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EXAMPLES OF HNN-EXTENSIONS WITH NONTRIVIAL QUASI–KERNELS

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We introduce some examples of HNN-extensions motivated by the problems of C^*-simplicity and unique trace property. Moreover, we prove that our examples are not inner amenable and identify a relatively large, simple, normal subgroup in each one.

Keywords: C^*-simplicity, HNN-extensions, inner amenability.

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1. INTRODUCTION AND PRELIMINARIES

1.1. INTRODUCTION

The questions of C^*-simplicity and unique trace property for a discrete group have been studied extensively. By definition, a discrete group G is C^*-simple if the C^*-algebra associated to the left regular representation, $C^*_r(G)$, is simple; likewise it has the unique trace property if $C^*_r(G)$ has a unique tracial state. An extensive introduction to that topic was given by de la Harpe ([6]). Recently, Kalantar and Kennedy ([10]) gave a necessary and sufficient condition for C^*-simplicity in terms of action on the Furstenberg boundary of the group in question. Later, Breuillard, Kalantar, Kennedy, and Ozawa ([2]) studied further the question of C^*-simplicity and also showed that a group has the unique trace property if and only if its amenable radical is trivial. They also showed that C^*-simplicity implies the unique trace property. The reverse implication was disproven by examples given by Le Boudec ([11]). In the case of group amalgamations and HNN-extensions, the kernel controls the uniqueness of trace, and the quasi-kernels control the C^*-simplicity.

The notion of inner amenability for discrete groups was introduced by Effros ([5]) as an analogue to Property Γ for II_1 factors that was introduced by Murray and von Neumann ([12]). By definition, a discrete group G is inner amenable if there exist a conjugation invariant, positive, finitely additive, probability measure on $G\backslash\{1\}$. Effros showed that Property Γ implies inner amenability, but the reverse implication doesn't hold, as demonstrated by Vaes ([14]).

Our examples (all of which being HNN-extensions) stem from the questions of C^*-simplicity and the unique trace properties for groups. In particular, all of our examples have the unique trace property, and we also determine the C^*-simple ones and the non-C^*-simple ones. The examples of section 2 generalize the example given in [3, Section 5] (which corresponds to the group $\Lambda[Sym(2), Sym(2)]$ of section 2). There is a resemblance to the groups introduced by Le Boudec in [11] since they all act on trees. The main benefit is that our groups are given concretely by generators and relations, which makes them more tractable to investigate some further properties they possess.

We study some additional analytic properties of our examples. We show that they are all non-inner-amenable by showing that they are finitely fledged - a property that we introduce in [8].

We also explore some of the group-theoretical properties of our groups. We remark that they are not finitely presented. Also, under some mild natural assumptions, we show that each group has a relatively large, simple, normal subgroup.

1.2. PRELIMINARIES

For a group Γ acting on a set X, we denote the set-wise stabilizer of a subset $Y \subset X$ by

$$\Gamma_{\{Y\}} \equiv \{ g \in \Gamma \mid gY = Y \}$$

and the point-wise stabilizer of a subset $Y \subset X$ by

$$\Gamma_{(Y)} \equiv \{ g \in \Gamma \mid gy = y, \ \forall y \in Y \}.$$

For a point $x \in X$, we denote its stabilizer by

$$\Gamma_x = \{ g \in \Gamma \mid gx = x \}.$$

Note that, $\Gamma_{\{Y\}}$, $\Gamma_{(Y)}$, and Γ_x are all subgroups of Γ. Also note that,

$$g\Gamma_{\{Y\}}g^{-1} = \Gamma_{\{gY\}}, \ g\Gamma_x g^{-1} = \Gamma_{gx} \text{ , and } g\Gamma_{(Y)}g^{-1} = \Gamma_{(gY)}.$$

For a group G and its subgroup H, by $\langle\langle H\rangle\rangle_G$ or by $\langle\langle H\rangle\rangle$, we denote the normal closure of H in G.

For some general references on group amalgamations and HNN-extensions see, e.g., [1], [4], [13], [7], etc.

Let $G = \langle X \mid R\rangle$ be a group; let H be a subgroup of G; and let $\theta : H \hookrightarrow G$ be a monomorphism. Then an HNN-extension of this data (named after G. Higman, B. Neumann, H. Neumann) is the group

$$HNN(G,H,\theta) \equiv G*_\theta \equiv \langle X \sqcup \{\tau\} \mid R \sqcup \{\theta(h) = \tau^{-1}h\tau \mid h \in H\}\rangle.$$

It is convenient to denote $H_{-1} \equiv H$ and $H_1 \equiv \theta(H)$. Every element $\gamma \in HNN(G,H,\theta)$ can be written in reduced form as

$$\gamma = g_1\tau^{\varepsilon_1}\cdots g_n\tau^{\varepsilon_n}g_{n+1}, \text{ where } n \in \mathbb{N},\ g_1,\ldots,g_{n+1} \in G,\ \varepsilon_1,\ldots,\varepsilon_n = \pm 1,$$
$$\text{and where if } \varepsilon_{i+1} = -\varepsilon_i \text{ for } 1 \le i \le n-1, \text{ then } g_{i+1} \notin H_{\varepsilon_i}.$$

If S_ε is a set of left coset representatives for G/H_ε, where $\varepsilon = \pm 1$, satisfy $S_{-1} \cap S_1 = \{1\}$, then every element $\gamma \in HNN(G,H,\theta)$ can be uniquely written in normal form as

$$\gamma = s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}g, \text{ where } n \in \mathbb{N}_0,\ g \in G,\ \varepsilon_i = \pm 1,\ s_i \in S_{-\varepsilon_i},\ \forall 1 \le i \le n,$$
$$\text{and where if } \varepsilon_{i-1} = -\varepsilon_i \text{ for } 2 \le i \le n, \text{ then } s_i \ne 1.$$

The HNN-extension $HNN(G,H,\theta)$ is called nondegenerate if either $H \ne G$ or $\theta(H) \ne G$ and is called non-ascending if $H \ne G \ne \theta(G)$.
The Bass-Serre tree $T(HNN(G,H,\theta))$ of $HNN(G,H,\theta)$ is the graph, that can be shown to be a tree, consisting of a vertex set

$$\begin{aligned}&\mathrm{Vertex}(HNN(G,H,\theta)) = \\ &\{G\} \cup \{s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}G \mid n \in \mathbb{N},\ \ s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n} \text{ is in normal form}\}\end{aligned}$$

and an edge set

$$\begin{aligned}&\mathrm{Edge}(HNN(G,H,\theta)) = \\ &\{H\} \cup \{s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}H \mid n \in \mathbb{N},\ s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n} \text{ is in normal form}\}.\end{aligned}$$

The group $HNN(G,H,\theta)$ acts on $T(HNN(G,H,\theta))$ by left multiplication.

The vertex $v = s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}G$ is adjacent to the vertex $w = s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\tau^{\varepsilon_{n+1}}G$ with connecting edge

$$e = \begin{cases} s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\tau^{\varepsilon_{n+1}}H \text{ if } \varepsilon_{n+1} = -1, \\ s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}H \text{ if } \varepsilon_{n+1} = 1. \end{cases}$$

To see the reason for this, we need to look at the stabilizers. The stabilizer of v is

$$HNN(G,H,\theta)_v = s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}G(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n})^{-1}$$

and the stabilizer of w is

$$HNN(G,H,\theta)_w = s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\tau^{\varepsilon_{n+1}}G(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\tau^{\varepsilon_{n+1}})^{-1}.$$

Therefore the stabilizer of e is

$$\begin{gathered}
HNN(G,H,\theta)_e \;=\; HNN(G,H,\theta)_v \;\cap\; HNN(G,H,\theta)_w \;= \\
s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\;[G \;\cap\; \tau^{\varepsilon_{n+1}}G\tau^{-\varepsilon_{n+1}}]\;(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1})^{-1} \;= \\
s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\;H_{-\varepsilon_{n+1}}\;(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1})^{-1} \;= \\
\begin{cases} s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}H(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1})^{-1} \text{ if } \varepsilon_{n+1}=1, \\ s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1}\tau^{\varepsilon_{n+1}}H\tau^{-\varepsilon_{n+1}}(s_1\tau^{\varepsilon_1}s_2\tau^{\varepsilon_2}\cdots s_n\tau^{\varepsilon_n}s_{n+1})^{-1} \text{ if } \varepsilon_{n+1}=-1. \end{cases}
\end{gathered}$$

Finally, since $HNN(G,H,\theta)$ can be expressed as

$$HNN(G,H,\theta) \;=\; (G * \langle\tau\rangle)/\langle\langle\tau^{-1}h\tau\theta(h^{-1}) \mid h \in H\rangle\rangle,$$

it has the following universal property (see, e.g., [4], page 36):

Remark 1.1. *Let C be a group; let $\alpha : G \longrightarrow C$ be a group homomorphism; and let $t \in C$ be an element for which the following holds: $t^{-1}\alpha(h)t = \alpha(\theta(h))$ for each $h \in H$. Then there is a unique group homomorphism $\beta : HNN(G,H,\theta) \longrightarrow C$ satisfying $\beta|_G = \alpha$ and $\beta(\tau) = t$.*

To conclude this section, we recall that we called a group amenablish if it has no nontrivial C^*-simple quotients ([9, Definition 7.1]). We showed in [9] that the class on amenablish groups is a radical class, so every group has a unique maximal normal amenablish subgroup, the amenablish radical. Also, the class of amenablish groups is closed under extensions. The amenablish radical "detects" C^*-simplicity the same way as the amenable radical "detects" the unique trace property (see [9, Corollary 7.3] and [2, Theorem 1.3]).

2. HNN-EXTENSIONS

2.1. NOTATION, DEFINITIONS, QUASI-KERNELS

We use the following notations, some of which appear in [3]:

$$T_\varepsilon = \{\gamma = g_0\tau^\varepsilon g_1\tau^{\varepsilon_1}\cdots g_n\tau^{\varepsilon_n}g_{n+1} \mid n \geq 0, \gamma \in \Lambda \text{ is reduced}\},$$
$$T_\varepsilon^\dagger = \{\gamma = \tau^\varepsilon g_1\tau^{\varepsilon_1}\cdots g_n\tau^{\varepsilon_n}g_{n+1} \mid n \geq 0, \gamma \in \Lambda \text{ is reduced}\}.$$

For $\varepsilon = \pm 1$, consider also the quasi-kernels defined in [3]:

$$K_\varepsilon \equiv \bigcap_{r \in \Lambda \setminus T_\varepsilon^\dagger} rHr^{-1}. \tag{1}$$

They satisfy the relation $\ker \Lambda = K_1 \cap K_{-1}$, where, by definition,

$$\ker \Lambda \equiv \bigcap_{r \in \Lambda} rHr^{-1}.$$

It follows from [3, Theorem 4.19] that Λ has the unique trace property if and only if $\ker \Lambda$ has the unique trace property. It also follows from [3, Theorem 4.20] that Λ is C^*-simple if and only if K_{-1} or K_1 is trivial or non-amenable provided Λ is a non-ascending HNN-extension and $\ker \Lambda$ is trivial.

We need the following results.

