ГОДИШНИК НА СОФИЙСКИЯ УНИВЕРСИТЕТ „СВ. КЛИМЕНТ ОХРИДСКИ“

ФАКУЛТЕТ ПО МАТЕМАТИКА И ИНФОРМАТИКА
Том 99

ANNUAIRE DE L'UNIVERSITE DE SOFIA „ST. KLIMENT OHRIDSKI“

FACULTE DE MATHEMATIQUES ET INFORMATIQUE
Tome 99

MONADIC SECOND-ORDER LOGIC ON EQUIVALENCE RELATIONS [1]

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This paper is devoted to exploring expressible power of monadic second-order sentences over the class of all relational structures containing only finite number of equivalence relations which are in local agreement (i.e. for any point of the universe the corresponding equivalence classes with set theoretic inclusion form linear order). Using the pebble games we prove the finite model property and establish an effective translation of these sentences in the first-order language preserving the models. So, the monadic second-order language over the considered class of relational structures has the same expressible power as the first-order language and the monadic second-order theory of this class of structures is decidable.

Keywords: MSO sentences, equivalence relations, finite model property, elimination of second-order quantifiers, decidability.

2000 MSC: Main 03C52; Secondary 03C13, 03C85

We consider purely relational finite languages for the first-order predicate calculus with only unary and binary predicate symbols. Let $\mathcal{L} = (P_1, \dots P_r, R_1 \dots R_n)$ be such a language, with $P_1, \dots, P_r$ and $R_1, \dots, R_n$ being the unary and the binary predicate symbols, respectively. Take the class of structures where the interpretations of the binary predicate symbols are equivalence relations. In [3] Ershov announces that the monadic second-order logic of this class of structures is decidable for $n = 1$. Furthermore, in [2] Janiczak shows that the first-order logic of this class is undecidable for $n \geq 2$.

We further restrict the equivalence relations and consider the class of structures in which the binary relations are interpreted by equivalence relations in local

[1] Partially supported by Contract 27 with Sofia University.

agreement. We then show the decidability of the resulting monadic second order logic and demonstrate that there is a translation of every MSO sentence ψ to a first-order sentence ψ' such that ψ and ψ' have exactly the same models.

From now on, unless explicitly stated otherwise, we consider all languages to be finite and purely relational with only unary and binary predicate symbols. We also fix a language $\mathcal{L} = (P_1, \ldots, P_r, R_1, \ldots, R_n)$, with $P_1, \ldots, P_r$ and $R_1, \ldots, R_n$ being the unary and the binary predicate symbols, respectively. We assume that we always have equality in the structures and for convenience treat the equality as one of the binary predicates of the structures. Thus we always have $n > 0$.

In what follows we extensively use a kind of bisimulation games called "pebble games" to show similarity between structures. For complete details refer to [1]. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for $\mathcal{L}$ and let $s \in \mathbf{N}, s > 0$. The infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$ is played by two players on a board which consists of the two structures $\mathfrak{A}$ and $\mathfrak{B}$. Each player has s pebbles, numbered from 1 to s. Players take turns. The first player chooses a pebble from her set of pebbles and a structure ($\mathfrak{A}$ or $\mathfrak{B}$) and places the selected pebble on some element of the structure. The second player answers by placing his pebble with the same number on some element of the other structure. The game continues indefinitely. Each time, after Player II has made his move, there is an even number of pebbles on the board. Half of them are in $\mathfrak{A}$ and the other half – in $\mathfrak{B}$. For example let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be the elements of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, on which the players have pebbles, and let for all $i = 1, \ldots, k$, a_i and b_i are under equal-numbered pebbles. Before each move of Player I the players review the configuration on the board, and if they find that the mapping $f : a_i \to b_i, i = 1, \ldots, k$ is not a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, Player I wins. Player II wins only if Player I does not win at any move.

Definition 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures for the language $\mathcal{L}$ and let $s \in \mathbf{N}, s > 0$. We say that Player II has a winning strategy for the infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$ iff Player II can win the game, no matter how Player I plays.*

A similar definition can be given for winning strategy for Player I. Obviously, for a given game exactly one player has a winning strategy.

