac{3}{h}(K_x(y))_{\mathring y} + h\left\{\frac{r}{p}\right\}_{x_i} K'_x(y_j) - h\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{x_i,j}\right).\end{aligned}$$

If $[k]_{y_j} = 0$, $k_j = k_{j+1} = k$, i.e. the mesh in y is uniform, we have

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar x})_x - (qU_{\bar y})_{y\breve x} + r_{\breve x} U - \frac{k^2}{12}(pU_{\bar x})_{x\bar y y} - \frac{h^2}{12}(qU_{\bar y})_{y\bar x x} \\ &+ \frac{h^2}{12}(rU_{\bar x})_x + \frac{k^2}{12}(rU_{\bar y})_{y\breve x} = f_{\breve x} + \frac{h^2}{12} f_{\bar x x} + \frac{k^2}{12} f_{\bar y y \breve x} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i} \qquad (2.6) \\ &- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right) K_x(y_j) + \frac{1}{12}\left(h\left\{\frac{q}{p}\right\}_{x_i} - \frac{k^2}{h}\right)(K_x(y))_{\bar y \hat y_j}.\end{aligned}$$

Similarly if $y_j = \eta$, $x_i \neq \xi$ we obtain

$$\begin{aligned}\Lambda' U_{ij} &= -(pU_{\bar x})_{x\breve y} - (qU_{\bar y})_y + r_{\breve y} U - \frac{h^2}{12}(pU_{\bar x})_{\hat x \bar y y} - \frac{1}{12}\left(h^2 (qU_{\bar y})_{y\bar x}\right)_{\hat x} \\ &+ \frac{1}{12}(h^2 rU_{\bar x})_{\hat x \breve y} + \frac{h^2}{12}(rU_{\bar y})_y + \frac{[h]_{x_i}}{6}\left(\left\{ru_{\mathring x}\right\}_{y_j} - (qu_{\bar y})_{y\mathring x}\right) \qquad (2.7) \\ &= f_{\breve y} + \frac{1}{12}\left(h^2 f_{\bar x}\right)_{\hat x \breve y} + \frac{h^2}{12} f_{\bar y y} + \frac{[h]_{x_i}}{6} f_{\mathring x \breve y} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j,i} \\ &- \left(\frac{h}{12}\left\{\frac{r}{q}\right\}_{y_j} + \frac{1}{h}\right) K_y(x_i) - \frac{1}{12h}\left(h^2 K_y(x)\right)_{\bar x \hat x} + \frac{h}{12}\left\{\frac{p}{q}\right\}_{y_j} (K_y(x))_{\bar x \hat x}\end{aligned}$$

$$- \frac{[h]_{x_i}}{18}\left(\frac{3}{h}\left(K_y(x)\right)_{\mathring{x}} + h\left\{\frac{r}{q}\right\}_{y_j} K'_y(x_i) - h\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{y_j,i}\right).$$

In the particular case $[h]_{x_i} = 0,\ h_i = h_{i+1} = \hbar_i = \hbar$ we obtain

$$\begin{aligned}
\Lambda' U_{ij} &= -(pU_{\bar{x}})_{x\breve{y}} - (qU_{\bar{y}})_y + r_{\breve{y}} U - \frac{h^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{\hbar^2}{12}(qU_{\bar{y}})_{y\bar{x}x} \qquad (2.8)\\
&+ \frac{\hbar^2}{12}(rU_{\bar{x}})_{x\breve{y}} + \frac{h^2}{12}(rU_{\bar{y}})_y = f_{\breve{y}} + \frac{\hbar^2}{12} f_{\bar{x}x\breve{y}} + \frac{h^2}{12} f_{\bar{y}y} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j}\\
&- \left(\frac{h}{12}\left(\frac{r}{q}\right)_{\breve{y}} + \frac{1}{h}\right) K_y(x_i) + \frac{1}{12}\left(h\left(\frac{p}{q}\right)_{\breve{y}} - \frac{\hbar^2}{h}\right)(K_y(x))_{\bar{x}\hat{x}}.
\end{aligned}$$

Case 2.3. *Point of type •*

$$\begin{aligned}
\Lambda' U_{ij} &= -(pU_{\bar{x}})_{x\breve{y}} - (qU_{\bar{y}})_{y\breve{x}} + r_{\breve{x}\breve{y}} U - \frac{h^2}{12}(pU_{\bar{x}})_{x\bar{y}y} - \frac{h^2}{12}(qU_{\bar{y}})_{y\bar{x}x}\\
&+ \frac{h^2}{12}(rU_{\bar{x}})_{x\breve{y}} + \frac{h^2}{12}(rU_{\bar{y}})_{y\breve{x}} = f_{\breve{x}\breve{y}} + \frac{h^2}{12}\left(f_{\bar{x}x\breve{y}} + f_{\bar{y}y\breve{x}}\right)\\
&+ \frac{h}{12}\left(\left[\frac{\partial f}{\partial x}\right]_{x\breve{y}} + \left[\frac{\partial f}{\partial y}\right]_{y\breve{x}}\right) - \frac{h}{12}\left((K_x)_{\bar{y}y} + (K_y)_{\bar{x}x}\right) \qquad (2.9)\\
&- \left(\left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{\breve{x}} + \frac{1}{h}\right) K_x(y)\right)_{\breve{y}} - \left(\left(\frac{h}{12}\left\{\frac{r}{q}\right\}_{\breve{y}} + \frac{1}{h}\right) K_y(x)\right)_{\breve{x}}\\
&+ \frac{h}{12}\left(\left\{\frac{q}{p}\right\}_{\breve{x}} K''_x(y)\right)_{\breve{y}} + \frac{h}{12}\left(\left\{\frac{p}{q}\right\}_{\breve{y}} K''_y(x)\right)_{\breve{x}}\\
&+ \frac{h^2}{72}\left[\left[\frac{\partial^2 f}{\partial x \partial y}\right]_{x_i} - \left\{\frac{r}{p}\right\}_{x_i} K'_x(y)\right]_{y_j}.
\end{aligned}$$

Remark 2.1. *If one omits the last term in (2.9) the resulting scheme is already of order lower than four, see also Table 5 in [1].*

The following assertion was proved in [1].

Proposition 2.2. *Suppose that* $p,\ q \in \mathcal{H}_\Omega^{5+\alpha}$, $r,\ f \in \mathcal{H}_\Omega^{4+\alpha}$, $K_x \in C^{2+\alpha}[0,\eta]\bigcup[\eta,1]$, $K_y \in C^{2+\alpha}[0,\xi]\bigcup[\xi,1]$, $u \in C(\overline{\Omega})\bigcap\mathcal{H}_\Omega^{6+\alpha}$, $\alpha \in (0,1)$. *Then the truncation error of the scheme (2.4)-(2.9) is of third order.*

Let us introduce the scalar products and the corresponding norms:

$$\begin{aligned}
(U,V)_{\bar\omega} &= \sum_{i=0}^{N}\sum_{j=0}^{M}\hbar_i\bar{k}_j(UV)_{\breve{x}_i\breve{y}_j},\ \|U\|_0^2=(U,U)_{\bar\omega},\\
(U,V)_{\omega_1^+\times\bar\omega_2} &= \sum_{i=1}^{N}\sum_{j=0}^{M}h_i\bar{k}_j(UV)_{\breve{y}_j,i},\ \|U_{\bar x}\|_0^2=(U_{\bar x},U_{\bar x})_{\omega_1^+\times\bar\omega_2},\\
(U,V)_{\bar\omega_1\times\omega_2^+} &= \sum_{i=0}^{N}\sum_{j=1}^{M}\hbar_ik_j(UV)_{\breve{x}_i,j},\ \|U_{\bar y}\|_0^2=(U_{\bar y},U_{\bar y})_{\bar\omega_1\times\omega_2^+},\\
(U,V)_{\omega_1^+\times\omega_2^+} &= \sum_{i=1}^{N}\sum_{j=1}^{M}h_ik_j(UV)_{i,j},\ \|U_{\bar x\bar y}\|_0^2=(U_{\bar x},U_{\bar x})_{\omega_1^+\times\omega_2^+},\\
(U,V)_{\omega_1\times\bar\omega_2} &= \sum_{i=1}^{N-1}\sum_{j=0}^{M}\hbar_i\bar{k}_j(UV)_{\breve{x}_i\breve{y}_j},\ \|U_{\bar x\hat x}\|_0^2=(U_{\bar x\hat x},U_{\bar x\hat x})_{\omega_1\times\bar\omega_2},\\
(U,V)_{\bar\omega_1\times\omega_2} &= \sum_{i=0}^{N}\sum_{j=1}^{M-1}\hbar_i\bar{k}_j(UV)_{\breve{x}_i\breve{y}_j},\ \|U_{\bar y\hat y}\|_0^2=(U_{\bar y\hat y},U_{\bar y\hat y})_{\bar\omega_1\times\omega_2},\\
\|\nabla U\|_0^2 &= \|U_{\bar x}\|_0^2+\|U_{\bar y}\|_0^2,\\
\|U\|_1^2 &= \|U\|_0^2+\|\nabla U\|_0^2,\\
\|U\|_2^2 &= \|U_{\bar x\hat x}\|_0^2+\|U_{\bar y\hat y}\|_0^2+2\|U_{\bar x\bar y}\|_0^2,
\end{aligned}$$

where $\bar\omega_1=\bar\omega_h,\ \bar\omega_2=\bar\omega_k,\ \bar\omega=\bar\omega_1\times\bar\omega_2,\ \gamma=\partial\Omega\cap\bar\omega.$