Remark 2.1. *Consider the Bass-Serre tree $\Theta = \Theta[\Lambda]$ of the group*

$$\Lambda = \mathrm{HNN}(G, H, \theta) = \langle G, \tau \mid \tau^{-1} h \tau = \theta(h) \text{ for all } h \in H \rangle,$$

and consider the edge H connecting vertices G and τG. Denote by Θ_1 the full subtree of Θ consisting of all vertices $v \in \Theta$ satisfying $\mathrm{dist}(v, G) < \mathrm{dist}(v, \tau G)$. Also, denote by $\bar{\Theta}_1$ the full subtree of Θ consisting of all vertices $v \in \Theta$ satisfying $\mathrm{dist}(v, G) > \mathrm{dist}(v, \tau G)$. Likewise, consider the edge $\tau^{-1}H$ connecting vertices G and $\tau^{-1}G$. Then, denote by Θ_{-1} the full subtree of Θ consisting of all vertices $v \in \Theta$ satisfying $\mathrm{dist}(v, G) < \mathrm{dist}(v, \tau^{-1} G)$, and denote by $\bar{\Theta}_{-1}$ the full subtree of Θ consisting of all vertices $v \in \Theta$ satisfying $\mathrm{dist}(v, G) > \mathrm{dist}(v, \tau^{-1} G)$.

It is easy to see that $\bar{\Theta}_\varepsilon = \tau^\varepsilon \Theta_{-\varepsilon}$,

$$\Theta_\varepsilon = \{G\} \ \cup \ \{ \ t_\varepsilon G \mid t_\varepsilon \in \Lambda \setminus T_\varepsilon^\dagger \ \}, \text{ and } \bar{\Theta}_\varepsilon = \{ \ t_\varepsilon^\dagger G \mid t_\varepsilon^\dagger \in T_\varepsilon^\dagger \ \}.$$

Proposition 2.2. *With the notation from the previous Remark, the following hold for each $\varepsilon = \pm 1$:*

(i) $K_\varepsilon = \Lambda_{(\Theta_\varepsilon)}$.

(ii) $K_\varepsilon < H \cap \theta(H)$.

(iii) $\gamma K_\varepsilon \gamma^{-1} = \Lambda_{(\gamma \Theta_\varepsilon)}$ *for every* $\gamma \in \Lambda$.

In particular $\Lambda_{(\bar{\Theta}_\varepsilon)} = \tau^\varepsilon K_{-\varepsilon} \tau^{-\varepsilon}$.

Proof. (i)

$$\begin{aligned} g \in K_\varepsilon \ &\Longleftrightarrow r^{-1} g r \in H, \quad \forall r \in \Lambda \setminus T_\varepsilon^\dagger \quad \Longleftrightarrow \quad gr \in rH, \quad \forall r \in \Lambda \setminus T_\varepsilon^\dagger \\ &\Longleftrightarrow \quad grH = rH, \quad \forall r \in \Lambda \setminus T_\varepsilon^\dagger \quad \Longleftrightarrow \quad g \text{ fixes every edge of } \Theta_\varepsilon \\ &\Longleftrightarrow \quad g \in \Lambda_{(\Theta_\varepsilon)}. \end{aligned}$$

(ii) From (i), we know that every element $g \in K_\varepsilon$ fixes all vertices adjacent to G except for the vertex $\tau^\varepsilon G$, eventually. Therefore it also fixes $\tau^\varepsilon G$, so g fixes all edges around G. In particular, g fixes the edge H, so $g \in H$. Likewise, g fixes the edge $\tau^{-1}H$, so $g \in \tau^{-1}H\tau = \theta(H)$.

(iii) As in (i), we have

$$\begin{aligned} g \in \gamma K_\varepsilon \gamma^{-1} &\iff \gamma^{-1}g\gamma \in K_\varepsilon \iff \gamma^{-1}g\gamma \in \Lambda_{(\Theta_\varepsilon)} \\ &\iff g \in \gamma\Lambda_{(\Theta_\varepsilon)}\gamma^{-1} \iff g \in \Lambda_{(\gamma\Theta_\varepsilon)}. \end{aligned}$$

□

Lemma 2.3. *For $\varepsilon = \pm 1$, K_ε is a normal subgroup of $H_{-\varepsilon}$, and a normal subgroup of $H \cap \theta(H)$. Moreover, if $\ker \Lambda$ is trivial, then K_{-1} and K_1 have a trivial intersection and mutually commute.*

Proof. From Proposition 2.2 (ii), it follows that K_1 and K_{-1} are subgroups of $H \cap \theta(H)$. Take $h \in H_{-\varepsilon}$. Then

$$\begin{aligned} &h \cdot T_\varepsilon^\dagger = \{h\tau^\varepsilon g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \mid n \geq 0,\ \tau^\varepsilon g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \text{ is reduced}\} = \\ &\{\tau^\varepsilon \theta^\varepsilon(h) g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \mid n \geq 0,\ \tau^\varepsilon g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \text{ is reduced}\} = T_\varepsilon^\dagger. \end{aligned}$$

This gives the first assertion. For the second assertion, take $k_\varepsilon \in K_\varepsilon$ for each $\varepsilon = \pm 1$. Then, from $K_\varepsilon \triangleleft H \cap \theta(H)$, it follows that $k_{-1}k_1^{-1}k_{-1}^{-1} \in K_1$ and $k_1 k_{-1} k_1^{-1} \in K_{-1}$. Thus

$$K_{-1} \ni (k_1 k_{-1} k_1^{-1}) k_{-1}^{-1} = k_1 (k_{-1} k_1^{-1} k_{-1}^{-1}) \in K_1,$$

and therefore $k_1 k_{-1} k_1^{-1} k_{-1}^{-1} \in K_1 \cap K_{-1} = \ker \Lambda = \{1\}$. □

Lemma 2.4.

(i) *Let $\gamma = \tau^{\varepsilon_n} g_n \cdots g_2 \tau^{\varepsilon_1} g_1 \tau^\varepsilon \in \Lambda$ be reduced. Then $\gamma \cdot T_{-\varepsilon}^\dagger \supset T_{-\varepsilon_n}^\dagger$. In particular, $K_{-\varepsilon_n} < \gamma K_{-\varepsilon} \gamma^{-1}$.*

(ii) *Let $\gamma \in G \setminus H_\varepsilon$. Then $\gamma T_{-\varepsilon}^\dagger \cap T_{-\varepsilon}^\dagger = \emptyset$. In particular, $\gamma K_{-\varepsilon} \gamma^{-1} \cap K_{-\varepsilon} = \ker \Lambda$.*

(iii) *Let $\gamma \in \Lambda$ be a reduced word starting and ending with τ^ε. Then $T_{-\varepsilon}^\dagger \cap \gamma T_\varepsilon^\dagger = \emptyset$. In particular, $K_{-\varepsilon} \cap \gamma K_\varepsilon \gamma^{-1} = \ker \Lambda$.*

Proof. (i) Observe that

$$\begin{aligned} &\gamma \cdot T_{-\varepsilon} \\ &\quad \supset \{\gamma \cdot \tau^{-\varepsilon} g_1^{-1} \tau^{-\varepsilon_1} \cdots g_n^{-1} \tau^{-\varepsilon_n} \cdot \tau^{-\varepsilon_n} \cdot g_{n+1} \tau^{\varepsilon_{n+1}} g_{n+2} \tau^{\varepsilon_{n+2}} \cdots g_{n+m} \tau^{\varepsilon_{n+m}} g_{n+m+1} \mid \\ &\qquad m \geq 0,\ \tau^{-\varepsilon_n} g_{n+1} \tau^{\varepsilon_{n+1}} g_{n+2} \cdots g_{n+m} \tau^{\varepsilon_{n+m}} g_{n+m+1} \text{ is reduced}\} \\ &\quad = \{\lambda = \tau^{-\varepsilon_n} g_{n+1} \tau^{\varepsilon_{n+1}} g_{n+2} \cdots g_{n+m} \tau^{\varepsilon_{n+m}} g_{n+m+1} \mid m \geq 0,\ \lambda \text{ is reduced}\} \\ &\quad = T_{-\varepsilon_n}. \end{aligned}$$

The second statement follows from the observation

$$\gamma \cdot (\Lambda \setminus T^{\dagger}_{-\varepsilon}) = \Lambda \setminus \gamma T^{\dagger}_{-\varepsilon} \subset \Lambda \setminus T^{\dagger}_{-\varepsilon_n}.$$

(ii) and (iii) follow easily. $\square$

Lemma 2.5. *Let* $\gamma = g_{n+1}\tau^{\varepsilon_n} g_n \cdots g_2 \tau^{\varepsilon_1} g_1 \tau^{\varepsilon}$, $\gamma' = g'_{n+1}\tau^{\varepsilon'_n} g'_n \cdots g'_2 \tau^{\varepsilon'_1} g'_1 \tau^{\varepsilon}$, *and* $\gamma'' = g''_{n+1}\tau^{\varepsilon''_n} g'_n \cdots g''_2 \tau^{\varepsilon''_1} g''_1 \tau^{-\varepsilon}$ *be reduced, where* $n \geq 0$ *and* $\varepsilon = \pm 1$. *Then:*

(i) *If* $(\gamma')^{-1}\gamma \in H_{-\varepsilon}$, *then* $\gamma K_\varepsilon \gamma^{-1} = \gamma' K_\varepsilon (\gamma')^{-1}$.

(ii) *If* $\ker \Lambda$ *is trivial and if* $(\gamma')^{-1}\gamma \notin H_{-\varepsilon}$, *then* $\gamma K_\varepsilon \gamma^{-1}$ *and* $\gamma' K_\varepsilon (\gamma')^{-1}$ *have a trivial intersection and mutually commute.*

(iii) *If* $\ker \Lambda$ *is trivial, then* $\gamma K_\varepsilon \gamma^{-1}$ *and* $\gamma'' K_{-\varepsilon} (\gamma'')^{-1}$ *have a trivial intersection and mutually commute.*

Proof. (i) $(\gamma')^{-1}\gamma K_\varepsilon \gamma^{-1}\gamma' = K_\varepsilon$ by Lemma 2.3.

(ii) If $(\gamma')^{-1}\gamma$ is an element of $G \setminus H_{-\varepsilon}$, then the assertion follows from Lemma 2.4 (ii). If $(\gamma')^{-1}\gamma$ starts with $\tau^{-\varepsilon}$ and ends with τ^{ε}, then, by Lemma 2.4 (i), it follows that

$$(\gamma')^{-1}\gamma K_\varepsilon \gamma^{-1}\gamma' < K_{-\varepsilon},$$

which, combined with $K_\varepsilon \cap K_{-\varepsilon} = \ker \Lambda = \{1\}$, proves the assertion.

(iii) Observe that the reduced form of $(\gamma'')^{-1}\gamma$ starts and ends with τ^{ε}, therefore the assertion follows from Lemma 2.4 (iii). $\square$

Assume that $\ker \Lambda = \{1\}$. Let S_ε be a left coset representatives of G/H_ε for $\varepsilon = \pm 1$.

It follows from Lemma 2.5 that, for two reduced words

$$\gamma = s_{n+1}\tau^{\varepsilon_n} s_n \cdots s_2 \tau^{\varepsilon_1} s_1 \tau^{\varepsilon} \text{ and } \gamma' = t_{n+1}\tau^{\varepsilon'_n} t_n \cdots t_2 \tau^{\varepsilon'_1} t_1 \tau^{\varepsilon}$$

with $s_i, t_i \in S_{-1} \cup S_1$ and $\varepsilon, \varepsilon_i, \varepsilon'_i \in \{-1, 1\}$,

$$\gamma K_\varepsilon \gamma^{-1} = \gamma' K_\varepsilon (\gamma')^{-1}$$

if and only if $\gamma = \gamma'$, and this happens if and only if $\varepsilon_i = \varepsilon'_i$ and $s_i = t_i$, $\forall i$. In the case $\gamma \neq \gamma'$, $\gamma K_\varepsilon \gamma^{-1}$ and $\gamma' K_\varepsilon (\gamma')^{-1}$ have a trivial intersection and mutually commute.
If $\gamma'' = r_{n+1}\tau^{\varepsilon''_n} r_n \cdots r_2 \tau^{\varepsilon''_1} s_1 \tau^{-\varepsilon}$ is another reduced word, where $r_i \in S_{-1} \cup S_1$ and $\varepsilon''_i \in \{-1, 1\}$, then $\gamma K_\varepsilon \gamma^{-1}$ and $\gamma'' K_{-\varepsilon} (\gamma'')^{-1}$ have a trivial intersection and mutually commute.