Definition 2. *We say that two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language $\mathcal{L}$ are s-partially isomorphic (and write it $\mathfrak{A} \cong^s_{part} \mathfrak{B}$) iff Player II has a winning strategy for the infinite pebble game $G^s_\infty(\mathfrak{A}, \mathfrak{B})$.*

The main result about pebble games is given by the following:

Theorem 1. *([1]) For any two structures $\mathfrak{A}$ and $\mathfrak{B}$ for the language $\mathcal{L}$ and for any $s \in \mathbf{N}, s > 0$, $\mathfrak{A} \cong^s_{part} \mathfrak{B}$ iff $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulas of no more than s variables of the infinitary logic $\mathcal{L}_{\infty\omega}$.*

Definition 3. *Let $R_1, \ldots, R_n$ be equivalence relations with common domain. $R_1, \ldots, R_n$ are in local agreement iff for all x in the domain of the relations, the set $\{|x|_{R_1}, \ldots, |x|_{R_n}\}$ of the equivalence classes of x according to $R_1, \ldots, R_n$ is linearly ordered according to the set theoretic inclusion.*

Definition 4. *Let $\mathcal{L}$ be a language of the considered type. We denote by $\mathcal{K}_\mathcal{L}$ the class of structures for $\mathcal{L}$ in which the binary predicates are interpreted with equivalence relations in local agreement. Sometimes, when the language $\mathcal{L}$ can be determined from the context we write $\mathcal{K}$ instead of $\mathcal{K}_\mathcal{L}$*

Definition 5. *Let $\mathfrak{A} \in \mathcal{K}$ and let $C \subseteq |\mathfrak{A}|$. We say that C is a maximal equivalence class in $\mathfrak{A}$ iff*

$$(\exists 1 \leq i \leq n)(|x|_{R_i} = C \,\&\, (\forall x \in C)(\forall 1 \leq j \leq n)(|x|_{R_j} \subseteq |x|_{R_i}))$$

Note that C is a maximal equivalence class iff C is a equivalence class and it is not a proper subset of any other equivalence class.

For brevity we sometimes use the term 'class' instead of the long form 'maximal equivalence class'.

Let $\mathfrak{A} = (|\mathfrak{A}|, P_1, \ldots, P_r, R_1, \ldots R_n), \mathfrak{A} \in K$ and C is a maximal equivalence class in $\mathfrak{A}$. Let $P_i' \rightleftharpoons P_{i|C}$, for $1 \leq i \leq r$ and $R_i' \rightleftharpoons R_{i|C}$ for $1 \leq i \leq n$. Then $\mathfrak{C} = (C, P_1', \ldots, P_r', R_1', \ldots, R_n')$ is a substructure of $\mathfrak{A}$. We say that $\mathfrak{C}$ is the substructure of $\mathfrak{A}$ generated by C.

Since the maximal equivalence classes do not intersect and also cover the whole set $|\mathfrak{A}|$, we get that the structure $\mathfrak{A}$ can be represented as a direct sum of the substructures generated by its maximal equivalence classes. As there is one-to-one mapping from maximal equivalence classes and the substructures generated by them we shall use these two terms interchangeably. Whether we speak about an equivalence class or a substructure will be clear from the context.

Definition 6. *Let $s \in \mathbf{N}, s > 0$, and let k and l be cardinals (finite or infinite). We say that k and l are s-equal iff:*

$$k = l \vee (k \geq s \,\&\, l \geq s)$$

Note that for any $s \in \mathbf{N}, s > 0$, any two cardinals greater or equal to s are s-equal.

Proposition 1. *Let $s \in \mathbf{N}, s > 0$. Two structures $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B} \in \mathcal{K}$ are s-partially isomorphic if and only if for each maximal equivalence class in one of the structures. the number of the classes s-partially isomorphic to it in $\mathfrak{A}$ is s-equal to the number of the classes s-partially isomorphic to it in $\mathfrak{B}$.*

Proof: We use Pebble games to show the equivalence.