Using Green's formula it is easily to check the identity [18]

$$\|U\|_2^2=\|\triangle U\|_0^2=\|U_{\bar x\hat x}+U_{\bar y\hat y}\|_0^2.$$

When deriving a priori error estimates of the difference schemes the following negative mesh norm will be often used

$$\|\Psi\|_{-1}=\sup_{v|_\gamma=0}\frac{|(\Psi,v)_\omega|}{\|v\|_1}.$$

Lemma 3.1. *For every mesh function $v(x,y)$ with zero boundary values Friedrichs inequality holds*

$$\|\nabla v\|_0^2\ge 16\|v\|_0^2. \tag{3.1}$$

The following equalities will also be used:

$$-\left((pU_{\bar{x}})_{\hat{x}\check{y}},U\right)_{\omega} = \left(p_{\check{y}},U_{\bar{x}}^{2}\right)_{\omega_1^+\times\omega_2}, \tag{3.2}$$

$$-\left((qU_{\bar{y}})_{\hat{y}\check{x}},U\right)_{\omega} = \left(q_{\check{x}},U_{\bar{y}}^{2}\right)_{\omega_1\times\omega_2^+}. \tag{3.3}$$

3.1. SECOND ORDER DIFFERENCE SCHEME

Theorem 3.1. *The problem*

$$\Lambda U = -\left(pU_{\bar{x}}\right)_{\hat{x}\check{y}} - \left(qU_{\bar{y}}\right)_{\hat{y}\check{x}} + r_{\check{x}\check{y}}U = \varphi,\ U|_{\gamma} = 0 \tag{3.4}$$

has a unique solution that satisfies the estimate

$$\|U\|_1 \le C\|\varphi\|_{-1}.$$

Proof. We take the scalar product of (3.4) and U and sum on up the mesh w. Using (3.2), (3.3) we obtain

$$\begin{aligned}(\Lambda U,U)_{\omega} &= \left(p_{\check{y}},U_{\bar{x}}^{2}\right)_{\omega_1^+\times\omega_2} + \left(q_{\check{x}},U_{\bar{y}}^{2}\right)_{\omega_1\times\omega_2^+} + \left(r_{\check{x}\check{y}},U^2\right)_{\omega}\\ &\ge \left(p_{\check{y}},U_{\bar{x}}^{2}\right)_{\omega_1^+\times\omega_2} + \left(q_{\check{x}},U_{\bar{y}}^{2}\right)_{\omega_1\times\omega_2^+} \ge c_1\|\nabla U\|_0^2,\end{aligned}$$

where $c_1 = \min\{p,q\}$.

Therefore

$$\|\nabla U\|_0^2 \le \frac{1}{c_1}\left(\Lambda U,U\right)_{\omega}. \tag{3.5}$$

From (3.1) and (3.5) we get

$$\|U\|_1^2 \le \frac{17}{16c_1}\left(\Lambda U,U\right)_{\omega} = \frac{17}{16c_1}\left(\varphi,U\right)_{\omega}, \tag{3.6}$$

which implies the existence and uniqueness of the solution of (3.4). Further, from the inequality $|(\varphi,U)_{\omega}| \le \|\varphi\|_{-1}\|U\|_1$ and (3.6), follows the estimate in the theorem □.

Theorem 3.2. *Suppose that the assumptions in Proposition 2.1 are fulfilled. Then for the error $z_{ij} = U_{ij} - u(x_i,y_j)$ of the difference scheme (2.1) the error estimate holds*

$$\|z\|_1 \le C\left(\|h^2\|_0 + \|k^2\|_0\right). \tag{3.7}$$

Proof. We have

$$\Lambda z_{ij} = \Psi_{ij} = \left((pu_{\bar{x}})_{\hat{x}_i} - \frac{1}{\hbar_i}[w_1]_{x_i} - \left(\frac{\partial w_1}{\partial x}\right)_{\hat{x}_i} \right)_{\check{y}_j} + \left((qu_{\bar{y}})_{\hat{y}_j} - \frac{1}{\bar{k}_j}[w_2]_{y_j} - \left(\frac{\partial w_2}{\partial y}\right)_{\hat{y}_j} \right)_{\check{x}_i} = (\varphi_i(y))_{\check{y}_j} + (\psi_j(x))_{\check{x}_i}.$$

$$\varphi_i(y) = \eta^x_{\hat{x}_i} + O(\hbar_i^2), \text{ where } \eta^x_i(y) = \frac{h_i^2}{6}\left(\frac{1}{4}p\frac{\partial^3 u}{\partial x^3} + \frac{3}{4}\frac{\partial^2}{\partial x^2}\left(p\frac{\partial u}{\partial x}\right)\right)(x_{i-1/2}, y),$$

$$\psi_j(x) = \eta^y_{\hat{y}_j} + O(\bar{k}_i^2), \text{ where } \eta^y_j(x) = \frac{k_j^2}{6}\left(\frac{1}{4}q\frac{\partial^3 u}{\partial y^3} + \frac{3}{4}\frac{\partial^2}{\partial y^2}\left(q\frac{\partial u}{\partial y}\right)\right)(x, y_{j-1/2}).$$

$$\Lambda z_{ij} = \eta^x_{\hat{x}_i\check{y}_j} + \eta^y_{\hat{y}_j\check{x}_i} + \Psi^*_{ij}, \tag{3.8}$$

where $\Psi^*_{ij} = O(\hbar_i^2 + \bar{k}_j^2)$. (3.8) whit z_{ij} and . Using Green's formula, we obtain

$$(\Psi, z)_\omega = -\left(\eta^x_{\check{y}}, z_{\bar{x}}\right)_{\omega_1^+\times\omega_2} - \left(\eta^y_{\check{x}}, z_{\bar{y}}\right)_{\omega_1\times\omega_2^+} + (\Psi^*, z)_{\bar{\omega}}. \tag{3.9}$$

Hence

$$\begin{aligned} |(\Psi, z)_\omega| &\le C\left(\|h^2\|_0\|z_{\bar{x}}\|_0 + \|k^2\|_0\|z_{\bar{y}}\|_0 + (\|h^2\|_0 + \|k^2\|_0)\|z\|_0\right) \\ &\le C\left(\|h^2\|_0 + \|k^2\|_0\right)\|z\|_1, \end{aligned} \tag{3.10}$$

where $\|h^2\|_0^2 = \sum_{i=0}^N \hbar_i^5$, $\|k^2\|_0^2 = \sum_{j=0}^M \bar{k}_j^5$, and C is a constant independent of h, k. Therefore

$$\|z\|_1 \le C\left(\|h^2\|_0 + \|k^2\|_0\right). \quad \square \tag{3.11}$$

3.2. FOURTH ORDER DIFFERENCE SCHEME

Lemma 3.2. *Let the mesh ω_h satisfy $\frac{1}{\alpha} \le \frac{h_i+1}{h_i}, \frac{k_j+1}{k_j} \le \alpha$, $i = 1, ..., N-1$; $j = 1, .., M-1$, where $1 \le \alpha \le 1.45$ and V is a mesh function defined on the mesh $\overline{\omega}$ with zero boundary conditions $V|_\gamma = 0$. Then*

$$\left|\left([k]\,(pV)_{\bar{x}\mathring{y}}, V_{\bar{x}}\right)_{\omega_1^+\times\omega_2}\right| \le \left(p_{\check{y}}, V_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2},$$

$$\left|\left([h]\,(qV)_{\bar{y}\mathring{x}}, V_{\bar{y}}\right)_{\omega_1\times\omega_2^+}\right| \le \left(q_{\check{x}}, V_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+},$$

$$\left|\left([h]\,(rV)_{\mathring{x}\check{y}} + [k]\,(rV)_{\mathring{y}\check{x}} + \frac{2}{3}[h]\,[k]\,(rV)_{\mathring{x}\mathring{y}}, V\right)_\omega\right| \le 2\left(r_{\check{x}\check{y}}, V^2\right)_\omega.$$

Proof. Let us denote
$[h]_{x_i}=[h]_i,\ [k]_{y_j}=[k]_j,\ [h]_0=0,\ [h]_N=0,\ [k]_0=0,\ [k]_M=0$. Then

$$\begin{aligned}
&\left([h]\,(pV)_{\overset{\circ}{x}},U\right)_{\omega_1}\\
=\;& \frac{1}{2}\sum_{i=1}^{N-1}[h]_iV_i\left(\frac{h_{i+1}}{h_i}p(x_i-)\,(V_i-V_{i-1})+\frac{h_i}{h_{i+1}}p(x_i+)\,(V_{i+1}-V_i)\right)\\
=\;& \sum_{i=1}^{N-1}\left(\left([h]_{i-1}\frac{h_{i-1}}{h_i}-[h]_i\frac{h_{i+1}}{h_i}\right)p(x_i-)V_{i-1}V_i\right.\\
+\;& \left.\frac{[h]_i}{2}\left(\frac{h_{i+1}}{h_i}p(x_i-)-\frac{h_i}{h_{i+1}}p(x_i+)\right)V_i^2\right)
\end{aligned}$$