From these considerations, it follow that

$$\mathcal{K}(0) \equiv \bigoplus_{s \in S_{-1}} s K_1 s^{-1} \;\oplus\; \bigoplus_{t \in S_1} t K_{-1} t^{-1} \tag{2}$$

and, for $n \geq 0$,

$$\mathcal{K}(n+1) \equiv \bigoplus_{\substack{\varepsilon=\pm 1 \\ s_i \in S_{-1}\cup S_1,\ \varepsilon_i=\pm 1 \\ s_{n+1}\tau^{\varepsilon_n}s_n\cdots s_2\tau^{\varepsilon_1}s_1\tau^{\varepsilon} \text{ reduced}}} s_{n+1}\tau^{\varepsilon_n}s_n\cdots s_2\tau^{\varepsilon_1}s_1\tau^{\varepsilon}K_\varepsilon\tau^{-\varepsilon}s_1^{-1}\tau^{-\varepsilon_1}s_2^{-1}\cdots s_n^{-1}\tau^{-\varepsilon_n}s_{n+1}^{-1} \tag{3}$$

are normal subgroups of G. Also, consider the groups

$$\mathcal{K}(0,\varepsilon) \equiv \bigoplus_{s\in S_{-\varepsilon}} sK_1s^{-1} \ \oplus \ \bigoplus_{t\in S'_\varepsilon} tK_{-1}t^{-1},$$

which are normal in H_ε for $\varepsilon = \pm 1$.

Remark 2.6. *The group G acts transitively on the vertices $s\tau G$, where $s \in S_{-1}$. It also acts transitively on the vertices $s\tau^{-1}G$, where $s \in S_1$. This fact is an important ingredient in the examples below.*

Remark 2.7. *It follows from Lemma 2.4 that K_{-1} is isomorphic to a subgroup of K_1 and vice-versa. Consequently, $K_{-1} = \{1\}$ if and only if $K_1 = \{1\}$. In this situation, $\mathcal{K}(n) = \{1\}\ \forall n \geq 0$.*

2.2. A FAMILY OF EXAMPLES

For $\varepsilon = \pm 1$, consider nonempty sets I'_ε, and let $I_\varepsilon \equiv I'_\varepsilon \sqcup \{\iota_\varepsilon\}$. Also, let Σ_ε be transitive permutation groups on I_ε, and let $\Gamma = \Sigma_{-1}\cdot\Sigma_1$ be the corresponding permutation group on $I_{-1}\sqcup I_1$. Let $\Sigma'_\varepsilon \equiv (\Sigma_\varepsilon)_{\iota_\varepsilon}$ be the respective stabilizer groups, and define $\Gamma_\varepsilon \equiv \Gamma_{\iota_\varepsilon} = \Sigma'_\varepsilon \cdot \Sigma_{-\varepsilon}$. Define

$$\begin{aligned}\Lambda[\Sigma_{-1},\Sigma_1] &\equiv \Lambda[I_{-1},I_1,\iota_{-1},\iota_1;\Sigma_{-1},\Sigma_1] \\ &\equiv \mathrm{HNN}(G,H,\theta) = \langle G,\tau \mid \tau^{-1}h\tau = \theta(h) \text{ for all } h\in H\rangle,\end{aligned}$$

where

$$\begin{aligned}\underline{H} \equiv \langle\{h(i_1,\varepsilon_1\dots,i_n,\varepsilon_n;\sigma_n) \mid n\in\mathbb{N},\ \varepsilon_t\in\{-1,1\},\ i_t\in I_{-\varepsilon_t},\text{ and } \sigma_n\in\Gamma_{\varepsilon_n} \\ \text{satisfy } i_t\in I'_{-\varepsilon_t} \text{ whenever } \varepsilon_t\varepsilon_{t-1}=-1;\ \}\rangle \text{ and}\end{aligned}$$

$$H_\varepsilon = \langle \underline{H}\cup\{h(\sigma_\varepsilon) \mid \sigma_\varepsilon\in\Gamma_\varepsilon\}\rangle,\ \varepsilon=\pm 1.$$

Finally, define

$$G \ = \ \langle H_{-1},H_1\rangle \ = \ \langle \underline{H}\cup\{\ h(\sigma) \mid \sigma\in\Gamma\}\rangle,$$

where the following relations hold (there are redundancies):

(R1) Elements $h(\sigma_{-1})$'s and $h(\sigma_1)$'s commute for all $\sigma_\varepsilon\in\Sigma_\varepsilon$, where $\varepsilon=\pm 1$.

(R2) Let $1\leq m<n$, $\sigma_n\in\Gamma_{\varepsilon_n}$, and $\sigma'_m\in\Gamma_{e_m}$. If $(i_1,\varepsilon_1\dots,i_m,\varepsilon_m)\neq(j_1,e_1\dots,j_m,e_m)$, the elements

$$h(j_1,e_1\dots,j_m,e_m;\sigma'_m) \text{ and } h(i_1,\varepsilon_1\dots,i_m,\varepsilon_m,\dots,i_n,\varepsilon_n;\sigma_n)$$

commute.

(R3) For $1 \le m < n$ and $\sigma_t \in \Gamma_{\varepsilon_t}$, the following holds

$$h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m; \sigma_m) h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m, i_{m+1}, \varepsilon_{m+1}, \dots, i_n, \varepsilon_n; \sigma_n) h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m; \sigma_m)^{-1}$$
$$= h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m, \sigma_m(i_{m+1}), \varepsilon_{m+1}, \dots, i_n, \varepsilon_n; \sigma_n).$$

(R4) For $\sigma_m, \sigma'_m \in \Gamma_{\varepsilon_m}$, the following holds

$$h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m; \sigma_m) h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m; \sigma'_m) = h(i_1, \varepsilon_1 \dots, i_m, \varepsilon_m; \sigma_m \sigma'_m).$$

(R5) For $\sigma, \sigma' \in \Gamma$, the following holds

$$h(\sigma) h(\sigma') = h(\sigma\sigma').$$

(R6) For $n \in \mathbb{Z}$, $\sigma \in \Gamma$, and $\sigma_n \in \Gamma_{\varepsilon_n}$, the following holds

$$h(\sigma) h(i_1, \varepsilon_1 \dots, i_n, \varepsilon_n; \sigma_n) h(\sigma)^{-1} = h(\sigma(i_1), \varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n).$$

(R7) For $\varepsilon = \pm 1$ and $\sigma_\varepsilon \in \Gamma_\varepsilon$, the following holds

$$\theta^{-\varepsilon}(h(\sigma_\varepsilon)) = (\tau^\varepsilon h(\sigma_\varepsilon) \tau^{-\varepsilon}) = h(\iota_{-\varepsilon}, \varepsilon; \sigma_\varepsilon).$$

(R8) For $\varepsilon = \pm 1$, $n \in \mathbb{N}$, and $\sigma_n \in \Gamma_{\varepsilon_n}$, the following holds

$$\begin{aligned} \theta^{-\varepsilon}(h(i_1, \varepsilon, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n)) &= (\tau^\varepsilon h(i_1, \varepsilon, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n) \tau^{-\varepsilon}) \\ &= h(\iota_{-\varepsilon}, \varepsilon, i_1, \varepsilon, i_2, \varepsilon_2 \dots, i_n, \varepsilon_n; \sigma_n). \end{aligned}$$

(R9) For $\varepsilon = \pm 1$, $n \in \mathbb{N}$, and $\sigma_n \in \Gamma_{\varepsilon_n}$, the following holds

$$\begin{aligned} \theta^{\varepsilon}(h(i_1, \varepsilon \dots, i_n, \varepsilon_n; \sigma_n)) &= (\tau^{-\varepsilon} h(i_1, \varepsilon \dots, i_n, \varepsilon_n; \sigma_n) \tau^{\varepsilon}) \\ &= \begin{cases} h(i_2, \varepsilon_2 \dots, i_n, \varepsilon_n; \sigma_n), & \text{if } i_1 = \iota_{-\varepsilon}, \\ h(\iota_\varepsilon, -\varepsilon, i_1, \varepsilon \dots, i_n, \varepsilon_n; \sigma_n), & \text{if } i_1 \neq \iota_{-\varepsilon}. \end{cases} \end{aligned}$$

2.3. SOME BASIC PROPERTIES OF THE EXAMPLES AND THEIR QUASI-KERNELS

In this subsection we fix a group $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$.

First, let's note that $\mathrm{Index}[G : H_\varepsilon] = \#(I_\varepsilon)$ for $\varepsilon = \pm 1$. To see this, recall that Σ_ε acts transitively on I_ε, and for $i \in I_\varepsilon$, choose $\mu^i_\varepsilon \in \Sigma_\varepsilon$ satisfying $\mu^i_\varepsilon(\iota_\varepsilon) = i$. Let's denote $\lambda^i_\varepsilon = h(\mu^i_\varepsilon)$. If $\sigma \in \Sigma_\varepsilon \setminus \Sigma'_\varepsilon$ satisfies $\sigma(\iota_\varepsilon) = i$, then $(\mu^i_\varepsilon)^{-1} \circ \sigma(\iota_\varepsilon) = \iota_\varepsilon$. Therefore $(\mu^i_\varepsilon)^{-1} \circ \sigma \in \Sigma'_\varepsilon$, so $h((\mu^i_\varepsilon)^{-1} \circ \sigma) \in H_\varepsilon$. It follows that $h(\sigma) \in h(\mu^i_\varepsilon) H_\varepsilon = \lambda^i_\varepsilon H$. Consequently, for each $\varepsilon = \pm 1$,

$$G = H_\varepsilon \sqcup \bigsqcup_{i \in I'_\varepsilon} \lambda^i_\varepsilon H_\varepsilon. \tag{4}$$

It is easy to see in these notations that for $\varepsilon = \pm 1$, the set

$$S_\varepsilon \;=\; \{\; \lambda_\varepsilon^i \mid i \in I'_\varepsilon \;\} \;\cup\; \{\; 1 \;\}$$

is a left coset representative of H_ε in G.

Next, consider the action of Λ on its Bass-Serre tree $\Theta = \Theta[\Lambda]$. The set of all adjacent vertices to the vertex G is

$$\{\; \tau G \;\} \;\cup\; \{\; \lambda_{-1}^i \tau G \mid i \in I'_{-1} \;\} \;\cup\; \{\; \tau^{-1} G \;\} \;\cup\; \{\; \lambda_1^i \mid i \in I'_1 \;\}.$$

This set can be indexed by the set $I_{-1} \cup I_1$ in the obvious way: Denote by $v(\emptyset)$ the vertex G, by $v(\iota_{-1}, 1)$ the vertex τG, by $v(\iota_1, -1)$ the vertex $\tau^{-1}G$, by $v(i_{-1}, 1)$ the vertex $\lambda_{-1}^{i_{-1}} \tau G$, where $i_{-1} \in I'_{-1}$, and by $v(i_1, -1)$ the vertex $\lambda_1^{i_1} \tau^{-1} G$, where $i_1 \in I'_1$. Denote a general vertex

$$\lambda_{-\varepsilon_1}^{i_1} \tau^{\varepsilon_1} \cdots \lambda_{-\varepsilon_n}^{i_n} \tau^{\varepsilon_n} G$$

by $v(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$ for an element $\lambda_{-\varepsilon_1}^{i_1} \tau^{\varepsilon_1} \cdots \lambda_{-\varepsilon_n}^{i_n} \tau^{\varepsilon_n} \in \Lambda$ in its normal form, i.e., $i_t \in I_{-\varepsilon_t}$ and if $\varepsilon_{t-1} \cdot \varepsilon_t = -1$, then $i_t \in I'_{-\varepsilon_t}$.