First, let the condition be true. We show that $\mathfrak{A} \cong^s_{part} \mathfrak{B}$ by showing there is a winning strategy for Player II in the infinite pebble game with s pebbles.

Suppose we have k pebbles, $a_1, \ldots, a_k$ in $\mathfrak{A}$ and k pebbles, $b_1, \ldots, b_k$ in $\mathfrak{B}$, $k \leq s$ and the mapping $a_1, \ldots, a_k \to b_1, \ldots, b_k$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. We will show that whatever the first player does, the second player can preserve the partial isomorphism. Obviously, this implies that the two structures are s-partially isomorphic.

When $k < s$, the first player has two choices – she can either move one of her pebbles already placed in one of the structures, or she can place a new pebble in one of the structures. Without loss of generality we shall consider the case when a new pebble is placed in one of the structures.

Thus, let a_{k+1} be the new pebble placed in $\mathfrak{A}$. The pebble goes to one of the classes in $\mathfrak{A}$. Denote that class by A. Clearly, if A contains any other pebbles, then the class B where Player II should place his answer is determined. Otherwise, Player II should pick a class B of $\mathfrak{B}$ which is s-partially isomorphic to A and does not contain any pebbles in it. This can always be done because the number of classes in $\mathfrak{A}$ which are s-partially isomorphic to A is s-equal to the number of class in $\mathfrak{B}$ which are s-partially isomorphic to A. A similar argument can be used when Player I places the pebble in $\mathfrak{B}$. Thus Player II has a winning strategy for the game and therefore $\mathfrak{A}\cong^s_{part}\mathfrak{B}$.

Now suppose $\mathfrak{A}\cong^s_{part}\mathfrak{B}$ and suppose there is a class C in one of the structures, such that the number of classes of $\mathfrak{A}$ which are s-partially isomorphic to C is not s-equal to the number of classes in $\mathfrak{B}$ which are s-partially isomorphic to C. There are several cases, but without loss of generality we shall consider only one of them – when there are finite number of classes s-partially isomorphic to C in both structures. So, let $A_1, \ldots, A_k$ and $B_1, \ldots, B_l$ are all the classes in $\mathfrak{A}$ and $\mathfrak{B}$ respectively, which are s-partially isomorphic to C (the class C, of course, is among them). As k is not s-equal to l it follows that either $k < l \,\&\, k < s$ or $l < k \,\&\, l < s$.

1. $k < l \,\&\, k < s$.

   In that case the following is a winning strategy for the first player:

   Start placing one pebble in each class from $B_1, \ldots, B_l$. Since $k < l \,\&\, k < s$ there will be a move in which Player II will place his pebble in some class A after Player I has placed her pebble in some class B and the class A will either has already a pebble in it (in which case Player II immediately loses) or A will not be s-partially isomorphic to C (and B). At this point Player I can restrict the game to the classes A and B only. As these two classes are not s-partially isomorphic Player I has a winning strategy for the rest of the game.

2. $l < k \,\&\, l < s$

   This case is resolved by symmetry (this time Player I starts to place pebbles in classes of $\mathfrak{A}$)

□

Definition 7. *Let φ be a formula of the form $\exists x_1 \ldots \exists x_k \forall y \psi$, where ψ is a formula. Let x be a variable and R be a binary predicate symbol. We call the formula*

$$\varphi^R_x \rightleftharpoons \exists x_1 \ldots \exists x_k \forall y \left(\bigwedge_{i=1}^{k} x_i R x \,\&\, (yRx \Rightarrow \psi) \right)$$

a shallow relativization of φ (w.r.t. x and R).

Theorem 2. *For each structure $\mathfrak{A} \in \mathcal{K}$ and for each $s \in \mathbf{N}, s > 0$, there is a FO-formula $\psi^s_{\mathfrak{A}}$, such that for every structure $\mathfrak{B} \in \mathcal{K}$:*

$$\mathfrak{B} \models \psi^s_{\mathfrak{A}} \iff \mathfrak{A} \cong^s_{part} \mathfrak{B}$$

Moreover, for a given $s \in \mathbf{N}, s > 0$, there are only finite number of different formulae $\psi^s_{\mathfrak{A}}$ (modulo logical equivalence). In other words, for a given $s \in \mathbf{N}, s > 0$, the relation $\cong^s_{part}$ has finitely many equivalence classes.