The inequalities

$$\begin{aligned}
|V_{i-1}V_i| &\le \frac{1}{2}\sqrt{\frac{\hbar_{i-1}}{\hbar_i}}V_{i-1}^2+\frac{1}{2}\sqrt{\frac{\hbar_i}{\hbar_{i-1}}}V_i^2,\\
|V_iV_{i+1}| &\le \frac{1}{2}\sqrt{\frac{\hbar_{i+1}}{\hbar_i}}V_{i+1}^2+\frac{1}{2}\sqrt{\frac{\hbar_i}{\hbar_{i+1}}}V_i^2.
\end{aligned}$$

imply

$$\begin{aligned}
\left|\left([h]\,(pV)_{\overset{\circ}{x}},V\right)_{\omega_1}\right| \le\;& \max_i\left\{\frac{[h]_i^2}{h_ih_{i+1}}+\frac{1}{4}\sqrt{\frac{\hbar_{i-1}}{\hbar_i}}\left|\frac{[h]_{i-1}}{\hbar_{i-1}}\frac{h_{i-1}}{h_i}-\frac{[h]_i}{\hbar_i}\frac{h_{i+1}}{h_i}\right|\right.\\
+\;& \left.\frac{1}{4}\sqrt{\frac{\hbar_{i+1}}{\hbar_i}}\left|\frac{[h]_i}{\hbar_i}\frac{h_i}{h_{i+1}}-\frac{[h]_{i+1}}{\hbar_{i+1}}\frac{h_{i+2}}{h_{i+1}}\right|\right\}\left(p_{\breve{x}},V^2\right)_{\omega_1}.
\end{aligned}$$

Since

$$\frac{[h]_i}{2}\left(\frac{h_{i+1}}{h_i}r(x_i-)-\frac{h_i}{h_{i+1}}r(x_i+)\right)=\begin{cases}0,\ r(x_i-)\ne r(x_i+),\\ \hbar_i\frac{[h]_i^2}{h_ih_{i+1}}r(x_i),\ [r]_{x_i}=0.\end{cases}$$

$$\frac{[h]_i^2}{h_ih_{i+1}}\le\alpha-2+\frac{1}{\alpha},\ \frac{\hbar_i}{\hbar_{i-1}}\le\alpha,\ \frac{|[h]_i|}{\hbar_i}\le\frac{2(\alpha-1)}{\alpha+1},$$

we have

$$\left|\left([h]\,(pV)_{\overset{\circ}{x}\breve{y}},V\right)_{\omega}\right|\le\left(\alpha-2+\frac{1}{\alpha}+2\alpha\sqrt{\alpha}\frac{\alpha-1}{\alpha+1}\right)\left(p_{\breve{x}\breve{y}},V^2\right)_{\omega}\le\left(p_{\breve{x}\breve{y}},V^2\right)_{\omega}.$$

Analogously one can prove that

$$\left|\left([k]\,(qV)_{\overset{\circ}{y}\check{x}},V\right)_\omega\right| \le \left(q_{\check{x}\check{y}},V^2\right)_\omega .$$

Next, we have

$$\left([h]\,[k]\,(rV)_{\overset{\circ}{x}\overset{\circ}{y}},V\right)_\omega = \frac{1}{4}\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}[h]_i[k]_jV_{ij}\left(h_{i+1}k_{j+1}r(x_i-,y_j-)V_{\bar{x}_i\bar{y}_j}\right.$$
$$+\, h_ik_{j+1}r(x_i+,y_j-)V_{\bar{x}_{i+1}\bar{y}_j}+h_ik_jr(x_i+,y_j+)V_{\bar{x}_{i+1}\bar{y}_{j+1}}+\left.h_{i+1}k_jr(x_i-,y_j+)V_{\bar{x}_i\bar{y}_{j+1}}\right)$$
$$=\sum_{i=1}^{N-1}\sum_{j=1}^{M-1}\left(\frac{[h]_i^2}{h_ih_{i+1}}\frac{[k]_j^2}{k_jk_{j+1}}\hbar_i\bar{k}_jr_{\check{x}_i\check{y}_j}V_{ij}^2\right.$$
$$+\hbar_i\frac{[h]_i^2}{h_ih_{i+1}}\left([k]_{j-1}\frac{k_{j-1}}{k_j}-[k]_j\frac{k_{j+1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{ij-1}V_{ij}$$
$$+\bar{k}_j\frac{[k]_j^2}{k_jk_{j+1}}\left([h]_{i-1}\frac{h_{i-1}}{h_i}-[h]_i\frac{h_{i+1}}{h_i}\right)r_{\check{x}_i\check{y}_j}V_{i-1j}V_{ij}$$
$$+\frac{1}{2}\left([h]_{i-1}[k]_{j-1}\frac{h_{i-1}}{h_i}\frac{k_{j-1}}{k_j}+[h]_i[k]_j\frac{h_{i+1}}{h_i}\frac{k_{j+1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{i-1j-1}U_{ij}$$
$$-\frac{1}{4}\left([h]_{i-1}[k]_{j+1}\frac{h_{i-1}}{h_i}\frac{k_{j+2}}{k_{j+1}}+[h]_i[k]_j\frac{h_{i+1}}{h_i}\frac{k_j}{k_{j+1}}\right)r_{\check{x}_i\check{y}_j}V_{i-1j+1}U_{ij}$$
$$-\frac{1}{4}\left.\left([h]_i[k]_j\frac{h_i}{h_{i+1}}\frac{k_{j+1}}{k_j}+[h]_{i+1}[k]_{j-1}\frac{h_{i+2}}{h_{i+1}}\frac{k_{j-1}}{k_j}\right)r_{\check{x}_i\check{y}_j}V_{i+1j-1}V_{ij}\right)$$
$$\le\max_i\left\{\frac{[h]_i^2}{h_ih_{i+1}}\frac{[k]_j^2}{k_jk_{j+1}}+\sqrt{\frac{\hbar_i\bar{k}_j}{\hbar_{i+1}\bar{k}_{j+1}}}\left|\frac{[h]_i[k]_j}{\hbar_i\bar{k}_j}\frac{h_ik_j}{h_{i+1}k_{j+1}}\right|\right\}\left(r_{\check{x}\check{y}},V^2\right)_\omega$$
$$\le\left(\left(\alpha-2+\frac{1}{\alpha}\right)^2+4\alpha^3\frac{(\alpha-1)^2}{(\alpha+1)^2}\right)\left(r_{\check{x}\check{y}},V^2\right)_\omega .$$

Therefore

$$\left|\left([h]\,(rV)_{\overset{\circ}{x}\check{y}}+[k]\,(rV)_{\overset{\circ}{y}\check{x}}+\frac{2}{3}\,[h]\,[k]\,(rV)_{\overset{\circ}{x}\overset{\circ}{y}},V\right)_\omega\right|$$
$$\le\left(2\left(\alpha-2+\frac{1}{\alpha}\right)+4\alpha\sqrt{\alpha}\frac{\alpha-1}{\alpha+1}+\frac{2}{3}\left(\alpha-2+\frac{1}{\alpha}\right)^2+\frac{8}{3}\alpha^3\frac{(\alpha-1)^2}{(\alpha+1)^2}\right)\left(r_{\check{x}\check{y}},V^2\right)_\omega$$
$$\le 2\left(r_{\check{x}\check{y}},V^2\right)_\omega . \qquad \square$$

Theorem 3.3. *The solution of the problem*

$$\begin{aligned}
\Lambda' U &= -\left(pU_{\bar x}\right)_{\hat x \breve y} - \left(qU_{\bar y}\right)_{\hat y \breve x} + r_{\breve x \breve y} U - \frac{1}{12}\left(\left(k^2\left(pU_{\bar x}\right)_{\hat x \bar y}\right)_{\hat y} + \left(h^2\left(qU_{\bar y}\right)_{\hat y \bar x}\right)_{\hat x}\right) \\
&- \frac{1}{6}\left(\left[k\right]_{y_j}\left(pU_{\bar x}\right)_{\hat x \mathring{y}} + \left[h\right]_{x_i}\left(qU_{\bar y}\right)_{\hat y \mathring{x}}\right) + \frac{1}{12}\left(\left(h^2 rU_{\bar x}\right)_{\hat x \breve y} + \left(k^2 rU_{\bar y}\right)_{\hat y \breve x}\right) \quad (3.12) \\
&+ \frac{1}{6}\left(\left[h\right]_{x_i}\left(rU\right)_{\mathring{x}\breve y} + \left[k\right]_{y_j}\left(rU\right)_{\mathring{y}\breve x} + \frac{2}{3}\left[h\right]_{x_i}\left[k\right]_{y_j}\left(rU\right)_{\mathring{x}\mathring{y}}\right) = \varphi',\ U|_{\gamma} = 0,
\end{aligned}$$

where φ' stands for the right hand-sides in (2.4)-(2.9). respectively. exists and is unique. It satisfies the a priori estimates:

$$\|U\|_1 \le C\|\varphi'\|_{-1}, \quad (3.13)$$

$$\|U\|_2 \le C\|\varphi'\|_0, \quad (3.14)$$

where the constant C doesn't depend on the mesh.