With the notation of Remark 2.1, for $\varepsilon = \pm 1$, Θ_ε is the full subtree of Θ containing the vertex $v(\emptyset) = G$ and vertices $v(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$, where $n \geq 1$ and $(i_1, \varepsilon_1) \neq (\iota_{-\varepsilon}, \varepsilon)$, and $\bar{\Theta}_\varepsilon$ is the full subtree of Θ containing the vertices $v(\iota_{-\varepsilon}, \varepsilon, i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$, where $n \geq 0$.

Remark 2.8. *It follows from [1, Exercise VI.3] that our examples are never finitely presented since H is never finitely generated.*

We continue with

Lemma 2.9. (i) *Let $m \geq 1$, $\sigma_m \in \Gamma_{\varepsilon_m}$, $i_t \in I_{-\varepsilon_t}$, and $\varepsilon \in \{-1, 1\}$ satisfy $\varepsilon_t \varepsilon_{t-1} = -1 \Rightarrow i_t \in I'_{-\varepsilon_t}$. Then*

$$\begin{aligned} &h(i_1, eps_1 \dots, i_m, \varepsilon_m; \sigma_m) \\ &\qquad = \lambda_{-\varepsilon_1}^{i_1} \tau^{\varepsilon_1} \cdots \lambda_{-\varepsilon_m}^{i_m} \tau^{\varepsilon_m} h(\sigma_m) \tau^{-\varepsilon_m} (\lambda_{-\varepsilon_m}^{i_m})^{-1} \cdots \tau^{-\varepsilon_1} (\lambda_{-\varepsilon_1}^{i_1})^{-1}. \end{aligned}$$

(ii) *Every element h of G can be written as*

$$h = h(\sigma) \prod_{k=1}^{m} h(i_1^k, \varepsilon_{k,1}, \dots, i_{n_k}^k, \varepsilon_{k,n_k}; \sigma_k),$$

where $m \geq 1$, $\sigma_k \in \Gamma_{\varepsilon_{k,n_k}}$, $1 \leq n_1 \leq \cdots \leq n_m$, and $\sigma \in \Gamma$ satisfy the condition: if $n_k = n_{k+a}$ for some $1 \geq k \geq m$ and some $a \geq 1$, then

$$(i_1^k, \varepsilon_{k,1}, \dots, i_{n_k}^k, \varepsilon_{k,n_k}) \neq (i_1^{k+a}, \varepsilon_{k+a,1}, \dots, i_{n_{k+a}}^{k+a}, \varepsilon_{k+a,n_{k+a}}).$$

(ii) *Every element* $g \in T_\varepsilon$ *can be written as*

$$g = \lambda^{i}_{-\varepsilon}\tau^{\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h,$$

where $h \in G$ *and* $m \geq 0$.

Proof. (i) follows by repeated applications of relations (R7), (R8), and (R6).
(ii) follows by repeated applications of relations (R3) and (R6).
(iii) follows by equation (4) and the structure of HNN-extensions. □

Lemma 2.10. *Let* $n > m \geq 1$ *and* $\sigma_k \in \Gamma_{\varepsilon_k}$. *Then the following hold*

(i) $$h(i_1, \varepsilon_1 \ldots, i_m, \varepsilon_m; \sigma_m)v(i_1, \varepsilon_1, \ldots, i_m, \varepsilon_m, i_{m+1}, \varepsilon_{m+1}, \ldots, i_n, \varepsilon_n) \\ = v(i_1, \varepsilon_1, \ldots, i_m, \varepsilon_m, \sigma_m(i_{m+1}), \varepsilon_{m+1}, \ldots, i_n, \varepsilon_n).$$

(ii) $h(i_1, \varepsilon_1 \ldots, i_m, \varepsilon_m; \sigma_m) \in \Lambda_{v(i_1,\varepsilon_1\ldots,i_m,\varepsilon_m)}$ *and* $h(\sigma) \in \Lambda_{v(\emptyset)}$ *for* $\sigma \in \Gamma$.

(iii) *If* $\sigma_\varepsilon \in \Gamma_\varepsilon$, *then* $h(\sigma_\varepsilon) \in \Lambda_{(\bar{\Theta}_{-\varepsilon})} = \tau^{-\varepsilon}K_\varepsilon\tau^{\varepsilon}$.

(iv) *Let* $m \leq n$ *and let* $h(i_1, \varepsilon_1 \ldots, i_n, \varepsilon_n; \sigma_n), h(j_1, e_1 \ldots, j_m, e_m; \delta_m) \in \Lambda$. *If* $(i_1, \varepsilon_1 \ldots, i_m, \varepsilon_m) \neq (j_1, e_1 \ldots, j_m, e_m)$, *then* $h(i_1, \varepsilon_1 \ldots, i_n, \varepsilon_n; \sigma_n) \in \Lambda_{v(j_1,e_1\ldots,j_m,e_m)}$ *and* $h(j_1, e_1 \ldots, j_m, e_m; \delta_m) \in \Lambda_{v(i_1,\varepsilon_1\ldots,i_n,\varepsilon_n)}$.

(iv) $h(i_1, \varepsilon_1 \ldots, i_n, \varepsilon_n; \sigma_n) \in \Lambda_{(\bar{\Theta}_\varepsilon)} \iff (i_1, \varepsilon_1) \neq (\iota_{-\varepsilon}, \varepsilon)$.

Proof. (i) First, note that

$$\sigma \equiv (\lambda^{\sigma_m(i_{m+1})}_{-\varepsilon_{m+1}})^{-1} \circ h(\sigma_m)\lambda^{i_{m+1}}_{-\varepsilon_{m+1}} \in \Gamma_{-\varepsilon_{m+1}}$$

since it fixes $\iota_{-\varepsilon_{m+1}}$. It follows by Lemma 2.9 (i) and (iii) that there are $k_t \in I_{\varepsilon_t}$ and a $\chi \in H_{\varepsilon_n}$ that satisfy $(\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n})^{-1} = \chi\tau^{-\varepsilon_n}\lambda^{k_{n-1}}_{\varepsilon_{n-1}}\cdots\lambda^{k_{m+1}}_{\varepsilon_{m+1}}\tau^{-\varepsilon_{m+1}}$. Therefore

$$\begin{aligned}(\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n})^{-1}h(\sigma)\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}\\ &= \chi\tau^{-\varepsilon_n}\lambda^{k_{n-1}}_{\varepsilon_{n-1}}\cdots\lambda^{k_{m+1}}_{\varepsilon_{m+1}}\tau^{-\varepsilon_{m+1}}h(\sigma)\tau^{\varepsilon_{m+1}}(\lambda^{k_{m+1}}_{\varepsilon_{m+1}})^{-1}\cdots(\lambda^{k_{n-1}}_{\varepsilon_{n-1}})^{-1}\tau^{\varepsilon_n}\chi^{-1}\\ &= \chi h(\iota_{\varepsilon_n}, -\varepsilon_n, k_{n-1}, -\varepsilon_{n-1}, \ldots, k_{m+2}, -\varepsilon_{m+2}, \iota_{\varepsilon_m+1}, -\varepsilon_{m+1}; \sigma)\chi^{-1}.\end{aligned}$$

Then Lemma 2.9 (i) implies

$$\begin{aligned}&h(i_1, \varepsilon_1 \ldots, i_m, \varepsilon_m; \sigma_m)v(i_1, \varepsilon_1, \ldots, i_m, \varepsilon_m, i_{m+1}, \varepsilon_{m+1}, \ldots, i_n, \varepsilon_n)\\ &= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\sigma_m)\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\cdot\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G\\ &= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\sigma_m)\lambda^{i_{m+1}}_{-\varepsilon_{m+1}}\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G\\ &= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{\sigma_m(i_{m+1})}_{-\varepsilon_{m+1}}h(\sigma)\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G\end{aligned}$$

$$= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{\sigma_m(i_{m+1})}_{-\varepsilon_{m+1}}\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}$$
$$\cdot(\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n})^{-1}h(\sigma)\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G$$
$$= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{\sigma_m(i_{m+1})}_{-\varepsilon_{m+1}}\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}$$
$$\cdot\chi h(\iota_{\varepsilon_n},-\varepsilon_n,k_{n-1},-\varepsilon_{n-1},\dots,k_{m+2},-\varepsilon_{m+2},\iota_{\varepsilon_m+1},-\varepsilon_{m+1};\sigma)\chi^{-1}G$$
$$= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{\sigma_m(i_{m+1})}_{-\varepsilon_{m+1}}\tau^{\varepsilon_{m+1}}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G$$
$$= v(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m,\sigma_m(i_{m+1}),\varepsilon_{m+1},\dots,i_n,\varepsilon_n).$$

(ii) The second claim is obvious. For the first claim,

$$h(i_1,\varepsilon_1\dots,i_m,\varepsilon_m;\sigma_m)v(i_1,\varepsilon_1\dots,i_m,\varepsilon_m)$$
$$= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\sigma_m)\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\cdot\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_n}\tau^{\varepsilon_m}G$$
$$= \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\sigma_m)G = v(i_1,\varepsilon_1\dots,i_m,\varepsilon_m).$$

(iii) The fact $\Lambda_{(\bar{\Theta}_{-\varepsilon})} = \tau^{-\varepsilon}K_\varepsilon\tau^\varepsilon$ is stated in Proposition 2.2. Let $n\ge 0$ and let $v(\iota_\varepsilon,-\varepsilon,i_1,\varepsilon_1,\dots,i_n,\varepsilon_n)\in\bar{\Theta}_{-\varepsilon}$. By the argument at the beginning of the proof of (i), there are $k_t\in I_{\varepsilon_t}$ and a $\chi\in H_{\varepsilon_n}$ satisfying

$$(\tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n})^{-1}h(\sigma_\varepsilon)\tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}$$
$$= \chi h(\iota_{\varepsilon_n},-\varepsilon_n,k_{n-1},-\varepsilon_{n-1},\dots,i_{\varepsilon_1},-\varepsilon_1,\varepsilon,\iota_{-\varepsilon};\sigma_\varepsilon)\chi^{-1}.$$

Therefore

$$h(\sigma_\varepsilon)v(\iota_\varepsilon,-\varepsilon,i_1,\varepsilon_1,\dots,i_n,\varepsilon_n) = h(\sigma_\varepsilon)\tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_m}\tau^{\varepsilon_n}G$$
$$= \tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_m}\tau^{\varepsilon_n}\cdot(\tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_m}\tau^{\varepsilon_n})^{-1}h(\sigma_\varepsilon)\tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_m}\tau^{\varepsilon_n}G$$
$$= \tau^{-\varepsilon}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_m}\tau^{\varepsilon_n}\cdot\chi h(\iota_{\varepsilon_n},-\varepsilon_n,k_{n-1},-\varepsilon_{n-1},\dots,i_{\varepsilon_1},-\varepsilon_1,\varepsilon,\iota_{-\varepsilon};\sigma_\varepsilon)\chi^{-1}G$$
$$= v(\iota_\varepsilon,-\varepsilon,i_1,\varepsilon_1,\dots,i_n,\varepsilon_n).$$

Consequently $h(\sigma_\varepsilon)\in\bar{\Theta}_{-\varepsilon}$.