Proof: Let M be the set of all substructures of $\mathfrak{A}$ which are generated by the maximal equivalence classes of $\mathfrak{A}$. The relation $\cong^s_{part}$ partitions M to equivalence classes. Let J be the factorization of M by $\cong^s_{part}$ and $I_j, j \in J$ be the equivalence classes of M. For $j \in J$ and $i \in I_j$, denote by $\mathfrak{A}^i_j$ the structure i. Then $\mathfrak{A}$ can be represented in the following way:

$$\mathfrak{A} = \bigcup_{j \in J, i \in I_j} \mathfrak{A}^i_j$$

We have:

$$\mathfrak{A}^{i_1}_{j_1} \cong^s_{part} \mathfrak{A}^{i_2}_{j_2} \iff j_1 = j_2$$

Let $m_j = \min\{\overline{\overline{I}}_j, s\}$, for $j \in J$ and $m = \overline{\overline{J}}$.

The proof is by induction on the number n of the binary predicate symbols in the language of the structure $\mathfrak{A}$.

1. $n = 1$

   In this case each of the structures $\mathfrak{A}^i_j$ consists of a single element. Consider the following formula:

   $$\psi^{\mathfrak{A}^i_j}(x) \rightleftharpoons \lambda^1_j P_1(x) \,\&\, \ldots \,\&\, \lambda^r_j P_r(x)$$

   where λ^k_j is the empty word, when $\mathfrak{A}^i_j \models \exists x P_k(x)$, and λ^k_j is the negation sign otherwise. Obviously, if two one-element structures satisfy the same formula of the above mentioned type then the structures are s-partially isomorphic. Note that $\psi^{\mathfrak{A}^i_j}(x)$ depends only on j, but not on i. For that reason we may omit the upper index of $\mathfrak{A}$ and just write $\psi^{\mathfrak{A}_j}(x)$.

   Now take the formula:

   $$\begin{aligned}\psi^s_{\mathfrak{A}} \rightleftharpoons \quad & \exists x^1_1 \ldots \exists x^1_{m_1} \\ & \exists x^2_1 \ldots \exists x^2_{m_2} \\ & \ldots \\ & \exists x^m_1 \ldots \exists x^m_{m_m}\end{aligned}$$

$$\begin{aligned}\forall y(&\bigwedge_{i_1 \neq i_2 \vee j_1 \neq j_2} x_{i_1}^{j_1} \neq x_{i_2}^{j_2} \,\&\\ &\& \,\tau_1(x_1^1, \ldots, x_{m_1}^1) \,\&\\ &\ldots\\ &\& \,\tau_m(x_1^m, \ldots, x_{m_m}^m) \,\&\\ &\& \,(\bigvee_{j \in J} \psi^{\mathfrak{A}_j}(y)))\end{aligned}$$

where the formulae τ are defined as follows:

$$\tau_j(x_1, \ldots, x_k)$$
$$\rightleftharpoons \forall z(\bigwedge_{i=1}^{k} z \neq x_i \Rightarrow \neg\psi^{\mathfrak{A}_j}(z) \,\&\, \psi^{\mathfrak{A}_j}(x_1) \,\&\, \ldots \,\&\, \psi^{\mathfrak{A}_j}(x_k)), \text{ when } k < s$$

and

$$\tau_j(x_1, \ldots, x_k) \rightleftharpoons \psi^{\mathfrak{A}_j}(x_1) \,\&\, \ldots \,\&\, \psi^{\mathfrak{A}_j}(x_k), \text{ when } k = s$$

The meaning of the formula is evident - it just guarantees the desired condition from Proposition 1.