Proof. Let us organize the scalar product

$$\begin{aligned}
\left(\Lambda' U, U\right)_{\omega} &= -\left(\left(pU_{\bar x}\right)_{\hat x \breve y}, U\right)_{\omega} - \left(\left(qU_{\bar y}\right)_{\hat y \breve x}, U\right)_{\omega} + \left(r_{\breve x \breve y} U, U\right)_{\omega} \\
&- \frac{1}{12}\left(\left(k^2\left(pU_{\bar x}\right)_{\hat x \bar y}\right)_{\hat y}, U\right)_{\omega} - \frac{1}{12}\left(\left(h^2\left(qU_{\bar y}\right)_{\hat y \bar x}\right)_{\hat x}, U\right)_{\omega} \\
&+ \frac{1}{12}\left(\left(h^2 rU_{\bar x}\right)_{\hat x \breve y}, U\right)_{\omega} + \frac{1}{12}\left(\left(k^2 rU_{\bar y}\right)_{\hat y \breve x}\right)_{\omega} \quad (3.15) \\
&- \frac{1}{6}\left(\left[k\right]\left(pU_{\bar x}\right)_{\hat x \mathring{y}}, U\right)_{\omega} - \frac{1}{6}\left(\left[h\right]\left(qU_{\bar y}\right)_{\hat y \mathring{x}}, U\right)_{\omega} \\
&+ \frac{1}{6}\left(\left[h\right]_{x_i} rU_{\mathring{x}\breve y}, U\right)_{\omega} + \frac{1}{6}\left(\left[k\right]_{y_j} rU_{\mathring{y}\breve x}, U\right)_{\omega} \\
&+ \frac{1}{9}\left(\left[h\right]_{x_i}\left[k\right]_{y_j} rU_{\mathring{x}\mathring{y}}, U\right)_{\omega}.
\end{aligned}$$

In view of formulas (3.2), (3.3)

$$\begin{aligned}
\left(\left(k^2\left(pU_{\bar x}\right)_{\hat x \bar y}\right)_{\hat y}, U\right)_{\omega} &= \left(k^2 p, U^2_{\bar x \bar y}\right)_{\omega_1^+ \times \omega_2^+}, \\
\left(\left(h^2\left(qU_{\bar y}\right)_{\hat y \bar x}\right)_{\hat x}, U\right)_{\omega} &= \left(h^2 q, U^2_{\bar x \bar y}\right)_{\omega_1^+ \times \omega_2^+} \\
-\left(\left(h^2 rU_{\bar x}\right)_{\hat x \breve y}, U\right)_{\omega} &= \left(k^2 r_{\breve x}, U^2_{\bar y}\right)_{\omega_1 \times \omega_2^+}, \\
-\left(\left(k^2 rU_{\bar y}\right)_{\hat y \breve x}, U\right)_{\omega} &= \left(h^2 r_{\breve y}, U^2_{\bar x}\right)_{\omega_1^+ \times \omega_2}, \\
-\left(\left[k\right]\left(pU_{\bar x}\right)_{\hat x \mathring{y}}, U\right)_{\omega} &= \left(\left[k\right]\left(pU\right)_{\bar x \mathring{y}}, U^2_{\bar x}\right)_{\omega_1^+ \times \omega_2}, \\
-\left(\left[h\right]\left(qU\right)_{\mathring{x}\bar y}, U\right)_{\omega} &= \left(\left[h\right]\left(qU\right)_{\bar y \mathring{x}}, U^2_{\bar y}\right)_{\omega_1 \times \omega_2^+};
\end{aligned}$$

we can cultivate (3.18) in the form:

$$\begin{aligned}
& (\Lambda' U, U)_\omega \\
= & \left(p_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} + \left(q_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} + \left(r_{\check{x}\check{y}} U, U\right)_\omega \\
- & \frac{1}{12}\left(k^2 p + h^2 q, U_{\bar{x}\bar{y}}^2\right)_{\omega_1^+\times\omega_2^+} - \frac{1}{12}\left(h^2 r_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} - \frac{1}{12}\left(k^2 r_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} \\
+ & \frac{1}{6}\left([k]\,(pU)_{\bar{x}\mathring{y}}, U_{\bar{x}}\right)_{\omega_1^+\times\omega_2} + \frac{1}{6}\left([h]\,(qU)_{\bar{y}\mathring{x}}, U_{\bar{y}}\right)_{\omega_1\times\omega_2^+} \\
+ & \frac{1}{6}\left([h]_{x_i}(rU)_{\mathring{x}\check{y}} + [k]_{y_j}(rU)_{\mathring{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j}(rU)_{\mathring{x}\mathring{y}}, U\right)_\omega .
\end{aligned}$$

Further, we will use the inequalities

$$\begin{aligned}
\left(k^2 p, U_{\bar{x}\bar{y}}^2\right)_{\omega_1^+\times\omega_2^+} &\leq 4\left(p_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2}, \\
\left(h^2 q, U_{\bar{x}\bar{y}}^2\right)_{\omega_1^+\times\omega_2^+} &\leq 4\left(q_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} \\
\left(h^2 r_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} &\leq 4\left(r_{\check{x}\check{y}}, U^2\right)_\omega \\
\left(k^2 r_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+} &\leq 4\left(r_{\check{x}\check{y}}, U^2\right)_\omega
\end{aligned}$$

We will check only the first one.The others can proved analogously.

$$\begin{aligned}
0 &\leq \left(k^2 p, U_{\bar{x}\bar{y}}^2\right)_{\omega_1^+\times\omega_2^+} = \sum_{i=1}^{N}\sum_{j=1}^{M} h_i k_j p(x_i-, y_j-)\left(U_{\bar{x}_{i,j}} - U_{\bar{x}_{i,j-1}}\right)^2 \\
&\leq 2\sum_{i=1}^{N}\sum_{j=1}^{M} h_i k_j p(x_i-, y_j-)\left(U_{\bar{x}_{i,j}}^2 + U_{\bar{x}_{i,j-1}}^2\right) = 4\left(p_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2}.
\end{aligned}$$

Using these inequalities and Lemma(3.2), we obtain

$$(\Lambda' U, U)_\omega \geq \frac{1}{2}\left(\left(p_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2} + \left(q_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+}\right) \geq \frac{c_0}{2}\|\nabla U\|_0^2.$$

Therefore

$$\|\nabla U\|_0^2 \leq \frac{2}{c_0}(\Lambda' U, U)_\omega . \tag{3.16}$$

It follows from (3.1) and (3.16) that

$$\|U\|_1^2 \leq \frac{17}{8c_0}(\Lambda U, U)_\omega = \frac{17}{8c_0}(\varphi, U)_\omega , \tag{3.17}$$

which implies existence and uniqueness of the solution of (3.12).Next, from the inequality $|(\varphi, U)_\omega| \leq \|\varphi\|_{-1}\|U\|_1$ and (3.17) follows the desired estimate (3.13).

Let us organize the scalar product

$$
\begin{aligned}
-\left(\Lambda' U, \triangle U\right)_\omega \; = \; & \left(\left(pU_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(qU_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega - \left(r_{\check{x}\check{y}} U, \triangle U\right)_\omega \\
& + \frac{1}{12}\left(\left(k^2\left(pU_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2\left(qU_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}}, \triangle U\right)_\omega \\
& - \frac{1}{12}\left(\left(h^2 rU_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(k^2 rU_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega \qquad (3.18)\\
& + \frac{1}{6}\left([k]\left(pU_{\bar{x}}\right)_{\hat{x}\overset{\circ}{y}} + [h]\left(qU_{\bar{y}}\right)_{\hat{y}\overset{\circ}{x}}, \triangle U\right)_\omega \\
& - \frac{1}{6}\left([h]_{x_i} rU_{\overset{\circ}{x}\check{y}} + [k]_{y_j} rU_{\overset{\circ}{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j} rU_{\overset{\circ}{x}\overset{\circ}{y}}, \triangle U\right)_\omega .
\end{aligned}
$$