(iv) Note that the element $\gamma = \tau^{-e_m}(\lambda^{j_m}_{-e_m})^{-1}\cdots\tau^{-e_1}(\lambda^{j_1}_{-e_1})^{-1}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}$ belongs to $T^\dagger_{-e_m}$ because of the condition $(i_1,\varepsilon_1\dots,i_m,\varepsilon_m)\neq(j_1,e_1\dots,j_m,e_m)$. It follows from Lemma 2.9 (iii) that $\gamma = \tau^{-e_m}\lambda^{k_1}_{-l_1}\tau^{l_1}\lambda^{k_2}_{-l_2}\tau^{l_2}\cdots\lambda^{k_s}_{-l_s}\tau^{l_s}h$, where $h\in G$ and where $k_t\in I_{-l_t}$, $\forall t$. Then

$$h(j_1,e_1\dots,j_m,e_m;\delta_m)\in\Lambda_{v(i_1,\varepsilon_1\dots,i_n,\varepsilon_n)}$$
$$\iff \lambda^{j_1}_{-e_1}\tau^{e_1}\cdots\lambda^{j_m}_{-e_m}\tau^{e_m}h(\delta_m)\tau^{-e_m}(\lambda^{j_m}_{-e_m})^{-1}\cdots\tau^{-e_1}(\lambda^{j_1}_{-e_1})^{-1}\in\Lambda_{v(i_1,\varepsilon_1\dots,i_n,\varepsilon_n)}$$
$$\iff h(\delta_m)\in\tau^{-e_m}(\lambda^{j_m}_{-e_m})^{-1}\cdots\tau^{-e_1}(\lambda^{j_1}_{-e_1})^{-1}\Lambda_{v(i_1,\varepsilon_1\dots,i_n,\varepsilon_n)}\lambda^{j_1}_{-e_1}\tau^{e_1}\cdots\lambda^{j_m}_{-e_m}\tau^{e_m}$$
$$\iff h(\delta_m)\in\Lambda_{\tau^{-e_m}(\lambda^{j_m}_{-e_m})^{-1}\cdots\tau^{-e_1}(\lambda^{j_1}_{-e_1})^{-1}v(i_1,\varepsilon_1\dots,i_n,\varepsilon_n)}$$
$$\iff h(\delta_m)\in\Lambda_{\tau^{-e_m}(\lambda^{j_m}_{-e_m})^{-1}\cdots\tau^{-e_1}(\lambda^{j_1}_{-e_1})^{-1}\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_n}_{-\varepsilon_n}\tau^{\varepsilon_n}G}$$
$$\iff h(\delta_m)\in\Lambda_{\tau^{-e_m}\lambda^{k_1}_{-l_1}\tau^{l_1}\lambda^{k_2}_{-l_2}\tau^{l_2}\cdots\lambda^{k_s}_{-l_s}\tau^{l_s}hG}$$
$$\iff h(\delta_m)\in\Lambda_{v(\iota_{e_m},-e_m,k_1,l_1,\dots,k_s,l_s)}.$$

The last equivalence holds according to (iii). The inclusion $h(i_1,\varepsilon_1\dots,i_n,\varepsilon_n;\sigma_n)\in\Lambda_{v(j_1,e_1\dots,j_m,e_m)}$ is proven analogously.

(v) Every vertex of $\Lambda_{(\bar\Theta_\varepsilon)}$ is of the form $v(\iota_{-\varepsilon},\varepsilon,j_1,e_1,\dots,j_m,e_m)$, so if tuples $(i_1,\varepsilon_1\dots,i_n,\varepsilon_n)$ and $(\iota_{-\varepsilon},\varepsilon,j_1,e_1,\dots,j_m,e_m)$ satisfy the assumptions of (iv), then $h(i_1,\varepsilon_1\dots,i_n,\varepsilon_n;\sigma_n)\in\Lambda_{(\bar\Theta_\varepsilon)}$. By (i), $h(\iota_{-\varepsilon},\varepsilon,j_1,e_1,\dots,j_m,e_m;\sigma_m)\notin\Lambda_{(\bar\Theta_\varepsilon)}$, and the statement follows. □

Proposition 2.11. *For a group $\Lambda=\Lambda[I_{-1},I_1,\iota_{-1},\iota_1;\Sigma_{-1},\Sigma_1]$ and for $\varepsilon=\pm1$, the following hold*

(i) $\Lambda_{(\bar\Theta_\varepsilon)}=\langle\ \{\ h(\sigma_{-\varepsilon})\mid\sigma_{-\varepsilon}\in\Gamma_{-\varepsilon}\ \}\ \cup$
$\{\,h(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m;\sigma_m)\mid m\geq1, h(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m;\sigma_m)\in H_{-\varepsilon},$ *and* $(i_1,\varepsilon_1)\neq(\iota_{-\varepsilon},\varepsilon)\,\}\ \rangle$;

(ii) $|K_\varepsilon|=\langle\{\ h(\iota_\varepsilon,-\varepsilon;\sigma_{-\varepsilon})\mid\sigma_{-\varepsilon}\in\Gamma_{-\varepsilon}\ \}\sqcup$
$\{\ h(\iota_\varepsilon,-\varepsilon,i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)\mid n\geq1,\sigma_n\in\Gamma_{\varepsilon_n}\}\rangle$;

(iii) $\ker\Lambda=\{1\}$.

Proof. (i) Denote the group on the right-hand-side by Δ. The inclusion $\Delta<\Lambda_{(\bar\Theta_\varepsilon)}$ follows from Lemma 2.10 (iii) and (v). Take an element $h\in\Lambda_{(\bar\Theta_\varepsilon)}$. Proposition 2.2 (iv) implies that $h\in H_{-\varepsilon}$. If we assume $h=h(\sigma)$, then $\sigma\in\Gamma_{-\varepsilon}$, and therefore $h(\sigma)\in\Delta$. If h is not of the form $h(\sigma)$, Lemma 2.9 (ii) can be applied to $h^{-1}\in H_{-\varepsilon}$. It follows that

$$h=\prod_{k=1}^{m}h(i_1^k,\varepsilon_{k,1},\dots,i_{n_k}^k,\varepsilon_{k,n_k};\sigma_k)\cdot h(\sigma_{-\varepsilon}),$$

where $m\geq0$, $\sigma_k\in\Gamma_{\varepsilon_{k,n_k}}$, $n_1\geq n_2\geq\dots\geq n_m\geq1$, and $\sigma_{-\varepsilon}\in\Gamma_{-\varepsilon}$. Assume $h(i_1^l,\varepsilon_{l,1},\dots,i_{n_l}^l,\varepsilon_{l,n_l};\sigma_l)\notin\Delta$ for some $1\leq l\leq m$ and that l is the biggest number with this property. We will derive a contradiction below. Then it is clear that $i_1^l=\iota_{-\varepsilon}$ and $\varepsilon_{l,1}=\varepsilon$. Also, $\sigma_l\in\Gamma_{\varepsilon_{l,n_l}}$ is not the identity, so there exist two different elements $\kappa,\rho\in I_{-1}\sqcup I_1$, such that $\sigma_l(\kappa)=\rho$. Let h act on

$$v=v(i_1^l,\varepsilon_{l,1},\dots,i_{n_l}^l,\varepsilon_{l,n_l},\kappa,\varepsilon_{l,n_l},\alpha_1,e_1,\dots,\alpha_{n_1},e_{n_1}),$$

where α's and e's are arbitrary and allowed. The terms $h(\sigma_{-\varepsilon})$ and $\prod_{k=l+1}^m h(i_1^k,\varepsilon_{k,1},\dots,i_{n_k}^k,\varepsilon_{k,n_k};\sigma_k)$ leave v fixed by the choice of l. From the final condition of Lemma 2.9 (ii) and from Lemma 2.10 (iv), it follows that the terms with length equal to n_l also leave v fixed. Finally, from Lemma 2.10 (i), it follows that the remaining terms act on v by eventually changing only the α's. Therefore we conclude that

$$\begin{aligned}hv(i_1^l,\varepsilon_{l,1},\dots,i_{n_l}^l,\varepsilon_{l,n_l},\kappa,\varepsilon_{l,n_l},\alpha_1,e_1,\dots,\alpha_{n_1},e_{n_1})&\\=v(i_1^l,\varepsilon_{l,1},\dots,i_{n_l}^l,\varepsilon_{l,n_l},\rho,\varepsilon_{l,n_l},\beta_1,e_1,\dots,\beta_{n_1},e_{n_1})&\end{aligned}$$

for some β's. This shows that $h \notin \Lambda_{(\bar{\Theta}_\varepsilon)}$, a contradiction that proves (i).

(ii) From Proposition 2.2 (iii), it follows that

$$K_\varepsilon = \tau^{-\varepsilon} K_\varepsilon (\tau^{-\varepsilon}) \tau^\varepsilon = \tau^{-\varepsilon} \Lambda_{(\bar{\Theta}_\varepsilon)} \tau^\varepsilon = \theta^\varepsilon (\Lambda_{(\bar{\Theta}_\varepsilon)}).$$

The assertion follows from relation (R7) and Lemma 2.9 (i).

(iii) is obvious. □

Now, we want to explore the structure of the quasi-kernels of $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$, in particular, that of $\Lambda_{(\bar{\Theta}_\varepsilon)}$.

First, we note that Proposition 2.11 (ii) and relation (R6) imply that for $i \in I_\varepsilon$,

$$\begin{aligned}
&\lambda^i_\varepsilon \tau^{-\varepsilon} \Lambda_{(\bar{\Theta}_\varepsilon)} \tau^\varepsilon (\lambda^i_\varepsilon)^{-1} = \lambda^i_\varepsilon K_\varepsilon (\lambda^i_\varepsilon)^{-1} \\
&\quad = \langle \{ h(i, -\varepsilon, i_1, \varepsilon_1, \dots, i_m, \varepsilon_m; \sigma_m) \mid m \geq 0,\ h(i, -\varepsilon, i_1, \varepsilon_1, \dots, i_m, \varepsilon_m; \sigma_m) \in \underline{H} \} \rangle.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\Lambda_{(\bar{\Theta}_\varepsilon)} \\
&\quad = \langle \{ h(\sigma_{-\varepsilon}) \mid \sigma_{-\varepsilon} \in \Gamma_{-\varepsilon} \} \cup \bigcup_{i \in I_\varepsilon} \lambda^i_\varepsilon \tau^{-\varepsilon} \Lambda_{(\bar{\Theta}_\varepsilon)} \tau^\varepsilon (\lambda^i_\varepsilon)^{-1} \cup \bigcup_{i \in I'_{-\varepsilon}} \lambda^i_{-\varepsilon} \tau^\varepsilon \Lambda_{(\bar{\Theta}_{-\varepsilon})} \tau^{-\varepsilon} (\lambda^i_{-\varepsilon})^{-1} \rangle \\
&\quad = \langle\ \{\ h(\sigma_{-\varepsilon})\ \mid\ \sigma_{-\varepsilon} \in \Gamma_{-\varepsilon}\ \}\ \cup \mathcal{K}(0, -\varepsilon)\ \rangle.
\end{aligned}$$

In other words,

$$\Lambda_{(\bar{\Theta}_\varepsilon)} \cong \mathcal{K}(0, -\varepsilon) \rtimes \Gamma_{-\varepsilon}.$$

This can be written "recursively" as

$$K_\varepsilon \cong [\bigoplus_{\#(S'_{-\varepsilon})} K_{-\varepsilon} \oplus \bigoplus_{\#(S_\varepsilon)} K_\varepsilon] \rtimes \Gamma_{-\varepsilon}. \tag{5}$$

This is in a sense a "wreath product" representation.