2. $n > 1$

   Let $j \in J$ is fixed. Since in each of the structures $\mathfrak{A}_j^i$ at least one of the binary relations coincide with the universal relation, we can drop one relation and use the induction hypothesis. Without loss of generality suppose that R_n is interpreted by the universal relation in each structure $\mathfrak{A}_j^i$. Let $\mathcal{L}' = \mathcal{L} \setminus \{R_n\}$. We can use the induction hypothesis and see that for the language $\mathcal{L}'$ there is a formula $\psi_{\mathfrak{A}_j^i}^s$ such that:

$$\mathfrak{B}_{|\mathcal{L}'} \models \psi_{\mathfrak{A}_j^i}^s \iff \mathfrak{B}_{|\mathcal{L}'} \cong_{part}^s \mathfrak{A}_j^i \text{ for } i \in I_j$$

   From here we see that $\psi_{\mathfrak{A}_j^i}^s$ does not depend on i and hence we again can use the short notation $\psi_{\mathfrak{A}_j}^s$. From the induction hypothesis we also get that all these formulae are finitely many (modulo logical equivalence), and hence J is finite.

   Note that $\psi_{\mathfrak{A}_j}^s$ is a formula in both $\mathcal{L}$ and $\mathcal{L}'$. Now let the formulae $\overline{\psi_{\mathfrak{A}_j}^s}$ are shallow relativizations of $\psi_{\mathfrak{A}_j}^s$ w.r.t. R_n (The variable w.r.t. which we relativize will be clear from the context).

   The formula we are seeking is:

$$\begin{aligned}\psi_{\mathfrak{A}}^s \rightleftharpoons \quad &\exists x_1^1 \ldots \exists x_{m_1}^1\\ &\exists x_1^2 \ldots \exists x_{m_2}^2\\ &\ldots\end{aligned}$$

$$
\begin{aligned}
&\exists x_1^m \dots \exists x_{m_m}^m \\
&\forall y(\bigwedge_{\substack{i_1 \neq i_2 \vee j_1 \neq j_2 \\ k=1\dots n}} \neg x_{i_1}^{j_1} R_k x_{i_2}^{j_2} \; \& \\
&\& \; \tau_1(x_1^1, \dots, x_{m_1}^1) \; \& \\
&\dots \\
&\& \; \tau_m(x_1^m, \dots, x_{m_m}^m) \; \& \\
&\& \; (\bigvee_{j \in J} \overline{\psi_{\mathfrak{A}_j}^s}(y)))
\end{aligned}
$$

where the formulae τ are defined as:

$$
\begin{aligned}
&\tau_j(x_1, \dots, x_k) \\
&\rightleftharpoons \forall z(\bigwedge_{i=1}^{k} z \neq x_i \Rightarrow \neg\overline{\psi_{\mathfrak{A}_j}^s}(z) \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_1) \,\&\, \dots \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_k)), \text{ when } k < s
\end{aligned}
$$

and as:

$$
\tau_j(x_1, \dots, x_k) \rightleftharpoons \overline{\psi_{\mathfrak{A}_j}^s}(x_1) \,\&\, \dots \,\&\, \overline{\psi_{\mathfrak{A}_j}^s}(x_k), \text{ when } k = s
$$

Since the number of elements of J and the numbers m_j are bounded we get that there are only finite number of formulae $\psi_{\mathfrak{A}}^s$, for all structures of $\mathcal{K}$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{K}$ and $\mathfrak{B}_i$ is a maximal equivalence class in $\mathfrak{B}$ relatively the binary predicate symbol R. Let $\overline{\psi_{\mathfrak{A}}^s}$ be a shallow relativisation of $\psi_{\mathfrak{A}}^s$ w.r.t. R and x.

Observe that:

$$
\mathfrak{B}_i \models \psi_{\mathfrak{A}}^s \iff \mathfrak{B} \models \overline{\psi_{\mathfrak{A}}^s}[a/x], \text{ for } a \in |\mathfrak{B}_i|
$$

From the observation it is easy to verify that the formula we gave guarantees the condition from Proposition 1.