From the inequalities:

$$
\begin{aligned}
\left(\left(pU_{\bar{x}}\right)_{\hat{x}\check{y}}, U_{\bar{x}\hat{x}}\right)_\omega &\geq c_0\|U_{\bar{x}\hat{x}}\|_0^2, \\
\left(\left(qU_{\bar{y}}\right)_{\hat{y}\check{x}}, U_{\bar{y}\hat{y}}\right)_\omega &\geq c_0\|U_{\bar{y}\hat{y}}\|_0^2, \\
\left(\left(pU_{\bar{x}}\right)_{\hat{x}\check{y}}, U_{\bar{y}\hat{y}}\right)_\omega &\geq c_0\|U_{\bar{x}\bar{y}}\|_0^2, \\
\left(\left(qU_{\bar{y}}\right)_{\hat{y}\check{x}}, U_{\bar{x}\hat{x}}\right)_\omega &\geq c_0\|U_{\bar{x}\bar{y}}\|_0^2, \\
\left(\left(pU_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(qU_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega &\geq c_0\|U\|_2^2, \\
-\left(r_{\check{x}\check{y}} U, U_{\bar{x}\hat{x}}\right)_\omega &= \left(r_{\check{y}}, U_{\bar{x}}^2\right)_{\omega_1^+\times\omega_2}, \\
-\left(r_{\check{x}\check{y}} U, U_{\bar{y}\hat{y}}\right)_\omega &= \left(r_{\check{x}}, U_{\bar{y}}^2\right)_{\omega_1\times\omega_2^+}, \\
-\left(r_{\check{x}\check{y}} U, \triangle U\right)_\omega &\geq r_0\|U\|_1^2, \\
\left(\left(k^2\left(pU_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} + \left(h^2\left(qU_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}}, \triangle U\right)_\omega &\geq -4c_0\|U\|_2^2, \\
-\left(\left(h^2 rU_{\bar{x}}\right)_{\hat{x}\check{y}} + \left(k^2 rU_{\bar{y}}\right)_{\hat{y}\check{x}}, \triangle U\right)_\omega &\geq -8r_0\|U\|_1^2, \\
\left([k]\left(pU_{\bar{x}}\right)_{\hat{x}\overset{\circ}{y}} + [h]\left(qU_{\bar{y}}\right)_{\hat{y}\overset{\circ}{x}}, \triangle U\right)_\omega &\geq -c_0\|U\|_2^2, \\
-\left([h]_{x_i} rU_{\overset{\circ}{x}\check{y}} + [k]_{y_j} rU_{\overset{\circ}{y}\check{x}} + \frac{2}{3}[h]_{x_i}[k]_{y_j} rU_{\overset{\circ}{x}\overset{\circ}{y}}, \triangle U\right)_\omega &\geq -2r_0\|U\|_1^2,
\end{aligned}
$$

we obtain

$$
-\left(\Lambda' U, \triangle U\right)_\omega \geq \frac{c_0}{2}\|U\|_2^2. \qquad (3.19)
$$

Therefore

$$
\|U\|_2^2 \leq -\frac{2}{c_0}\left(\varphi', \triangle U\right)_\omega \leq \frac{2}{c_0}\|\varphi'\|_0\|U\|_2. \qquad (3.20)
$$

Now (3.14) follow from (3.20). □

Now we turn our attention to the convergence of the scheme.

Theorem 3.4. *Suppose that the assumptions in Proposition 2.2 are fulfilled. Then for the error $z_{ij} = U_{ij} - u(x_i, y_j)$ of the difference scheme (3.12) the error estimates hold:*

$$\begin{aligned} & \textit{if the mesh is uniform} \\ & \|z\|_1 \leq Ch^4, \\ & \textit{else} \\ & \|z\|_1 \leq Ch^3. \end{aligned} \tag{3.21}$$

Proof. We will derive the approximation error, considering several cases.

Case 3.1. Mesh points of type $\diamond$.

In this case

$$\begin{aligned} \Psi &= f - ru + (pu_{\bar{x}})_{\hat{x}} + (qu_{\bar{y}})_{\hat{y}} + \frac{1}{12}\left(h^2(f-ru)_{\bar{x}}\right)_{\hat{x}} + \frac{1}{12}\left(k^2(f-ru)_{\bar{y}}\right)_{\hat{y}} \\ &+ \frac{1}{12}\left(k^2\left(pu_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} + \frac{1}{12}\left(h^2\left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}} + \frac{1}{9}[h]_{x_i}[k]_{y_j}(f-ru)_{\mathring{x}\mathring{y}} \\ &+ \frac{[h]_{x_i}}{6}\left(f - ru + (qu_{\bar{y}})_{\hat{y}}\right)_{\mathring{x}} + \frac{[k]_{y_j}}{6}\left(f - ru + (pu_{\bar{x}})_{\hat{x}}\right)_{\mathring{y}} \end{aligned}$$

Using the differential equation (1.1), and formulas (20), (21), (31), (32) in [1], we obtain

$$\begin{aligned} (pu_{\bar{x}})_{\hat{x}} &= p\frac{\partial^2 u}{\partial x^2} + \frac{[h^2]_i}{6\hbar_i}p\frac{\partial^3 u}{\partial x^3} + \frac{(h^2)_{\breve{x}_i}}{12}p\frac{\partial^4 u}{\partial x^4} + \frac{[h^4]_i}{120\hbar_i}p\frac{\partial^5 u}{\partial x^5} + O(\hbar_i^4), \\ (qu_{\bar{y}})_{\hat{y}} &= q\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j}q\frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12}q\frac{\partial^4 u}{\partial y^4} + \frac{[k^4]_j}{120\bar{k}_j}q\frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4), \\ \left(h^2 g_{\bar{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i}\frac{\partial g}{\partial x} + (h^2)_{\breve{x}_i}\frac{\partial^2 g}{\partial x^2} + \frac{[h^4]_i}{6\hbar_i}\frac{\partial^3 u}{\partial x^3} + O(\hbar_i^4), \\ \left(k^2 g_{\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}\frac{\partial g}{\partial y} + (k^2)_{\breve{y}_j}\frac{\partial^2 u}{\partial y^2} + \frac{[k^4]_j}{6\bar{k}_j}\frac{\partial^3 g}{\partial y^3} + O(\bar{k}_j^4), \\ \left(k^2\left(pu_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}p\frac{\partial^3 u}{\partial x^2\partial y} + \frac{[h^2]_i[k^2]_j}{6\hbar_i\bar{k}_j}p\frac{\partial^4 u}{\partial x^3\partial y} + (k^2)_{\breve{y}_j}p\frac{\partial^4 u}{\partial x^2\partial y^2} \\ &+ \frac{[k^4]_j}{6\bar{k}_j}p\frac{\partial^5 u}{\partial x^2\partial y^3} + \frac{(h^2)_{\breve{x}_i}[k^2]_j}{12\bar{k}_j}p\frac{\partial^5 u}{\partial x^4\partial y} \\ &+ \frac{[h^2]_i(k^2)_{\breve{y}_j}}{6\hbar_i}p\frac{\partial^5 u}{\partial x^3\partial y^2} + O(\hbar_i + \bar{k}_j)^4, \\ \left(h^2\left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}} &= \frac{[h^2]_i}{\hbar_i}q\frac{\partial^3 u}{\partial x\partial y^2} + \frac{[h^2]_i[k^2]_j}{6\hbar_i\bar{k}_j}q\frac{\partial^4 u}{\partial x\partial y^3} + (h^2)_{\breve{x}_i}q\frac{\partial^4 u}{\partial x^2\partial y^2} \\ &+ \frac{[h^4]_j}{6\hbar_i}q\frac{\partial^5 u}{\partial x^3\partial y^2} + \frac{[h^2]_i(k^2)_{\breve{y}_j}}{12\hbar_i}q\frac{\partial^5 u}{\partial x\partial y^4} \end{aligned}$$

$$+ \quad \frac{(h^2)_{\breve{x}_i}[k^2]_j}{6\bar{k}_j} q \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^4,$$

$$(pu_{\bar{x}})_{\hat{x}\mathring{y}} = p\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i}{6\hbar_i} p \frac{\partial^4 u}{\partial x^3 \partial y} + \frac{(h^2)_{\breve{x}_i}}{12} p \frac{\partial^5 u}{\partial x^4 \partial y}$$

$$+ \quad \frac{k_j k_{j+1}}{3} p \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^3,$$

$$(qu_{\bar{y}})_{\hat{y}\mathring{x}} = q\frac{\partial^3 u}{\partial x \partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q \frac{\partial^4 u}{\partial x \partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12} q \frac{\partial^5 u}{\partial x \partial y^4}$$

$$+ \quad \frac{h_i h_{i+1}}{3} q \frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^3,$$

$$g_{\mathring{x}} = \frac{\partial g}{\partial x} + \frac{h_i h_{i+1}}{6}\frac{\partial^3 u}{\partial x^3} + O(\hbar_i^3),$$

$$g_{\mathring{y}} = \frac{\partial g}{\partial y} + \frac{k_j k_{j+1}}{6}\frac{\partial^3 u}{\partial y^3} + O(\bar{k}_j^3),$$

$$g_{\mathring{x}\mathring{y}} = \frac{\partial^2 u}{\partial x \partial y} + O(\hbar_i + \bar{k}_j)^2.$$