Let's denote

$$\mathcal{H}_\varepsilon(0) = \langle\ \{\ h(\sigma_{-\varepsilon})\ \mid\ \sigma_{-\varepsilon} \in \Gamma_{-\varepsilon}\ \}\ \rangle.$$

For $n \geq 1$, let

$$\mathcal{H}_\varepsilon(n) = \langle \{ h(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n; \sigma_n) \mid h(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n; \sigma_n) \in H_{-\varepsilon} \text{ and } (i_1, \varepsilon_1) \neq (\iota_{-\varepsilon}, \varepsilon) \} \rangle.$$

Note that, each $\mathcal{H}_\varepsilon(n)$ is isomorphic to a direct sum of copies of Γ_1 and Γ_{-1}. Let us also denote

$$\mathcal{H}_\varepsilon[n]\ =\ \langle\ \mathcal{H}_\varepsilon(0) \cup \mathcal{H}_\varepsilon(1) \cup \dots \cup \mathcal{H}_\varepsilon(n)\ \rangle.$$

Relation (R3) implies that $\mathcal{H}_\varepsilon(n) \lhd \mathcal{H}_\varepsilon[n]$ and that there is an extension

$$\{1\} \longrightarrow \mathcal{H}_\varepsilon(n) \longrightarrow \mathcal{H}_\varepsilon[n] \longrightarrow \mathcal{H}_\varepsilon[n-1] \longrightarrow \{1\}. \tag{6}$$

The natural embeddings $\mathcal{H}_\varepsilon[m] \hookrightarrow \mathcal{H}_\varepsilon[n]$ give a representation of $\Lambda_{(\bar{\Theta}_\varepsilon)}$ as a direct limit of groups

$$\Lambda_{(\bar{\Theta}_\varepsilon)}\ =\ \varinjlim_n \mathcal{H}_\varepsilon[n]. \tag{7}$$

Lemma 2.12. *K_{-1} is amenable if and only if K_1 is amenable, if and only if Γ_{-1} and Γ_1 are both amenable, and if and only if Σ_{-1} and Σ_1 are both amenable.*

Proof. Assume that Γ_ε is not amenable for some $\varepsilon = \pm 1$. Then, by equation (5), it follows that $K_{-\varepsilon}$ is not amenable, so equation (5), applied once more, gives the nonamenability of K_ε.

Conversely, assume that Γ_{-1} and Γ_1 are both amenable. Then $\mathcal{H}_\varepsilon(n)$ is amenable as a direct sum of copies of Γ_{-1} and Γ_1. Also, $\mathcal{H}_\varepsilon[0] = \mathcal{H}_\varepsilon(0) \cong \Gamma_{-\varepsilon}$ is amenable for $\varepsilon = \pm 1$. Therefore an easy induction based on the extension (6), gives the amenability of $\mathcal{H}_\varepsilon[n]$ for each $\varepsilon = \pm 1$ and each $n \geq 0$. Finally, the direct limit representation (7) of $\Lambda_{(\bar{\Theta}_\varepsilon)}$ implies the amenability of $\Lambda_{(\bar{\Theta}_\varepsilon)}$ for and therefore that of $K_\varepsilon = \tau^{-\varepsilon}\Lambda_{(\bar{\Theta}_\varepsilon)}\tau^\varepsilon$ for $\varepsilon = \pm 1$. □

2.4. GROUP-THEORETIC STRUCTURE

We give a result about the structure of our groups.

Theorem 2.13. *Take $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$. Let's assume that:*

(i) *Σ_{-1} and Σ_1 are 2-transitive, that is, all stabilizers $(\Sigma_\varepsilon)_{i_\varepsilon}$ are transitive on the sets $I_\varepsilon \setminus \{i_\varepsilon\}$ for all $i_\varepsilon \in I_\varepsilon$ and $\varepsilon = \pm 1$;*

(ii) *For each $\varepsilon = \pm 1$, either $\Sigma_\varepsilon = \langle (\Sigma_\varepsilon)_{i_\varepsilon} \mid i_\varepsilon \in I_\varepsilon \rangle$ or $\Sigma_\varepsilon = Sym(2)$.*

Then Λ has a simple normal subgroup Ξ for which there is a group extension

$$1 \longrightarrow \Xi \longrightarrow \Lambda \xrightarrow{\eta} (\Gamma/[\Gamma,\Gamma]) \wr_{\mathbb{Z}} \mathbb{Z} \longrightarrow 1,$$

where η is defined on the generators by

$$\eta(h(\sigma)) = ((\ldots,0,\ldots,0,([\sigma],0),0,\ldots,0,\ldots),0), \quad \eta(\tau) = ((\ldots,0,\ldots),1), \quad \text{and}$$

$$\eta(h(i_1,\varepsilon_1,\ldots,i_n,\varepsilon_n;\sigma_n)) = ((\ldots,0,\ldots,0,([\sigma_n],\varepsilon_1+\cdots+\varepsilon_n),0,\ldots,0,\ldots),0).$$

Here $[\sigma]$ denotes the image of the permutation $\sigma \in \Gamma$ in $\Gamma/[\Gamma,\Gamma]$.

Proof. It follows from relations (R7), (R8), and (R9) that the action of θ on an element $h(i_1,\varepsilon_1,\ldots,i_n,\varepsilon_n;\sigma_n)$ is consistent with the definition of η and the multiplication in the wreath product, that is,

$$\begin{aligned}\eta(\theta(h(i_1,\varepsilon_1,\ldots,i_n,\varepsilon_n;\sigma_n))) &= \eta(\tau^{-1}h(i_1,\varepsilon_1,\ldots,i_n,\varepsilon_n;\sigma_n)\tau) \\ &= ((\ldots,0,\ldots,0,([\sigma_n],\varepsilon_1+\cdots+\varepsilon_n-1),0,\ldots,0,\ldots),0).\end{aligned}$$

It is easy to see that, since the commutant is in the kernel, the homomorphism $\eta : G \to (\Gamma/[\Gamma,\Gamma]) \wr_{\mathbb{Z}} \mathbb{Z}$ is well defined by

$$\eta(g) = ((\ldots,(\prod_{\varepsilon_1+\cdots+\varepsilon_n=m}[\sigma_n],m),\ldots),0),$$

where the products are taken over all the factors $h(i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)$ of g. These two observations together with the universal property of the HNN-extensions (Remark 1.1) enable us to extend η to the entire group Λ.

Now, notice that if $\lambda = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}g_3\tau^{\varepsilon_3}\cdots g_n\tau^{\varepsilon_n}g_{n+1}\in\Xi$, then $\varepsilon_1+\cdots+\varepsilon_n=0$. Thus

$$\lambda \;=\; g_1(\tau^{\varepsilon_1}g_2\tau^{-\varepsilon_1})(\tau^{\varepsilon_1+\varepsilon_2}g_3\tau^{-\varepsilon_1-\varepsilon_2})\cdots(\tau^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{n-1}}g_n\tau^{-\varepsilon_1-\varepsilon_2-\cdots-\varepsilon_{n-1}})g_{n+1}$$

can be represented as products of τ-conjugates of elements from G.

Using Lemma 2.9 (ii), we see that every $\lambda=\tau^n g\tau^{-n}$ can be written as a product of elements of the form $\tau^n h(\sigma)\tau^{-n}$ and $\tau^n h(i_1,\varepsilon_1\dots,i_m,\varepsilon_m;\sigma_m)\tau^{-n}$. The second element equals either $\tau^{n-m}h(\sigma_m)\tau^{m-n}$ or $h(j_1,\varepsilon'_1,\dots,j_k,\varepsilon'_k;\sigma_m)$ for some j_p's and ε'_p's. Therefore, it is easy to see that Ξ is generated by the following set

$$\begin{aligned}
&\{h(i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)h(i'_1,\varepsilon_1,\dots,i'_n,\varepsilon_n;\sigma_n^{-1})\mid \varepsilon_k=\pm1,\ i_k,i'_k\in I_{-\varepsilon_k},\ \forall k;\ n\ge 2,\ \sigma_n\in\Gamma_{\varepsilon_n}\}\\
&\ \cup\ \{i,\varepsilon,i_0,-\varepsilon,i_1,\varepsilon,i_2,\varepsilon_2,\dots,i_n,\varepsilon_n;\sigma_n)h(\bar{i},\varepsilon,i'_2,\varepsilon_2,\dots,i_n,\varepsilon_n;\sigma_n^{-1})\mid\\
&\qquad n\ge 2,,\ i_0\in I_\varepsilon,\ i'_2\in I_{-\varepsilon_2},\ i,\bar{i}\in I_{-\varepsilon};\ i_k\in I_{-\varepsilon_k},\ \varepsilon,\varepsilon_k=\pm1,\ \forall k\}\\
&\ \cup\ \{h(\sigma_\varepsilon)h(i_\varepsilon,-\varepsilon,i_{-\varepsilon},\varepsilon;\sigma_\varepsilon^{-1})\mid \sigma_\varepsilon\in\Gamma_\varepsilon,\ i_{-\varepsilon}\in I'_\varepsilon,\ i_\varepsilon\in I_{-\varepsilon},\ \varepsilon=\pm1\}\\
&\ \cup\ \{h(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m,i,\varepsilon,j,-\varepsilon,j_1,\varepsilon'_1,\dots,j_n,\varepsilon'_n;\sigma)\\
&\qquad\cdot h(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m,j',-\varepsilon,i',\varepsilon,j_1,\varepsilon'_1,\dots,j_n,\varepsilon'_n;\sigma^{-1})\mid\\
&\qquad m,n\in\mathbb{N}_0,\ i,i',\in I_{-\varepsilon},j,j'\in I_\varepsilon,\ \sigma\in\Gamma_{\varepsilon'_n};\ \varepsilon,\varepsilon_k,\varepsilon'_k=\pm1,i_k\in I_{-\varepsilon_k},\ j_k\in I_{-\varepsilon'_k},\ \forall k\}\\
&\ \cup\ \{\tau^{\varepsilon n}h(\sigma_{-\varepsilon})\tau^{-\varepsilon n}h(\underbrace{\iota_{-\varepsilon},\varepsilon,\dots,\iota_{-\varepsilon},\varepsilon}_{n\text{ times}};\sigma_{-\varepsilon}^{-1})\mid\sigma_{-\varepsilon}\in\Gamma_{-\varepsilon},\varepsilon=\pm1,\ n\in\mathbb{N}\}\\
&\ \cup\ \{\tau^{\varepsilon n}h(\sigma_{-\varepsilon})\tau^{-\varepsilon n}\mid n\in\mathbb{N},\ \sigma_{-\varepsilon}\in\Gamma_{-\varepsilon}\cap[\Gamma,\Gamma],\ \varepsilon=\pm1\}\ \cup\ \{h(\sigma)\mid\sigma\in[\Gamma,\Gamma]\}\,.
\end{aligned}\tag{8}$$