□

Definition 8. *Let $h(s, n, r)$ denote the number of equivalence classes of the relation $\cong_{part}^s$ for a language with n binary and r unary predicate symbols.*

Corollary 1. *For each structure $\mathfrak{A}$ in $\mathcal{K}$ there exists a finite structure $\mathfrak{A}_{\mathsf{fin}}$, such that:*

$$
\mathfrak{A} \cong_{part}^s \mathfrak{A}_{\mathsf{fin}}
$$

and the structure $\mathfrak{A}_{\mathsf{fin}}$ can be selected to be of cardinality bounded from above by a computable function.

Proof: The proof easy follows from Proposition 1 by induction on the number of binary predicate symbols in the language and the fact that $\cong^s_{part}$ has only finite number of equivalence classes. The cardinality of $\mathfrak{A}_{fin}$ is bounded by $s^n.\prod_{k=1}^{n} h(s,k,r)$.
□

Proposition 2. *Let $\mathfrak{A}$ u $\mathfrak{B}$ be structures from $\mathcal{K}$ and let $\mathfrak{A}\cong^{s.h(s,n,r+1)}_{part}\mathfrak{B}$. Then for any $P \subseteq |\mathfrak{A}|$ exists $Q \subseteq |\mathfrak{B}|$, such that: $(\mathfrak{A}, P) \cong^s_{part} (\mathfrak{B}, Q)$*

Proof: The proof is by induction on the number n of the binary predicate symbols in the language.

As in the proof of Theorem 2, let the structure $\mathfrak{A}$ be represented as a direct sum of its maximal equivalence classes:

$$\mathfrak{A} = \bigcup_{j\in J^{\mathfrak{A}}, i\in I_j} \mathfrak{A}^i_j$$

Where $J^{\mathfrak{A}}$ is the factorization of the set of maximal equivalence classes of $\mathfrak{A}$ over the relation $\cong^{s.h(s,n,r+1)}_{part}$. For $j \in J^{\mathfrak{A}}$, I_j is the appropriate equivalence class of $\cong^{s.h(s,n,r+1)}_{part}$ over the set of the maximal equivalence classes of $\mathfrak{A}$ and $\mathfrak{A}^i_j$ is more verbose notation for i.

Similarly, $\mathfrak{B}$ can be represented as:

$$\mathfrak{B} = \bigcup_{j\in J^{\mathfrak{B}}, i\in I^{\mathfrak{B}}_j} \mathfrak{B}^i_j$$

Note that due to the fact that the structures $\mathfrak{A}$ and $\mathfrak{B}$ are $s.h(s,n,r+1)$-partially isomorphic we have $J^{\mathfrak{A}} = J^{\mathfrak{B}}$.

For convenience we assume that the two representations are compatible in the following sense:

$$(\forall j_1 \in J^{\mathfrak{A}})(\forall j_2 \in J^{\mathfrak{B}})(j_1 = j_2 \iff (\forall i_1 \in I^{\mathfrak{A}}_{j_1})(\forall i_2 \in I^{\mathfrak{B}}_{j_2})(\mathfrak{A}^{i_1}_{j_1} \cong^{s.h(s,n,r+1)}_{part} \mathfrak{B}^{i_2}_{j_2}))$$

From Proposition 1 we get

$$\overline{\overline{I^{\mathfrak{A}}_j}} =_{s.h(s,n,r+1)} \overline{\overline{I^{\mathfrak{B}}_j}} \tag{1}$$

for all $j \in J^{\mathfrak{B}}$.

Let now $P \subseteq |\mathfrak{A}|$. We introduce a new predicate symbol P_{r+1}, which will be interpreted in $\mathfrak{A}$ by the predicate P. Thus we obtain a new structure $\mathfrak{C} \rightleftharpoons (\mathfrak{A}, P)$ for the enriched language $\mathcal{L} \cup \{P_{r+1}\}$.