Then

$$(pu_{\bar{x}})_{\hat{x}} = p\frac{\partial^2 u}{\partial x^2} + \frac{[h^2]_i}{6\hbar_i} p \frac{\partial^3 u}{\partial x^3} + \frac{(h^2)_{\breve{x}_i}}{12} p \frac{\partial^4 u}{\partial x^4} + \frac{[h^4]_i}{120\hbar_i} p \frac{\partial^5 u}{\partial x^5} + O(\hbar_i^4),$$

$$(qu_{\bar{y}})_{\hat{y}} = q\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q \frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12} q \frac{\partial^4 u}{\partial y^4} + \frac{[k^4]_j}{120\bar{k}_j} q \frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4),$$

$$\left(h^2 g_{\bar{x}}\right)_{\hat{x}} = \frac{[h^2]_i}{\hbar_i}\frac{\partial g}{\partial x} + (h^2)_{\breve{x}_i}\frac{\partial^2 g}{\partial x^2} + \frac{[h^4]_i}{6\hbar_i}\frac{\partial^3 u}{\partial x^3} + O(\hbar_i^4),$$

$$\left(k^2 g_{\bar{y}}\right)_{\hat{y}} = \frac{[k^2]_j}{\bar{k}_j}\frac{\partial g}{\partial y} + (k^2)_{\breve{y}_j}\frac{\partial^2 u}{\partial y^2} + \frac{[k^4]_j}{6\bar{k}_j}\frac{\partial^3 g}{\partial y^3} + O(\bar{k}_j^4),$$

$$\left(k^2 \left(pu_{\bar{x}}\right)_{\hat{x}\bar{y}}\right)_{\hat{y}} = \frac{[k^2]_j}{\bar{k}_j} p \frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i[k^2]_j}{6\hbar_i \bar{k}_j} p \frac{\partial^4 u}{\partial x^3 \partial y} + (k^2)_{\breve{y}_j} p \frac{\partial^4 u}{\partial x^2 \partial y^2}$$

$$+ \quad \frac{[k^4]_j}{6\bar{k}_j} p \frac{\partial^5 u}{\partial x^2 \partial y^3} + \frac{(h^2)_{\breve{x}_i}[k^2]_j}{12\bar{k}_j} p \frac{\partial^5 u}{\partial x^4 \partial y}$$

$$+ \quad \frac{[h^2]_i (k^2)_{\breve{y}_j}}{6\hbar_i} p \frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^4,$$

$$\left(h^2 \left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}}\right)_{\hat{x}} = \frac{[h^2]_i}{\hbar_i} q \frac{\partial^3 u}{\partial x \partial y^2} + \frac{[h^2]_i[k^2]_j}{6\hbar_i \bar{k}_j} q \frac{\partial^4 u}{\partial x \partial y^3} + (h^2)_{\breve{x}_i} q \frac{\partial^4 u}{\partial x^2 \partial y^2}$$

$$+ \quad \frac{[h^4]_j}{6\hbar_i} q \frac{\partial^5 u}{\partial x^3 \partial y^2} + \frac{[h^2]_i (k^2)_{\breve{y}_j}}{12\hbar_i} q \frac{\partial^5 u}{\partial x \partial y^4}$$

$$+ \quad \frac{(h^2)_{\breve{x}_i}[k^2]_j}{6\bar{k}_j} q \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^4,$$

$$
\begin{aligned}
(pu_{\bar{x}})_{\hat{x}\mathring{y}} &= p\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{[h^2]_i}{6\hbar_i} p \frac{\partial^4 u}{\partial x^3 \partial y} + \frac{(h^2)_{\breve{x}_i}}{12} p \frac{\partial^5 u}{\partial x^4 \partial y} \\
&+ \frac{k_j k_{j+1}}{3} p \frac{\partial^5 u}{\partial x^2 \partial y^3} + O(\hbar_i + \bar{k}_j)^3, \\
(qu_{\bar{y}})_{\hat{y}\mathring{x}} &= q\frac{\partial^3 u}{\partial x \partial y^2} + \frac{[k^2]_j}{6\bar{k}_j} q \frac{\partial^4 u}{\partial x \partial y^3} + \frac{(k^2)_{\breve{y}_j}}{12} q \frac{\partial^5 u}{\partial x \partial y^4} \\
&+ \frac{h_i h_{i+1}}{3} q \frac{\partial^5 u}{\partial x^3 \partial y^2} + O(\hbar_i + \bar{k}_j)^3, \\
g_{\mathring{x}} &= \frac{\partial g}{\partial x} + \frac{h_i h_{i+1}}{6} \frac{\partial^3 u}{\partial x^3} + O(\hbar_i^3), \\
g_{\mathring{y}} &= \frac{\partial g}{\partial y} + \frac{k_j k_{j+1}}{6} \frac{\partial^3 u}{\partial y^3} + O(\bar{k}_j^3), \\
g_{\mathring{x}\mathring{y}} &= \frac{\partial^2 u}{\partial x \partial y} + O(\hbar_i + \bar{k}_j)^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Psi_{ij} &= \frac{1}{72\hbar_i} \left[h^2 k^2 \left(r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right) \right]_{x_i \breve{y}_j} \\
&+ \frac{1}{72\bar{k}_j} \left[h^2 k^2 \left(r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right) \right]_{y_j \breve{x}_i} \\
&- \frac{1}{180\hbar_i} \left[h^4 p \frac{\partial^5 u}{\partial x^5} \right]_{x_i \breve{y}_j} - \frac{1}{180\bar{k}_j} \left[k^4 q \frac{\partial^5 u}{\partial y^5} \right]_{y_j \breve{x}_i} - \Psi_{ij}^{*} + \Psi_{ij}^{**} \\
&= \frac{(k^2)_{\breve{y}_j}}{72} \left(h^2 \mu^x \right)_{\hat{x}_i, j} + \frac{(h^2)_{\breve{x}_i}}{72} \left(k^2 \mu^y \right)_{\hat{y}_j, i} - \frac{1}{180} \left(h^4 \eta^x \right)_{\hat{x}_i, j} \\
&- \frac{1}{180} \left(k^4 \eta^y \right)_{\hat{y}_j, i} - \Psi_{ij}^{*} + \Psi_{ij}^{**},
\end{aligned}
$$

where

$$
\mu_i^x(y) = \left(r \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2} \right) (x_{i-1/2}, y),\ \mu_j^y(x) = \left(r \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y} \right) (x, y_{j-1/2}),
$$

$$
\eta_i^x(y) = \left(p \frac{\partial^5 u}{\partial x^5} \right) (x_{i-1/2}, y),\ \eta_j^y(x) = \left(q \frac{\partial^5 u}{\partial y^5} \right) (x, y_{j-1/2}),
$$

$$
\Psi_{ij}^{*} = \frac{h_i h_{i+1}}{72\hbar_i} \left[h^2 p \frac{\partial^5 u}{\partial x^5} \right]_{x_i} + \frac{k_j k_{j+1}}{72\bar{k}_j} \left[k^2 p \frac{\partial^5 u}{\partial y^5} \right]_{y_j},\ \Psi_{ij}^{**} = O(\hbar_i + \bar{k}_j)^4.
$$

Case 3.2. Points $x_i = \xi$, $y_j \neq \eta$ of type $*$. Now

$$\begin{aligned}
\Psi &= f_{\check{x}} - r_{\check{x}}u + (pu_{\bar{x}})_x + (qu_{\bar{y}})_{\hat{y}\check{x}} + \frac{h^2}{12}(f-ru)_{\bar{x}x} + \frac{1}{12}\left(k^2(f-ru)_{\bar{y}}\right)_{\hat{y}\check{x}} \\
&+ \frac{1}{12}\left(k^2(pu_{\bar{x}})_{\hat{x}\bar{y}}\right)_{\hat{y}} + \frac{h^2}{12}(qu_{\bar{y}})_{\hat{y}\bar{x}x} + \frac{[k]_{y_j}}{6}\left(f_{\check{x}} - r_{\check{x}}u + (pu_{\bar{x}})_{\hat{x}}\right)_{\mathring{y}} \\
&- \left(\frac{h}{12}\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right)K_x(y_j) - \frac{1}{12h}\left(k^2K_x(y)\right)_{\bar{y}\hat{y}} + \frac{h}{12}\left\{\frac{q}{p}\right\}_{x_i}(K_x(y))_{\bar{y}\hat{y}} \\
&+ \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i} - \frac{[k]_{y_j}}{18}\left(\frac{3}{h}(K_x(y))_{\mathring{y}} + h\left\{\frac{r}{p}\right\}_{x_i}K_x'(y_j) - h\left[\frac{\partial^2 f}{\partial x\partial y}\right]_{x_i,j}\right)
\end{aligned}$$