Take any element $a\in\Xi\backslash\{1\}$. It remains to show that $\langle\langle a\rangle\rangle_\Xi=\Xi$. Relations (R3), (R8), and (R9) and Lemma 2.9 (iii) imply that we can find a big enough n and i_k's so that the element $h(i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)$ does not commute with a and does not modify a. Moreover, if we take

$$v\;\equiv\;h(i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)h(i'_1,\varepsilon_1,\dots,i'_n,\varepsilon_n;\sigma_n^{-1})\;\in\;\Xi\setminus\{1\},$$

for any i'_k's (not all equal to i_k's), we will have

$$\begin{aligned}
\langle\langle a\rangle\rangle_\Xi\ni b&\equiv ava^{-1}v\\
&=h(p_1,l_1,\dots,p_m,l_m;\sigma_n)h(p'_1,l'_1,\dots,p'_d,l'_d;\sigma_n^{-1})\\
&\qquad\cdot h(i_1,\varepsilon_1,\dots,i_n,\varepsilon_n;\sigma_n)h(i'_1,\varepsilon_1,\dots,i'_n,\varepsilon_n;\sigma_n^{-1})
\end{aligned}$$

for some m, d, p_k's, p'_k's, l_k's, and l'_k's.
Now, it is clear that we can find big enough s and appropriate e_k's, e''_k's, j_k's, and j''_k's, so that $h(j''_1,e''_1,\dots,j''_s,e''_s;\sigma^{-1})$ commutes with b and $h(j_1,e_1,\dots,j_s,e_s;\sigma)$

does not. Then,

$$\begin{aligned}\langle\langle a\rangle\rangle_\Xi \ni b' \equiv\ & bh(j_1,e_1,\dots,j_s,e_s;\sigma)h(j_1'',e_1'',\dots,j_s'',e_s'';\sigma^{-1})b^{-1}\\ & h(j_1'',e_1'',\dots,j_s'',e_s'';\sigma)h(j_1,e_1,\dots,j_s,e_s;\sigma^{-1})\\ =\ & h(j_1',e_1,\dots,j_s',e_s;\sigma)h(j_1,e_1,\dots,j_s,e_s;\sigma^{-1})\neq 1\end{aligned}$$

for some j_k''s, from relation (R3). We can take s to be big enough and adjust the 'tail' of $(j_1,e_1,\dots,j_s,e_s)$ so that $e_1+\dots+e_n=0$. Since the tuples $(j_1,e_1,\dots,j_s,e_s)$ and $(j_1',e_1,\dots,j_s',e_s)$ are different, it follows from Lemma 2.9 (i) and from the assumption $\varepsilon_1+\dots+\varepsilon_n=0$ that

$$\beta b'\beta^{-1}=h(p_1'',e_1''',\dots,p_k'',e_k''',p'',e_s;\sigma)h(\sigma^{-1})\ \in\ \langle\langle a\rangle\rangle_\Xi$$

for some $k\in\mathbb{N}$, p_l'''s, and e_l''''s, where

$$\begin{aligned}&\Xi\ \ni\ \beta\ =\ \tau^{-e_s}(\lambda_{-e_s}^{j_s})^{-1}\cdots\tau^{-e_1}(\lambda_{-e_s}^{j_1})^{-1}\cdot\\ &\cdot\prod_{e_k=-1}h(\rho_1^k,w_1^k,\dots,\rho_{t_k}^k,w_{t_k}^k,w,1;\mu_{-e_k}^{j_k})\cdot\prod_{e_k=1}h(\bar\rho_1^k,\bar w_1^k,\dots,\bar\rho_{t_k'}^k,\bar w_{t_k'}^k,\bar w,-1;\mu_{-e_k}^{j_k}),\end{aligned}$$

and where the last two factors are chosen appropriately to bring β into Ξ. This argument does not depend on the 'tail' of $(p_1,e_1,\dots,p_s,e_s)$, therefore we can take e_s to be either 1 or -1.

We conclude that the following are elements of $\langle\langle a\rangle\rangle_\Xi$:

$$\begin{aligned}&c=h(\sigma_1)h(\iota_1,-1,p_1,e_1,\dots,p_k,e_k,p,1;\sigma_1^{-1})\text{ and}\\ &d=h(\sigma_{-1})h(\iota_{-1},1,q_1,l_1,\dots,q_k,l_k,q,-1;\sigma_{-1}^{-1})\end{aligned}$$

for any big enough even number k, for any $\sigma_1\in\Gamma_1$ and $\sigma_{-1}\in\Gamma_{-1}$, and for some p_m's, q_m's, e_m's, and l_m's.

We claim that, in the tuples $(\iota_1,-1,p_1,e_1,\dots,p_k,e_k,p,1)$ and $(\iota_{-1},1,q_1,l_1,\dots,q_k,l_k,q,-1)$, the indices p, q, p_t's, and q_t's can be chosen arbitrary. To see this, consider

$$\Xi\ \ni\ f\ =\ h(\iota_1,-1,p_1,e_1,\dots,p_t,e_t;\omega_t)h(q_0,-1,q_1,o_1,\dots,q_r,o_r,q,e_t;\omega_t^{-1}),$$

where $q_0\neq\iota_1$ and where the second factor is chosen appropriately. Then by relation (R3),

$$fcf^{-1}\ =\ h(\sigma_1)h(\iota_1,-1,p_1,e_1,\dots,\omega_t(p_{t+1}),\dots,p_k,e_k,p,1;\sigma_1^{-1})\ \in\ \langle\langle a\rangle\rangle_\Xi\,.$$

Because of the transitivity and 2-transitivity of Σ_{-1} and Σ_1, the claim is proven. The element d can be manipulated similarly.

Now, consider

$$\Xi\ \ni\ s\ =\ h(\iota_{-1},1,i_2,\varepsilon_2,\dots,i_t,\varepsilon_t;\omega_t)h(\iota_1,-1,q_1',o_1',\dots,q_r',o_r',q',e_t;\omega_t^{-1})$$

for an appropriate choice of q_l''s and p_l's so it commutes with $h(\iota_1, -1, p_1, e_1, \dots, p_k, e_k, p, 1; \sigma_1^{-1})$. Therefore

$$scs^{-1}c^{-1} = h(\iota_{-1}, 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t)h(\sigma_1(\iota_{-1}), 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t^{-1}) \in \langle\langle a\rangle\rangle_\Xi,$$

so by the transitivity of the group Σ_{-1}, we see that every element of the form

$$h(\iota_{-1}, 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t)h(i_1, 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t^{-1})$$

belongs to $\langle\langle a\rangle\rangle_\Xi$. Products of such elements yield

$$h(i_1', 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t)h(i_1, 1, i_2, \varepsilon_2, \dots, i_t, \varepsilon_t; \omega_t^{-1}) \ \in \ \langle\langle a\rangle\rangle_\Xi$$

for any $i_1, i_1' \in I_{-1}$. By making the same argument that uses transitivity and 2-transitivity, we see that we can change the i_l indices of the first factor, so we infer that the first set of (8) belongs to $\langle\langle a\rangle\rangle_\Xi$.

Consider an integer $n \geq 2$, an even number $k \geq 2$, and an appropriate $h(j_1, \varepsilon_1', \dots, j_k, \varepsilon_k'; \sigma)$ that commutes with $h(i_1, \varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n)$ and with $h(\iota_{-\varepsilon_1}, \varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n^{-1})$ and has the property that

$$\delta' \equiv \tau^{\varepsilon_1}h(\sigma)\tau^{-\varepsilon_1}h(j_1, \varepsilon_1', \dots, j_k, \varepsilon_k'; \sigma^{-1})$$

belongs to Ξ. Then

$$\begin{aligned}\delta' h(i_1,&\varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n)h(\iota_{-\varepsilon_1}, \varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n^{-1})(\delta')^{-1}\\ =&h(\iota_{-\varepsilon_1}, \varepsilon_1, \sigma(\iota_{\varepsilon_1}), -\varepsilon_1, i_1, \varepsilon_1, i_2, \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n)\\ &\quad h(\iota_{-\varepsilon_1}, \varepsilon_1, \sigma(i_2), \varepsilon_2, \dots, i_n, \varepsilon_n; \sigma_n^{-1}) \in \langle\langle a\rangle\rangle_\Xi\,.\end{aligned}$$

Products of those elements with elements from the first set give all the elements from the second set of (8), so it is included in $\langle\langle a\rangle\rangle_\Xi$.

The third set of (8) belongs to $\langle\langle a\rangle\rangle_\Xi$ since its elements are products of the elements c and d above with elements from the second set.

A generic element of the fourth set of (8) can be written as

$$\begin{aligned}h(i_1, \varepsilon_1, \dots, i_m, \varepsilon_m, i, \varepsilon, j, -\varepsilon, \bar{i}, \varepsilon, j_2, \varepsilon_2', \dots, j_n, \varepsilon_n'; \sigma)\cdot&\\ h(i_1, \varepsilon_1, \dots, i_m, \varepsilon_m, j', -\varepsilon, i', \varepsilon, \bar{i}, \varepsilon, j_2, \varepsilon_2', \dots, j_n, \varepsilon_n'; \sigma^{-1}),&\quad (9)\end{aligned}$$

where we have written $\varepsilon_1' = \varepsilon$. We must show that this element belongs to $\langle\langle a\rangle\rangle_\Xi$.

First, we start with the following element from the first set of (8)

$$\begin{aligned}\langle\langle a\rangle\rangle_\Xi \ \ni \ z \ = \ &h(i_1, \varepsilon_1, \dots, i_m, \varepsilon_m, i, \varepsilon, \iota_{-\varepsilon}, \varepsilon, q, -\varepsilon, j, -\varepsilon, \bar{i}, \varepsilon, j_2, \varepsilon_2', \dots, j_n, \varepsilon_n'; \sigma)\cdot\\ &h(i_1, \varepsilon_1, \dots, i_m, \varepsilon_m, i, \varepsilon, \iota_{-\varepsilon}, \varepsilon, q, -\varepsilon, \iota_\varepsilon, -\varepsilon, \bar{i}, \varepsilon, j_2, \varepsilon_2', \dots, j_n, \varepsilon_n'; \sigma^{-1}),\end{aligned}$$

where $q \in I_\varepsilon'$.

Next, using Lemma 2.9 (i) and adopting the notations thereof, we define

$$\begin{aligned}\Xi \ \ni \ \gamma = {} & \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{i}_{-\varepsilon}\tau^{2\varepsilon}(\lambda^{q}_{\varepsilon})^{-1}\tau^{-2\varepsilon}(\lambda^{i}_{-\varepsilon})^{-1}\\ & \cdot\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\cdot h(r_1,e_1,\ldots,r_{2l-1},e_{2l-1},\bar r_{-\varepsilon},\varepsilon;\mu^{q}_{\varepsilon})\end{aligned}$$

for appropriate r_k's and e_k's satisfying $e_1+\cdots+e_{2l-1}+\varepsilon=0$ and for which the last factor commutes with everything in the next expressions. Then

$$\gamma z\gamma^{-1} \ = \ h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,i,\varepsilon,j,-\varepsilon,\bar i,\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)\cdot\bar h,$$

where

$$\begin{aligned}\bar h &\equiv \gamma h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,i,\varepsilon,\iota_{-\varepsilon},\varepsilon,q,-\varepsilon,\iota_{\varepsilon},-\varepsilon,\bar i,\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1})\gamma^{-1}\\ &=\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{i}_{-\varepsilon}\lambda^{\bar i}_{-\varepsilon}\tau^{\varepsilon}\cdots\lambda^{j_n}_{-\varepsilon'_n}\tau^{\varepsilon'_n}h(\sigma^{-1})\\ &\quad\cdot\tau^{-\varepsilon'_n}(\lambda^{j_n}_{-\varepsilon'_n})^{-1}\cdots\tau^{-\varepsilon}(\lambda^{\bar i}_{-\varepsilon})^{-1}(\lambda^{i}_{-\varepsilon})^{-1}\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\\ &=\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{i}_{-\varepsilon}h(\bar i,\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1})(\lambda^{i}_{-\varepsilon})^{-1}\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\\ &=\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1})\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}.\end{aligned}$$