The structure $\mathfrak{C}$ can be represented as a direct sum of maximal equivalence classes as well:

$$\mathfrak{C} = \bigcup_{j\in J^{\mathfrak{C}}, i\in I^{\mathfrak{C}}_j} \mathfrak{C}^i_j$$

Now let $J_j^{\mathfrak{C}} \rightleftharpoons \{\mathfrak{c}_{|\mathcal{L}} \mid \mathfrak{c} \in I_j^{\mathfrak{C}}\}$, for $j \in J^{\mathfrak{C}}$. For the above representation,evidently, for each $j \in J^{\mathfrak{A}}$, there are $j_1, \ldots, j_r \in J^{\mathfrak{C}}$, with $r \leqq h(s, n, r+1)$ such that:

$$I_j^{\mathfrak{A}} = J_{j_1}^{\mathfrak{C}} \cup \ldots \cup J_{j_r}^{\mathfrak{C}}$$

Now fix a $j \in J^{\mathfrak{A}}$. Because of equation (1) and since $r \leqq h(s, n, r+1)$ it is easy to see that $I_j^{\mathfrak{B}}$ can be represented as:

$$I_j^{\mathfrak{B}} = I_{j_1} \cup \ldots \cup I_{j_r}$$

and

$$(\forall 1 \leqq i \leqq r)(\overline{\overline{I_{j_i}}} =_s \overline{\overline{I_{j_i}^{\mathfrak{C}}}})$$

Let $1 \leqq i \leqq r$ be fixed, $\mathfrak{B}_{j_i}^{\ell}$ be arbitrary structure from I_{j_i} and $\mathfrak{C}_{j_i}^{t} \in I_{j_i}^{\mathfrak{C}}$. We shall define a predicate Q in $\mathfrak{B}_{j_i}^{\ell}$, with the desired properties. The definition of Q can be carried in the same way for any structure of I_{j_i}.

We distinguish two cases:

1. $n = 1$:

   Without loss of generality we can assume that the only binary relation in the language is the equality. In this case $\mathfrak{C}_{j_i}^{t}$ and $\mathfrak{B}_{j_i}^{\ell}$ are one-point structures and Q can be defined on $\mathfrak{B}_{j_i}^{\ell}$, in the same way as P is defined in $\mathfrak{C}_{j_i}^{t}$.

2. $n > 1$:

   Let $\mathfrak{A}_j^t \rightleftharpoons \mathfrak{C}_{j_i|\mathcal{L}}^{t}$. From the fact that $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ are maximal equivalence classes for $\mathfrak{A}$ and $\mathfrak{B}$ respectively, and from $\mathfrak{A}_j^t \cong_{part}^{s.h(s,n,r+1)} \mathfrak{B}_{j_i}^{\ell}$, it follows that there exists a binary predicate symbol R from the language $\mathcal{L}$, such that it is interpreted in $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ with the universal relation. Let $\mathcal{L}' \rightleftharpoons \mathcal{L} \setminus \{R\}$ and consider the restrictions of $\mathfrak{A}_j^t$ and $\mathfrak{B}_{j_i}^{\ell}$ to the language $\mathcal{L}'$. We have:

   $$\mathfrak{A}_j^t|_{\mathcal{L}'} \cong_{part}^{s.h(s,n,r+1)} \mathfrak{B}_{j_i}^{\ell}|_{\mathcal{L}'}$$

   Since $h(s, n, r+1) \geqq h(s, n-1, r+1)$, from the induction hypothesis we get:

   $$\left(\exists Q \subseteq |\mathfrak{B}|_{j_i}^{\ell}\right)\left(\left(\mathfrak{A}_j^t, P\right)_{|\mathcal{L}'} \cong_{part}^{s} \left(\mathfrak{B}_{j_i}^{\ell}, Q\right)_{|\mathcal{L}'}\right)$$

   As R is interpreted in both structures with the universal relation we get:

   $$\left(\mathfrak{A}_j^t, P\right) \cong_{part}^{s} \left(\mathfrak{B}_{j_i}^{\ell}, Q\right)$$

Thus we show that the predicate Q can be defined on any maximal equivalence class. Now, taking the union of these predicates we define the interpretation of Q on the structure $\mathfrak{B}$. □