Using the differential equation (1.1), and formulas (20), (21), (31), (32) in [1], we obtain

$$\begin{aligned}
(pu_{\bar{x}})_{\hat{x}} &= \frac{1}{h}K_x(y_j) + \left(p\frac{\partial^2 u}{\partial x^2}\right)_{x_i} + \frac{h}{6}\left[p\frac{\partial^3 u}{\partial x^3}\right] + \frac{h^2}{12}\left(p\frac{\partial^4 u}{\partial x^4}\right)_{x_i} \\
&+ \frac{h^3}{120}\left[p\frac{\partial^5 u}{\partial x^5}\right]_{x_i} + O(h^4), \\
(qu_{\bar{y}})_{\hat{y}\check{x}} &= \{q\}_{x_i}\frac{\partial^2 u}{\partial y^2} + \frac{[k^2]_j}{6\bar{k}_j}\{q\}_{x_i}\frac{\partial^3 u}{\partial y^3} + \frac{(k^2)_{\bar{y}_j}}{12}\{q\}_{x_i}\frac{\partial^4 u}{\partial y^4} \\
&+ \frac{[k^4]_j}{120\bar{k}_j}\{q\}_{x_i}\frac{\partial^5 u}{\partial y^5} + O(\bar{k}_j^4), \\
h^2 g_{\bar{x}x} &= h\left[\frac{\partial g}{\partial x}\right]_{x_i} + h^2\left\{\frac{\partial^2 g}{\partial x^2}\right\}_{x_i} + \frac{h^3}{6}\left[\frac{\partial^3 g}{\partial x^3}\right]_{x_i} + O(h^4), \\
(k^2 g_{\bar{y}})_{\hat{y}\check{x}} &= \frac{[k^2]_j}{\bar{k}_j}\left\{\frac{\partial g}{\partial y}\right\}_{x_i} + (k^2)_{\bar{y}_j}\left\{\frac{\partial^2 u}{\partial y^2}\right\}_{x_i} + \frac{[k^4]_j}{6\bar{k}_j}\left\{\frac{\partial^3 g}{\partial y^3}\right\}_{x_i} + O(\bar{k}_j^4), \\
\left(k^2(pu_{\bar{x}})_{x\bar{y}}\right)_{\hat{y}} &= \frac{[k^2]_j}{\bar{k}_j}\left\{p\frac{\partial^3 u}{\partial x^2\partial y}\right\}_{x_i} + \frac{h[k^2]_j}{6\bar{k}_j}\left[p\frac{\partial^4 u}{\partial x^3\partial y}\right]_{x_i} + (k^2)_{\bar{y}_j}\left\{p\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}_{x_i} \\
&+ \frac{[k^4]_j}{6\bar{k}_j}\left\{p\frac{\partial^5 u}{\partial x^2\partial y^3}\right\}_{x_i} + \frac{h^2[k^2]_j}{12\bar{k}_j}\left\{p\frac{\partial^5 u}{\partial x^4\partial y}\right\}_{x_i} \\
&+ \frac{h(k^2)_{\bar{y}_j}}{6}\left[p\frac{\partial^5 u}{\partial x^3\partial y^2}\right]_{x_i} + \frac{[k^2]_j}{h\bar{k}_j}K_x'(y_j) + \frac{(k^2)_{y_j}}{h}K_x''(y_j) \\
&+ \frac{[k^4]_j}{6h\bar{k}_j}K_x'''(y_j) + \frac{(k^4)_{y_j}}{12h}K_x^{IV}(y_j) + O(h+\bar{k}_j)^4, \\
h^2(qu_{\bar{y}})_{\hat{y}\bar{x}x} &= h\left\{\frac{q}{p}\right\}_{x_i}\left(K_x''(y_j) + \frac{[k^2]_j}{6\bar{k}_j}K_x'''(y_j) + \frac{(k^2)_{y_j}}{12}K_x^{IV}(y_j)\right) \\
&+ h^2\left\{q\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}_{x_i} + \frac{h^3}{6}\left[q\frac{\partial^5 u}{\partial x^3\partial y^2}\right]_{x_i}
\end{aligned}$$

$$+ \quad \frac{h^2[k^2]_j}{6\bar{k}_j}\left\{q\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\}_{x_i} + O(h+\bar{k}_j)^4,$$

$$\begin{aligned}
(pu_{\bar{x}})_{x\mathring{y}} &= \frac{1}{h}\left(K_x(y)\right)_{\mathring{y}} + \left\{p\frac{\partial^3 u}{\partial x^2 \partial y}\right\}_{x_i} + \frac{h}{6}\left[p\frac{\partial^4 u}{\partial x^3 \partial y}\right]_{x_i} + \frac{h^2}{12}\left\{p\frac{\partial^5 u}{\partial x^4 \partial y}\right\}_{x_i} \\
&+ \frac{k_j k_{j+1}}{3}\left\{p\frac{\partial^5 u}{\partial x^2 \partial y^3}\right\}_{x_i} + O(h+\bar{k}_j)^3, \\
g_{\mathring{y}\check{x}} &= \left\{\frac{\partial g}{\partial y}\right\}_{x_i} + \frac{k_j k_{j+1}}{6}\left\{\frac{\partial^3 u}{\partial y^3}\right\}_{x_i} + O(\bar{k}_j^3).
\end{aligned}$$

Then

$$\begin{aligned}
\Psi_{ij} &= \frac{1}{72h}\left[h^2 k^2\left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{x_i\check{y}_j} \\
&+ \frac{1}{72\bar{k}_j}\left[h^2 k^2\left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)\right]_{y_j\check{x}_i} \\
&- \frac{1}{180\hbar_i}\left[h^4 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i\check{y}_j} - \frac{1}{180\bar{k}_j}\left[k^4 q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\check{x}_i} - \Psi_{ij}^* + \Psi_{ij}^{**} \\
&= \frac{\left(k^2\right)_{\check{y}_j}}{72}\left(h^2\mu^x\right)_{\hat{x}_i,j} + \frac{\left(h^2\right)_{\check{x}_i}}{72}\left(k^2\mu^y\right)_{\hat{y}_j,i} - \frac{1}{180}\left(h^4\eta^x\right)_{\hat{x}_i,j} - \frac{1}{180}\left(k^4\eta^y\right)_{\hat{y}_j,i} \\
&- \Psi_{ij}^* + \Psi_{ij}^{**},
\end{aligned}$$

where

$$\mu_i^x(y) = \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)(x_{i-1/2}, y),\ \mu_j^y(x) = \left(r\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}\right)(x, y_{j-1/2}),$$

$$\eta_i^x(y) = \left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2}, y),\ \eta_j^y(x) = \left(q\frac{\partial^5 u}{\partial y^5}\right)(x, y_{j-1/2}),$$

$$\Psi_{ij}^* = \frac{k_j k_{j+1}}{72\bar{k}_j}\left[k^2 q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\check{x}_i},\ \Psi_{ij}^{**} = O(h+\bar{k}_j)^4.$$

One can analogously obtain the error in the case $y_j = \eta,\ x_i \neq \xi$.

$$\begin{aligned}
\Psi_{ij} &= \frac{\left(k^2\right)_{\check{y}_j}}{72}\left(h^2\mu^x\right)_{\hat{x}_i,j} + \frac{\left(h^2\right)_{\check{x}_i}}{72}\left(k^2\mu^y\right)_{\hat{y}_j,i} - \frac{1}{180}\left(h^4\eta^x\right)_{\hat{x}_i,j} - \frac{1}{180}\left(k^4\eta^y\right)_{\hat{y}_j,i} \\
&- \Psi_{ij}^* + \Psi_{ij}^{**},
\end{aligned}$$

where

$$\mu_i^x(y) = \left(r\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 f}{\partial x \partial y^2}\right)(x_{i-1/2}, y),\ \mu_j^y(x) = \left(r\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 f}{\partial x^2 \partial y}\right)_{\breve{y}}(x, y_{j-1/2}),$$

$$\eta_i^x(y) = \left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2}, y),\ \eta_j^y(x) = \left(q\frac{\partial^5 u}{\partial y^5}\right)(x, y_{j-1/2}),$$

$$\Psi_{ij}^* = \frac{h_i h_{i+1}}{72\bar{k}_j}\left[h^2 p\frac{\partial^5 u}{\partial x^5}\right]_{x_i \breve{y}_j},\ \Psi_{ij}^{**} = O(\hbar_i + h)^4.$$

Case 3.3. Points of type •. Now we have

$$\begin{aligned}
\Psi &= f_{\breve{x}\breve{y}} - r_{\breve{x}\breve{y}}u + (pu_{\bar{x}})_{x\breve{y}} + (qu_{\bar{y}})_{y\breve{x}} + \frac{h^2}{12}(f - ru)_{\bar{x}x\breve{y}} + \frac{h^2}{12}(f - ru)_{\bar{y}y\breve{x}} \\
&+ \frac{h^2}{12}(pu_{\bar{x}})_{x\bar{y}y} + \frac{h^2}{12}(qu_{\bar{y}})_{y\bar{x}x} + \frac{h}{12}\left[\frac{\partial f}{\partial x}\right]_{x_i\breve{y}} + \frac{h}{12}\left[\frac{\partial f}{\partial y}\right]_{y_j\breve{x}} \\
&- \left(\frac{h}{12}\left(\left\{\frac{r}{p}\right\}_{x_i} + \frac{1}{h}\right)K_x(y)\right)_{\breve{y}} - \left(\frac{h}{12}\left(\left\{\frac{r}{q}\right\}_{y_j} + \frac{1}{h}\right)K_y(x)\right)_{\breve{x}} \\
&- \frac{h}{12}(K_x(y))_{\bar{y}y} - \frac{h}{12}(K_y(x))_{\bar{x}x} + \frac{h}{12}\left(\left\{\frac{q}{p}\right\}_{x_i} K_x''(y)\right)_{\breve{y}} \\
&+ \frac{h}{12}\left(\left\{\frac{p}{q}\right\}_{y_j} K_y''(x)\right)_{\breve{x}} + \frac{h^2}{12}\left[\left[\frac{\partial^2 f}{\partial x \partial y} - r\frac{\partial^2 u}{\partial x \partial y}\right]_{x_i}\right]_{y_j}.
\end{aligned}$$