Likewise, we consider the following element from the first set of (8)

$$\begin{aligned}\langle\langle a\rangle\rangle_{\Xi}\ni z' = {} & h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j',-\varepsilon,\iota_{\varepsilon},-\varepsilon,p,\varepsilon,\iota_{-\varepsilon},\varepsilon,\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)\\ & \cdot h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j',-\varepsilon,\iota_{\varepsilon},-\varepsilon,p,\varepsilon,i',\varepsilon,\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1}),\end{aligned}$$

where $p\in I'_{-\varepsilon}$ and define

$$\begin{aligned}\Xi\ni\gamma' = {} & \lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{j'}_{\varepsilon}\tau^{-2\varepsilon}(\lambda^{p}_{-\varepsilon})^{-1}\tau^{2\varepsilon}(\lambda^{j'}_{\varepsilon})^{-1}\\ & \cdot\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\cdot\cdot h(r'_1,e_1,\ldots,r'_{2l-1},e_{2l-1},\bar r_{-\varepsilon},\varepsilon;\mu^{p}_{-\varepsilon})\end{aligned}$$

for appropriate r'_k's. Then,

$$\gamma' z'(\gamma')^{-1} \ = \ \bar{\bar h}\cdot h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j',-\varepsilon,i',\varepsilon,\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1}),$$

where

$$\begin{aligned}\bar{\bar h} &\equiv\gamma' h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j',-\varepsilon,\iota_{\varepsilon},-\varepsilon,p,\varepsilon,\iota_{-\varepsilon},\varepsilon,\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)(\gamma')^{-1}\\ &=\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}\lambda^{j'}_{\varepsilon}h(\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)(\lambda^{j'}_{\varepsilon})^{-1}\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\\ &=\lambda^{i_1}_{-\varepsilon_1}\tau^{\varepsilon_1}\cdots\lambda^{i_m}_{-\varepsilon_m}\tau^{\varepsilon_m}h(\mu^{j'}_{\varepsilon}(\mu^{i}_{-\varepsilon}(\bar i)),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)\tau^{-\varepsilon_m}(\lambda^{i_m}_{-\varepsilon_m})^{-1}\cdots\tau^{-\varepsilon_1}(\lambda^{i_1}_{-\varepsilon_1})^{-1}\\ &=(\bar h)^{-1},\end{aligned}$$

since $\mu^{j'}_{\varepsilon}(\mu^{i}_{-\varepsilon}(\bar i))=\mu^{i}_{-\varepsilon}(\bar i)$, due to relation (R6) and $\mu^{i}_{-\varepsilon}(\bar i)\in I_{-\varepsilon}$. Finally,

$$\begin{aligned}\langle\langle a\rangle\rangle_{\Xi}\ni\gamma z\gamma^{-1}\cdot\gamma' z'(\gamma')^{-1} = {} & h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,i,\varepsilon,j,-\varepsilon,\bar i,\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma)\\ & \cdot h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j',-\varepsilon,i',\varepsilon,\mu^{i}_{-\varepsilon}(\bar i),\varepsilon,j_2,\varepsilon'_2,\ldots,j_n,\varepsilon'_n;\sigma^{-1}),\end{aligned}$$

and after a multiplication with an element from the first set of (8), we get the element (9).

Therefore the fourth set of (8) is in $\langle\langle a\rangle\rangle_\Xi$.

Repeating almost verbatim the corresponding part of the proof of Theorem [8, Theorem 3.16] gives us that the seventh set of (8) belongs to $\langle\langle a\rangle\rangle_\Xi$. Note that if $\Sigma_\varepsilon = \mathrm{Sym}(2)$, then $[\Sigma_\varepsilon, \Sigma_\varepsilon]$ is the trivial group.

Next, we take numbers $m > n$ and

$$\gamma'' \;=\; \tau^{\varepsilon m} h(\sigma'_{-\varepsilon})\tau^{-\varepsilon m} h(j_1, \varepsilon, \ldots, j_{m+1}, \varepsilon, j, -\varepsilon; (\sigma'_{-\varepsilon})^{-1}) \;\in\; \Xi,$$

where $\sigma'_{-\varepsilon} \in \Gamma_{-\varepsilon}$, $j_k \in I'_{-\varepsilon}$, $\forall k$, and $j \in I'_\varepsilon$, with the relation $(\sigma'_{-\varepsilon})^{-1}(\iota_\varepsilon) = q$ for some $q \in I'_\varepsilon$.
After that, we take the following element of $\langle\langle a\rangle\rangle_\Xi$ (it is a product of elements from the second and fourth set)

$$\begin{aligned} x \;\equiv\; & h(\underbrace{\iota_{-\varepsilon}, \varepsilon, \ldots, \iota_{-\varepsilon}, \varepsilon}_{m \text{ times}}, q, -\varepsilon, \underbrace{\iota_\varepsilon, -\varepsilon, \ldots, \iota_\varepsilon, -\varepsilon}_{m-n-1 \text{ times}}; \sigma_{-\varepsilon})\cdot \\ & \cdot h(\underbrace{\iota_{-\varepsilon}, \varepsilon, \ldots, \iota_{-\varepsilon}, \varepsilon}_{m \text{ times}}, q, -\varepsilon, \underbrace{\iota_\varepsilon, -\varepsilon, \ldots, \iota_\varepsilon, -\varepsilon}_{m \text{ times}}, p, \varepsilon, \underbrace{\iota_{-\varepsilon}, \varepsilon, \ldots, \iota_{-\varepsilon}, \varepsilon}_{n-1 \text{ times}}; \sigma_{-\varepsilon}), \end{aligned}$$

where $p \in I'_{-\varepsilon}$. Then

$$\gamma'' x (\gamma'')^{-1} \;= \tau^{\varepsilon n} h(\sigma_{-\varepsilon})\tau^{-\varepsilon n} \cdot h(p, \varepsilon, \underbrace{\iota_{-\varepsilon}, \varepsilon, \ldots, \iota_{-\varepsilon}, \varepsilon}_{n-1 \text{ times}}; \sigma_{-\varepsilon}) \;\in\; \langle\langle a\rangle\rangle_\Xi.$$

Therefore upon a multiplication by an element from the first set of (8), we infer that the fifth set of (8) belongs to $\langle\langle a\rangle\rangle_\Xi$.

Finally, the argument from Theorem [8, Theorem 3.16] can be used for the sixth set of (8) the same way it was used for the seventh set.

This completes the proof. $\square$

Remark 2.14. *The example introduced in [3, Section 5] corresponds to the case $\Sigma_{-1} \cong \Sigma_1 \cong Sym(2)$. Theorem 2.13 corresponds to [3, Proposition 5.11] in this particular case.*

2.5. ANALYTIC STRUCTURE

In this section, we use some results from [8, Section 2].

Lemma 2.15. *The group $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$ is a non-ascending HNN-extension and its action on its Bass-Serre tree is minimal and of general type.*

Proof. Since the action is transitive, it is minimal. Since $H \neq G \neq \theta(H)$, then Λ is nondegenerate and non-ascending. The result now follows from [7, Proposition 20]. $\square$

Theorem 2.16. *The HNN-extension* $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$ *has a unique trace. It is* C^**-simple if and only if either one of the groups* Σ_{-1} *and* Σ_1 *is non-amenable.*

Proof. Lemma 2.15 enables us to apply [3, Theorem 4.19] to conclude that Λ has the unique trace property since $\ker \Lambda$ is trivial. It also enables us to apply [3, Theorem 4.20] to conclude that Λ is C^*-simple if and only if K_{-1} and K_1 are non-amenable, which, by Lemma 2.12, is equivalent to the requirement that some of the groups Σ_{-1} and Σ_1 is non-amenable. □

Finally, we prove

Theorem 2.17. *The HNN-extension* $\Lambda = \Lambda[\Sigma_{-1}, \Sigma_1]$ *in not inner amenable.*

Proof. Lemma 2.15 allows us to apply [8, Proposition 2.3], so we need to show that the action of $\Lambda = \Lambda[I_{-1}, I_1, \iota_{-1}, \iota_1; \Sigma_{-1}, \Sigma_1]$ on its Bass-Serre is finitely fledged.

For this, take any elliptic element $g \in \Lambda \setminus \{1\}$. Since g fixes some vertex, it is a conjugate of an element of G. The finite fledgedness property is conjugation invariant, so we can assume $g \in G \setminus \{1\}$.

From Lemma 2.9 (ii), we can write $g = h(\sigma)h_{-1}h_1$, where $\sigma \in \Gamma$,

$$h_{-1} = \prod_{k=1}^{m} h(i_1^k, -1, i_2^k, \varepsilon_{k,2}, \dots, i_{n_k}^k, \varepsilon_{k,n_k}; \sigma_k)\,,$$
$$h_1 = \prod_{l=m+1}^{r} h(i_1^l, 1, i_2^l, \varepsilon_{l,2}, \dots, i_{n_l}^l, \varepsilon_{l,n_l}; \theta_l)\,,$$

$r \geq m \geq 0$, $\sigma_k \in \Gamma_{\varepsilon_{k,n_k}}$, $\theta_l \in \Gamma_{\varepsilon_{l,n_l}}$, and $i_z^p \in I'_{\varepsilon_{p,z}}$. We also require $0 \leq n_1 \leq \ldots \leq n_m$ and $0 \leq n_{m+1} \leq \cdots \leq n_r$.

Let us assume that g fixes a vertex $v = v(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n)$, where $n \geq \max\{n_m, n_r\} + 1$, and let's take $w = v(i_1, \varepsilon_1, \dots, i_n, \varepsilon_n, \dots, i_{n+d}, \varepsilon_{n+d})$ for any $d \geq 1$. We note that $h_{-\varepsilon_1}$ fixes w and $h(\sigma)h_{\varepsilon_1}$ modifies only indices with numbers no greater than $\{n_m, n_r\} + 1 \leq n$. Therefore

$$\begin{aligned}&h(\sigma)h_{\varepsilon_1}v = v(i'_1, \varepsilon_1, \dots, i'_n, \varepsilon_n) \text{ and}\\ &h(\sigma)h_{\varepsilon_1}w = v(i'_1, \varepsilon_1, \dots, i'_n, \varepsilon_n, i_{n+1}, \varepsilon_{n+1}, \dots, i_{n+d}, \varepsilon_{n+d})\end{aligned}$$

for some $i'_k \in I'_{-\varepsilon_k}$. By our assumption, it follows that

$$v \;=\; gv \;=\; h(\sigma)h_{\varepsilon_1}v \;=\; v(i'_1, \varepsilon_1, \dots, i'_n, \varepsilon_n).$$

Thus $i'_k = i_k$ for all $1 \leq k \leq n$, and therefore $gw = w$.

This concludes the proof. □

Corollary 2.18. *Theorems 2.16 and 2.13 imply:*
If either Σ_{-1} or Σ_1 is non-amenable, then the amenablish radical of Λ is trivial. If Σ_{-1} and Σ_1 are both amenable, then Λ is amenablish.

Proof. If we show that the centralizer $C_\Lambda(\Xi)$ is trivial, [2, Theorem 4.1] will imply that Λ is C^*-simple if and only if Ξ is C^*-simple. Since Ξ is simple, if it is not C^*-simple, then it is amenablish, and therefore Λ is also amenablish because $(\Gamma/[\Gamma,\Gamma]) \wr_{\mathbb{Z}} \mathbb{Z}$ is amenable. If Ξ is C^*-simple, then so is Λ, thus both of their amenablish radicals are trivial.

To illustrate that $C_\Lambda(\Xi)$ is trivial, assume that there is a nontrivial $g \in C_\Lambda(\Xi)$. Then g can be written as in Lemma 2.9 (iii), and using relations (R3), (R7), and (R8), we can find a non-trivial element of Ξ

$$h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j_1,\varepsilon'_1,\ldots,j_n,\varepsilon'_n;\sigma)\cdot h(i_1,\varepsilon_1,\ldots,i_m,\varepsilon_m,j'_1,\varepsilon''_1,\ldots,j'_n,\varepsilon''_n;\sigma^{-1})$$

that does not commute with g, a contradiction. □

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