Definition 9. *Let φ be a monadic second-order formula. We say that φ is first-order definable over $\mathcal{K}$ iff there is a first-order formula ψ such that for every structure $\mathfrak{A} \in \mathcal{K}$:*

$$\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \psi$$

Theorem 3. *Every MSO sentence is first-order definable over $\mathcal{K}$.*

Proof: Let $\varphi \rightleftharpoons K_1 P_1 \ldots K_m P_m \varphi_1$ be a MSO formula, where $K_1, \ldots, K_m$ are quantifiers, $P_1, \ldots, P_m$ are monadic second-order variables, and φ_1 is a FO formula. We shall prove that there exists a FO formula ψ, such that for each structure $\mathfrak{A} \in \mathcal{K}$:

$$\mathfrak{A} \models \varphi \iff \mathfrak{A} \models \psi$$

It is sufficient to show how to remove from the formula φ a single quantifier over a second-order variable.

1. Let $\varphi \rightleftharpoons \exists P \varphi_1$, where φ_1 is a first-order formula and let $\mathfrak{A} \in \mathcal{K}$. Let s be the number of the first-order variables that appear in φ_1. Consider the formula:

$$\psi \rightleftharpoons \bigvee_{\substack{\mathfrak{A} \in \mathcal{K} \\ (\exists P \subseteq |\mathfrak{A}|)((\mathfrak{A}, P) \models \varphi_1)}} \psi_{\mathfrak{A}}^{s.h(s,n,r+1)}$$

   Because of the finite number of the formulae $\psi_{\mathfrak{A}}^{m}$ (modulo logical equivalence), for all $m \in \mathbf{N}, m > 0$, the disjunction is a finite and hence ψ is a first-order formula.

   Note that the above disjunction can be empty. In that case the normal conventions apply: that is, we consider the empty disjunctions to be false (replacing the disjunction with $\neg \forall x (x = x)$, for example)

   Apparently, for each $\mathfrak{B} \in \mathcal{K}$ if $\mathfrak{B} \models \exists P \varphi_1$, then $\mathfrak{B} \models \psi$.

   In the opposite direction, take $\mathfrak{B} \in \mathcal{K}$ and $\mathfrak{B} \models \psi$. We shall demonstrate that $\mathfrak{B} \models \exists P \varphi_1$. Since $\mathfrak{B} \models \psi$ there exists a structure $\mathfrak{A} \in \mathcal{K}$ such that $\mathfrak{B} \models \psi_{\mathfrak{A}}^{s.h(s,n,k+1)}$ and hence $\mathfrak{A} \cong_{part}^{s.h(s,n,k+1)} \mathfrak{B}$. Moreover, for $\mathfrak{A}$ we have:

$$(\exists P \subseteq |\mathfrak{A}|)\, ((\mathfrak{A}, P) \models \varphi_1)$$

   From Proposition 2 we obtain that there exists $Q \subseteq |\mathfrak{B}|$ such that:

$$(\mathfrak{A}, P) \cong_{part}^{s} (\mathfrak{B}, Q)$$

   Thus we get $(\mathfrak{B}, Q) \models \varphi_1$, from where we conclude the desired $\mathfrak{B} \models \varphi$.

2. Let $\varphi \rightleftharpoons \forall P \varphi_1$, where φ_1 is a first-order formula. Consider the formula:

$$\psi \rightleftharpoons \bigvee_{\substack{\mathfrak{A} \in \mathcal{K} \\ (\forall P \subseteq |\mathfrak{A}|)((\mathfrak{A}, P) \models \varphi_1)}} \psi_{\mathfrak{A}}^{s.h(s,n,k+1)}$$

   By an argument similar to the above we obtain $\mathfrak{B} \models \psi \iff \mathfrak{B} \models \varphi$.

□

Corollary 2 *The monadic second-order logic of $\mathcal{K}$ is decidable.*

Proof: Immediately from Corollary 1 and Theorem 3. □

Finally, we would like to mention some further directions for research on the subject. One interesting area is adding functional symbols to the language. We already know that adding one functional symbol under some simple restrictions does produce a theory which is not decidable, but have not investigated other interesting cases. Also, one can try to expand this result to arbitrary formulae, not just sentences. We do not know yet if this can be done. Another interesting perspective is research on some possible connections with Data Analysis Logic.

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Received on October 26, 2006

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