Using the equation (1.1) and the equalities

$$\begin{aligned}
(pu_{\bar{x}})_{x\breve{y}} &= \frac{1}{h}K_x(y)_{\breve{y}} + \left(p\frac{\partial^2 u}{\partial x^2}\right)_{\breve{x}\breve{y}} + \frac{h}{6}\left[p\frac{\partial^3 u}{\partial x^3}\right]_{x_i\breve{y}} + \frac{h^2}{12}\left(p\frac{\partial^4 u}{\partial x^4}\right)_{\breve{x}\breve{y}} \\
&+ \frac{h^3}{120}\left[p\frac{\partial^5 u}{\partial x^5}\right]_{x_i\breve{y}} + O(h^4), \\
(qu_{\bar{y}})_{\hat{y}\breve{x}} &= \frac{1}{h}K_y(x)_{\breve{x}} + \left(q\frac{\partial^2 u}{\partial y^2}\right)_{\breve{x}\breve{y}} + \frac{h}{6}\left[q\frac{\partial^3 u}{\partial y^3}\right]_{y_j\breve{x}} + \frac{h^2}{12}\left(q\frac{\partial^4 u}{\partial y^4}\right)_{\breve{x}\breve{y}} \\
&+ \frac{h^3}{120}\left[q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\breve{x}} + O(h^4), \\
h(K_y)_{\bar{x}x} &= [K_y']_{x_i} + h\{K_y''\}_{x_i} + \frac{h^2}{6}[K_y''']_{x_i} + \frac{h^3}{12}\{K_y^{IV}\}_{x_i} + O(h^4), \\
h(K_x)_{\bar{y}y} &= [K_x']_{y_j} + h\{K_x''\}_{y_j} + \frac{h^2}{6}[K_x''']_{y_j} + \frac{h^3}{12}\{K_x^{IV}\}_{y_j} + O(h^4), \\
h^2(pu_{\bar{x}})_{x\bar{y}y} &= \left[\left[p\frac{\partial^u}{\partial x \partial y}\right]_{x_i}\right]_{y_j} + h\left[p\frac{\partial^3 u}{\partial x \partial y^2}\right]_{x_i\breve{y}} + h\left[p\frac{\partial^3 u}{\partial x^2 \partial y}\right]_{y_j\breve{x}}
\end{aligned}$$

$$
\begin{aligned}
&+\quad h^2\left\{p\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}_{x_i y_j}+\frac{h^2}{6}\left[\left[p\frac{\partial^4 u}{\partial x^3\partial y}+p\frac{\partial^4 u}{\partial x\partial y^3}\right]_{x_i}\right]_{y_j}\\
&+\quad \frac{h^3}{12}\left[p\frac{\partial^5 u}{\partial x\partial y^4}\right]_{x_i\breve{y}}+\frac{h^3}{12}\left[p\frac{\partial^5 u}{\partial x^4\partial y}\right]_{y_j\breve{x}}+\frac{h^3}{6}\left[p\frac{\partial^5 u}{\partial x^3\partial y^2}\right]_{x_i\breve{y}}\\
&+\quad \frac{h^3}{6}\left[p\frac{\partial^5 u}{\partial x^2\partial y^3}\right]_{y_j\breve{x}}+O(h^4),
\end{aligned}
$$

$$
\begin{aligned}
h^2\left(qu_{\bar{y}}\right)_{\hat{y}\bar{x}x} &= \left[\left[q\frac{\partial^u}{\partial x\partial y}\right]_{x_i}\right]_{y_j}+h\left[q\frac{\partial^3 u}{\partial x\partial y^2}\right]_{x_i\breve{y}}+h\left[q\frac{\partial^3 u}{\partial x^2\partial y}\right]_{y_j\breve{x}}\\
&+\quad h^2\left\{q\frac{\partial^4 u}{\partial x^2\partial y^2}\right\}_{x_i y_j}+\frac{h^2}{6}\left[\left[q\frac{\partial^4 u}{\partial x^3\partial y}+q\frac{\partial^4 u}{\partial x\partial y^3}\right]_{x_i}\right]_{y_j}\\
&+\quad \frac{h^3}{12}\left[q\frac{\partial^5 u}{\partial x\partial y^4}\right]_{x_i\breve{y}}+\frac{h^3}{12}\left[q\frac{\partial^5 u}{\partial x^4\partial y}\right]_{y_j\breve{x}}\\
&+\quad \frac{h^3}{6}\left[q\frac{\partial^5 u}{\partial x^3\partial y^2}\right]_{x_i\breve{y}}+\frac{h^3}{6}\left[q\frac{\partial^5 u}{\partial x^2\partial y^3}\right]_{y_j\breve{x}}+O(h^4),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\Psi_{ij} &= \frac{h^3}{72}\left[r\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial^3 f}{\partial x\partial y^2}\right]_{x_i\breve{y}_j}+\frac{h^3}{72}\left[r\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial^3 f}{\partial x\partial y^2}\right]_{y_j\breve{x}_i}\\
&-\quad \frac{h^3}{180}\left[p\frac{\partial^5 u}{\partial x^5}\right]_{x_i\breve{y}_j}-\frac{h^3}{180}\left[q\frac{\partial^5 u}{\partial y^5}\right]_{y_j\breve{x}_i}-\Psi^*_{ij}+\Psi^{**}_{ij}\\
&= \frac{h^4}{72}\left(\mu^x\right)_{\hat{x}_i\breve{y}_j}+\frac{h^4}{72}\left(\mu^y\right)_{\hat{y}_j\breve{x}_i}-\frac{h^4}{180}\left(\eta^x\right)_{\hat{x}_i\breve{y}_j}-\frac{h^4}{180}\left(\eta^y\right)_{\hat{y}_j\breve{x}_i}\\
&-\quad \Psi^*_{ij}+\Psi^{**}_{ij},
\end{aligned}
$$

where

$$
\mu^x_i(y)=\left(r\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial^3 f}{\partial x\partial y^2}\right)(x_{i-1/2},y),\ \mu^y_j(x)=\left(r\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial^3 f}{\partial x^2\partial y}\right)(x,y_{j-1/2})
$$

$$
\eta^x_i(y)=\left(p\frac{\partial^5 u}{\partial x^5}\right)(x_{i-1/2},y),\ \eta^y_j(x)=\left(q\frac{\partial^5 u}{\partial y^5}\right)(x,y_{j-1/2}),
$$

$$
\Psi^*_{ij}=\frac{h^3}{144}\left[q\frac{\partial^5 u}{\partial x\partial y^4}\right]_{x_i\breve{y}_j}+\frac{h^3}{144}\left[p\frac{\partial^5 u}{\partial x^4\partial y}\right]_{y_j\breve{x}_i},\ \Psi^{**}_{ij}=O(h^4).
$$

Therefore

$$
\begin{aligned}
(\Psi,z)_\omega &= -\frac{1}{72}\left(h^2k^2\mu^x,z_{\bar{x}}\right)_{\omega_1^+\times\omega_2}-\frac{1}{72}\left(h^2k^2\mu^y,z_{\bar{y}}\right)_{\omega_1\times\omega_2^+}\\
&+\quad \frac{1}{180}\left(h^4\eta^x,z_{\bar{x}}\right)_{\omega_1^+\times\omega_2}+\frac{1}{180}\left(k^4\eta^y,z_{\bar{y}}\right)_{\omega_1\times\omega_2^+}-(\Psi^*,z)_\omega+(\Psi^{**},z)_\omega .
\end{aligned}
$$

Hence

$$\begin{aligned} |(\Psi, z)_\omega| &\le C\|h^2\|_0\|k^2\|_0\left(\|z_{\bar{x}}\|_0 + \|z_{\bar{y}}\|_0\right) + C\|h^4\|_0\|z_{\bar{x}}\|_0 + C\|k^4\|_0\|z_{\bar{y}}\|_0 \\ &+ \|\Psi^*\|_{-1}\|z\|_1 + \|\Psi^{**}\|_0\|z\|_0 \\ &\le C\left(\|h^4\|_0 + \|k^4\|_0 + \|\Psi^*\|_{-1}\right)\|z\|_1, \end{aligned}$$

where C is a constant, independent of the mesh. Therefore if the mesh is uniform

$$\begin{aligned} \|z\|_1 &\le Ch^4, \\ &\text{else} \\ \|z\|_1 &\le Ch^3. \quad \square \end{aligned} \tag{3.22}$$

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